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TWO-SIDED ESTIMATES OF THE LEBESGUE CONSTANTS WITH RESPECT TO VILENKIN SYSTEMS AND APPLICATIONS

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Abstract. In this paper, we derive two-sided estimates of the Lebesgue constants for bounded Vilenkin systems, we also present some applications of importance, e.g., we obtain a characterization for the boundedness of a subsequence of partial sums with respect to Vilenkin–Fourier series of H_1 martingales in terms of n's variation. The conditions given in this paper are in a sense necessary and sufficient.

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1. Introduction. It is known that for every Vilenkin systems

$$L_n := \|D_n\|_1 \leq c \log n$$

holds. For the definitions of D_n, the Vilenkin systems and other objects in this section (e.g., $v(n)$ and $v^*(n)$), we refer to our Section 2.

For some concrete systems, it is possible to write two-sided estimations of Lebesgue constants L_{n_k}. In particular, for every bounded Vilenkin systems, Lukyanenko [**4**] proved two-sided estimates for the Lebesgue constants L_{n_k} for some concrete indices $n_k \in \mathbb{N}$. Lukomskii [**3**] generalised this result and proved two-sided estimates for the Lebesgue constants L_n without the conditions on the indexes. He showed that for $n = \sum_{j=0}^{\infty} n_j M_j$ and every bounded Vilenkin systems, we have the following two-sided estimates of Lebesgue constants:

$$\frac{1}{4\lambda} v(n) + \frac{1}{\lambda} v^*(n) + \frac{1}{2\lambda} \leq L_n \leq \frac{3}{2} v(n) + 4v^*(n) - 1. \quad (1)$$

It is well known that (see, e.g., [**1**] and [**2**]) Vilenkin systems do not form bases in the space L_1. Moreover, there exists a function in the dyadic Hardy space H_1, such that the partial sums of f are not bounded in L_1-norm. Onneweer [**6**] showed that if the modulus of continuity of $f \in L_1[0, 1)$ satisfies the condition

$$\omega_1(\delta, f) = o\left(\frac{1}{\log(1/\delta)}\right), \quad \text{as } \delta \to 0, \tag{2}$$

then its Vilenkin–Fourier series converges in L_1-norm. He also proved that condition (2) cannot be improved.

In [**8**] (see also [**9**]), it was proved that if $f \in H_1$ and

$$\omega_{H_1}\left(\frac{1}{M_n}, f\right) = o\left(\frac{1}{n}\right), \text{ as } n \to \infty, \tag{3}$$

then $S_k f$ converge to f in L_1-norm. Moreover, there was showed that condition (3) cannot be improved.

It is also known that any subsequence S_{n_k} is bounded from L_1 to L_1 if and only if n_k has uniformly bounded variation and as a corollary the subsequence S_{2^n} of partial sums is bounded from Hardy space H_p to the Hardy space H_p, for all $p > 0$.

In this paper, we improve the upper bound in (1) and also prove a new similar lower bound by using a completely different new method. By applying this results, we also find the characterizations of boundedness (or even the ratio of divergence of the norm) of the subsequence of partial sums of the Vilenkin–Fourier series of H_1 martingales in terms of n's variation. We also derive a relationship of the ratio of convergence of the partial sum of the Vilenkin series with the modulus of continuity of a martingale. The conditions given in the paper are in a sense necessary and sufficient.

Our main results (Theorem 1) is presented and proved in Section 3. The mentioned applications especially Theorems 2 and 3 can be found in Section 4. Section 2 is reserved for necessary definitions, notations and some Lemmas (Lemmas 2 and 3 are new).

2. Preliminaries. Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Let $m := (m_0, m_1, \dots)$ denote a sequence of the positive numbers not less than 2. Denote by

$$Z_{m_k} := \{0, 1, \dots, m_k - 1\}$$

the additive group of integers modulo m_k, $k \in \mathbb{N}$.

Define the group G_m as the complete direct product of the group Z_{m_k} with the product of the discrete topologies of Z_{m_k}'s.

The direct product μ of the measures

$$\mu_k(\{j\}) := 1/m_k, (j \in Z_{m_k})$$

is the Haar measure on G_m, with $\mu(G_m) = 1$.

In this paper, we discuss bounded Vilenkin groups only, that is

$$\sup_{n\in\mathbb{N}} m_n < \infty.$$

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The elements of G_m are represented by sequences

$$x := (x_0, x_1, \dots, x_k, \dots),\ (x_k \in Z_{m_k}).$$

It is easy to give a base for the neighbourhood of G_m :

$$I_0(x) := G_m,$$

$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \dots, y_{n-1} = x_{n-1}\}, (x \in G_m, n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$, for $n \in \mathbb{N}$ and $\overline{I_n} := G_m \backslash I_n$.

The norm (or quasi-norm) of the spaces $L_p(G_m)$ is defined by

$$\|f\|_p := \left(\int_{G_m} |f|^p\, d\mu\right)^{1/p} \quad (0 < p < \infty).$$

If we define the so-called generalised number system based on m in the following way:

$$M_0 := 1,\ M_{k+1} := m_k M_k \qquad (k \in \mathbb{N}),$$

then every $n \in \mathbb{N}$ can be uniquely expressed as $n = \sum_{k=0}^{\infty} n_k M_k$, where $n_k \in Z_{m_k}$ $(k \in \mathbb{N})$ and only a finite number of n_k's differ from zero. Let $|n| := \max\{k \in \mathbb{N} :\ n_k \neq 0\}$.

