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Qualitative analysis of a ratio-dependent predator–prey system with diffusion

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Ratio-dependent predator–prey models are favoured by many animal ecologists recently as they better describe predator–prey interactions where predation involves a searching process. When densities of prey and predator are spatially homogeneous, the so-called Michaelis–Menten ratio-dependent predator–prey system, which is an ordinary differential system, has been studied by many authors. The present paper deals with the case where densities of prey and predator are spatially inhomogeneous in a bounded domain subject to the homogeneous Neumann boundary condition. Its main purpose is to study qualitative properties of solutions to this reaction-diffusion (partial differential) system. In particular, we will show that even though the unique positive constant steady state is globally asymptotically stable for the ordinary-differential-equation dynamics, non-constant positive steady states exist for the partial-differential-equation model. This demonstrates that *stationary patterns* arise as a result of diffusion.

1. Introduction

There is growing biological and physiological evidence [1–4, 9, 15, 16] that in many situations, such as when predators have to search, share and compete for food, a more suitable general predator–prey model should be based on the so-called ratio-dependent theory, which asserts that the per capita predator growth rate should be a function of the ratio of prey to predator abundance.

Let u and v represent the densities of prey and predator, respectively. In general, a ratio-dependent predator–prey model takes the form

$$u'(t) = ug(u) - vp\left(\frac{u}{v}\right), \qquad v'(t) = v\left(q\left(\frac{u}{v}\right) - d\right),$$

where the function g represents the growth rate of prey in the absence of predator, the function p is the so-called predator functional response, the function q is the rate of conversion of prey to predator, also called the birth rate of the predator, and the positive constant d is the death rate of the predator. The functions p and q are assumed to satisfy the usual non-negative and increasing properties, as well as being equal to zero at zero. In Gause's model, $q = \eta p$, where η is a positive constant

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sometimes referred to as the conversion rate. A typical model of g is the logistic growth $g(u) = a(1 - u/K)$, where a and K are positive constants.

Based on the prevalent Lotka–Volterra-type predator–prey model with Michaelis–Menten-type functional response, the following ratio-dependent analogue has been studied by many authors:

$$\left.\begin{aligned} u'(t) &= au\left(1 - \frac{u}{K}\right) - \frac{cuv}{u + mv}, \\ v'(t) &= v\left(-d + \frac{c\eta u}{u + mv}\right), \end{aligned}\right\} \tag{1.1}$$

where a, c, d, η and K are positive constants. Under the scaling

$$t \mapsto at, \qquad u \mapsto \frac{u}{K}, \qquad v \mapsto \frac{v}{K},$$

system (1.1) takes the form

$$\left.\begin{aligned} u'(t) &= u(1 - u) - \frac{buv}{u + mv}, \\ v'(t) &= rv\left(-k + \frac{u}{u + mv}\right), \end{aligned}\right\} \tag{1.2}$$

where $b = c/a$, $r = c\eta/a$, $k = d/(c\eta)$. The qualitative properties of (1.2) and related models have been studied by many authors [5, 18–20, 23, 33].

We note that (1.2) has at most one positive steady state $(\tilde{u}, \tilde{v})$ given by

$$\tilde{u} = 1 - \frac{b(1 - k)}{m}, \quad \tilde{v} = \frac{1 - k}{mk}\tilde{u} \quad \text{if } k < 1,\ b(1 - k) < m. \tag{1.3}$$

Furthermore, if $k < 1$ and

$$\left.\begin{aligned} &b < \frac{m}{1 - k} && \text{when } \frac{1}{1 - k} \leqslant r < \infty \quad \text{or} \\ &b < m(1 + rk) && \text{when } r < \frac{1}{1 - k}, \end{aligned}\right\} \tag{1.4}$$

then $(\tilde{u}, \tilde{v})$ is globally asymptotically stable in R^2_+ [18, 23]. We note that condition (1.4) implies $b(1 - k) < m$.

Now, if the predator and prey are confined to a fixed bounded domain Ω in $\boldsymbol{R}^n$ with smooth boundary, and their densities are spatially inhomogeneous, from (1.2), we are led to consider the following reaction-diffusion system:

$$\left.\begin{aligned} u_t - d_1\Delta u &= u(1 - u) - \frac{buv}{u + mv}, && x \in \Omega, \quad t > 0, \\ v_t - d_2\Delta v &= rv\left(\frac{u}{u + mv} - k\right), && x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega, \quad t > 0, \\ u(x, 0) &= u_0(x) \geqslant 0, \quad v(x, 0) = v_0(x) \geqslant 0, && x \in \Omega. \end{aligned}\right\} \tag{1.5}$$

In the above, ν is the outward unit normal vector of the boundary $\partial\Omega$ and the homogeneous Neumann boundary condition is being considered. The constants d_1 and d_2, which are the diffusion coefficients, are positive, and the initial data u_0, v_0 are continuous functions. We note that (1.5) has a unique non-negative global solution (u, v). In addition, if $u_0 \not\equiv 0$, $v_0 \not\equiv 0$, then the solution is positive, i.e. $u(x, t) > 0$, $v(x, t) > 0$ on $\bar{\Omega}$ for all $t > 0$. It is obvious that if $k < 1$ and $b(1-k) < m$, then $(\tilde{u}, \tilde{v})$ given by (1.3) is also a positive constant steady state of (1.5).

The main aim of this paper is to study the dissipation, persistence and stability of non-negative constant steady states, as well as the existence of non-constant positive steady states of (1.5). In particular, we will prove that if

$$k < 1, \quad \max\{b(1-k), (\sqrt{1+b} - \sqrt{k})^2\} < m < b(1-k^2), \tag{1.6}$$

then (1.5) has non-constant positive steady states for suitable ranges of d_1 and d_2 (see theorems 5.2 and 5.4 for details). This implies that if (1.4) and (1.6) hold simultaneously, the positive constant equilibrium $(\tilde{u}, \tilde{v})$ is globally asymptotically stable for the ordinary-differential-equation (ODE) dynamics (1.2), but the corresponding partial-differential-equation dynamics (1.5) will have non-constant positive steady states, thus showing that *stationary patterns* arise as a result of diffusion.

Remark 1.1. If we choose $b = 3$, $k = \frac{1}{2}$ and $r \geqslant 2$, then (1.4) and (1.6) hold simultaneously, provided that $(2 - 1/\sqrt{2})^2 < m < \frac{9}{4}$.

We point out that the study of non-constant steady-state solutions involves the analysis of non-trivial solutions to elliptic systems (see (3.1) below). Typically, there are two methods to establish the existence of such non-trivial solutions. One is singular perturbation [21, 22]. The other, which will be used in this paper, is a bifurcation technique. We refer the reader to [7, 10, 11, 14, 26] for applications of this method to a variety of problems in mathematical biology. A variation of the bifurcation technique makes use of the powerful Leray–Schauder degree theory [8, 12, 13, 27–29, 34]. As a general background for diffusive predator–prey models, we refer to [6, 24, 32].

The organization of this paper is as follows. In § 2, we first study the dissipation, persistence and stability of non-negative constant steady states for (1.5). In § 3, *a priori* upper and lower bounds for positive steady states of (1.5) are established. Section 4 deals with non-existence of non-constant positive steady states of (1.5) for certain ranges of the parameters. In § 5, we consider the global existence of non-constant positive steady states. Bifurcation is discussed in § 6, while stability is relegated to the last section.

2. Large-time behaviour of solutions to (1.5)

First, we note that (1.5) has two trivial non-negative constant steady states, namely, $E_0 = (0, 0)$ and $E_1 = (1, 0)$.

2.1. Dissipation

Theorem 2.1. *For any solution (u, v) of (1.5),*

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot, t) \leqslant 1, \qquad \limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot, t) \leqslant \max\{0, \ (1-k)/(mk)\}. \tag{2.1}$$

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Thus, for any $\varepsilon > 0$, the rectangle $[0, 1+\varepsilon) \times [0, \max\{0,\ (1-k)/(mk)\} + \varepsilon)$ is a global attractor of (1.5) in R^2_+.

Proof. Since u satisfies

$$\begin{aligned} u_t - d_1 \Delta u &\leqslant u(1-u), && x \in \Omega, && t > 0, \\ \frac{\partial u}{\partial \nu} &= 0, && x \in \partial\Omega, && t > 0, \\ u(x,0) &= u_0(x) \geqslant 0, && x \in \Omega, \end{aligned}$$

the first inequality of (2.1) follows.

