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SKEW DERIVATIONS IN BANACH ALGEBRAS

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Abstract We investigate the global versions of the Kleinecke–Shirokov theorem for skew derivations in Banach algebras. Centralizing skew derivations on Banach algebras are also studied.

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1. Introduction

Throughout, unless specially stated, $\mathcal{A}$ always denotes a complex Banach algebra with centre $Z(\mathcal{A})$. By $\mathrm{rad}(\mathcal{A})$ and $Q(\mathcal{A})$, we denote the Jacobson radical of $\mathcal{A}$ and the set of all quasinilpotent elements of $\mathcal{A}$, respectively. For $a, b \in \mathcal{A}$, we denote by $[a, b] = ab - ba$ the commutator of a and b. A linear map $d\colon \mathcal{A} \to \mathcal{A}$ is called a derivation of $\mathcal{A}$ if $d(ab) = d(a)b + ad(b)$ for all $a, b \in \mathcal{A}$. For $a \in \mathcal{A}$, the map $d_a\colon b \in \mathcal{A} \mapsto [a, b]$ defines a derivation of $\mathcal{A}$ called the inner derivation of $\mathcal{A}$ induced by a.

The classical Kleinecke–Shirokov theorem [**19**, **33**] states that if a and b are elements in $\mathcal{A}$ such that $[b, [b, a]] = 0$, then $[b, a]$ is quasinilpotent. A reformulation of the Kleinecke–Shirokov theorem says that if an inner derivation d_b of $\mathcal{A}$ satisfies $d_b^2(a) = 0$ for $a \in \mathcal{A}$, then $d_b(a)$ is quasinilpotent. This result has been generalized to continuous derivations (see, for example, [**26**]) and to arbitrary derivations by Thomas [**38**]. In [**31**] Pták gave a global version of the Kleinecke–Shirokov theorem and proved that if d is an inner derivation of $\mathcal{A}$ such that $d^2(a)$ is quasinilpotent for every $a \in \mathcal{A}$, then $d^2(a)^2$ lies in the radical of $\mathcal{A}$ for every $a \in \mathcal{A}$. Later it was also generalized to arbitrary derivations by Turovskii and Shul'man [**39**]. On the other hand, according to the Kleinecke–Shirokov theorem, we see that if an inner derivation d_a of $\mathcal{A}$ satisfies $[d_a(b), b] = 0$ for $b \in \mathcal{A}$, then $d_a(b)$ is quasinilpotent. This result has also been extended to continuous derivations (see, for example, [**26**]) but it is still unknown for discontinuous derivations. In [**10**], Brešar and Vukman gave another global version of the Kleinecke–Shirokov theorem and proved that if d is a continuous derivation of $\mathcal{A}$ such that $[d(a), a] \in \mathrm{rad}(\mathcal{A})$ for every $a \in \mathcal{A}$, then $d(a)$ lies in the radical of $\mathcal{A}$ for every $a \in \mathcal{A}$. Later, Brešar [**5**] showed that if d is a continuous derivation of $\mathcal{A}$ such that $[d(a), a]$ is quasinilpotent for every $a \in \mathcal{A}$, then $d(a)$

lies in the radical of $\mathcal{A}$ for every $a \in \mathcal{A}$. Recently, Lee [**20**] proved that if d is a derivation of $\mathcal{A}$ such that $[d(a), a]$ is quasinilpotent for every $a \in \mathcal{A}$, then $d(\mathcal{I}_{\mathcal{A}}) \subseteq \mathrm{rad}(\mathcal{A})$, where $\mathcal{I}_{\mathcal{A}}$ is the ideal of $\mathcal{A}$ generated by all commutators of $\mathcal{A}$.

Let σ be a linear automorphism of $\mathcal{A}$ and let $1_{\mathcal{A}}$ denote the identity automorphism of $\mathcal{A}$. By a σ-derivation of $\mathcal{A}$ we mean a linear map $\delta\colon \mathcal{A} \to \mathcal{A}$ such that $\delta(ab) = \sigma(a)\delta(b)+\delta(a)b$ for all $a, b \in \mathcal{A}$. Generally, we call σ-derivations *skew derivations*. Clearly, the map $\sigma - 1_{\mathcal{A}}$ is a σ-derivation and $1_{\mathcal{A}}$-derivations are just ordinary derivations. Thus, the concept of σ-derivations can be viewed as an extension of derivations and automorphisms. The skew derivations appear in q-Weyl algebras, enveloping algebras of solvable Lie superalgebras and coordinate rings of quantum matrices [**17**]. See [**1**, **6**, **8**, **9**, **12**, **14**, **18**, **21**–**24**, **28**] for some recent results concerning skew derivations in Banach algebras. Brešar and Villena [**9**] proved that if δ is a continuous σ-derivation of $\mathcal{A}$ satisfying $\delta^2(a) = 0$ for some $a \in \mathcal{A}$, where σ is a continuous automorphism of $\mathcal{A}$ such that $\delta\sigma = \sigma\delta$, then $\delta(a)$ is quasinilpotent. In this paper, we investigate global versions of the Kleinecke–Shirokov theorem for skew derivations in Banach algebras. Our main results are as follows.

Theorem 1.1. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. If $\delta^2(a)$ is quasinilpotent for every $a \in \mathcal{A}$, then $\delta^2(a)^2$ lies in the radical of $\mathcal{A}$ for every $a \in \mathcal{A}$.*

An element $a \in \mathcal{A}$ is said to be central modulo the radical of $\mathcal{A}$ if $[a, b] \in \mathrm{rad}(\mathcal{A})$ for all $b \in \mathcal{A}$. Some spectral characterizations of elements that are central modulo the radical have been studied in [**7**, **30**].

Theorem 1.2. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. If $[\delta(a), a]$ is quasinilpotent for every $a \in \mathcal{A}$, then $\delta(a)$ is central modulo the radical of $\mathcal{A}$ for every $a \in \mathcal{A}$.*

In [**11**], Brešar and Vukman proved that if d is a continuous derivation of $\mathcal{A}$ such that $[d(a), a]^2 \in \mathrm{rad}(\mathcal{A})$ for every $a \in \mathcal{A}$, then $d(A) \subseteq \mathrm{rad}(\mathcal{A})$. An automorphism σ of $\mathcal{A}$ is said to be inner if there exists a unit u in $\mathcal{A}$ such that $\sigma(a) = uau^{-1}$ for all $a \in \mathcal{A}$. As an application of Theorem 1.2, we have the following corollary.

Corollary 1.3. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. If $[\delta(a), a]^{n(a)} \in \mathrm{rad}(\mathcal{A})$ for every $a \in \mathcal{A}$, where $n(a) \geqslant 1$ is an integer depending on a, then $[\delta(\mathcal{A}), \mathcal{A}] \subseteq \mathrm{rad}(\mathcal{A})$. Moreover, $\delta(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$ if σ is inner and δ is continuous.*

In [**6**] Brešar proved that if σ is an automorphism of $\mathcal{A}$ such that $[\sigma(a), a] \in \mathrm{rad}(\mathcal{A})$ for every $a \in \mathcal{A}$, then $(\sigma - 1_{\mathcal{A}})(a)$ is central modulo the radical of $\mathcal{A}$ for every $a \in \mathcal{A}$. As an application of Theorem 1.2, we have the following corollary.