For the natural number $n = \sum_{j=0}^{\infty} n_j M_j$, we define

$$\delta_j := \operatorname{sign}(n_j) = \operatorname{sign}\left(\ominus n_j\right), \quad \delta_j^* := \left|\ominus n_j - 1\right| \delta_j,$$

where $\ominus$ is the inverse operation for

$$a_k \oplus b_k = (a_k + b_k) \mod m_k.$$

We define functions v and v^* by

$$v(n) := \sum_{j=0}^{\infty} \left|\delta_{j+1} - \delta_j\right| + \delta_0,\ v^*(n) := \sum_{j=0}^{\infty} \delta_j^*,$$

Next, we introduce on G_m an orthonormal system, which is called the Vilenkin system. At first define the complex-valued functions $r_k(x) : G_m \to \mathbb{C}$, the generalised Rademacher functions, by

$$r_k(x) := \exp(2\pi \imath x_k / m_k),\ \left(\imath^2 = -1,\ x \in G_m, k \in \mathbb{N}\right).$$

Let $x \in G_m$. It is well known that

$$\sum_{k=0}^{m_n - 1} r_n^k(x) = \begin{cases} 0 & x_n \neq 0, \\ m_n & x_n = 0. \end{cases} \tag{4}$$

Now, define the Vilenkin systems $\psi := (\psi_n : n \in \mathbb{N})$ on G_m as

$$\psi_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x),\ (n \in \mathbb{N}).$$

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Specifically, we call this system the Walsh-Paley one if $m \equiv 2$.

The Vilenkin systems are orthonormal and complete in $L_2(G_m)$ (see, e.g., [**1, 10**]).

Next, we introduce analogues of the usual definitions in Fourier analysis. If $f \in L_1(G_m)$, we can establish the Fourier coefficients, the partial sums, the Dirichlet kernels, with respect to Vilenkin systems in the usual manner:

$$\widehat{f}(n) := \int_{G_m} f \overline{\psi}_n d\mu, \ (k \in \mathbb{N}),$$

$$S_n f := \sum_{k=0}^{n-1} \widehat{f}(k) \psi_k, \quad (k \in \mathbb{N}),$$

and

$$D_n := \sum_{k=0}^{n-1} \psi_k, \qquad (k \in \mathbb{N}).$$

Let $n \in \mathbb{N}$. Then,

$$D_{M_n}(x) = \prod_{k=0}^{n-1} \left(\sum_{s=0}^{m_k-1} r_k^s(x) \right) \tag{5}$$

$$= \begin{cases} M_n & x \in I_n, \\ 0 & x \notin I_n, \end{cases}$$

and

$$D_n = \psi_n \left(\sum_{j=0}^{\infty} D_{M_j} \sum_{u=m_j-n_j}^{m_j-1} r_j^u \right). \tag{6}$$

The σ-algebra generated by the intervals $\{I_n(x) : x \in G_m\}$ is denoted by $\digamma_n \ (n \in \mathbb{N})$. Let $f := \left(f^{(n)}, n \in \mathbb{N}\right)$ be a martingale with respect to $\digamma_n \ (n \in \mathbb{N})$. (for details see, e.g., [**12**]).

The maximal function of a martingale f is defined by

$$f^* := \sup_{n \in \mathbb{N}} \left| f^{(n)} \right|.$$

In the case $f \in L_1(G_m)$ the maximal functions are also be given by

$$f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{|I_n(x)|} \left| \int_{I_n(x)} f(u) \mu(u) \right|.$$

For $0 < p < \infty$, the Hardy martingale spaces H_p consist of all martingales for which

$$\|f\|_{H_p} := \|f^*\|_p < \infty.$$

The martingale $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is said to be L_p-bounded ($0 < p \leq \infty$) if $f^{(n)} \in L_p$ and

$$\|f\|_p := \sup_{n\in\mathbb{N}} \left\|f^{(n)}\right\|_p < \infty.$$

If $f \in L_1(G_m)$, then it is easy to show that the sequence $F = (S_{M_n}f : n \in \mathbb{N})$ is a martingale. This type of martingales is called regular. If $1 \leq p \leq \infty$ and $f \in L_p(G_m)$, then $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is L_p-bounded and

$$\lim_{n\to\infty} \left\|S_{M_n}f - f\right\|_p = 0,$$

consequently $\|F\|_p = \|f\|_p$, (see **[5]**). The converse of the latest statement holds also if $1 < p \leq \infty$ (see **[5]**): For an arbitrary L_p-bounded martingale $f = \left(f^{(n)}, n \in \mathbb{N}\right)$, there exists a function $f \in L_p(G_m)$ for which $f^{(n)} = S_{M_n}f$. If $p = 1$, then there exists a function $f \in L_1(G_m)$ of the preceding type if and only if f is uniformly integrable (see **[5]**) namely if

$$\lim_{y\to\infty} \sup_{n\in\mathbb{N}} \int_{\{|f^{(n)}|>y\}} \left|f^{(n)}(x)\right| d\mu(x) = 0.$$

Thus, the map $f \to f := (S_{M_n}f : n \in \mathbb{N})$ is isometric from L_p onto the space of L_p-bounded martingales when $1 < p \leq \infty$. Consequently, these two spaces can be identified with each other. Similarly, the space $L_1(G_m)$ can be identified with the space of uniformly integrable martingales.

A bounded measurable function a is a p-atom if there exists an interval I such that

$$\int_I a d\mu = 0, \ \ \|a\|_\infty \leq \mu(I)^{-1/p}, \ \ \operatorname{supp}(a) \subset I.$$

If $f = \left(f^{(n)}, n \in \mathbb{N}\right)$ is a martingale, then the Vilenkin–Fourier coefficients must be defined in a slightly different manner:

$$\widehat{f}(i) := \lim_{k\to\infty} \int_{G_m} f^{(k)} \overline{\psi}_i d\mu.$$

The best approximation of $f \in L_p(G_m)$ $(1 \leq p \in \infty)$ is defined as

$$E_n\left(f, L_p\right) := \inf_{P\in P_n} \|f - P\|_p,$$

where P_n is the set of all Vilenkin polynomials of order less than $n \in \mathbb{N}$.