As a result, for any $\varepsilon > 0$, there exists $T > 0$ such that $u(x,t) \leqslant 1+\varepsilon$ for all $x \in \bar{\Omega}$ and $t \geqslant T$. It then follows that v satisfies

$$\begin{aligned} v_t - d_2 \Delta v &\leqslant rv\biggl(-k + \frac{1+\varepsilon}{1+\varepsilon+mv}\biggr) \\ &= rv \frac{(1-k)(1+\varepsilon) - mkv}{1+\varepsilon+mv}, && x \in \Omega, && t \geqslant T, \\ \frac{\partial v}{\partial \nu} &= 0, && x \in \partial\Omega, && t \geqslant T, \\ v(x,T) &> 0, && x \in \bar{\Omega}. \end{aligned}$$

If $k < 1$, let $z(t)$ be a solution of the ODE

$$\begin{aligned} z'(t) &= rz \frac{(1-k)(1+\varepsilon) - mkz}{1+\varepsilon+mz}, \quad t \geqslant T, \\ z(T) &= \max_{\bar{\Omega}} v(\cdot, T) > 0. \end{aligned}$$

Then

$$\lim_{t\to\infty} z(t) = \frac{(1-k)(1+\varepsilon)}{mk}.$$

It follows from the comparison principle that $v(x,t) \leqslant z(t)$, and hence

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot,t) \leqslant \frac{(1-k)(1+\varepsilon)}{mk}.$$

If $k \geqslant 1$, we have the differential inequality

$$v_t - d_2 \Delta v \leqslant \frac{-mrkv^2}{1+\varepsilon+mv},$$

and the same argument above yields

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot,t) \leqslant 0.$$

In either case, the second inequality of (2.1) holds. □

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2.2. Persistence

Definition 2.2. Problem (1.5) is said to have the persistence property if, for any non-negative initial data $(u_0(x), v_0(x))$ with $u_0(x) \not\equiv 0$, $v_0(x) \not\equiv 0$, there exists a positive constant $\varepsilon = \varepsilon(u_0, v_0)$ such that the corresponding solution (u, v) of (1.5) satisfies

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot, t) \geqslant \varepsilon, \qquad \liminf_{t\to\infty} \min_{\bar{\Omega}} v(\cdot, t) \geqslant \varepsilon.$$

Theorem 2.3. *If $k < 1$ and $b < m$, then (1.5) has the persistence property.*

Proof. Suppose $u_0(x) \geqslant 0$ ($\not\equiv 0$) and $v_0(x) \geqslant 0$ ($\not\equiv 0$). By the first equation of (1.5), we have

$$\begin{aligned} u_t - d_1 \Delta u &\geqslant u\left(1 - \frac{b}{m} - u\right), && x \in \Omega, \quad t > 0,\\ \frac{\partial u}{\partial \nu} &= 0, && x \in \partial\Omega, \quad t > 0,\\ u(x, 0) &= u_0(x) \geqslant 0 \ (\not\equiv 0), && x \in \Omega. \end{aligned}$$

Since $b/m < 1$, it follows by a comparison argument that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot, t) \geqslant 1 - \frac{b}{m}.$$

Thus there exists $T > 0$ such that

$$u(x, t) \geqslant \frac{1}{2}\left(1 - \frac{b}{m}\right) \triangleq \eta > 0 \quad \forall x \in \bar{\Omega}, \quad t \geqslant T.$$

As a result, v satisfies

$$\begin{aligned} v_t - d_2 \Delta v &\geqslant rv\left(-k + \frac{\eta}{mv + \eta}\right) = rv\frac{(1-k)\eta - mkv}{mv + \eta}, && x \in \Omega, \quad t \geqslant T,\\ \frac{\partial v}{\partial \nu} &= 0, && x \in \partial\Omega, \quad t \geqslant T,\\ v(x, T) &> 0, && x \in \bar{\Omega}. \end{aligned}$$

Let $w(t)$ be a solution of the ODE

$$\begin{aligned} w'(t) &= rw\frac{(1-k)\eta - mkw}{mw + \eta}, \quad t \geqslant T,\\ w(x, T) &= \min_{\bar{\Omega}} v(\cdot, T) > 0. \end{aligned}$$

Then

$$\lim_{t\to\infty} w(t) = \frac{(1-k)\eta}{mk},$$

since $k < 1$. The comparison principle gives $v(x, t) \geqslant w(t)$, and, in turn,

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} v(\cdot, t) \geqslant \frac{(1-k)\eta}{mk} > 0.$$

The proof is complete. ∎

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THEOREM 2.4. *Suppose that $d_1 = d_2$. If there exists a positive constant α such that $(\alpha + m)(1 + rk) \leqslant b + \alpha r$, then the sector $\mathcal{R} = \{(u, v) \mid u, v \geqslant 0,\ u \leqslant \alpha v\}$ is an invariant domain of (1.5), i.e. if the initial data $(u_0(x), v_0(x)) \in \mathcal{R}$ for all $x \in \bar{\Omega}$, then $(u(x,t), v(x,t)) \in \mathcal{R}$ for all $x \in \bar{\Omega}$ and $t \geqslant 0$. Furthermore, if this α satisfies either $\alpha + m \leqslant b$ or $\alpha(1 - k) < mk$, and the initial data (u_0, v_0) lie in $\mathcal{R}$, then $(u, v) \to (0, 0)$ uniformly on $\bar{\Omega}$ as $t \to \infty$. In particular, in this case, system (1.5) does not have the persistence property.*

Proof. Let $G(u, v) = u - \alpha v$ and

$$f(u,v) = u(1-u) - \frac{buv}{u+mv}, \qquad g(u,v) = rv\left(\frac{u}{u+mv} - k\right).$$

Then

$$\mathcal{R} = \{(u,v) \mid u, v \geqslant 0,\ G(u,v) \leqslant 0\} \quad \text{and} \quad \mathrm{d}G = (1, -\alpha).$$

As $(\alpha+m)(1+rk) \leqslant b+\alpha r$, it follows by direct computation that, on the boundary $u = \alpha v$,

$$\mathrm{d}G \cdot (f,g)^T = f - \alpha g = u\left(1 + rk - \frac{b + \alpha r}{\alpha + m}\right) - u^2 \leqslant 0.$$

The invariance of $\mathcal{R}$ then follows from corollary 14.8 of [32].

Now we assume that either $\alpha + m \leqslant b$ or $\alpha < mk/(1-k)$, and that the initial data (u_0, v_0) lie in $\mathcal{R}$. Then $(u(x,t), v(x,t))$ lies in $\mathcal{R}$. Therefore, u satisfies

$$\left.\begin{aligned} &u_t - d_1 \Delta u \leqslant u\left(1 - \frac{b}{\alpha+m} - u\right), && x \in \Omega, && t > 0, \\ &\frac{\partial u}{\partial \nu} = 0, && x \in \partial\Omega, && t > 0, \\ &u(x,0) \geqslant 0, && x \in \Omega. && \end{aligned}\right\} \tag{2.2}$$

CASE 1. If $\alpha + m \leqslant b$, then $u \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$ by the same argument as in the proof of theorem 2.1. Furthermore, for any $\varepsilon > 0$, there exists T such that $u(x,t) \leqslant \varepsilon$ for all $x \in \bar{\Omega}$ and $t \geqslant T$. Thus v satisfies

$$\begin{aligned} &v_t - d_2 \Delta v \leqslant rv\left(\frac{\varepsilon}{\varepsilon + mv} - k\right) = rv\frac{\varepsilon(1-k) - mkv}{\varepsilon + mv}, && x \in \Omega, && t \geqslant T, \\ &\frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega, && t \geqslant T, \\ &v(x,T) \geqslant 0, && x \in \Omega. && \end{aligned}$$

The standard comparison argument shows that

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot, t) \leqslant \frac{\varepsilon(1-k)}{mk}.$$

We conclude from the arbitrariness of $\varepsilon > 0$ that $v \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

CASE 2. If $\alpha + m > b$ and $\alpha(1-k) < mk$, when $k \geqslant 1$, theorem 2.1 implies that $v \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$, and hence so does u, since $u \leqslant \alpha v$. Now, when

$k < 1$, we deduce from (2.2) that

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} u(\cdot,t) \leqslant 1 - \frac{b}{\alpha+m} \triangleq \sigma.$$

Thus, for any $\varepsilon > 0$, there exists $T_0 > 0$ such that $u(x,t) \leqslant \varepsilon + \sigma$ for all $x \in \bar{\Omega}$ and $t \geqslant T_0$. Consequently, v satisfies

$$\begin{aligned} v_t - d_2\Delta v &\leqslant rv\left(\frac{\varepsilon+\sigma}{\varepsilon+\sigma+mv} - k\right) = rv\frac{(1-k)(\varepsilon+\sigma) - mkv}{\varepsilon+\sigma+mv}, && x \in \Omega, \quad t \geqslant T_0, \\ \frac{\partial v}{\partial \nu} &= 0, && x \in \partial\Omega, \quad t \geqslant T_0, \\ v(x,T_0) &\geqslant 0, && x \in \Omega. \end{aligned}$$

As above,

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot,t) \leqslant \frac{(1-k)(\varepsilon+\sigma)}{mk},$$

and hence there exists $T_1 > T_0$ such that

$$v(x,t) \leqslant \varepsilon + \frac{(1-k)(\varepsilon+\sigma)}{mk} \quad \forall x \in \bar{\Omega}, \quad t \geqslant T_1,$$

and, in turn,

$$u(x,t) \leqslant \alpha\left[\varepsilon + \frac{(1-k)(\varepsilon+\sigma)}{mk}\right] \triangleq \varphi(\varepsilon) \quad \forall x \in \bar{\Omega}, \quad t \geqslant T_1.$$

Let $\eta = \frac{1}{2}[1 + \alpha(1-k)/(mk)] < 1$. Since $\varphi(0) = \sigma\alpha(1-k)/(mk) < \eta\sigma$, one can choose $\varepsilon \ll 1$ such that $\varphi(\varepsilon) < \eta\sigma$, and consequently,

$$u(x,t) \leqslant \alpha\left[\varepsilon + \frac{(1-k)(\varepsilon+\sigma)}{mk}\right] < \eta\sigma \quad \forall x \in \bar{\Omega}, \quad t \geqslant T_1.$$

Thus v satisfies

$$\begin{aligned} v_t - d_2\Delta v &\leqslant rv\left(\frac{\eta\sigma}{\eta\sigma+mv} - k\right) = rv\frac{(1-k)\eta\sigma - mkv}{\eta\sigma+mv}, && x \in \Omega, \quad t \geqslant T_1, \\ \frac{\partial v}{\partial \nu} &= 0, && x \in \partial\Omega, \quad t \geqslant T_1, \\ v(x,T_1) &\geqslant 0, && x \in \Omega. \end{aligned}$$

Repeating the arguments above,

$$\limsup_{t\to\infty} \max_{\bar{\Omega}} v(\cdot,t) \leqslant \frac{(1-k)\eta\sigma}{mk},$$

there exists $T_2 > T_1$ such that

$$v(x,t) \leqslant \varepsilon + \frac{(1-k)\eta\sigma}{mk} \qquad \forall x \in \bar{\Omega}, \quad t \geqslant T_2$$

and

$$u(x,t) \leqslant \alpha\varepsilon + \frac{\alpha(1-k)\eta\sigma}{mk} \leqslant \eta^2\sigma \quad \forall x \in \bar{\Omega}, \quad t \geqslant T_2,$$

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provided that $\varepsilon \leqslant (\eta\sigma/\alpha)[\eta - \alpha(1-k)/(mk)]$. Inductively, there exists an increasing sequence $\{T_n\}$ with $T_n \to \infty$ such that

$$u(x,t) \leqslant \eta^n \sigma \quad \forall x \in \bar{\Omega}, \quad t \geqslant T_n.$$

Since $\eta < 1$, it follows that $u \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$, and hence so does v, as in case 1. The proof is complete. □

2.3. Stability

We discuss the stability of the constant steady states $(1,0)$ and $(\tilde{u},\tilde{v})$.