Corollary 1.4. *Let $\mathcal{A}$ be a complex Banach algebra and let σ be an automorphism of $\mathcal{A}$. If $[\sigma(a), a] \in Q(\mathcal{A})$ for every $a \in \mathcal{A}$, then $(\sigma - 1_{\mathcal{A}})(a)$ is central modulo the radical of $\mathcal{A}$ for every $a \in \mathcal{A}$.*

In 1955 Singer and Wermer [**36**] showed that every continuous derivation on a commutative Banach algebra $\mathcal{A}$ has its range in $\mathrm{rad}(\mathcal{A})$. They also conjectured that the

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continuity assumption for the derivations was superfluous. It was more than 30 years before this conjecture was finally proved by Thomas [**37**]. In [**27**] Mathieu and Runde gave a noncommutative version of the Singer–Wermer theorem and proved that if d is a derivation of $\mathcal{A}$ such that $[d(a), a] \in Z(\mathcal{A})$ for every $a \in \mathcal{A}$, then $d(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$. Using Theorem 1.2, we obtain the following theorem.

Theorem 1.5. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. If $[\delta(a), a] \in Z(\mathcal{A})$ for every $a \in \mathcal{A}$, then $[\delta(\mathcal{A}), \mathcal{A}] \subseteq \mathrm{rad}(\mathcal{A})$. Moreover, $\delta(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$ if σ is inner.*

It is noteworthy to mention that our approaches to the proofs of this paper are quite different from those in [**5**,**10**,**11**,**20**,**25**,**31**] and are based on the extended Jacobson density theorems for rings with automorphisms and skew derivations. Such density theorems connect the concept of a dense action of irreducible representations with the concept of outerness of automorphisms and skew derivations recently developed by Beidar and Brešar [**2**,**6**] and by Chuang and Liu [**14**], respectively. It is also our aim here to present a new possible technique that can be used in the study of skew derivations in Banach algebras.

2. Preliminaries

Throughout this section, $\mathcal{A}$ denotes a complex Banach algebra. By $\mathrm{Prim}(\mathcal{A})$ we denote the set of all primitive ideals of $\mathcal{A}$. The (Jacobson) radical $\mathrm{rad}(\mathcal{A})$ of $\mathcal{A}$ is defined to be the intersection of all primitive ideals of A and, by the usual convention, $\mathrm{rad}(\mathcal{A}) = \mathcal{A}$ if there are no primitive ideals of $\mathcal{A}$. For a complex Banach space X, we denote by $L(X)$ the algebra of all linear operators on X and $B(X)$ by the Banach algebra of all bounded linear operators on X. We say that π is a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X if π is a continuous algebra homomorphism from $\mathcal{A}$ into $B(X)$ such that the only invariant subspaces of X under $\pi(\mathcal{A})$ are $\{0\}$ and X. It is known that the kernel of a continuous irreducible representation of $\mathcal{A}$ is a primitive ideal of $\mathcal{A}$ and for each primitive ideal P of $\mathcal{A}$ there exists a continuous irreducible representation π_P of $\mathcal{A}$ on a complex Banach space X_P such that $\ker \pi_P = P$ and $\pi_P(\mathcal{A}) \cong \mathcal{A}/P$ acts densely on X_P. We write $\mathrm{sp}(x)$ for the spectrum of $x \in \mathcal{A}$. If $\mathrm{Prim}(A) \neq \emptyset$, we have the following result [**32**, Theorem 2.2.9]:

$$\mathrm{sp}(x) = \begin{cases} \displaystyle\bigcup_{P \in \mathrm{Prim}(\mathcal{A})} \mathrm{sp}(\pi_P(x)) & \text{if } \mathcal{A} \text{ is unital,} \\ \displaystyle\bigcup_{P \in \mathrm{Prim}(\mathcal{A})} \mathrm{sp}(\pi_P(x)) \cup \{0\} & \text{if } \mathcal{A} \text{ is non-unital.} \end{cases}$$

Throughout this section, π always denotes a continuous irreducible representation of $\mathcal{A}$ on the complex Banach space X. Following [**6**], we call an automorphism σ of $\mathcal{A}$ π-inner if there exists an invertible $S \in L(X)$ such that $\pi\sigma(a) = S\pi(a)S^{-1}$ for all $a \in \mathcal{A}$. An automorphism that is not π-inner is called π-outer. Two automorphisms σ and τ of $\mathcal{A}$ are called π-dependent if $\sigma\tau^{-1}$ is π-inner, that is, there exists an invertible $S \in L(X)$ such that $\pi\sigma(a) = S\pi\tau(a)S^{-1}$ for all $a \in \mathcal{A}$. Otherwise, they are called π-independent. Clearly, an automorphism σ of $\mathcal{A}$ and the identity automorphism $1_{\mathcal{A}}$ of $\mathcal{A}$ are π-independent if and only if σ is π-outer.

To prove our results, we need the notion generalized in [**14**] (or see [**21**]) to σ-derivations of $\mathcal{A}$ into $L(X)$. Let σ be an automorphism of $\mathcal{A}$. By a σ-derivation $\tilde{\delta}\colon \mathcal{A} \to L(X)$, we mean that $\tilde{\delta}$ is a linear map satisfying $\tilde{\delta}(ab) = \pi\sigma(a)\tilde{\delta}(b) + \tilde{\delta}(a)\pi(b)$ for all $a, b \in \mathcal{A}$. Clearly, if $\delta\colon \mathcal{A} \to \mathcal{A}$ is a σ-derivation of $\mathcal{A}$, then the map $\tilde{\delta} = \pi\delta\colon \mathcal{A} \to L(X)$ is a σ-derivation. A σ-derivation $\tilde{\delta}\colon \mathcal{A} \to L(X)$ is called π-inner if there exists $T \in L(X)$ such that $\tilde{\delta}(a) = \pi\sigma(a)T - T\pi(a)$ for all $a \in \mathcal{A}$. Otherwise, it is called π-outer. Note that a σ-derivation δ of $\mathcal{A}$ is called π-inner if $\tilde{\delta} = \pi\delta$ is π-inner. We have the extended Jacobson density theorems on π-outer σ-derivations and automorphisms as follows.

Theorem 2.1 (Chuang and Liu [14, Theorem 2.7]). *Let $\delta\colon \mathcal{A} \to L(X)$ be a π-outer σ-derivation, where σ is π-outer. Then, for any $\mathbb{C}$-independent $x_1, \dots, x_n \in X$ and arbitrary $y_1, \dots, y_n$, $z_1, \dots, z_n$, $w_1, \dots, w_n \in X$, there exists $a \in \mathcal{A}$ such that $\delta(a)x_i = y_i$, $\pi\sigma(a)x_i = z_i$ and $\pi(a)x_i = w_i$ for all $i = 1, \dots, n$.*

Theorem 2.2 (Chuang and Liu [14, Theorem 2.6]). *Let $\delta\colon \mathcal{A} \to L(X)$ be a π-outer σ-derivation, where σ is π-inner. Then, for any $\mathbb{C}$-independent $x_1, \dots, x_n \in X$ and arbitrary $y_1, \dots, y_n$, $z_1, \dots, z_n \in X$, there exists $a \in \mathcal{A}$ such that $\delta(a)x_i = y_i$ and $\pi(a)x_i = z_i$ for all $i = 1, \dots, n$.*

Theorem 2.3 (Brešar [6, Theorem 1.2]). *Suppose that $\sigma_1, \dots, \sigma_m$ are automorphisms of A such that σ_i and σ_j are π-independent for all $i \neq j$. Then, for any $\mathbb{C}$-independent $x_1, \dots, x_n \in X$ and arbitrary $y_{ij} \in X$, there exists $a \in \mathcal{A}$ such that $\pi\sigma_i(a)x_j = y_{ij}$ for all $i = 1, \dots, m$ and $j = 1, \dots, n$.*