The integrated modulus of continuity of $f \in L_p$ is defined by

$$\omega_p\left(\frac{1}{M_n}, f\right) := \sup_{h\in I_n} \|f(\cdot + h) - f(\cdot)\|_p.$$

The concept of modulus of continuity in H_p $(0 < p \leq 1)$ can be defined in the following way:

$$\omega_{H_p}\left(\frac{1}{M_n}, f\right) := \left\|f - S_{M_n}f\right\|_{H_p}.$$

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Watari [**11**] showed that there are strong connections between

$$\omega_p\left(\frac{1}{M_n},f\right),\ E_{M_n}\left(f,L_p\right)$$

and

$$\left\|f-S_{M_n}f\right\|_p,\ \ p\geq 1,\ n\in\mathbb{N}.$$

In particular,

$$\frac{1}{2}\omega_p\left(\frac{1}{M_n},f\right)\leq\left\|f-S_{M_n}f\right\|_p\leq\omega_p\left(\frac{1}{M_n},f\right) \tag{7}$$

and

$$\frac{1}{2}\left\|f-S_{M_n}f\right\|_p\leq E_{M_n}\left(f,L_p\right)\leq\left\|f-S_{M_n}f\right\|_p.$$

The Hardy martingale spaces $H_p\left(G_m\right)$ for $0<p\leq 1$ have atomic characterizations (see [**12, 13**]):

LEMMA 1. *A martingale* $f=\left(f^{(n)},n\in\mathbb{N}\right)\in H_p\,(0<p\leq 1)$ *if and only if there exist a sequence* $(a_k,k\in\mathbb{N})$ *of p-atoms and a sequence* $(\mu_k,k\in\mathbb{N})$ *of real numbers such that, for every* $n\in\mathbb{N}$,

$$\sum_{k=0}^{\infty}\mu_k S_{M_n}a_k=f^{(n)},\quad a.e. \tag{8}$$

$$\sum_{k=0}^{\infty}\left|\mu_k\right|^p<\infty.$$

Moreover,

$$\|f\|_{H_p}\backsim\inf\left(\sum_{k=0}^{\infty}\left|\mu_k\right|^p\right)^{1/p},$$

where the infimum is taken over all decomposition of f of the form (8).

For the proof of main result, we also need the following new Lemmas of independent interest:

LEMMA 2. *Let* $k,s\in\mathbb{N}$ *and* $x\in G_m$. *Then,*

$$\sum_{u=1}^{s_k-1}r_k^u(x)=\frac{\cos\left(\pi s_k x_k/m_k\right)\sin\left(\pi\left(s_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}\imath+\frac{\sin\left(\pi s_k x_k/m_k\right)\sin\left(\pi\left(s_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}.$$

Proof. Since

$$\sum_{u=1}^{s_k-1} r_k^u(x) = \sum_{u=1}^{s_k-1} \cos\left(\frac{2\pi u x_k}{m_k}\right) + \sum_{u=1}^{s_k-1} \imath \sin\left(\frac{2\pi u x_k}{m_k}\right),$$

if we apply the following well-known identities

$$\sum_{k=1}^{n} \cos kx = \frac{\sin\frac{nx}{2}\cos\frac{(n+1)x}{2}}{\sin\frac{x}{2}} \tag{9}$$

and

$$\sum_{k=1}^{n} \sin kx = \frac{\sin\frac{nx}{2}\sin\frac{(n+1)x}{2}}{\sin\frac{x}{2}}. \tag{10}$$

We immediately get the proof. □

LEMMA 3. *Let k, $\mathbb{N}$, $2 \leq s_k \leq m_k$ and $x_k = 1$. Then,*

$$\left|\sum_{n=1}^{s_k-1} r_k^n(x)\right| = \frac{\sin(\pi(s_k-1)x_k/m_k)}{\sin(\pi x_k/m_k)} \geq 1.$$

Proof. Since

$$\frac{\sin(\pi(m_k-1)/m_k)}{\sin(\pi/m_k)} = \frac{\sin(\pi/m_k)}{\sin(\pi/m_k)} = 1,$$

if we take graph of $\sin x$ into account, we obtain that

$$\frac{\sin(\pi(s_k-1)/m_k)}{\sin(\pi/m_k)} \geq 1, \text{ for } 2 \leq s_k \leq m_k.$$

Let $x_k = 1$. By using Lemma 2, we get that

$$\begin{aligned}\left|\sum_{u=1}^{s_k-1} r_k^u(x)\right| &= \left(\frac{\cos^2(\pi s_k x_k/m_k)\sin^2(\pi(s_k-1)x_k/m_k)}{\sin^2(\pi x_k/m_k)}\right.\\ &\quad\left.+\frac{\sin^2(\pi s_k x_k/m_k)\sin^2(\pi(s_k-1)x_k/m_k)}{\sin^2(\pi x_k/m_k)}\right)^{1/2}\\ &= \frac{\sin(\pi(s_k-1)x_k/m_k)}{\sin(\pi x_k/m_k)} = \frac{\sin(\pi(s_k-1)/m_k)}{\sin(\pi/m_k)} \geq 1. \end{aligned} \tag{11}$$

The proof is complete. □

3. The main result. Our main result reads:

THEOREM 1. *Let $n = \sum_{j=0}^{\infty} n_j M_j$. Then,*

$$\frac{1}{4\lambda} v(n) + \frac{1}{\lambda^2} v^*(n) \leq L_n \leq v(n) + v^*(n), \tag{12}$$

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where $\lambda := \sup_{n\in\mathbb{N}} m_n$.