Theorem 2.5. *If $k \geqslant 1$ and $b \leqslant m$, then*

$$\lim_{t\to\infty} (u(\cdot,t), v(\cdot,t)) = (1,0) \quad \textit{uniformly on } \bar{\Omega},$$

provided that $u_0 \not\equiv 0$. Thus $(1,0)$ is globally asymptotically stable in R^2_+.

Proof. From the proof of theorem 2.1, we see that there exists a positive function $f(t)$ satisfying $\lim_{t\to\infty} f(t) = 0$ such that $\max_{\bar{\Omega}} v(\cdot,t) \leqslant f(t)$. Thus $v \to 0$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

If $b < m$, from the proof of theorem 2.3, we see that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot,t) \geqslant 1 - \frac{b}{m}.$$

For any ε, $0 < \varepsilon \ll 1$, there exists T, $0 < T < \infty$, such that

$$v(x,t) \leqslant \varepsilon, \quad u(x,t) \geqslant \frac{1}{2}\left(1 - \frac{b}{m}\right) \triangleq \eta > 0 \quad \forall x \in \bar{\Omega}, \quad t \geqslant T.$$

Therefore,

$$\begin{aligned} u_t - d_1 \Delta u &\geqslant u\left[1 - u - \frac{b\varepsilon}{m\varepsilon + \eta}\right], && x \in \Omega, && t \geqslant T, \\ \frac{\partial u}{\partial \nu} &= 0, && x \in \partial\Omega, && t \geqslant T, \\ u(x,T) &> 0, && x \in \bar{\Omega}. \end{aligned}$$

An application of the comparison principle gives

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot,t) \geqslant 1 - \frac{b\varepsilon}{m\varepsilon + \eta}.$$

The arbitrariness of ε then implies that

$$\liminf_{t\to\infty} \min_{\bar{\Omega}} u(\cdot,t) \geqslant 1. \tag{2.3}$$

This, along with the first inequality of (2.1), implies that $u \to 1$ uniformly on $\bar{\Omega}$ as $t \to \infty$.

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If $b = m$, then, since $v(x,t) \leqslant f(t)$, we have

$$\begin{aligned} u_t - d_1\Delta u = \frac{1-u-mv}{u+mv}u^2 \geqslant \frac{1-u-mf(t)}{u+mf(t)}u^2, &\quad x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \nu} = 0, &\quad x \in \partial\Omega, \quad t > 0, \\ u(x,0) \geqslant 0 \ (\not\equiv 0), &\quad x \in \bar{\Omega}. \end{aligned}$$

Let $w(t)$ be a solution of the ODE

$$\begin{aligned} w'(t) &= \frac{1-w-mf(t)}{w+mf(t)}w^2, \quad t > t_0 > 0, \\ w(t_0) &= \min_{\bar{\Omega}} u(\cdot, t_0) > 0. \end{aligned}$$

Then $\lim_{t\to\infty} w(t) = 1$, and, by the comparison principle, (2.3) follows. The proof can now be completed as above. □

Next we turn to the stability of $(\tilde{u}, \tilde{v})$. In this discussion, we will always assume that $k < 1$ and $b(1-k) < m$, which imply that $(\tilde{u}, \tilde{v})$ given by (1.3) is the unique positive constant steady state of (1.5).

Let $0 = \mu_0 < \mu_1 < \mu_2 < \mu_3 < \cdots$ be the eigenvalues of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Set

$$\boldsymbol{X} = \left\{ (u,v) \in [C^1(\bar{\Omega})]^2 \;\middle|\; \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0 \text{ on } \partial\Omega \right\}$$

and consider the decomposition $\boldsymbol{X} = \bigoplus_{i=0}^{\infty} \boldsymbol{X}_i$, where $\boldsymbol{X}_i$ is the eigenspace corresponding to μ_i.

THEOREM 2.6. *If $k < 1$ and $m \geqslant b(1-k^2)$, then the positive constant solution $(\tilde{u}, \tilde{v})$ of (1.5) is uniformly asymptotically stable* (*in the sense of [17]*).

Proof. Let

$$\left. \begin{aligned} \theta &= \tilde{u}\left(-1 + \frac{b\tilde{v}}{(\tilde{u}+m\tilde{v})^2}\right), & \beta &= \frac{b\tilde{u}^2}{(\tilde{u}+m\tilde{v})^2}, \\ \lambda &= \frac{mr\tilde{v}^2}{(\tilde{u}+m\tilde{v})^2}, & \delta &= \frac{mr\tilde{u}\tilde{v}}{(\tilde{u}+m\tilde{v})^2} \end{aligned} \right\} \tag{2.4}$$

and

$$\mathcal{L} = \begin{pmatrix} d_1\Delta + \theta & -\beta \\ \lambda & d_2\Delta - \delta \end{pmatrix}.$$

Then the linearization of (1.5) at $(\tilde{u}, \tilde{v})$ can be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} u \\ v \end{pmatrix} = \mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} f_1(u-\tilde{u}_1, v-\tilde{v}_1) \\ f_2(u-\tilde{u}_1, v-\tilde{v}_1) \end{pmatrix},$$

where $f_i(z_1, z_2) = O(z_1^2 + z_2^2)$, $i = 1, 2$. A straightforward calculation yields

$$\left.\begin{aligned} \theta &= -1 + \frac{b(1-k^2)}{m}, \\ \beta &= bk^2 > 0, \\ \lambda &= \frac{r(1-k)^2}{m} > 0, \\ \delta &= rk(1-k) > 0, \\ \beta\lambda - \theta\delta &= \frac{rk(1-k)[m - b(1-k)]}{m} > 0. \end{aligned}\right\} \tag{2.5}$$

Since $m \geqslant b(1-k^2)$, $\theta \leqslant 0$. For each i, $i = 0, 1, 2, \dots$, $\boldsymbol{X}_i$ is invariant under the operator $\mathcal{L}$, and ξ is an eigenvalue of $\mathcal{L}$ on $\boldsymbol{X}_i$ if and only if ξ is an eigenvalue of the matrix

$$A_i = \begin{pmatrix} -d_1\mu_i + \theta & -\beta \\ \lambda & -d_2\mu_i - \delta \end{pmatrix}.$$

Noting that

$$\begin{aligned} \det A_i &= d_1 d_2 \mu_i^2 + (\delta d_1 - \theta d_2)\mu_i + \beta\lambda - \theta\delta > 0, \\ \operatorname{tr} A_i &= -(d_1 + d_2)\mu_i + \theta - \delta < 0, \end{aligned}$$

we conclude that the two eigenvalues ξ_i^+ and ξ_i^- of A_i have negative real parts. More specifically, we have the following.

(i) For $i = 0$, if $(\delta - \theta)^2 \leqslant 4(\beta\lambda - \theta\delta)$, then $\operatorname{Re}\xi_0^\pm = \frac{1}{2}(\theta - \delta) < 0$, and if $(\delta - \theta)^2 > 4(\beta\lambda - \theta\delta)$, then

$$\begin{aligned} \operatorname{Re}\xi_0^+ &= \tfrac{1}{2}\big[\theta - \delta + \sqrt{(\delta-\theta)^2 - 4(\beta\lambda - \theta\delta)}\big] < 0, \\ \operatorname{Re}\xi_0^- &= \tfrac{1}{2}\big[\theta - \delta - \sqrt{(\delta-\theta)^2 - 4(\beta\lambda - \theta\delta)}\big] < 0. \end{aligned}$$

(ii) For $i \geqslant 1$, as μ_i is increasing with respect to i and $\mu_i \to \infty$ as $i \to \infty$, it follows that if $(\operatorname{tr} A_i)^2 - 4\det A_i \leqslant 0$, then

$$\begin{aligned} \operatorname{Re}\xi_i^\pm = \tfrac{1}{2}\operatorname{tr} A_i &= \tfrac{1}{2}[-(d_1+d_2)\mu_i + \theta - \delta] \\ &\leqslant \tfrac{1}{2}[-(d_1+d_2)\mu_1 + \theta - \delta] < 0, \end{aligned}$$

and if $(\operatorname{tr} A_i)^2 - 4\det A_i > 0$, since $\det A_i > 0$ and $\operatorname{tr} A_i < 0$,

$$\begin{aligned} \operatorname{Re}\xi_i^- &= \tfrac{1}{2}\big(\operatorname{tr} A_i - \sqrt{(\operatorname{tr} A_i)^2 - 4\det A_i}\big) \leqslant \tfrac{1}{2}\operatorname{tr} A_i \\ &\leqslant \tfrac{1}{2}[-(d_1+d_2)\mu_1 + \theta - \delta] < 0, \\ \operatorname{Re}\xi_i^+ &= \tfrac{1}{2}\big(\operatorname{tr} A_i + \sqrt{(\operatorname{tr} A_i)^2 - 4\det A_i}\big) = \frac{2\det A_i}{\operatorname{tr} A_i - \sqrt{(\operatorname{tr} A_i)^2 - 4\det A_i}} \\ &\leqslant \frac{\det A_i}{\operatorname{tr} A_i} < -\tilde{\varepsilon} \end{aligned}$$

for some positive constant $\tilde{\varepsilon}$ that does not depend on i.