Lemma 2.4 (Chebotar *et al.* [13, Lemma 2.7]). *Let X be a vector space over $\mathbb{C}$ and let $T\colon X \to X$ and $U\colon X \to X$ be linear operators. Suppose that $Tx \in \mathbb{C}Ux$ for all $x \in X$. Then $T = \alpha U$ for some $\alpha \in \mathbb{C}$.*

3. Proof of Theorem 1.1

Let $\mathcal{A}$ be a complex Banach algebra and let δ be a σ-derivation of $\mathcal{A}$, where σ is an automorphism of $\mathcal{A}$. For $a, b \in \mathcal{A}$, we have

$$\delta^2(ab) = \delta(\sigma(a)\delta(b) + \delta(a)b) = \sigma^2(a)\delta^2(b) + (\delta\sigma + \sigma\delta)(a)\delta(b) + \delta^2(a)b,$$

and hence

$$\delta^2(\sigma^{-1}(a)b) = \sigma(a)\delta^2(b) + (\delta + \sigma\delta\sigma^{-1})(a)\delta(b) + \delta^2\sigma^{-1}(a)b.$$

Let π be a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X. Then,

$$\pi\delta^2(ab) = \pi\sigma^2(a)\pi\delta^2(b) + \pi(\delta\sigma + \sigma\delta)(a)\pi\delta(b) + \pi\delta^2(a)\pi(b) \tag{3.1}$$

and

$$\pi\delta^2(\sigma^{-1}(a)b) = \pi\sigma(a)\pi\delta^2(b) + \pi(\delta + \sigma\delta\sigma^{-1})(a)\pi\delta(b) + \pi\delta^2\sigma^{-1}(a)\pi(b) \tag{3.2}$$

for all $a, b \in \mathcal{A}$.

Lemma 3.1. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. Suppose that $\delta^2(a)$ is quasinilpotent for every $a \in \mathcal{A}$. If π is a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X such that δ is π-outer, then $\pi\delta^2(a) = 0$ and $\pi(\delta\sigma + \sigma\delta)(a) = 0$ for every $a \in \mathcal{A}$.*

Proof. Note that $\sigma\delta\sigma^{-1}$ and $\delta + \sigma\delta\sigma^{-1}$ are both σ-derivations of $\mathcal{A}$. We divide the proof into two cases.

Case 1 ($\boldsymbol{\delta + \sigma\delta\sigma^{-1}}$ is $\boldsymbol{\pi}$-outer). Choose $0 \neq x \in X$. By Theorem 2.1 and 2.2, there is $b \in \mathcal{A}$ such that $\pi\delta(b)x = x$ and $\pi(b)x = 0$. Assume first that σ is π-outer. Let $Y = \mathbb{C}x + \mathbb{C}\pi\delta^2(b)x$. By Theorem 2.1, there is $a \in \mathcal{A}$ such that $\pi(\delta + \sigma\delta\sigma^{-1})(a)x = x$ and $\pi\sigma(a)Y = 0$. Thus, $\pi\sigma(a)\pi\delta^2(b)x = 0$. In view of (3.2), $\pi\delta^2(\sigma^{-1}(a)b)x = x$, a contradiction. Assume next that σ is π-inner. Then there is an invertible $S \in L(X)$ such that $\pi\sigma(a) = S\pi(a)S^{-1}$ for all $a \in \mathcal{A}$. Let $Y = \mathbb{C}x + \mathbb{C}S^{-1}\pi\delta^2(b)x$. By Theorem 2.2, there is an $a \in \mathcal{A}$ such that $\pi(\delta+\sigma\delta\sigma^{-1})(a)x = x$ and $\pi(a)Y = 0$. Thus, $\pi(a)S^{-1}\pi\delta^2(b)x = 0$. In particular, $\pi\sigma(a)\pi\delta^2(b)x = S\pi(a)S^{-1}\pi\delta^2(b)x = 0$. In view of (3.2), $\pi\delta^2(\sigma^{-1}(a)b)x = x$, a contradiction.

Case 2 ($\boldsymbol{\delta+\sigma\delta\sigma^{-1}}$ is $\boldsymbol{\pi}$-inner). So, there is a $T \in L(X)$ such that $\pi(\delta+\sigma\delta\sigma^{-1})(a) = \pi\sigma(a)T - T\pi(a)$ for all $a \in \mathcal{A}$. In this case, (3.2) becomes

$$\begin{aligned}\pi\delta^2(\sigma^{-1}(a)b) &= \pi\sigma(a)\pi\delta^2(b) + (\pi\sigma(a)T - T\pi(a))\pi\delta(b) + \pi\delta^2\sigma^{-1}(a)\pi(b)\\ &= \pi\sigma(a)(\pi\delta^2(b) + T\pi\delta(b)) - T\pi(a)\pi\delta(b) + \pi\delta^2\sigma^{-1}(a)\pi(b)\end{aligned}\tag{3.3}$$

for all $a, b \in \mathcal{A}$.

Assume first that $T = 0$. Then $\pi(\delta + \sigma\delta\sigma^{-1}) = 0$, and hence $\pi(\delta\sigma + \sigma\delta) = 0$. By (3.1),

$$\pi\delta^2(ab) = \pi\sigma^2(a)\pi\delta^2(b) + \pi\delta^2(a)\pi(b)$$

for all $a, b \in \mathcal{A}$. This implies that $\pi\delta^2\colon \mathcal{A} \to L(X)$ is a σ^2-derivation. Moreover, $\pi\delta^2$ must be π-inner; otherwise, by Theorem 2.1 and 2.2, for any $0 \neq x \in X$ there would exist $a \in \mathcal{A}$ such that $\pi\delta^2(a)x = x$, a contradiction. Let $U \in L(X)$ be such that $\pi\delta^2(a) = \pi\sigma^2(a)U - U\pi(a)$ for all $a \in \mathcal{A}$. If $U = 0$, then $\pi\delta^2 = 0$ and we are done. Assume that $Ux \neq 0$ for some $x \in X$. Then σ^2 is π-inner; otherwise, by Theorem 2.3, there would exist $a \in \mathcal{A}$ such that $\pi\sigma^2(a)Ux = x$ and $\pi(a)x = 0$, and so $\pi\delta^2(a)x = (\pi\sigma^2(a)U - U\pi(a))x = x$, a contradiction. Hence, there exists an invertible $S \in L(X)$ such that $\pi\sigma^2(a) = S\pi(a)S^{-1}$ and so $\pi\delta^2(a) = S\pi(a)S^{-1}U - U\pi(a)$ for all $a \in \mathcal{A}$. If $S^{-1}U \notin \mathbb{C}I$, there would exist $x \in X$ such that $S^{-1}Ux$ and x are $\mathbb{C}$-independent, letting $a \in \mathcal{A}$ be such that $\pi(a)x = 0$ and $\pi(a)S^{-1}Ux = S^{-1}x$, and then $\pi\delta^2(a)x = (S\pi(a)S^{-1}U - U\pi(a))x = x$, a contradiction. Hence, $S^{-1}U \in \mathbb{C}I$. Thus, $\pi\delta^2(a) = S\pi(a)S^{-1}U - U\pi(a) = S(S^{-1}U)\pi(a) - U\pi(a) = 0$ for all $a \in \mathcal{A}$ and we are done.