Proof. First, we choose indices $0 \le \ell_1 \le \alpha_1 < \ell_2 \le \alpha_2 < \cdots < \ell_s \le \alpha_s < \ell_{s+1} = \infty$, such that $\alpha_j + 1 < \ell_{j+1}$, for $j = 1, 2, \ldots, s$, $n_k = 0$, for $0 < k < \ell_1$, $n_k \in \{1, 2, \ldots, m_k - 1\}$, for $\ell_j \le k \le \alpha_j$ and $n_k = 0$, for $\alpha_j < k < \ell_{j+1}$. According to (6), we have that

$$D_n = \psi_n \left(\sum_{k=0}^{\infty} D_{M_k} \sum_{u=1}^{m_k-1} r_k^u \right) - \psi_n \left(\sum_{k=0}^{\infty} D_{M_k} \sum_{u=1}^{m_k-n_k-1} r_k^u \right) \tag{13}$$

$$= \psi_n \left(\sum_{j=1}^{s} \sum_{k=\ell_j}^{\alpha_j} D_{M_k} \sum_{u=1}^{m_k-1} r_k^u \right) - \psi_n \left(\sum_{j=1}^{s} \sum_{k=\ell_j}^{\alpha_j} D_{M_k} \sum_{u=1}^{\ominus n_k-1} r_k^u \right)$$

$$:= I - II.$$

Since

$$M_k - 1 = \sum_{j=0}^{k-1} (m_j - 1) M_j, \tag{14}$$

if we apply again (6), we get that

$$D_{M_k-1} = \psi_{M_k-1} \left(\sum_{j=0}^{k-1} D_{M_j} \sum_{u=1}^{m_j-1} r_j^u \right).$$

Hence,

$$I = \psi_n \left(\sum_{j=1}^{s} \left(\sum_{k=0}^{\alpha_j} D_{M_k} \sum_{u=1}^{m_k-1} r_k^u - \sum_{k=0}^{\ell_j-1} D_{M_k} \sum_{u=1}^{m_k-1} r_k^u \right) \right) \tag{15}$$

$$= \psi_n \left(\sum_{j=1}^{s} \left(\frac{D_{M_{\alpha_j+1}-1}}{\psi_{M_{\alpha_j+1}-1}} - \frac{D_{M_{\ell_j}-1}}{\psi_{M_{\ell_j}-1}} \right) \right)$$

$$= \psi_n \left(\sum_{j=1}^{s} \left(\frac{D_{M_{\alpha_j+1}} - \psi_{M_{\alpha_j+1}-1}}{\psi_{M_{\alpha_j+1}-1}} - \frac{D_{M_{\ell_j}} - \psi_{M_{\ell_j}-1}}{\psi_{M_{\ell_j}-1}} \right) \right)$$

$$= \psi_n \left(\sum_{j=1}^{s} \left(\frac{D_{M_{\alpha_j+1}}}{\psi_{M_{\alpha_j+1}-1}} - \frac{D_{M_{\ell_j}}}{\psi_{M_{\ell_j}-1}} \right) \right)$$

and

$$\|I\|_1 \le \sum_{j=1}^{s} \left(\left\| D_{M_{\alpha_j+1}} \right\|_1 + \left\| D_{M_{\ell_j}} \right\|_1 \right) = 2s \le v(n).$$

Moreover,

$$\|II\|_1 \le \sum_{j=1}^{s}\sum_{j=\ell_j}^{\alpha_j} \left|\ominus n_j - 1\right| \delta_j \left\|D_{M_j}\right\|_1$$
$$= \sum_{j=1}^{s}\sum_{j=\ell_j}^{\alpha_j} \left|\ominus n_j - 1\right| \delta_j \le v^*(n).$$

The proof of the upper estimate in (1) follows by combining the last two estimates.

Let $x \in I_{k+1}(x_k e_k)$, where $1 \le x_k \le n_k - 1$ and $e_k := (0, \dots, 0, 1, 0, \dots) \in G_m$, where only the kth coordinate is one, the others are zero. Then, by the definition of Vilenkin functions, if we apply (14) and equalities $x_0 = x_1 = \dots = x_{k-1} = 0$, we find that

$$\psi_{M_l - 1}(x) = 1, \tag{16}$$

for any $0 \le l \le k$.

Let $\ell_j \le k \le \alpha_j$ and $x \in I_{k+1}(x_k e_k)$, where $1 \le x_k \le n_k - 1$. Then, in view of (5) and (15), we get that

$$I = -\psi_n(x)\frac{D_{M_{\ell_j}}(x)}{\psi_{M_{\ell_j}-1}(x)}$$
$$+\psi_n(x)\left(\sum_{l=1}^{j-1}\left(\frac{D_{M_{\alpha_l+1}}(x)}{\psi_{M_{\alpha_l+1}-1}(x)} - \frac{D_{M_{\ell_l}}(x)}{\psi_{M_{\ell_l}-1}(x)}\right)\right)$$
$$= \psi_n(x)\left(-M_{\ell_j} + \sum_{l=1}^{j-1}\left(M_{\alpha_l+1} - M_{\ell_l}\right)\right).$$