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The above arguments show that there exists a positive constant ε, which does not depend on i, such that

$$\operatorname{Re}\xi_i^{\pm} < -\varepsilon \quad \forall i.$$

Consequently, the spectrum of $\mathcal{L}$, which consists of eigenvalues, lies in $\{\operatorname{Re}\xi < -\varepsilon\}$. Our result now follows from theorem 5.1.1 of [17]. □

3. *A priori* estimates on positive steady states of (1.5)

The main purpose of this section is to give *a priori* positive lower and upper bounds for the positive steady states of (1.5). The corresponding steady-state problem of (1.5) is the elliptic system

$$\left.\begin{aligned} -d_1\Delta u &= u(1-u) - \frac{buv}{u+mv}, && x \in \Omega, \\ -d_2\Delta v &= rv\left(\frac{u}{u+mv} - k\right), && x \in \Omega, \\ \frac{\partial u}{\partial \nu} &= \frac{\partial v}{\partial \nu} = 0, && x \in \partial\Omega. \end{aligned}\right\} \tag{3.1}$$

Without loss of generality, in the sequel, we will assume that $r = 1$. We first state the following Harnack inequality due to Lin *et al.* [25].

PROPOSITION 3.1 (Harnack inequality (cf. [25])). *Let $w \in C^2(\Omega) \cap C^1(\bar{\Omega})$ be a positive classical solution to $\Delta w(x) + c(x)w(x) = 0$ in Ω subject to the homogeneous Neumann boundary condition where $c \in C(\bar{\Omega})$. Then there exists a positive constant $C_* = C_*(n, \Omega, \|c\|_\infty)$ such that*

$$\max_{\bar{\Omega}} w \leqslant C_* \min_{\bar{\Omega}} w.$$

Throughout the rest of this paper, by classical solutions we mean solutions in $C^2(\Omega) \cap C^1(\bar{\Omega})$. For notational convenience, we shall write $\Lambda = (b, k, m)$ in the sequel.

THEOREM 3.2. *For any positive classical solution (u, v) of (3.1),*

$$\max_{\bar{\Omega}} u(x) \leqslant 1, \qquad \max_{\bar{\Omega}} v(x) \leqslant \frac{(1-k)}{mk}. \tag{3.2}$$

Proof. The result follows immediately from the maximum principle. □

THEOREM 3.3. *For any positive integer ℓ, the positive classical solution (u, v) of (3.1) lies in $C^\ell(\bar{\Omega}) \times C^\ell(\bar{\Omega})$. Furthermore, for any fixed positive constant d, there exists a positive constant $C = C(n, \Omega, d, \Lambda)$ such that, if $d_1, d_2 \geqslant d$,*

$$\|u\|_{C^{2,\alpha}(\bar{\Omega})} + \|v\|_{C^{2,\alpha}(\bar{\Omega})} \leqslant C \tag{3.3}$$

for some $0 < \alpha < 1$.

Proof. Since u and v are positive on $\bar{\Omega}$, the standard regularity theorem for elliptic equations yields $(u, v) \in C^\ell(\bar{\Omega}) \times C^\ell(\bar{\Omega})$.

Now, by (3.2), it is known that the right-hand sides of the first two equations of (3.1) are in $L^\infty(\Omega)$. The L^p theory (take $p > n$ here) for elliptic equations and the embedding theorem show that $u, v \in C^{1,\alpha}(\bar{\Omega})$ for some $0 < \alpha < 1$, and the $C^{1,\alpha}$ norms of u and v depend only on n, Ω, Λ and the lower bounds of d_1 and d_2. Let

$$f(x) = \frac{u(x)v(x)}{u(x) + mv(x)}.$$

Then a straightforward calculation yields

$$\nabla f = \frac{mv^2 \nabla u + u^2 \nabla v}{(u + mv)^2}, \qquad \|\nabla f\|_\infty \leqslant \frac{1 + 1/m}{\|\nabla u\|_\infty + \|\nabla v\|_\infty},$$

and an application of the Schauder theory for elliptic equations concludes the proof. □

THEOREM 3.4. *Let d and D be fixed positive constants. Assume that, for any fixed positive constants D_1 and D_2, the elliptic problem*

$$\left.\begin{aligned} -D_1 \Delta w &= w - \frac{bwz}{w + mz}, && x \in \Omega, \\ -D_2 \Delta z &= \frac{wz}{w + mz} - kz, && x \in \Omega, \\ \frac{\partial w}{\partial \nu} &= \frac{\partial z}{\partial \nu} = 0, && x \in \partial\Omega, \end{aligned}\right\} \tag{3.4}$$

has no positive solution. Then there exists a positive constant $\underset{\sim}{C} = \underset{\sim}{C}(n, \Omega, d, D, \Lambda)$ such that, if $(d_1, d_2) \in [d, \infty) \times [d, D]$, every positive classical solution (u, v) of (3.1) satisfies

$$\min_{\bar{\Omega}} u(x) > \underset{\sim}{C}, \qquad \min_{\bar{\Omega}} v(x) > \underset{\sim}{C}. \tag{3.5}$$

Proof. We first note that, by theorem 2.1, if (3.1) has a positive solution, then we must have $k < 1$. Let

$$c_1(x) = d_1^{-1}\left[(1 - u) - \frac{bv}{u + mv}\right], \qquad c_2(x) = d_2^{-1}\left[-k + \frac{u}{u + mv}\right].$$

In view of (3.2), there exists a positive constant $\hat{C} = \hat{C}(n, \Omega, d, \Lambda)$ such that $\|c_1\|_\infty$, $\|c_2\|_\infty \leqslant \hat{C}$ if $d_1, d_2 \geqslant d$. As u and v satisfy

$$\Delta u + c_1(x)u = 0 \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

$$\Delta v + c_2(x)v = 0 \quad \text{in } \Omega, \qquad \frac{\partial v}{\partial \nu} = 0 \quad \text{on } \partial\Omega,$$

proposition 3.1 shows that there exists a positive constant $C_* = C_*(n, \Omega, d, \Lambda)$ such that the Harnack inequality

$$\max_{\bar{\Omega}} u \leqslant C_* \min_{\bar{\Omega}} u, \qquad \max_{\bar{\Omega}} v \leqslant C_* \min_{\bar{\Omega}} v \tag{3.6}$$

holds when $d_1, d_2 \geqslant d$.

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Now, suppose, on the contrary, that (3.5) does not hold. Then, by (3.6), there exists a sequence $\{(d_{1,i}, d_{2,i})\}_{i=1}^{\infty}$ with $(d_{1,i}, d_{2,i}) \in [d, \infty) \times [d, D]$, and positive classical solutions (u_i, v_i) of (3.1) with $(d_1, d_2) = (d_{1,i}, d_{2,i})$, such that

$$\max_{\bar{\Omega}} u_i \to 0 \quad \text{or} \quad \max_{\bar{\Omega}} v_i \to 0 \quad \text{as } i \to \infty. \tag{3.7}$$

By the maximum principle, $u_i \leqslant 1$. Integrating by parts, we obtain

$$\int_{\Omega} u_i \biggl\{1 - u_i - \frac{bv_i}{u_i + mv_i}\biggr\} \,\mathrm{d}x = 0, \quad \int_{\Omega} v_i \biggl(\frac{u_i}{u_i + mv_i} - k\biggr) \,\mathrm{d}x = 0, \quad i = 1, 2, \dots. \tag{3.8}$$

Theorem 3.3 implies that there exists a subsequence of $\{(u_i, v_i)\}_{i=1}^{\infty}$, which we shall still denote by $\{(u_i, v_i)\}_{i=1}^{\infty}$, and two non-negative functions $u, v \in C^2(\bar{\Omega})$, such that $(u_i, v_i) \to (u, v)$ in $[C^2(\bar{\Omega})]^2$ as $i \to \infty$. By (3.7), we note that $u \equiv 0$ or $v \equiv 0$. Also, $u \leqslant 1$. Furthermore, since u_i, v_i satisfy (3.6), so do u, v. We consider the following three cases.

Case 1 ($u \equiv 0$ and $v \not\equiv 0$). Since v satisfies the second inequality of (3.6), $v > 0$ on $\bar{\Omega}$. Therefore,

$$-k + \frac{u_i}{u_i + mv_i} < 0 \quad \text{on } \bar{\Omega} \quad \forall i \gg 1.$$

This contradicts the second integral identity of (3.8) and the fact that $v_i > 0$.

Case 2 ($v \equiv 0$ and $u \not\equiv 0$). As above, $u > 0$ on $\bar{\Omega}$. As $(u_i, v_i) \to (u, 0)$, it follows from the first integral identity of (3.8) that $\int_{\Omega} u(1 - u) \,\mathrm{d}x = 0$. This fact combines with $0 < u \leqslant 1$ to yield $u \equiv 1$, which implies that $u_i/(u_i + mv_i) \to 1$ uniformly on $\bar{\Omega}$ as $i \to \infty$, since $v_i \to 0$ uniformly on $\bar{\Omega}$. As $k < 1$, this contradicts the second integral identity of (3.8) and the fact that $v_i > 0$.