Assume now that $T \neq 0$. Choose $x \in X$ such that $Tx \neq 0$. Suppose that σ is π-outer. By Theorem 2.1, there is $b \in \mathcal{A}$ such that $\pi\delta(b)Tx = x$ and $\pi(b)Tx = 0$. Let $Y = \mathbb{C}x + \mathbb{C}(\pi\delta^2(b)+T\pi\delta(b))Tx$. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi\sigma(a)Y = 0$ and $\pi(a)x = -x$. In particular, $\pi\sigma(a)(\pi\delta^2(b)+T\pi\delta(b))Tx = 0$. Then, by (3.3), $\pi\delta^2(\sigma^{-1}(a)b)Tx = Tx$, a contradiction. Hence, σ is π-inner. That is, there exists an invertible $S \in L(X)$ such that $\pi\sigma(a) = S\pi(a)S^{-1}$ and so (3.3) becomes

$$\pi\delta^2(\sigma^{-1}(a)b) = S\pi(a)S^{-1}(\pi\delta^2(b) + T\pi\delta(b)) - T\pi(a)\pi\delta(b) + \pi\delta^2\sigma^{-1}(a)\pi(b) \tag{3.4}$$

for all $a, b \in \mathcal{A}$. Suppose that $S^{-1}T \in \mathbb{C}I$. Then, $\pi(\delta + \sigma\delta\sigma^{-1})(a) = \pi\sigma(a)T - T\pi(a) = S\pi(a)S^{-1}T - T\pi(a) = S(S^{-1}T)\pi(a) - T\pi(a) = 0$, and thus $\pi(\delta\sigma + \sigma\delta) = 0$. With this and (3.1), we see that $\pi\delta^2\colon \mathcal{A} \to L(X)$ is a σ^2-derivation. By the same proof as above, we obtain $\pi\delta^2 = 0$, as desired. Hence, we may assume that $S^{-1}T \notin \mathbb{C}I$. Choose $x \in X$ such that $S^{-1}Tx$ and x are $\mathbb{C}$-independent. Then Tx and Sx are $\mathbb{C}$-independent. By Theorems 2.1 and 2.2, there exists $b \in \mathcal{A}$ such that $\pi(b)Tx = \pi(b)Sx = 0$, $\pi\delta(b)Tx = Tx$ and $\pi\delta(b)Sx = Sx$. Thus, for $\mu, \gamma \in \mathbb{C}$, $\pi(b)(\mu Tx + \gamma Sx) = 0$, $\pi\delta(b)(\mu Tx + \gamma Sx) = \mu Tx + \gamma Sx$, and by (3.4) we have

$$\begin{aligned}\pi\delta^2(\sigma^{-1}(a)b)(\mu Tx + \gamma Sx)& \\ &= S\pi(a)S^{-1}(\pi\delta^2(b) + T\pi\delta(b))(\mu Tx + \gamma Sx) - T\pi(a)(\mu Tx + \gamma Sx) \\ &= S\pi(a)U(\mu Tx + \gamma Sx) - T\pi(a)(\mu Tx + \gamma Sx)\end{aligned} \tag{3.5}$$

for all $a \in \mathcal{A}$, where $U = S^{-1}(\pi\delta^2(b) + T\pi\delta(b))$. If $U(\mu Tx + \gamma Sx)$ and $\mu Tx + \gamma Sx$ are $\mathbb{C}$-independent for some $\mu, \gamma \in \mathbb{C}$, letting $a \in \mathcal{A}$ such that $\pi(a)U(\mu Tx + \gamma Sx) = \gamma x$ and $\pi(a)(\mu Tx + \gamma Sx) = -\mu x$, then by (3.5), $\pi\delta^2(\sigma^{-1}(a)b)(\mu Tx + \gamma Sx) = \mu Tx + \gamma Sx$, a contradiction. Hence, we conclude that

$$U(\mu Tx + \gamma Sx) \text{ and } \mu Tx + \gamma Sx \text{ are } \mathbb{C}\text{-dependent for all } \mu, \gamma \in \mathbb{C}.$$

This implies that $UTx = \alpha Tx$, $USx = \beta Sx$ and $U(Tx + Sx) = \ell(Tx + Sx)$ for $\alpha, \beta, \ell \in \mathbb{C}$. Thus, $\ell(Tx + Sx) = U(Tx + Sx) = UTx + USx = \alpha Tx + \beta Sx$, implying that $(\ell - \alpha)Tx + (\ell - \beta)Sx = 0$. By the $\mathbb{C}$-independence of Tx and Sx, we obtain $\alpha = \beta = \ell$. This implies that $USx = \alpha Sx$. Thus, $U(Tx - \alpha Sx) = \alpha(Tx - \alpha Sx)$. With this, and setting $\mu = 1$ and $\gamma = -\alpha$ in (3.5), we obtain

$$\pi\delta^2(\sigma^{-1}(a)b)(Tx - \alpha Sx) = (\alpha S - T)\pi(a)(Tx - \alpha Sx) \tag{3.6}$$

for all $a \in \mathcal{A}$. Let $a \in \mathcal{A}$ be such that $\pi(a)(Tx - \alpha Sx) = -x$. By (3.6), we obtain $\pi\delta^2(\sigma^{-1}(a)b)(Tx - \alpha Sx) = Tx - \alpha Sx$, a contradiction. □

Lemma 3.2. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. Suppose that $\delta^2(a)$ is quasinilpotent for every $a \in \mathcal{A}$. If π is a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X such that δ and σ are both π-inner, then $\pi(\delta^2(a)^2) = 0$ for every $a \in \mathcal{A}$.*

Proof. By assumption, there exist $T \in L(X)$ and an invertible $S \in L(X)$ such that $\pi\delta(a) = \pi\sigma(a)T - T\pi(a)$ and $\pi\sigma(a) = S\pi(a)S^{-1}$ for all $a \in \mathcal{A}$. Thus, $\pi\delta(a) = S\pi(a)S^{-1}T - T\pi(a)$ for all $a \in A$. We then have

$$\begin{aligned}\pi\delta^2(a) &= S\pi(\delta(a))S^{-1}T - T\pi(\delta(a)) \\ &= S(S\pi(a)S^{-1}T - T\pi(a))S^{-1}T - T(S\pi(a)S^{-1}T - T\pi(a)) \\ &= S^2\pi(a)(S^{-1}T)^2 - (ST + TS)\pi(a)S^{-1}T + T^2\pi(a)\end{aligned} \tag{3.7}$$

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for all $a \in \mathcal{A}$. If there is $x \in X$ such that $(S^{-1}T)^2x$, $S^{-1}Tx$ and x are $\mathbb{C}$-independent, letting $a \in \mathcal{A}$ such that $\pi(a)(S^{-1}T)^2x = S^{-2}x$ and $\pi(a)S^{-1}Tx = \pi(a)x = 0$, then, by (3.7), $\pi\delta^2(a)x = x$, a contradiction. So $(S^{-1}T)^2x$, $S^{-1}Tx$ and x are $\mathbb{C}$-dependent for every $x \in X$. This implies that $(S^{-1}T)^2 = \mu S^{-1}T + \nu I$ for some $\mu, \nu \in \mathbb{C}$, where I denotes the identity operator on X. Then (3.7) reduces to

$$\begin{aligned} \pi\delta^2(a) &= (\mu S^2 - ST - TS)\pi(a)S^{-1}T + (\nu S^2 + T^2)\pi(a) \\ &= A\pi(a)B + C\pi(a) \end{aligned} \tag{3.8}$$

for all $a \in \mathcal{A}$, where $A = \mu S^2 - ST - TS$, $B = S^{-1}T$ and $C = \nu S^2 + T^2$. Suppose that $A = \lambda C$ for some $\lambda \in \mathbb{C}$. Then (3.8) becomes $\pi\delta^2(a) = C\pi(a)D$ for all $a \in \mathcal{A}$, where $D = \lambda B + I$. If $DCx \neq 0$ for some $x \in X$, letting $a \in \mathcal{A}$ such that $\pi(a)DCx = x$, then $\pi\delta^2(a)Cx = C\pi(a)DCx = Cx$, a contradiction. If $DC = 0$, then $\pi(\delta^2(a)^2) = (\pi\delta^2(a))^2 = 0$, proving the lemma. Hence, we may assume that $A \notin \mathbb{C}C$. Similarly, we may assume $C \notin \mathbb{C}A$.