By using Lemma 2, we have that

$$II = \psi_n(x)\left(D_{M_k}(x)\sum_{u=1}^{m_k - n_k - 1} r_k^u(x)\right)$$
$$+\psi_n(x)\left(\sum_{l=\ell_j}^{k-1} D_{M_l}(x)\sum_{u=1}^{\ominus n_l - 1} r_l^u(x) + \sum_{s=0}^{j-1}\sum_{l=\ell_s}^{\alpha_s} D_{M_l}(x)\sum_{u=1}^{\ominus n_l - 1} r_l^u(x)\right)$$
$$= \psi_n(x) M_k \frac{\cos(\pi(\ominus n_k)x_k/m_k)\sin(\pi(\ominus n_k - 1)x_k/m_k)}{\sin(\pi x_k/m_k)}\imath$$
$$+\psi_n(x) M_k \frac{\sin(\pi(\ominus n_k)x_k/m_k)\sin(\pi(\ominus n_k - 1)x_k/m_k)}{\sin(\pi x_k/m_k)}$$
$$+\psi_n(x)\sum_{l=\ell_j}^{k-1} M_l(\ominus n_l - 1) + \psi_n(x)\sum_{s=0}^{j-1}\sum_{l=\ell_s}^{\alpha_s} M_l(\ominus n_l - 1).$$

It is obvious that

$$|II - I| = \left|\frac{II - I}{\psi_n}\right| = \left(\mathrm{Re}^2\left(\frac{II - I}{\psi_n}\right) + \mathrm{Im}^2\left(\frac{II - I}{\psi_n}\right)\right)^{1/2}. \tag{17}$$

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On the other hand,

$$\operatorname{Im}\left(\frac{II-I}{\psi_n}\right)=M_k\frac{\cos\left(\pi\left(\ominus n_k\right)x_k/m_k\right)\sin\left(\pi\left(\ominus n_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)} \tag{18}$$

and

$$\begin{aligned}\operatorname{Re}\left(\frac{II-I}{\psi_n}\right)&=M_k\frac{\sin\left(\pi\left(\ominus n_k\right)x_k/m_k\right)\sin\left(\pi\left(\ominus n_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}\\&+\sum_{l=\ell_j}^{k-1}M_l\left(\ominus n_l-1\right)+\sum_{s=0}^{j-1}\sum_{l=\ell_s}^{\alpha_s}M_l\left(\ominus n_l-1\right)+M_{\ell_j}-\sum_{l=1}^{j-1}\left(M_{\alpha_l+1}-M_{\ell_l}\right).\end{aligned}$$

Let $x\in I_{k+1}\left(e_k\right)$ and $\lambda:=\sup_{n\in\mathbb{N}}m_n$. Since $x_k=1$ and

$$\frac{\sin\left(\pi\left(\ominus n_k\right)x_k/m_k\right)\sin\left(\pi\left(\ominus n_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}\geq 0,$$

$$\sum_{l=\ell_j}^{k-1}M_l\left(\ominus n_l-1\right)\geq 0,\quad \sum_{s=0}^{j-1}\sum_{l=\ell_s}^{\alpha_s}M_l\left(\ominus n_l-1\right)\geq 0,$$

$$M_{\ell_j}-\sum_{l=1}^{j-1}\left(M_{\alpha_l+1}-M_{\ell_l}\right)\geq 0,$$

we obtain that

$$\operatorname{Re}\left(\frac{II-I}{\psi_n}\right)\geq\frac{\sin\left(\pi\left(\ominus n_k\right)x_k/m_k\right)\sin\left(\pi\left(\ominus n_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}\geq 0. \tag{19}$$

If we apply (17)–(19) and Lemma 3, for $x\in I_{k+1}\left(e_k\right)$ we get that

$$\begin{aligned}|II-I|&=\left(\operatorname{Re}^2\left(\frac{II-I}{\psi_n}\right)+\operatorname{Im}^2\left(\frac{II-I}{\psi_n}\right)\right)^{1/2}\\&\geq\left(\left(\frac{M_k\cos\left(\pi\left(\ominus n_k\right)x_k/m_k\right)\sin\left(\pi\left(\ominus n_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}\right)^2\right.\\&\quad\left.+\left(\frac{M_k\sin\left(\pi\left(\ominus n_k\right)x_k/m_k\right)\sin\left(\pi\left(\ominus n_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}\right)^2\right)^{1/2}\\&\geq\frac{M_k\sin\left(\pi\left(\ominus n_k-1\right)x_k/m_k\right)}{\sin\left(\pi x_k/m_k\right)}\geq M_k\geq\frac{M_k\left|\ominus n_k-1\right|}{\lambda}.\end{aligned}$$

Let $x\in I_{\alpha_j+2}\left(x_{\alpha_j+1}e_{\alpha_j+1}\right)$, where $1\leq x_{\alpha_j+1}\leq m_{\alpha_j+1}-1$. Then, by using (6) if we invoke equalities (13), (15) and (16) we get that

$$|D_n|=$$

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$$= \left| \sum_{k=1}^{j} \left(\frac{D_{M_{\alpha_k+1}}}{\psi_{M_{\alpha_k+1}-1}} - \frac{D_{M_{\ell_k}}}{\psi_{M_{\ell_k}-1}} \right) - \left(\sum_{k=1}^{j} \sum_{l=\ell_k}^{\alpha_k} D_{M_l} \sum_{u=1}^{m_l-n_l-1} r_l^u \right) \right|$$

$$= \sum_{k=1}^{j} \left((M_{\alpha_k+1} - M_{\ell_k}) - \sum_{l=\ell_k}^{\alpha_k} |\ominus n_l - 1| \, M_l \right)$$

$$\geq \sum_{k=1}^{j} \left((M_{\alpha_k+1} - M_{\ell_k}) - \sum_{l=\ell_k}^{\alpha_k} (m_l - 2) \, M_l \right)$$

$$= \sum_{k=1}^{j} \left((M_{\alpha_k+1} - M_{\ell_k}) - \sum_{l=\ell_k}^{\alpha_k} M_{l+1} + 2 \sum_{l=\ell_k}^{\alpha_k} M_l \right)$$