Case 3 ($u \equiv 0$ and $v \equiv 0$). Let

$$w_i = \frac{u_i}{\|u_i\|_{\infty} + \|v_i\|_{\infty}}, \qquad z_i = \frac{v_i}{\|u_i\|_{\infty} + \|v_i\|_{\infty}}.$$

Then (w_i, z_i) satisfies the elliptic system

$$\left.\begin{aligned} -d_{1,i}\Delta w_i &= w_i(1 - u_i(x)) - \frac{bw_i z_i}{w_i + mz_i}, & x &\in \Omega, \\ -d_{2,i}\Delta z_i &= z_i\biggl(\frac{w_i}{w_i + mz_i} - k\biggr), & x &\in \Omega, \\ \frac{\partial w_i}{\partial \nu} &= \frac{\partial z_i}{\partial \nu} = 0, & x &\in \partial\Omega, \end{aligned}\right\} \tag{3.9}$$

and hence the following integral identities hold:

$$\int_{\Omega} w_i \biggl\{1 - u_i - \frac{bz_i}{w_i + mz_i}\biggr\} \,\mathrm{d}x = 0, \quad \int_{\Omega} z_i \biggl(\frac{w_i}{w_i + mz_i} - k\biggr) \,\mathrm{d}x = 0, \quad i = 1, 2, \dots. \tag{3.10}$$

Because $d_{1,i}, d_{2,i} \geqslant d$, theorem 3.3 implies the existence of a subsequence $(w_i, z_i) \to (w, z)$ in $[C^2(\bar{\Omega})]^2$ for some non-negative functions $w, z \in C^2(\bar{\Omega})$. Since $\|w_i\|_{\infty} + \|z_i\|_{\infty} = 1$, we have $\|w\|_{\infty} + \|z\|_{\infty} = 1$. On the other hand, applying proposition 3.1

to (3.9), we see that (w_i, z_i) satisfies the Harnack inequality (3.6), and hence so does (w, z) by taking $i \to \infty$. These observations show that $w(x)+z(x) > 0$ on $\bar{\Omega}$. Letting $i \to \infty$ in (3.10), we have

$$\int_\Omega w\left(1 - \frac{bz}{w+mz}\right) \mathrm{d}x = 0, \qquad \int_\Omega z\left(\frac{w}{w+mz} - k\right) \mathrm{d}x = 0. \tag{3.11}$$

By taking a subsequence if necessary, we may assume that $d_{1,i} \to D_1 \in [d, \infty]$, $d_{2,i} \to D_2 \in [d, D]$. We analyse the following two possibilities.

(i) If $D_1 = \infty$, then w satisfies

$$-\Delta w = 0, \quad x \in \Omega,$$
$$\frac{\partial w}{\partial \nu} = 0, \quad x \in \partial\Omega.$$

Therefore, $w \equiv \alpha$, a non-negative constant. Moreover, z satisfies

$$\left.\begin{aligned} -D_2 \Delta z = z\left(\frac{\alpha}{\alpha+mz} - k\right) = z\frac{\alpha(1-k) - mkz}{\alpha+mz}, &\quad x \in \Omega, \\ \frac{\partial z}{\partial \nu} = 0, &\quad x \in \partial\Omega. \end{aligned}\right\} \tag{3.12}$$

It follows that $\alpha > 0$, since $(\alpha, z) \not\equiv (0, 0)$. From the first integral identity of (3.11), we deduce that $z \not\equiv 0$, and, in turn, by $\max_{\bar{\Omega}} z \leqslant C_* \min_{\bar{\Omega}} z$, we have $z > 0$ on $\bar{\Omega}$. In view of (3.12), we conclude that $z \equiv \alpha(1-k)/(mk) \triangleq \beta$, which is a positive constant. Again, by (3.11),

$$1 - \frac{b\beta}{\alpha+m\beta} = 0, \qquad \frac{\alpha}{\alpha+m\beta} - k = 0.$$

This implies that (α, β) is a positive solution of (3.4)—a contradiction.

(ii) If $D_1 < \infty$, then (w, z) satisfies (3.4), since $w(x) + mz(x) > 0$ on $\bar{\Omega}$ and $u_i \to 0$. Again, as $w(x) + mz(x) > 0$ on $\bar{\Omega}$, it follows from (3.11) that neither $w \equiv 0$ nor $z \equiv 0$. The Harnack inequality then yields $w > 0$, $z > 0$ on $\bar{\Omega}$, which contradicts the assumption of the theorem.

The proof is complete. □

Corollary 3.5. *If the parameters a, k and m satisfy $\sqrt{m} + \sqrt{k} > \sqrt{1+b}$, then (3.4) has no positive solution. In particular, under this assumption, the assertion of theorem 3.4 holds.*

Proof. Suppose, on the contrary, that (w, z) is a positive solution of (3.4). Integrating by parts, we have (3.11) and hence

$$\int_\Omega \frac{w^2 + mkz^2 + (m+k-b-1)wz}{w+mz} \mathrm{d}x = 0. \tag{3.13}$$

Since $w, z > 0$, equation (3.13) implies $m + k < 1 + b$. Further, our assumption implies that $w^2 + mkz^2 + (m+k-b-1)wz$ is positive-definite. This contradiction to (3.13) completes the proof. □

4. Non-existence of non-constant positive steady states

From theorem 2.1, it is seen that if $k \geqslant 1$, then the solution (u, v) of (1.5) satisfies $\lim_{t\to\infty} v(\cdot, t) = 0$ uniformly on $\bar{\Omega}$. This implies that (3.1) has no positive classical solution if $k \geqslant 1$. In this section, we consider the case $k < 1$. Besides having an interest in its own right, the non-existence result derived in this section will facilitate the existence results of the next section.

THEOREM 4.1. *Let μ_1 be the smallest positive eigenvalue of the operator $-\Delta$ on Ω with the homogeneous Neumann boundary condition. Assume that $\mu_1 d_2 > 1 - k$ and let*

$$\tilde{d} = \min\{1, \mu_1 d_2 - (1-k)\}.$$

Then there exists a positive constant $d = d(\tilde{d}, n, \Omega, \Lambda)$ such that (3.1) has no non-constant positive classical solution for $d_1 \geqslant d$. In particular, we note that d does not depend on d_2 when d_2 is large.

Proof. Assume that (u, v) is a positive classical solution of (3.1). By (3.2), there exists a positive constant $C = C(m, k)$ such that $u(x), v(x) \leqslant C$. For ease of notation, we set

$$f(u, v) = u(1-u) - \frac{buv}{u+mv}, \qquad g(u, v) = \frac{uv}{u+mv}.$$

For any $\varphi \in L^1(\Omega)$, let $\bar{\varphi} = (1/|\Omega|)\int_\Omega \varphi\,\mathrm{d}x$. Multiplying the differential equation for u by $u - \bar{u}$, and then integrating over Ω by parts, we have

$$\begin{aligned} d_1 \int_\Omega |\nabla(u-\bar{u})|^2\,\mathrm{d}x &= \int_\Omega f(u,v)(u-\bar{u})\,\mathrm{d}x \\ &= \int_\Omega \{f(u,v) - f(\bar{u},\bar{v})\}(u-\bar{u})\,\mathrm{d}x \\ &= \int_\Omega \left\{[1-(u+\bar{u})](u-\bar{u})^2 - \frac{mbv\bar{v}(u-\bar{u})^2}{(u+mv)(\bar{u}+m\bar{v})} - \frac{bu\bar{u}(u-\bar{u})(v-\bar{v})}{(u+mv)(\bar{u}+m\bar{v})}\right\}\mathrm{d}x \\ &\leqslant \int_\Omega \{(u-\bar{u})^2 + b|u-\bar{u}||v-\bar{v}|\}\,\mathrm{d}x \\ &\leqslant \int_\Omega \{(1+C(\varepsilon))(u-\bar{u})^2 + \varepsilon|v-\bar{v}|^2\}\,\mathrm{d}x, \end{aligned} \tag{4.1}$$

where ε is an arbitrary small positive constant arising from Young's inequality. Similarly, we have

$$\begin{aligned} d_2 \int_\Omega |\nabla(v-\bar{v})|^2\,\mathrm{d}x &= \int_\Omega [-kv + g(u,v)](v-\bar{v})\,\mathrm{d}x \\ &= \int_\Omega \{-k(v-\bar{v}) + g(u,v) - g(\bar{u},\bar{v})\}(v-\bar{v})\,\mathrm{d}x \\ &\leqslant \int_\Omega \{(1-k+\varepsilon)(v-\bar{v})^2 + C(\varepsilon)(u-\bar{u})^2\}\,\mathrm{d}x. \end{aligned} \tag{4.2}$$

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Adding (4.1) and (4.2), we get

$$\int_\Omega (d_1|\nabla(u-\bar{u})|^2 + d_2|\nabla(v-\bar{v})|^2)\,\mathrm{d}x \leqslant \int_\Omega \{[1+2C(\varepsilon)](u-\bar{u})^2 + (1-k+2\varepsilon)(v-\bar{v})^2\}\,\mathrm{d}x. \quad (4.3)$$

It follows from the Poincaré inequality that

$$\mu_1 \int_\Omega (d_1(u-\bar{u})^2 + d_2(v-\bar{v})^2)\,\mathrm{d}x \leqslant \int_\Omega \{[1+2C(\varepsilon)](u-\bar{u})^2 + (1-k+2\varepsilon)(v-\bar{v})^2\}\,\mathrm{d}x. \quad (4.4)$$

Since $\mu_1 d_2 > 1-k$, we may choose $\varepsilon \ll 1$ such that $\mu_1 d_2 > 1-k+2\varepsilon$. Consequently, by (4.4),

$$\mu_1 d_1 \int_\Omega (u-\bar{u})^2\,\mathrm{d}x \leqslant (1+2C(\varepsilon)) \int_\Omega (u-\bar{u})^2\,\mathrm{d}x.$$

This implies that $u \equiv \bar{u} = \text{const.}$, and, in turn, $v \equiv \bar{v} = \text{const.}$, if

$$d_1 > d \triangleq \frac{1+2C(\varepsilon)}{\mu_1}.$$

It is obvious that d depends only on $\tilde{d}$, n, Ω and Λ. The proof is complete. □