Let $\xi, \eta \in \mathbb{C}$. If there is $x \in X$ such that $B(\xi A + \eta C)x$ and $(\xi A + \eta C)x$ are $\mathbb{C}$-independent, letting $a \in \mathcal{A}$ such that $\pi(a)B(\xi A+\eta C)x = \xi x$ and $\pi(a)(\xi A+\eta C)x = \eta x$, then, by (3.8), $\pi\delta^2(a)(\xi A + \eta C)x = (\xi A + \eta C)x$, a contradiction. So we conclude that

$$B(\xi A + \eta C)x \text{ and } (\xi A + \eta C)x \text{ are } \mathbb{C}\text{-dependent for all } \xi, \eta \in \mathbb{C} \text{ and } x \in X.$$

In particular, BAx and Ax are $\mathbb{C}$-dependent, BCx and Cx are $\mathbb{C}$-dependent and $B(A + C)x$ and $(A + C)x$ are $\mathbb{C}$-dependent for every $x \in X$. From Lemma 2.4, it follows that $BA = \alpha A$, $BC = \beta C$ and $B(A+C) = \gamma(A+C)$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. Then, $\gamma(A+C) = B(A + C) = BA + BC = \alpha A + \beta B$. Thus, $(\alpha - \gamma)A = (\gamma - \beta)C$. Recall that $A \notin \mathbb{C}C$ and $C \notin \mathbb{C}A$. This implies that $\alpha = \beta = \gamma$. Consequently, $BA = \alpha A$ and $BC = \alpha C$. Choose $x \in X$ such that $(\alpha A + C)x \neq 0$ and let $a \in \mathcal{A}$ be such that $\pi(a)(\alpha A + C)x = x$. By (3.8), $\pi\delta^2(a)(\alpha A + C)x = (\alpha A + C)x$, a contradiction. This proves the lemma. $\square$

Now we are ready to give the following proof.

Proof of Theorem 1.1. To prove that $\delta^2(a)^2$ lies in the radical of $\mathcal{A}$, it suffices to show that $\pi(\delta^2(a)^2) = (\pi\delta^2(a))^2 = 0$ for any continuous irreducible representation π of $\mathcal{A}$. Let π be a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X. By Lemma 3.1, we may assume that δ is π-inner. That is, there is $T \in L(X)$ such that $\pi\delta(a) = \pi\sigma(a)T - T\pi(a)$ for all $a \in \mathcal{A}$. Then (3.2) becomes

$$\begin{aligned} &\pi\delta^2(\sigma^{-1}(a)b) \\ &\quad = \pi\sigma(a)\pi\delta^2(b) + \pi(\delta + \sigma\delta\sigma^{-1})(a)\pi\delta(b) + \pi\delta^2\sigma^{-1}(a)\pi(b) \\ &\quad = \pi\sigma(a)\pi\delta^2(b) + (\pi\sigma(a)T - T\pi(a))\pi\delta(b) + \pi\sigma\delta\sigma^{-1}(a)\pi\delta(b) + \pi\delta^2\sigma^{-1}(a)\pi(b). \end{aligned} \tag{3.9}$$

Moreover, by Lemma 3.2, we may assume that σ is π-outer.

Assume first that $\sigma\delta\sigma^{-1}$ is π-outer. If $T = 0$, then $\pi\delta = 0$ and hence $\pi\delta^2 = 0$, as desired. So assume that $T \neq 0$ and let $x \in X$ be such that $Tx \neq 0$. Let $Y = \mathbb{C}x + \mathbb{C}Tx$. Since σ is π-outer, by Theorem 2.3 there is $b \in \mathcal{A}$ such that $\pi\sigma(b)Tx \neq 0$ and

$\pi(b)Y = 0$. This implies that $\pi(b)x = 0$ and $\pi\delta(b)x = (\pi\sigma(b)T - T\pi(b))x = \pi\sigma(b)Tx \neq 0$. Let $Z = \mathbb{C}\pi\delta(b)x + \mathbb{C}\pi\delta^2(b)x + \mathbb{C}T\pi\delta(b)x$. By Theorem 2.1, there is $a \in \mathcal{A}$ such that $\pi\sigma\delta\sigma^{-1}(a)\pi\delta(b)x = x$, $\pi\sigma(a)Z = 0$ and $\pi(a)Z = 0$. This implies that $\pi\sigma(a)\pi\delta^2(b)x = \pi\sigma(a)T\pi\delta(b)x = 0$ and $\pi(a)\pi\delta(b)x = 0$. By (3.9), $\pi\delta^2(\sigma^{-1}(a)b)x = x$, a contradiction.

Assume now that $\sigma\delta\sigma^{-1}$ is π-inner. That is, there exists $U \in L(X)$ such that $\pi\sigma\delta\sigma^{-1}(a) = \pi\sigma(a)U - U\pi(a)$ for all $a \in \mathcal{A}$. Thus, $\pi\sigma\delta(a) = \pi\sigma^2(a)U - U\pi\sigma(a)$ for all $a \in \mathcal{A}$. With this, we now have

$$\begin{aligned} \pi\delta^2(a) &= \pi\sigma(\delta(a))T - T\pi(\delta(a)) \\ &= \pi\sigma\delta(a)T - T\pi\delta(a) \\ &= (\pi\sigma^2(a)U - U\pi\sigma(a))T - T(\pi\sigma(a)T - T\pi(a)) \\ &= \pi\sigma^2(a)UT - (U+T)\pi\sigma(a)T + T^2\pi(a) \end{aligned} \tag{3.10}$$

for all $a \in \mathcal{A}$. We divide the proof into two cases.

Case 1 ($\boldsymbol{\sigma^2}$, $\boldsymbol{\sigma}$, and $\boldsymbol{1_{\mathcal{A}}}$ are pairwise $\boldsymbol{\pi}$-independent). Suppose that $UTx \neq 0$ for some $x \in X$. Let $Y = \mathbb{C}UTx + \mathbb{C}Tx + \mathbb{C}x$. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi\sigma^2(a)UTx = x$, $\pi\sigma(a)Y = 0$ and $\pi(a)Y = 0$. This implies that $\pi\sigma(a)Tx = 0$ and $\pi(a)x = 0$. By (3.10), $\pi\delta^2(a)x = x$, a contradiction. Hence, we assume that $UT = 0$. Then (3.10) becomes

$$\pi\delta^2(a) = -(U+T)\pi\sigma(a)T + T^2\pi(a) \tag{3.11}$$

for all $a \in \mathcal{A}$. Suppose that $T(U+T)x \neq 0$ for some $x \in X$. Let $Y = \mathbb{C}T(U+T)x + \mathbb{C}(U+T)x$. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi\sigma(a)T(U+T)x = x$ and $\pi(a)Y = 0$, implying that $\pi(a)(U+T)x = 0$. From (3.11), it follows that $\pi\delta^2(a)(U+T)x = -(U+T)x$, a contradiction. Thus, $T(U+T) = 0$. Suppose that $T^2x \neq 0$ for some $x \in X$. Let $Z = \mathbb{C}T^2x + \mathbb{C}T^3x$. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi(a)T^2x = x$ and $\pi\sigma(a)Z = 0$, implying that $\pi\sigma(a)T^3x = 0$. From (3.11), it follows that $\pi\delta^2(a)T^2x = T^2x$, a contradiction. Thus, $T^2 = 0$. Now, using $T(U+T) = T^2 = 0$ and (3.11), we have $(\pi\delta^2(a))^2 = \pi(\delta^2(a)^2) = 0$ for all $a \in \mathcal{A}$, proving the theorem.