$$\geq \sum_{k=1}^{j} \sum_{l=\ell_k}^{\alpha_k} M_l \geq M_{\alpha_j}.$$

Hence,

$$L_n \geq \sum_{l=0}^{s} \sum_{k=\ell_l+1}^{\alpha_l} \int_{I_{k+1}(e_k)} \frac{M_k \, |\ominus n_k - 1|}{\lambda} d\mu$$

$$+ \sum_{j=0}^{s} \sum_{x_{\alpha_j+1}=1}^{m_{\alpha_j+1}-1} \int_{I_{\alpha_j+2}\left(x_{\alpha_j+1} e_{\alpha_j+1}\right)} M_{\alpha_j} d\mu$$

$$\geq \sum_{l=0}^{s} \sum_{k=\ell_l}^{\alpha_l} \frac{M_k \, |\ominus n_k - 1|}{\lambda} \frac{1}{M_{k+1}} + \sum_{j=0}^{s} \frac{\left(m_{\alpha_j+1} - 1\right) M_{\alpha_j}}{M_{\alpha_j+2}}$$

$$\geq \sum_{l=0}^{s} \sum_{k=\ell_l}^{\alpha_l} \frac{|\ominus n_k - 1|}{\lambda^2} + \sum_{j=0}^{s} \frac{1}{2\lambda} \geq \frac{1}{\lambda^2} v^* (n) + \frac{1}{4\lambda} v (n).$$

The proof is complete. □

The next result for Vilenkin systems is known (see, e.g., [**1**]) but it also follows from our result.

Corollary 1. *Let* $q_n = M_{2n} + M_{2n-2} + \cdots + M_2$. *Then,*

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$$\frac{n}{2\lambda} \leq \left\| D_{q_n} \right\|_1 \leq \lambda n,$$

where $\lambda := \sup_{n\in\mathbb{N}} m_n$.

Proof. First, we observe that

$$v\left(q_n\right) = 2n. \tag{20}$$

By using Theorem 1, we get that

$$\left\| D_{q_n} \right\|_1 \geq \frac{1}{4\lambda} v\left(q_n\right) = \frac{n}{2\lambda}.$$

Moreover, since

$$v^*\left(q_n\right) = \sum_{j=0}^{n} \left(m_{2j} - 2\right) \leq (\lambda - 2) \sum_{j=0}^{n} 1 \leq (\lambda - 2)\, n$$

if we apply (20), we readily obtain that

$$\left\| D_{q_n} \right\|_1 \leq v^*\left(q_n\right) + v\left(q_n\right) \leq (\lambda - 2)\, n + 2n = \lambda n.$$

The proof is complete. □

Finally, we mention that the following well-known results for the Walsh systems (see the book [7]) also follow directly from our main result.

Corollary 2. *For the Walsh system, the inequality*

$$\frac{1}{8} v\left(n\right) \leq L_n \leq v\left(n\right)$$

holds.

4. Applications. First, we use our main result to find a characterization for the boundedness (or even the ratio of divergence of the norm) of a subsequence of partial sums of the Vilenkin–Fourier series of H_1 martingales.

Theorem 2.

(*a*) *Let* $f \in H_1$ *and* $M_k < n \leq M_{k+1}$. *Then, there exists an absolute constant* c *such that*

$$\left\| S_n f \right\|_{H_1} \leq c\left(v\left(n\right) + v^*\left(n\right)\right) \left\| f \right\|_{H_1}.$$

(*b*) *Let* $\{\Phi_n : n \in \mathbb{N}\}$ *be any non-decreasing and non-negative sequence satisfying condition*

$$\lim_{n\to\infty} \Phi_n = \infty$$

and $\{n_k \geq 2 : k \in \mathbb{N}\}$ *be a subsequence such that*

$$\lim_{k\to\infty} \frac{v\left(n_k\right) + v^*\left(n_k\right)}{\Phi_{n_k}} = \infty. \tag{21}$$

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Then, there exists a martingale $f \in H_1$ *such that*

$$\sup_{k\in\mathbb{N}}\left\|\frac{S_{n_k}f}{\Phi_{n_k}}\right\|_1 \to \infty, \text{ as } k \to \infty.$$

Proof. (a) In view of Theorem 1, we can conclude that

$$\begin{aligned}\|S_n f\|_1 &\le L_n \|f\|_1 \le L_n \|f\|_{H_1}\\ &\le c\,(v\,(n) + v^*\,(n))\,\|f\|_{H_1}.\end{aligned}$$

Let us consider the following martingale:

$$\begin{aligned}f_\# &:= (S_{M_k}S_n f,\ k \ge 1)\\ &= (S_{M_0}f, \dots, S_{M_k}f, \dots,\ S_n f, \dots, S_n f, \dots).\end{aligned}$$

It is easy to see that

$$\begin{aligned}\|S_n f\|_{H_1} \le \|f_\#\|_{H_1} &\le \left\|\sup_{0\le l\le k}\left|S_{M_l}f\right|\right\|_1 + \|S_n f\|_1 \le \|f\|_{H_1} + \|S_n f\|_1\\ &\le \|f\|_{H_1} + c\,(v\,(n) + v^*\,(n))\,\|f\|_{H_1} \le c\,(v\,(n) + v^*\,(n))\,\|f\|_{H_1}.\end{aligned}$$

(b) Under the conditions of Theorem 2, there exists an increasing sequence $\{\alpha_k : k \in \mathbb{N}_+\} \subset \{n_k : k \in \mathbb{N}_+\}$ of the positive integers such that

$$\sum_{k=1}^{\infty}\frac{\Phi_{\alpha_k}^{1/2}}{(v\,(\alpha_k) + v^*\,(\alpha_k))^{1/2}} < \infty. \tag{22}$$