5. Global existence of non-constant positive steady states

In this section, we discuss the global existence of non-constant positive classical solutions to (3.1) when the diffusion coefficients d_1 and d_2 vary while the parameters b, k and m are kept fixed. Theorem 2.1 implies that when $k \geqslant 1$, then (3.1) has no non-constant positive classical solutions. When $k < 1$ and $m > b(1-k)$, in which case (3.1) has a unique positive constant solution $(\tilde{u}, \tilde{v})$, theorem 2.6 says that if $m \geqslant b(1-k^2)$, then $(\tilde{u}, \tilde{v})$ is uniformly asymptotically stable. In this case, one cannot expect to obtain non-constant positive classical solutions bifurcating from the constant solution $(\tilde{u}, \tilde{v})$. In view of these reasons, we shall restrict this discussion to the case

$$k < 1, \qquad b(1-k) < m < b(1-k^2). \quad (5.1)$$

For simplicity, we write $\boldsymbol{u} = (u, v)$ and $\tilde{\boldsymbol{u}} = (\tilde{u}, \tilde{v})$. Let θ, β, λ and δ be given by (2.4), with $r = 1$, and set

$$D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \qquad \boldsymbol{F}(\boldsymbol{u}) = \begin{pmatrix} u(1-u) - buv/(u+mv) \\ -kv + uv/(u+mv) \end{pmatrix}, \qquad B = \begin{pmatrix} \theta & -\beta \\ \lambda & -\delta \end{pmatrix}.$$

Then $D_{\boldsymbol{u}}\boldsymbol{F}(\tilde{\boldsymbol{u}}) = B$ and (2.5) holds with $r = 1$. From (5.1), we see that $\theta > 0$. We note that (3.1) can be written as

$$\left.\begin{aligned} -\Delta \boldsymbol{u} &= D^{-1}\boldsymbol{F}(\boldsymbol{u}), && x \in \Omega, \\ \frac{\partial \boldsymbol{u}}{\partial \nu} &= 0, && x \in \partial\Omega. \end{aligned}\right\} \quad (5.2)$$

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Furthermore, $\boldsymbol{u}$ solves (5.2) if and only if it satisfies

$$f(d_1, d_2; \boldsymbol{u}) \triangleq \boldsymbol{u} - (\boldsymbol{I} - \Delta)^{-1}\{D^{-1}\boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{u}\} = 0 \quad \text{on } \boldsymbol{X}, \tag{5.3}$$

where $(\boldsymbol{I} - \Delta)^{-1}$ is the inverse of $\boldsymbol{I} - \Delta$ with the homogeneous Neumann boundary condition. Direct computation gives

$$D_{\boldsymbol{u}} f(d_1, d_2; \tilde{\boldsymbol{u}}) = \boldsymbol{I} - (\boldsymbol{I} - \Delta)^{-1}(D^{-1}B + \boldsymbol{I}).$$

As in the proof of theorem 2.6, we note that, for each $\boldsymbol{X}_i$, ξ is an eigenvalue of $D_{\boldsymbol{u}} f(d_1, d_2; \tilde{\boldsymbol{u}})$ on $\boldsymbol{X}_i$ if and only if $\xi(1 + \mu_i)$ is an eigenvalue of the matrix

$$M_i \triangleq \mu_i \boldsymbol{I} - D^{-1}B = \begin{pmatrix} \mu_i - \theta d_1^{-1} & \beta d_1^{-1} \\ -\lambda d_2^{-1} & \mu_i + \delta d_2^{-1} \end{pmatrix}.$$

It is straightforward to find

$$\begin{aligned} \det M_i &= \frac{1}{d_1 d_2}\{d_1 d_2 \mu_i^2 + (\delta d_1 - \theta d_2)\mu_i + \beta\lambda - \theta\delta\}, \\ \operatorname{tr} M_i &= 2\mu_i + \delta d_2^{-1} - \theta d_1^{-1}. \end{aligned}$$

Write

$$H(d_1, d_2; \mu) = d_1 d_2 \mu^2 + (\delta d_1 - \theta d_2)\mu + \beta\lambda - \theta\delta. \tag{5.4}$$

Then $H(d_1, d_2; \mu_i) = d_1 d_2 \det M_i$. If

$$(\delta d_1 - \theta d_2)^2 > 4 d_1 d_2 (\beta\lambda - \theta\delta), \tag{5.5}$$

then $H(d_1, d_2; \mu) = 0$ has two real roots, namely,

$$\begin{aligned} \mu_+(d_1, d_2) &= \frac{\theta d_2 - \delta d_1 + \sqrt{(\theta d_2 - \delta d_1)^2 - 4 d_1 d_2(\beta\lambda - \theta\delta)}}{2 d_1 d_2}, \\ \mu_-(d_1, d_2) &= \frac{\theta d_2 - \delta d_1 - \sqrt{(\theta d_2 - \delta d_1)^2 - 4 d_1 d_2(\beta\lambda - \theta\delta)}}{2 d_1 d_2}. \end{aligned}$$

Set

$$\begin{aligned} \mathcal{A} &= \mathcal{A}(d_1, d_2) = \{\mu \mid \mu \geqslant 0,\ \mu_-(d_1, d_2) < \mu < \mu_+(d_1, d_2)\}, \\ S_p &= \{\mu_0, \mu_1, \mu_2, \dots\}, \end{aligned}$$

and let $m(\mu_i)$ be the multiplicity of μ_i. In order to calculate the index of $f(d_1, d_2; , \cdot)$ at $\tilde{\boldsymbol{u}}$, we first prove the following lemma.

LEMMA 5.1. *Suppose $H(d_1, d_2; \mu_i) \neq 0$ for all $\mu_i \in S_p$. Then*

$$\operatorname{index}(f(d_1, d_2; \cdot), \tilde{\boldsymbol{u}}) = (-1)^{\sigma}, \tag{5.6}$$

where

$$\sigma = \begin{cases} \sum_{\mu_i \in \mathcal{A} \cap S_p} m(\mu_i) & \text{if } \mathcal{A} \cap S_p \neq \emptyset, \\ 0 & \text{if } \mathcal{A} \cap S_p = \emptyset. \end{cases}$$

In particular, if $H(d_1, d_2; \mu) > 0\ \forall \mu \geqslant 0$, then $\sigma = 0$.

Proof. Since $H(d_1, d_2; \mu_i) \neq 0$ for all $\mu_i \in S_p$, then 0 is not an eigenvalue of $D_{\boldsymbol{u}} f(d_1, d_2; \tilde{\boldsymbol{u}})$. This implies that $D_{\boldsymbol{u}} f(d_1, d_2; \tilde{\boldsymbol{u}})$ is a homeomorphism from $\boldsymbol{X}$ into itself and it follows from the implicit function theorem that $\boldsymbol{u} = \tilde{\boldsymbol{u}}$ is the isolated solution of $f(d_1, d_2; \boldsymbol{u}) = 0$. By the Leray–Schauder theorem (theorem 2.8.1 in [30]),

$$\operatorname{index}(f(d_1, d_2; \cdot), \tilde{\boldsymbol{u}}) = (-1)^{\gamma}, \qquad \gamma = \sum_{i \geqslant 0} \sum_{\xi_i < 0} m(\xi_i),$$

where ξ_i is the eigenvalue of $D_{\boldsymbol{u}} f(d_1, d_2; \tilde{\boldsymbol{u}})$ on $\boldsymbol{X}_i$, and $m(\xi_i)$ is its multiplicity. From the relation between the eigenvalues of the operator $D_{\boldsymbol{u}} f(d_1, d_2; \tilde{\boldsymbol{u}})$ and the matrix M_i, we see that

$$\operatorname{index}(f(d_1, d_2; \cdot), \tilde{\boldsymbol{u}}) = (-1)^{\gamma^*}, \qquad \gamma^* = \sum_{i \geqslant 0} \sum_{\tau_i < 0} m(\tau_i) m(\mu_i),$$

where τ_i is the eigenvalue of M_i and $m(\tau_i)$ its multiplicity. Modulo 2, the number of negative eigenvalues of M_i is given by

$$a_i \overset{\Delta}{=} \tfrac{1}{2}(1 - \operatorname{sgn}\{\det M_i\}) = \tfrac{1}{2}(1 - \operatorname{sgn}\{H(d_1, d_2; \mu_i)\}),$$

provided that $\det M_i \neq 0$. For any $\mu_i \in S_p$, if $\mu_i \in \mathcal{A}$, then

$$\det M_i = H(d_1, d_2; \mu_i) < 0,$$

and hence $a_i = 1$. If $\mu_i \notin \mathcal{A}$, then $\det M_i = H(d_1, d_2; \mu_i) > 0$, and hence $a_i = 0$. Consequently, modulo 2, $\gamma^* = \sigma$ and the proof is complete. □

From lemma 5.1, we see that to calculate the index of $f(d_1, d_2; \cdot)$ at $\tilde{\boldsymbol{u}}$, the key step is to determine the range of μ for which $H(d_1, d_2; \mu) < 0$.

Theorem 5.2. *Let the assertion of theorem 3.4 hold. If $\theta/d_1 \in (\mu_q, \mu_{q+1})$ for some $q \geqslant 1$, and $\sigma_q = \sum_{i=1}^{q} m(\mu_i)$ is odd, then there exists a positive constant d^* such that (3.1) has at least one non-constant positive classical solution if $d_2 \geqslant d^*$.*

Remark 5.3. If (1.6) holds, so does (5.1), and hence the assertion of theorem 3.4 holds by corollary 3.5.