Case 2 ($\boldsymbol{\sigma^2}$, $\boldsymbol{\sigma}$ and $\boldsymbol{1_{\mathcal{A}}}$ are not pairwise $\boldsymbol{\pi}$-independent). Since σ is π-outer, we see that σ^2 and $1_{\mathcal{A}}$ are π-dependent. That is, σ^2 is π-inner. So there exists an invertible $S \in L(X)$ such that $\pi\sigma^2(a) = S\pi(a)S^{-1}$ for all $a \in \mathcal{A}$. Then (3.10) becomes

$$\pi\delta^2(a) = S\pi(a)S^{-1}UT - (U+T)\pi\sigma(a)T + T^2\pi(a) \tag{3.12}$$

for all $a \in \mathcal{A}$. Suppose that $S^{-1}UT \notin \mathbb{C}I$. Then $S^{-1}UTx$ and x are $\mathbb{C}$-independent for some $x \in X$. Let $Y = \mathbb{C}S^{-1}UTx + \mathbb{C}x + \mathbb{C}Tx$. According to Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi(a)S^{-1}UTx = S^{-1}x$, $\pi(a)x = 0$ and $\pi\sigma(a)Y = 0$, implying that $\pi\sigma(a)Tx = 0$. By (3.12), $\pi\delta^2(a)x = x$, a contradiction. So $S^{-1}UT \in \mathbb{C}I$ and (3.12) reduces to

$$\pi\delta^2(a) = -(U+T)\pi\sigma(a)T + (T^2 + UT)\pi(a) \tag{3.13}$$

for all $a \in \mathcal{A}$. Suppose that $T(U+T)x \neq 0$ for some $x \in X$. Let $Y = \mathbb{C}T(U+T)x + \mathbb{C}(U+T)x$. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi\sigma(a)T(U+T)x = x$ and $\pi(a)Y = 0$, implying that $\pi(a)(U+T)x = 0$. From (3.13) it follows that $\pi\delta^2(a)(U+T)x = -(U+T)x$, a contradiction. Thus, $T(U+T) = 0$. Suppose that $(T^2+UT)x \neq 0$ for some $x \in X$. Let $Z = \mathbb{C}(T^2+UT)x + \mathbb{C}T(T^2+UT)x$. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi(a)(T^2+UT)x = x$ and $\pi\sigma(a)Z = 0$, implying that $\pi\sigma(a)T(T^2+UT)x = 0$. From (3.13) it follows that $\pi\delta^2(a)(T^2+UT)x = (T^2+UT)x$, a contradiction. Thus, $T^2+UT = 0$. Now, using $T(U+T) = T^2+UT = 0$ and (3.13), we see that $(\pi\delta^2(a))^2 = \pi(\delta^2(a)^2) = 0$, proving the theorem. □

4. Proof of Theorems 1.2 and 1.5

Lemma 4.1. *Let $\mathcal{A} = M_2(\mathbb{C})$, the 2×2 matrix algebra over the complex field. Suppose that $S, A \in \mathcal{A}$ and that S is invertible in $\mathcal{A}$. If $[S[A,a],a]^2 = 0$ for all $a \in \mathcal{A}$, then $A \in \mathbb{C}I_2$, where I_2 is the identity matrix in $\mathcal{A}$.*

Proof. Clearly, for any invertible element $P \in \mathcal{A}$ we have $[PSP^{-1}[PAP^{-1},a],a]^2 = 0$ for all $a \in \mathcal{A}$. Moreover, for any $\lambda \in \mathbb{C}$, $[PSP^{-1}[PAP^{-1} - \lambda I_2, a], a]^2 = 0$ for all $a \in \mathcal{A}$. Thus, writing A in its Jordan form modulo a scalar, we may assume that $A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)$ or $A = \left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right)$, where $\alpha \in \mathbb{C}$. Clearly, if $\alpha = 0$, then we are done. So we may assume that $\alpha \neq 0$. Also write

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad \text{where } s_{ij} \in \mathbb{C}.$$

Suppose that $x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \mathbb{C}^2$ such that Ax and x are $\mathbb{C}$-independent. Then, $\mathbb{C}^2 = \mathbb{C}Ax + \mathbb{C}x$. Write $Sx = \mu Ax + \nu x$ for $\mu, \nu \in \mathbb{C}$. Let $a \in \mathcal{A}$ such that $ax = 0$ and $aAx = x$. Then, $[S[A,a],a]x = (S(Aa - aA)a - aS(Aa - aA))x = aSaAx = \mu x$. From $0 = [S[A,a],a]^2x = \mu^2 x$ it follows that $\mu = 0$, and hence $Sx = \nu x$. So we conclude that

$$\text{if } Ax \text{ and } x \text{ are } \mathbb{C}\text{-independent for } x \in \mathbb{C}^2, \text{ then } Sx \in \mathbb{C}x. \tag{$*$}$$

Case 1 ($\boldsymbol{A = \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)}$). Let $x = \binom{1}{\gamma}$, where $0 \neq \gamma \in \mathbb{C}$. Then, $Ax = \binom{\alpha}{0}$ and $x = \binom{1}{\gamma}$ are $\mathbb{C}$-independent. By $(*)$, we have $Sx = \ell_\gamma x$, where $\ell_\gamma \in \mathbb{C}$ depending on γ. This implies that $s_{11} + \gamma s_{12} = \ell_\gamma$ and $s_{21} + \gamma s_{22} = \ell_\gamma \gamma$. Combining these two identities, we obtain $s_{12}r^2 + (s_{11} - s_{22})r - s_{21} = 0$ for all $0 \neq \gamma \in \mathbb{C}$. Consequently, $s_{12} = s_{21} = 0$ and $s_{11} = s_{22}$. So $S = s_{11}I_2$. Setting $a = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we have $[S[A,a],a] = 2s_{11}\alpha\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. Thus, $0 = [S[A,a],a]^2 = 4(s_{11}\alpha)^2 I_2$, a contradiction.

Case 2 ($\boldsymbol{A = \left(\begin{smallmatrix} 0 & \alpha \\ 0 & 0 \end{smallmatrix}\right)}$). Let $x = \binom{\gamma}{1}$, where $0 \neq \gamma \in \mathbb{C}$. Then, $Ax = \binom{\alpha}{0}$ and $x = \binom{\gamma}{1}$ are $\mathbb{C}$-independent. By $(*)$, we have $Sx = \ell_\gamma x$, where $\ell_\gamma \in \mathbb{C}$ depending on γ. This implies that $\gamma s_{11} + s_{12} = \ell_\gamma r$ and $\gamma s_{21} + s_{22} = \ell_\gamma$. Combining these two identities, we obtain $s_{21}r^2 + (s_{22} - s_{11})r - s_{12} = 0$ for all $0 \neq \gamma \in \mathbb{C}$. Consequently, $s_{12} = s_{21} = 0$ and $s_{11} = s_{22}$. So $S = s_{11}I_2$. Setting $a = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, we have $[S[A,a],a] = s_{11}\alpha\left(\begin{smallmatrix} 0 & 2 \\ -2 & 0 \end{smallmatrix}\right)$. Thus, $0 = [S[A,a],a]^2 = -4(s_{11}\alpha)^2 I_2$, a contradiction. This proves the lemma. □

Lemma 4.2. *Let $\mathcal{A}$ be a complex Banach algebra, let σ be an automorphism of $\mathcal{A}$ and let δ be a σ-derivation of $\mathcal{A}$. Suppose that $[\delta(a), a]$ is quasinilpotent for every $a \in \mathcal{A}$. If π is a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X with $\dim_{\mathbb{C}} X \geqslant 2$, then $\pi\delta = 0$.*

Proof. Clearly, we have

$$\pi([\delta(a), a]) = [\pi\delta(a), \pi(a)] = \pi\delta(a)\pi(a) - \pi(a)\pi\delta(a) \tag{4.1}$$

for all $a \in \mathcal{A}$.