Let

$$f^{(n)} := \sum_{\{k:\ |\alpha_k|<n\}} \lambda_k a_k,$$

where

$$\lambda_k = \frac{\Phi_{\alpha_k}^{1/2}}{(v\,(\alpha_k) + v^*\,(\alpha_k))^{1/2}}, \qquad a_k = D_{M_{|\alpha_k|+1}} - D_{M_{|\alpha_k|}}. \tag{23}$$

By combining (22) and Lemma 1, we conclude that the martingale $f \in H_1$.
It is easy to see that

$$\widehat{f}(j) = \begin{cases}\frac{\Phi_{\alpha_k}^{1/2}}{(v(\alpha_k)+v^*(\alpha_k))^{1/2}}, & \text{if } j \in \left\{M_{|\alpha_k|}, \dots, M_{|\alpha_k|+1} - 1\right\},\ k \in \mathbb{N}\\ 0, & \text{if } j \notin \bigcup_{k=0}^{\infty}\left\{M_{|\alpha_k|}, \dots, M_{|\alpha_k|+1} - 1\right\}.\end{cases} \tag{24}$$

It follows that

$$\frac{S_{\alpha_k}f}{\Phi_{\alpha_k}}$$

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$$= \frac{1}{\Phi_{\alpha_k}} \sum_{i=1}^{k-1} \frac{\Phi_{\alpha_i}^{1/2}}{\left(v\left(\alpha_i\right)+v^*\left(\alpha_i\right)\right)^{1/2}} \left(D_{M_{|\alpha_i|+1}} - D_{M_{|\alpha_i|}}\right)$$

$$+ \frac{D_{\alpha_k} - D_{M_{|\alpha_k|}}}{\Phi_{\alpha_k}^{1/2}\left(v\left(\alpha_k\right)+v^*\left(\alpha_k\right)\right)^{1/2}}.$$

Hence, if we invoke (21) and (22) for sufficiently large k, we can conclude that

$$\left\|\frac{S_{\alpha_k} f}{\Phi_{\alpha_k}}\right\|_1 \geq \frac{\left\|D_{\alpha_k}\right\|_1}{\Phi_{\alpha_k}^{1/2}\left(v\left(\alpha_k\right)+v^*\left(\alpha_k\right)\right)^{1/2}}$$

$$- \frac{\left\|D_{M_{|\alpha_k|}}\right\|_1}{\Phi_{\alpha_k}^{1/2}\left(v\left(\alpha_k\right)+v^*\left(\alpha_k\right)\right)^{1/2}}$$

$$- \frac{1}{\Phi_{\alpha_k}} \sum_{i=1}^{k-1} \frac{\Phi_{\alpha_i}^{1/2}}{\left(v\left(\alpha_i\right)+v^*\left(\alpha_i\right)\right)^{1/2}} \left\|D_{M_{|\alpha_i|+1}} - D_{M_{|\alpha_i|}}\right\|_1$$

$$\geq \frac{\left\|D_{\alpha_k}\right\|_1}{\Phi_{\alpha_k}^{1/2}\left(v\left(\alpha_k\right)+v^*\left(\alpha_k\right)\right)^{1/2}} - \frac{2}{\Phi_{\alpha_k}} \sum_{i=1}^{k} \frac{\Phi_{\alpha_i}^{1/2}}{\left(v\left(\alpha_i\right)+v^*\left(\alpha_i\right)\right)^{1/2}}$$

$$\geq \frac{c_1\left(v\left(\alpha_k\right)+v^*\left(\alpha_k\right)\right)^{1/2}}{\Phi_{\alpha_k}^{1/2}} - c_2 \to \infty, \text{ when } k \to \infty.$$

The proof is complete. □

At first we prove the following estimation:

COROLLARY 3. *Let $f \in H_1$ and $M_k < n \leq M_{k+1}$. Then, there exists an absolute constant c such that*

$$\|S_n f - f\|_{H_1} \leq c\left(v(n)+v^*(n)\right)\omega_{H_1}\left(\frac{1}{M_k}, f\right). \tag{25}$$

Proof of Theorem 3. By using Theorem 2 and obvious estimates, we find that

$$\|S_n f - f\|_{H_1} \leq \left\|S_n f - S_{M_k} f\right\|_{H_1} + \left\|S_{M_k} f - f\right\|_{H_1}$$

$$= \left\|S_n\left(S_{M_k} f - f\right)\right\|_{H_1} + \left\|S_{M_k} f - f\right\|_{H_1}$$

$$\leq (v(n) + v^*(n) + 1)\, \omega_{H_1}\left(\frac{1}{M_k}, f\right)$$

$$\leq c\,(v(n) + v^*(n))\, \omega_{H_1}\left(\frac{1}{M_k}, f\right).$$

Thus, the proof is complete. □

Next, we use Corollary 3 to derive necessary and sufficient conditions for the modulus of continuity of martingale Hardy spaces H_p, for which the partial sums of Vilenkin–Fourier series convergence in L_p-norm. We also point out the sharpness of this result.

THEOREM 3.

(*a*) *Let* $f \in H_1$ *and* $\{n_k : k \in \mathbb{N}\}$ *be a sequence of non-negative integers such that*

$$\omega_{H_1}\left(\frac{1}{M_{|n_k|}}, f\right) = o\left(\frac{1}{v(n_k) + v^*(n_k)}\right), \text{ as } k \to \infty.$$

Then,

$$\left\| S_{n_k} f - f \right\|_{H_1} \to 0, \text{ when } k \to \infty.$$

(*b*) *Let* $\{n_k : k \geq 1\}$ *be sequence of non-negative integers such that*

$$\sup_{k \in \mathbb{N}} (v(n_k) + v^*(n_k)) = \infty.$$

Then, there exists a martingale $f \in H_1$ *and a sequence* $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k :\in \mathbb{N}\}$, *for which*

$$\omega_{H_1}\left(\frac{1}{M_{|\alpha_k|}}, f\right) = O\left(\frac{1}{v(\alpha_k) + v^*(\alpha_k)}\right)$$

and

$$\limsup_{k \to \infty} \left\| S_{\alpha_k} f - f \right\|_1 > c > 0 \text{ when } k \to \infty. \tag{26}$$

Proof. The proof of part (a) follows immediately from (25) in Corollary 3.