Proof of theorem 5.2. Since $\theta > 0$, it follows that if d_2 is large enough, then (5.5) holds and $\mu_+(d_1, d_2) > \mu_-(d_1, d_2) > 0$. Also,

$$\lim_{d_2 \to \infty} \mu_+(d_1, d_2) = \frac{\theta}{d_1}, \qquad \lim_{d_2 \to \infty} \mu_-(d_1, d_2) = 0. \tag{5.7}$$

As $\theta/d_1 \in (\mu_q, \mu_{q+1})$, we see that there exists $d_0 \gg 1$ such that

$$\mu_+(d_1, d_2) \in (\mu_q, \mu_{q+1}), \quad 0 < \mu_-(d_1, d_2) < \mu_1 \quad \forall d_2 \geqslant d_0. \tag{5.8}$$

By theorem 4.1, we know that there exists $d > d_0$ such that (3.1) with $d_1 = d$ and $d_2 \geqslant d$ has no non-constant positive classical solution. Moreover, we can choose d so large that $\theta/d < \mu_1$. It follows that there exists $d^* > d$ such that

$$0 < \mu_-(d, d_2) < \mu_+(d, d_2) < \mu_1 \quad \forall d_2 \geqslant d^*. \tag{5.9}$$

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We shall prove that, for any $d_2 \geqslant d^*$, system (3.1) has at least one non-constant positive classical solution. On the contrary, suppose that this assertion is not true for some $d_2^* \geqslant d^*$. In the following, we will derive a contradiction by using a homotopy argument.

Fixing $d_2 = d_2^*$, for $t \in [0,1]$, we define

$$D(t) = \begin{pmatrix} td_1 + (1-t)d & 0 \\ 0 & td_2 + (1-t)d^* \end{pmatrix},$$

and consider the problem

$$\left.\begin{aligned} -\Delta \boldsymbol{u} &= D^{-1}(t)\boldsymbol{F}(\boldsymbol{u}), \quad && x \in \Omega, \\ \frac{\partial \boldsymbol{u}}{\partial \nu} &= 0, && x \in \partial\Omega. \end{aligned}\right\} \tag{5.10}$$

Note that $\boldsymbol{u}$ is a non-constant positive classical solution of (3.1) if and only if it is such a solution of (5.10) for $t = 1$. It is obvious that $\tilde{\boldsymbol{u}}$ is the unique positive constant solution of (5.10). For any $0 \leqslant t \leqslant 1$, $\boldsymbol{u}$ is a non-constant positive classical solution of (5.10) if and only if it is such a solution of the problem

$$h(\boldsymbol{u};t) := \boldsymbol{u} - (\boldsymbol{I} - \Delta)^{-1}\{D^{-1}(t)\boldsymbol{F}(\boldsymbol{u}) + \boldsymbol{u}\} = 0 \quad \text{on } \boldsymbol{X}. \tag{5.11}$$

Our earlier arguments have shown that (5.11) has no non-constant positive classical solution for $t = 0$, and we have assumed that there is no such solution for $t = 1$ (at $d_2 = d_2^*$). It is obvious that

$$h(\boldsymbol{u};1) = f(d_1, d_2; \boldsymbol{u}), \qquad h(\boldsymbol{u};0) = f(d, d^*; \boldsymbol{u}) \tag{5.12}$$

and

$$\left.\begin{aligned} D_{\boldsymbol{u}} f(d_1, d_2; \tilde{\boldsymbol{u}}) &= \boldsymbol{I} - (\boldsymbol{I} - \Delta)^{-1}(D^{-1}B + \boldsymbol{I}), \\ D_{\boldsymbol{u}} f(d, d^*; \tilde{\boldsymbol{u}}) &= \boldsymbol{I} - (\boldsymbol{I} - \Delta)^{-1}(\tilde{D}^{-1}B + \boldsymbol{I}), \end{aligned}\right\} \tag{5.13}$$

where $f(\cdot,\cdot;\cdot)$ has been defined in (5.3) and

$$\tilde{D} = \begin{pmatrix} d & 0 \\ 0 & d^* \end{pmatrix}.$$

In view of (5.8) and (5.9), it follows that

$$\mathcal{A}(d_1, d_2) \cap S_p = \{\mu_1, \mu_2, \dots, \mu_q\}, \qquad \mathcal{A}(d, d^*) \cap S_p = \emptyset.$$

Since σ_q is odd, lemma 5.1 gives

$$\left.\begin{aligned} &\operatorname{index}(h(\cdot;1), \tilde{\boldsymbol{u}}) = \operatorname{index}(f(d_1, d_2; \cdot), \tilde{\boldsymbol{u}}) = (-1)^{\sigma_q} = -1, \\ &\operatorname{index}(h(\cdot;0), \tilde{\boldsymbol{u}}) = \operatorname{index}(f(d, d^*; \cdot), \tilde{\boldsymbol{u}}) = (-1)^0 = 1. \end{aligned}\right\} \tag{5.14}$$

Now, by theorems 3.2 and 3.4, there exist positive constants

$$\underline{C} = \underline{C}(d_1, d, d^*, d_2^*, \Lambda) \quad \text{and} \quad \bar{C} = \bar{C}(m, k)$$

such that, for all $0 \leqslant t \leqslant 1$, the positive classical solutions of problem (5.11) satisfy $\underline{C} < u(x), v(x) < \bar{C}$ on $\bar{\Omega}$. Set

$$\Sigma = \{\boldsymbol{u} \in \boldsymbol{X} \mid \underline{C} < u(x),\ v(x) < \bar{C} \text{ on } \bar{\Omega}\}.$$

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Then $h(\boldsymbol{u};t) \neq 0$ for all $\boldsymbol{u} \in \partial\Sigma$ and $t \in [0,1]$. By the homotopy invariance of the Leray–Schauder degree [30],

$$\deg(h(\cdot;0),\Sigma,0) = \deg(h(\cdot;1),\Sigma,0). \tag{5.15}$$

Since both equations $h(\boldsymbol{u};0) = 0$ and $h(\boldsymbol{u};1) = 0$ have the unique positive classical solution $\tilde{\boldsymbol{u}}$ in Σ, by (5.14), we have

$$\begin{aligned}\deg(h(\cdot;0),\Sigma,0) &= \operatorname{index}(h(\cdot;0);\tilde{\boldsymbol{u}}) = 1,\\ \deg(h(\cdot;1),\Sigma,0) &= \operatorname{index}(h(\cdot;1),\tilde{\boldsymbol{u}}) = -1.\end{aligned}$$

This contradicts (5.15) and our proof is complete. □

Similarly, we have the following result whose proof we will omit.

THEOREM 5.4. *Assume that the pair* (d_1, d_2) *satisfies*

$$\mathcal{A}(d_1,d_2) \cap S_p = \{\mu_\ell, \mu_{\ell+1}, \dots, \mu_{\ell+q}\}$$

for some $\ell \geqslant 1$ *and* $q \geqslant 1$. *If* $\sigma_q = \sum_{j=0}^{q} m(\mu_{\ell+j})$ *is odd, then (3.1) has at least one non-constant positive classical solution.*

REMARK 5.5. It is easy to verify that

$$\lim_{d_1\to 0^+} \mu_+(d_1,d_2) = \frac{\beta\lambda - \theta\delta}{\theta d_2}, \qquad \lim_{d_1\to 0^+} \mu_+(d_1,d_2) = \infty.$$

If all μ_i, $i = 0,1,2,\dots$, are simple and $(\beta\lambda - \theta\delta)/(\theta d_2) \notin S_p$, then theorem 5.4 shows that there exists a sequence of intervals,

$$\{(d_-^{(j)}, d_+^{(j)})\}_{j=1}^{\infty}, \quad \text{with } d_+^{(j+1)} < d_-^{(j)} \text{ and } d_-^{(j)} \searrow 0^+ \text{ as } j \to \infty,$$

such that (3.1) has at least one non-constant positive classical solution for every $d_1 \in (d_-^{(j)}, d_+^{(j)})$.

6. Bifurcation of non-constant positive steady states

In this section, we consider the bifurcation of non-constant positive classical solutions with respect to the diffusion coefficients d_1 and d_2. It is assumed that b, k and m are fixed and (1.3) holds. We shall only consider the bifurcation with respect to the parameter d_2 when d_1 is kept fixed; the case where the roles of d_1 and d_2 are exchanged can be discussed similarly. The proofs of these results are based on topological-degree arguments used earlier in this paper; we shall omit them but refer readers to similar treatments in [31].

Recall that, for a constant solution $\hat{\boldsymbol{u}}_i$, $(\hat{d}_2;\hat{\boldsymbol{u}}_i) \in (0,\infty) \times \boldsymbol{X}$ is a *bifurcation point* of (3.1) if, for any $\delta \in (0,\hat{d}_2)$, there exists $d_2 \in [\hat{d}_2 - \delta, \hat{d}_2 + \delta]$ such that (3.1) has a non-constant positive solution. Otherwise, we say that $(\hat{d}_2;\hat{\boldsymbol{u}}_i)$ is a *regular point*.

Define $\mathcal{N}(d_2) = \{\mu > 0 \mid H(d_1,d_2;\mu) = 0\}$ for $d_2 > 0$, and consider the equilibrium $(\tilde{d}_2;\tilde{\boldsymbol{u}})$ of system (3.1) with $\tilde{d}_2 > 0$.

THEOREM 6.1 (local bifurcation).

(1) *If* $S_p \cap \mathcal{N}(\tilde{d}_2) = \emptyset$, *then* $(\tilde{d}_2;\tilde{\boldsymbol{u}})$ *is a regular point of (3.1).*

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(2) *Suppose* $S_p \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset$ *and* $(\delta d_1 - \theta \tilde{d}_2)^2 > 4 d_1 \tilde{d}_2 (\beta\lambda - \theta\delta)$. *If the sum* $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} m(\mu_i)$ *is odd, then* $(\tilde{d}_2; \tilde{\boldsymbol{u}})$ *is a bifurcation point of (3.1).*

THEOREM 6.2 (global bifurcation). *Suppose that*

$$S_p \cap \mathcal{N}(\tilde{d}_2) \neq \emptyset, \quad (\delta d_1 - \theta \tilde{d}_2)^2 > 4 d_1 \tilde{d}_2 (\beta\lambda - \theta\delta) \quad \textit{for some } \tilde{d}_2 > 0,$$

and that the assertion of theorem 3.4 holds. If the sum $\sum_{\mu_i \in \mathcal{N}(\tilde{d}_2)} m(\mu_i)$ *is odd, then there exists an interval* $(\alpha, \beta) \subset \boldsymbol{R}^+$ *such that, for every* $d_2 \in (\alpha, \beta)$*, system (3.1) admits a non-constant positive classical solution* $\boldsymbol{u} = \boldsymbol{u}(d_2)$*. Moreover, one of the following holds.*

(i) $\tilde{d}_2 = \alpha < \beta < \infty$ *and* $S_p \cap \mathcal{N}(\beta) \neq \emptyset$.