Suppose first that δ is π-outer. Choose $x, y \in X$ such that x and y are $\mathbb{C}$-independent. By Theorems 2.1 and 2.2, there is an $a \in \mathcal{A}$ such that $\pi\delta(a)x = y$, $\pi(a)x = 0$ and $\pi(a)y = -x$. Then, by (4.1), $\pi([\delta(a), a])x = (\pi\delta(a)\pi(a) - \pi(a)\pi\delta(a))x = x$, a contradiction. Hence, δ must be π-inner. That is, there exists $T \in L(X)$ such that $\pi\delta(a) = \pi\sigma(a)T - T\pi(a)$ for all $a \in \mathcal{A}$. Thus, (4.1) becomes

$$\pi([\delta(a), a]) = (\pi\sigma(a)T - T\pi(a))\pi(a) - \pi(a)(\pi\sigma(a)T - T\pi(a)) \tag{4.2}$$

for all $a \in \mathcal{A}$. If $T = 0$, then $\pi\delta = 0$, proving the lemma. So we may assume that $T \neq 0$.

Suppose next that σ is π-outer. Assume first that $T \notin \mathbb{C}I$. Then Tx and x are $\mathbb{C}$-independent for some $x \in X$. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi\sigma(a)Tx = Tx$, $\pi(a)Tx = -x$ and $\pi(a)x = 0$. By (4.2), $\pi([\delta(a), a])x = -\pi(a)\pi\sigma(a)Tx = x$, a contradiction. Assume now that $T \in \mathbb{C}I$ and write $T = \alpha I$, where $0 \neq \alpha \in \mathbb{C}$. Then (4.2) becomes

$$\pi([\delta(a), a]) = T(\pi\sigma(a)\pi(a) - \pi(a)\pi\sigma(a)) \tag{4.3}$$

for all $a \in \mathcal{A}$. Choose $x, y \in X$ such that x and y are $\mathbb{C}$-independent. By Theorem 2.3, there is $a \in \mathcal{A}$ such that $\pi\sigma(a)x = y$, $\pi(a)x = 0$ and $\pi(a)y = -x$. Then, by (4.3), $\pi([\delta(a), a])x = Tx = \alpha x$, a contradiction. Hence, σ must be π-inner. That is, there exists an invertible $S \in L(X)$ such that $\pi\sigma(a) = S\pi(a)S^{-1}$ for all $a \in \mathcal{A}$. Thus, (4.2) reduces to

$$\pi([\delta(a), a]) = (S\pi(a)S^{-1}T - T\pi(a))\pi(a) - \pi(a)(S\pi(a)S^{-1}T - T\pi(a)) \tag{4.4}$$

for all $a \in \mathcal{A}$. Note that

$$\pi\delta(a) = \pi\sigma(a)T - T\pi(a) = S\pi(a)S^{-1}T - T\pi(a) = S(\pi(a)S^{-1}T - S^{-1}T\pi(a)) \tag{4.5}$$

for all $a \in \mathcal{A}$. If $S^{-1}T \in \mathbb{C}I$, then, by (4.5), $\pi\delta = 0$, proving the lemma. So we may assume that $S^{-1}T \notin \mathbb{C}I$. Hence, $S^{-1}Tx$ and x are $\mathbb{C}$-independent for some $x \in X$.

Case 1 ($\dim_{\mathbb{C}} X \geqslant 3$). Choose $y \in X$ such that $S^{-1}Tx$, x and y are $\mathbb{C}$-independent. Let $a \in \mathcal{A}$ satisfy $\pi(a)x = 0$, $\pi(a)S^{-1}Tx = S^{-1}y$ and $\pi(a)y = -x$. Then, by (4.4), $\pi([\delta(a), a])x = -\pi(a)S\pi(a)S^{-1}Tx = x$, a contradiction.

Case 2 ($\dim_{\mathbb{C}} X = 2$). In this case, $\pi(\mathcal{A}) = B(X) \cong M_2(\mathbb{C})$. In view of (4.4), we have that $\pi([\delta(a), a]) = [S[A, \pi(a)], \pi(a)]$ is quasinilpotent in $\pi(\mathcal{A})$ for every $a \in \mathcal{A}$, where $A = -S^{-1}T$. By Lemma 4.1, $A = -S^{-1}T \in \mathbb{C}I$. This implies that $\pi\delta = 0$ by (4.5), proving the lemma. □

Proof of Theorem 1.2. Let π be a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X with $\ker\pi = P$. If $\dim_{\mathbb{C}} X \geqslant 2$, then, by Lemma 4.2, $\pi\delta = 0$ and thus $\pi([\delta(a), b]) = [\pi\delta(a), \pi(b)] = 0$ for all $a, b \in \mathcal{A}$. If $\dim_{\mathbb{C}} X = 1$, then $\pi(\mathcal{A}) = \mathbb{C}I$ and hence $\pi([\mathcal{A}, \mathcal{A}]) = [\pi(\mathcal{A}), \pi(\mathcal{A})] = 0$, implying that $\pi([\delta(a), b]) = 0$ for all $a, b \in \mathcal{A}$. Consequently, $[\delta(\mathcal{A}), \mathcal{A}] \subseteq \mathrm{rad}(\mathcal{A})$, proving the theorem. □

Let $\mathcal{I}_{\mathcal{A}}$ be the ideal of $\mathcal{A}$ generated by $[\mathcal{A},\mathcal{A}]$, where $[\mathcal{A},\mathcal{A}]$ denotes the subspace of $\mathcal{A}$ spanned by all commutators (that is, elements of the form $[a,b]=ab-ba$ where $a,b\in\mathcal{A}$) of $\mathcal{A}$. From $\mathcal{A}[\mathcal{A},\mathcal{A}]\mathcal{A}\subseteq\mathcal{A}[[\mathcal{A},\mathcal{A}],\mathcal{A}]+\mathcal{A}^2[\mathcal{A},\mathcal{A}]\subseteq\mathcal{A}[\mathcal{A},\mathcal{A}]$, it follows that $\mathcal{I}_{\mathcal{A}}=[\mathcal{A},\mathcal{A}]+\mathcal{A}[\mathcal{A},\mathcal{A}]$. For $b\in\mathcal{A}$, let $b_\ell\colon\mathcal{A}\to\mathcal{A}$ be the map defined by $b_\ell(a)=ba$ for all $a\in\mathcal{A}$.