Under the conditions of part (b) of Theorem 3, there exists a sequence $\{\alpha_k : k \in \mathbb{N}\} \subset \{n_k : k \in \mathbb{N}\}$ such that

$$v(\alpha_k) + v^*(\alpha_k) \uparrow \infty \text{ when } k \to \infty \tag{27}$$

and

$$(v(\alpha_k) + v^*(\alpha_k))^2 \leq v(\alpha_{k+1}) + v^*(\alpha_{k+1}). \tag{28}$$

Let

$$f^{(n)} := \sum_{\{k : |\alpha_k| < n\}} \lambda_k a_k,$$

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where

$$\lambda_k = \frac{1}{v(\alpha_k) + v^*(\alpha_k)}, \quad a_k = D_{M_{|\alpha_k|+1}} - D_{M_{|\alpha_k|}}.$$

By combining (27), (28) and Lemma 1, we conclude that the martingale $f \in H_1$. It is easy to see that

$$\widehat{f}(j) = \begin{cases} \frac{1}{v(\alpha_k)+v^*(\alpha_k)}, & \text{if } j \in \left\{M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1\right\}, \ k \in \mathbb{N}, \ldots \\ 0, & \text{if } j \notin \bigcup_{k=0}^{\infty} \left\{M_{|\alpha_k|}, \ldots, M_{|\alpha_k|+1} - 1\right\}. \end{cases} \tag{29}$$

It follows that

$$S_{\alpha_k} f = \sum_{i=1}^{k-1} \frac{D_{M_{|\alpha_i|+1}} - D_{M_{|\alpha_i|}}}{v(\alpha_i) + v^*(\alpha_i)} + \frac{D_{\alpha_k} - D_{M_{|\alpha_k|}}}{v(\alpha_k) + v^*(\alpha_k)}. \tag{30}$$

Since

$$S_{M_n} f = f^{(n)}, \quad \text{for } f = \left(f^{(n)} : n \in \mathbb{N}\right) \in H_p$$

and

$$\left(S_{M_k} f^{(n)} : k \in \mathbb{N}\right) = \left(S_{M_k} S_{M_n} f, k \in \mathbb{N}\right)$$

$$= \left(S_{M_0} f, \ldots, S_{M_{n-1}} f, S_{M_n} f, S_{M_n} f, \ldots\right)$$

$$= \left(f^{(0)}, \ldots, f^{(n-1)}, f^{(n)}, f^{(n)}, \ldots\right),$$

we obtain that

$$f - S_{M_n} f = \left(f^{(k)} - S_{M_k} f : k \in \mathbb{N}\right)$$

is a martingale for which

$$(f - S_{M_n} f)^{(k)} = \begin{cases} 0, & k = 0, \ldots, n, \\ f^{(k)} - f^{(n)}, & k \geq n+1. \end{cases} \tag{31}$$

According to Lemma 1, we get that

$$\left\|f - S_{M_n} f\right\|_{H_1} \leq \sum_{i=n+1}^{\infty} \frac{1}{v(\alpha_i) + v^*(\alpha_i)}$$

$$= O\left(\frac{1}{v(\alpha_n) + v^*(\alpha_n)}\right) \text{ when } n \to \infty.$$

By combining (5), (29) and (30) with Theorem 1, we obtain that

$$\|f - S_{\alpha_k} f\|_1$$

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$$\geq \left\| \frac{D_{M_{|\alpha_k|+1}} - D_{\alpha_k}}{v(\alpha_k) + v^*(\alpha_k)} + \sum_{i=k+1}^{\infty} \frac{D_{M_{|\alpha_i|+1}} - D_{M_{|\alpha_i|}}}{v(\alpha_i) + v^*(\alpha_i)} \right\|_1$$

$$\geq \frac{\|D_{\alpha_k}\|_1}{v(\alpha_k) + v^*(\alpha_k)} - \frac{\|D_{M_{|\alpha_k|+1}}\|_1}{v(\alpha_k) + v^*(\alpha_k)} - \sum_{i=k+1}^{\infty} \frac{\|D_{M_{|\alpha_i|+1}} - D_{M_{|\alpha_i|}}\|_1}{v(\alpha_i) + v^*(\alpha_i)}$$

$$\geq c - \frac{1}{v(\alpha_k) + v^*(\alpha_k)} - 3\sum_{i=k+1}^{\infty} \frac{1}{v(\alpha_i) + v^*(\alpha_i)}$$

$$\geq c - \frac{3}{v(\alpha_k) + v^*(\alpha_k)}.$$

Hence,

$$\limsup_{k\to\infty} \left\| S_{\alpha_k} f - f \right\|_1 > c > 0 \text{ as } k \to \infty.$$

The proof is complete. □

This known results can be found in [**8**].

COROLLARY 4. *(a) Let $f \in H_1$ and*

$$\omega_{H_1}\left(\frac{1}{M_n}, f\right) = o\left(\frac{1}{n}\right), \text{ when } n \to \infty.$$

Then,

$$\|S_k f - f\|_{H_1} \to 0, \text{ when } k \to \infty.$$

(b) Then there exists a martingale $f \in H_1$, for which

$$\omega_{H_1}\left(\frac{1}{M_n}, f\right) = O\left(\frac{1}{n}\right) \text{ when } n \to \infty$$

and

$$\|S_k f - f\|_1 \nrightarrow 0 \text{ when } k \to \infty.$$

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