(ii) $0 < \alpha < \beta = \tilde{d}_2$ *and* $S_p \cap \mathcal{N}(\alpha) \neq \emptyset$.

(iii) $(\alpha, \beta) = (\tilde{d}_2, \infty)$.

(iv) $(\alpha, \beta) = (0, \tilde{d}_2)$.

7. Stability of non-constant positive steady states

In § 5, we described two ways in which non-constant positive steady-state solutions of (1.5) arise, namely, first, by fixing d_1 and choosing d_2 sufficiently large (theorem 5.2), and second, by fixing d_2 and choosing d_1 sufficiently small (theorem 5.4). In this section, we discuss the stability of such solutions.

THEOREM 7.1. *Let* (u_s, v_s) *be a positive classical solution of (3.1). If it satisfies* $(u_s(x) + m v_s(x))^2 > 2 b v_s(x)$ *on* $\bar{\Omega}$*, then* (u_s, v_s) *is stable in the sense that there exists a small positive constant* σ *such that when the initial data* (u_0, v_0) *of (1.5) satisfy*

$$(1-\sigma) u_s(x) \leqslant u_0(x) \leqslant (1+\sigma) u_s(x), \qquad (1-\sigma) v_s(x) \leqslant v_0(x) \leqslant (1+\sigma) v_s(x) \quad (7.1)$$

$\forall x \in \bar{\Omega}$*, then the solution* (u, v) *of (1.5) satisfies*

$$\left.\begin{aligned} (1-\sigma) u_s(x) \leqslant u(x,t) \leqslant (1+\sigma) u_s(x), \quad x \in \bar{\Omega}, \quad t \geqslant 0, \\ (1-\sigma) v_s(x) \leqslant v(x,t) \leqslant (1+\sigma) v_s(x), \quad x \in \bar{\Omega}, \quad t \geqslant 0. \end{aligned}\right\} \quad (7.2)$$

Proof. For ease of presentation, we assume that $m = r = 1$. Define

$$f(u,v) = u(1-u) - \frac{buv}{u+mv}, \qquad g(u,v) = -kv + \frac{uv}{u+mv},$$

and set

$$\bar{u} = (1+\sigma) u_s, \qquad \underline{u} = (1-\sigma) u_s, \qquad \bar{v} = (1+\sigma) v_s, \qquad \underline{v} = (1-\sigma) v_s, \quad (7.3)$$

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where σ, $0 < \sigma \ll 1$, is to be determined later. Applying the mean-value theorem twice, we have

$$\begin{aligned}
&f(u_s, v_s) - f(\bar{u}, \underline{v}) \\
&\qquad = f_u(\xi, \eta)(u_s - \bar{u}) + f_v(\xi, \eta)(v_s - \underline{v}) \\
&\qquad = f_u(u_s, v_s)(u_s - \bar{u}) + f_v(u_s, v_s)(v_s - \underline{v}) \\
&\qquad\qquad + [f_u(\xi, \eta) - f_u(u_s, v_s)](u_s - \bar{u}) + [f_v(\xi, \eta) - f_v(u_s, v_s)](v_s - \underline{v}) \\
&\qquad = f_u(u_s, v_s)(u_s - \bar{u}) + f_v(u_s, v_s)(v_s - \underline{v}) \\
&\qquad\qquad + [f_{uu}(\xi_1, \eta_1)(\xi - u_s) + f_{uv}(\xi_1, \eta_1)(\eta - v_s)](u_s - \bar{u}) \\
&\qquad\qquad + [f_{uv}(\xi_2, \eta_2)(\xi - u_s) + f_{vv}(\xi_2, \eta_2)(\eta - v_s)](v_s - \underline{v}) \\
&\qquad \triangleq f_u(u_s, v_s)(u_s - \bar{u}) + f_v(u_s, v_s)(v_s - \underline{v}) + I,
\end{aligned}$$

where

$$u_s \leqslant \xi_1, \xi_2 \leqslant \xi \leqslant \bar{u} = (1 + \sigma)u_s, \qquad (1 - \sigma)v_s = \underline{v} \leqslant \eta \leqslant \eta_1, \eta_2 \leqslant v_s.$$

Thus $I = \sigma O(\sigma)$, and, in turn,

$$f(u_s, v_s) - f(\bar{u}, \underline{v}) = f_u(u_s, v_s)(u_s - \bar{u}) + f_v(u_s, v_s)(v_s - \underline{v}) + \sigma O(\sigma). \tag{7.4}$$

Similarly,

$$f(u_s, v_s) - f(\underline{u}, \bar{v}) = f_u(u_s, v_s)(u_s - \underline{u}) + f_v(u_s, v_s)(v_s - \bar{v}) + \sigma O(\sigma). \tag{7.5}$$

Since $(u_s(x) + v_s(x))^2 > 2bv_s(x)$ and $u_s(x) > 0$ on $\bar{\Omega}$, there exists $\sigma_0 > 0$ such that

$$u_s^2(x)\left[1 - \frac{2bv_s(x)}{(u_s(x) + v_s(x))^2}\right] + O(\sigma) > 0 \quad \forall x \in \bar{\Omega}, \quad 0 < \sigma \leqslant \sigma_0. \tag{7.6}$$

We are now in the position to verify that $(\bar{u}, \bar{v})$ and $(\underline{u}, \underline{v})$ defined by (7.3) are upper and lower solutions of (1.5), provided that (7.1) is satisfied. We note that (1.5) is a mixed quasi-monotone system for the positive u, v. In view of (7.4), we obtain by direct calculation that

$$\begin{aligned}
&\bar{u}_t - d_1 \Delta \bar{u} - f(\bar{u}, \underline{v}) \\
&\qquad = (1 + \sigma) f(u_s, v_s) - f(\bar{u}, \underline{v}) \\
&\qquad = \sigma f(u_s, v_s) + f(u_s, v_s) - f(\bar{u}, \underline{v}) \\
&\qquad = \sigma f(u_s, v_s) + f_u(u_s, v_s)(u_s - \bar{u}) + f_v(u_s, v_s)(v_s - \underline{v}) + \sigma O(\sigma) \\
&\qquad = \sigma f(u_s, v_s) - \sigma f_u(u_s, v_s) u_s + \sigma f_v(u_s, v_s) v_s + \sigma O(\sigma) \\
&\qquad = \sigma f(u_s, v_s) - \sigma [f_u(u_s, v_s) u_s + f_v(u_s, v_s) v_s] + 2\sigma f_v(u_s, v_s) v_s + \sigma O(\sigma) \\
&\qquad = \sigma f(u_s, v_s) - \sigma [f_u(u_s, v_s) - u_s^2] + 2\sigma f_v(u_s, v_s) v_s + \sigma O(\sigma).
\end{aligned} \tag{7.7}$$

As $f_v(u_s, v_s) = -bu_s^2/(u_s + v_s)^2$, in view of (7.6) and (7.7), we have

$$\bar{u}_t - d_1 \Delta \bar{u} - f(\bar{u}, \underline{v}) = \sigma \left\{ u_s^2 \left[1 - \frac{2bv_s}{(u_s + v_s)^2}\right] + O(\sigma) \right\} > 0, \quad x \in \bar{\Omega}, \quad t \geqslant 0. \tag{7.8}$$

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Similarly, by (7.5) and (7.6), we have

$$\underline{u}_t - d_1\Delta\underline{u} - f(\underline{u},\bar{v}) = -\sigma\left\{u_s^2\left[1-\frac{2bv_s}{(u_s+v_s)^2}\right]+O(\sigma)\right\} < 0, \quad x\in\bar{\Omega}, \quad t\geqslant 0. \tag{7.9}$$

Further, we can easily verify that

$$\left.\begin{aligned} \bar{v}_t - d_2\Delta\bar{v} - g(\bar{u},\bar{v}) &= (1+\sigma)g(u_s,v_s) - g(\bar{u},\bar{v}) = 0, & x\in\bar{\Omega}, \quad & t\geqslant 0,\\ \underline{v}_t - d_2\Delta\underline{v} - g(\underline{u},\underline{v}) &= (1-\sigma)g(u_s,v_s) - g(\underline{u},\underline{v}) = 0, & x\in\bar{\Omega}, \quad & t\geqslant 0,\\ \frac{\partial\bar{u}}{\partial\nu} = \frac{\partial\underline{u}}{\partial\nu} &= \frac{\partial\bar{v}}{\partial\nu} = \frac{\partial\underline{v}}{\partial\nu} = 0, & x\in\partial\Omega, \quad & t\geqslant 0. \end{aligned}\right\} \tag{7.10}$$

Now, equations (7.8)–(7.10) show that if the initial data (u_0,v_0) of problem (1.5) satisfy (7.1), then $(\bar{u},\bar{v})$ and $(\underline{u},\underline{v})$ defined by (7.3) are upper and lower solutions of (1.5), and hence (7.2) holds. This completes the proof. □

Remark 7.2. Theorem 7.1 leads us to believe that the solution (u_s,v_s) of (3.1) that satisfies $(u_s(x)+mv_s(x))^2 > 2bv_s(x)$ should be obtained by theorem 5.4 with $d_1 \ll 1$. However, the asymptotic stability of (u_s,v_s) remains open. Also, one would like to have a better understanding of the condition $(u_s(x)+mv_s(x))^2 > 2bv_s(x)$ on $\bar{\Omega}$.

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