Theorem 4.3. *Let $\mathcal{A}$ be a unital complex Banach algebra and let δ be a σ-derivation of $\mathcal{A}$, where σ is an inner automorphism of $\mathcal{A}$. Suppose that $[\delta(a),a]$ is quasinilpotent for every $a\in\mathcal{A}$. Then, $\delta(\mathcal{I}_{\mathcal{A}})\subseteq\operatorname{rad}(\mathcal{A})$. Moreover, $\delta(\mathcal{A})\subseteq\operatorname{rad}(\mathcal{A})$ if δ is continuous.*

Proof. By assumption, $\sigma(a)=uau^{-1}$ for all $a\in\mathcal{A}$, where u is a unit in $\mathcal{A}$. Let $d=u_\ell^{-1}\delta$. Then it is easy to see that d is a derivation of $\mathcal{A}$ and $\delta=u_\ell d$.

Let π be a continuous irreducible representation of $\mathcal{A}$ on a complex Banach space X with $\ker\pi=P$. Suppose first that $\dim_{\mathbb{C}}X\geqslant 2$. Then, by Lemma 4.2, $\pi\delta(A)=0$ and thus $\pi\delta(\mathcal{I}_{\mathcal{A}})=0$. Suppose next that $\dim_{\mathbb{C}}X=1$. In this case, $\pi(A)=\mathbb{C}I$ and $\mathcal{A}/P\cong\mathbb{C}$. Clearly, $\pi([A,A])=[\pi(A),\pi(A)]=0$ and thus $\pi(\mathcal{I}_{\mathcal{A}})=0$. Using $d([a,b])=[d(a),b]+[a,d(b)]$, we see that $d(\mathcal{I}_{\mathcal{A}})\subseteq\mathcal{I}_{\mathcal{A}}$. So $\pi d(\mathcal{I}_{\mathcal{A}})=0$, implying that $\pi\delta(\mathcal{I}_{\mathcal{A}})=0$. Note that $d(1)=0$ and hence $d(\mathbb{C})=0$. Suppose that δ is continuous; then so is d. By [**34**, Theorem 2.2], $d(P)\subseteq P$. So d naturally induces a derivation d_P of $\mathcal{A}/P$ by the rule: $d_P(a+P)=d(a)+P$ for all $a\in\mathcal{A}$. Since $\mathcal{A}/P\cong\mathbb{C}$, we have $d_P(\mathcal{A}/P)=0+P$. This implies that $d(\mathcal{A})\subseteq P$. Thus, $\pi d(\mathcal{A})=0$, implying that $\pi\delta(\mathcal{A})=0$. Consequently, $\delta(\mathcal{I}_{\mathcal{A}})\subseteq\operatorname{rad}(\mathcal{A})$ and if δ is continuous, then $\delta(\mathcal{A})\subseteq\operatorname{rad}(\mathcal{A})$. □

The famous result of Halmos asserts that if H is a complex infinite-dimensional separable Hilbert space, then every element of $\mathcal{A}=B(H)$ is a sum of two commutators. Consequently, $\mathcal{A}=\mathcal{I}_{\mathcal{A}}$. In [**4**, Lemma 2.6], Brešar proved that if A is a von Neumann algebra with no non-zero central abelian summand, then $\mathcal{A}=\mathcal{I}_{\mathcal{A}}$. Moreover, if A is a unital properly infinitely C^*-algebra or a unital stable C^*-algebra [**16**] or a unital C^*-algebra without tracial states [**29**], then $\mathcal{A}=\mathcal{I}_{\mathcal{A}}$.

As an immediate consequence of Theorem 4.3, we have the following corollary.

Corollary 4.4. *Let $\mathcal{A}$ be a unital complex Banach algebra with $\mathcal{A}=\mathcal{I}_{\mathcal{A}}$ and let δ be a σ-derivation of $\mathcal{A}$, where σ is an inner automorphism of $\mathcal{A}$. Suppose that $[\delta(a),a]$ is quasinilpotent for every $a\in\mathcal{A}$. Then, $\delta(\mathcal{A})\subseteq\operatorname{rad}(\mathcal{A})$.*

Proof of Corollary 1.3. Clearly, if $a\in\mathcal{A}$ with $a^n\in\operatorname{rad}(\mathcal{A})$, then a and a^n are both quasinilpotent. By Theorem 1.2 and Theorem 4.3, we are done. □

Proof of Corollary 1.4. Note that $\delta=\sigma-1_{\mathcal{A}}$ is a σ-derivation of $\mathcal{A}$ and $[\delta(a),a]=[(\sigma-1_{\mathcal{A}})(a),a]=[\sigma(a)-a,a]=[\sigma(a),a]$ for all $a\in\mathcal{A}$. By Theorem 1.2, $[\delta(\mathcal{A}),\mathcal{A}]\subseteq\operatorname{rad}(\mathcal{A})$. Thus, $\delta(a)=(\sigma-1_{\mathcal{A}})(a)$ is central modulo the radical for every $a\in\mathcal{A}$, as desired. □

To prove Theorem 1.5, we need the following result.

Theorem 4.5 (Beidar *et al.* [3, Theorem 2]). *Let $\mathcal{A}$ be a prime algebra and δ a σ-derivation of $\mathcal{A}$, where σ is an automorphism of $\mathcal{A}$. If $[\delta(a), a] \in Z(\mathcal{A})$ for every $a \in \mathcal{A}$, then $\delta = 0$ or $\mathcal{A}$ is commutative.*

Now we are ready for the following proof.

Proof of Theorem 1.5. Clearly, $[[\delta(a), a], a] = 0$ for every $a \in \mathcal{A}$. By the Kleinecke–Shirokov theorem, $[\delta(a), a]$ is quasinilpotent for every $a \in \mathcal{A}$. So $[\delta(\mathcal{A}), \mathcal{A}] \subseteq \mathrm{rad}(\mathcal{A})$ by Theorem 1.2.

Suppose that σ is inner and $\sigma(a) = uau^{-1}$ for all $a \in \mathcal{A}$, where u is a unit in $\mathcal{A}$. Clearly, $d = u_\ell^{-1}\delta$ is a derivation of $\mathcal{A}$. Let Q be a primitive ideal of $\mathcal{A}$. Using Zorn's lemma, we can find a minimal prime ideal $P \subseteq Q$. By [**27**, Lemma], $d(P) \subseteq P$ and hence $\delta(P) \subseteq P$. Clearly, $\sigma(P) \subseteq P$.

Case 1 (P is closed in $\mathcal{A}$). Then d naturally induces a derivation d_P of $\mathcal{A}/P$ defined by $d_P(a+P) = d(a)+P$ for all $a \in \mathcal{A}$. In particular, $\delta_P = (u+P)_\ell d_P$ is a σ_P-derivation of $\mathcal{A}/P$ such that $[\delta_P(a+P), a+P] \in Z(\mathcal{A}/P)$ for all $a \in \mathcal{A}$, where σ_P is an inner automorphism of $\mathcal{A}/P$ defined by $\sigma_P(a+P) = (u+P)\cdot(a+P)\cdot(u+P)^{-1}$ for all $a \in \mathcal{A}$. By Theorem 4.5, $\delta_P = 0$ or $\mathcal{A}/P$ is commutative. In the first case, $\delta(\mathcal{A}) \subseteq P$ and thus $\delta(\mathcal{A}) \subseteq Q$. In the latter case, by [**37**] $d_P(\mathcal{A}/P) \subseteq \mathrm{rad}(\mathcal{A}/P)$. Using $\mathrm{rad}(\mathcal{A}/P) \subseteq Q/P$, we obtain $d(\mathcal{A}) \subseteq Q$ and hence $\delta(\mathcal{A}) \subseteq Q$. So in both cases we have $\delta(\mathcal{A}) \subseteq Q$.

Case 2 (P is not closed in $\mathcal{A}$). By [**15**, Lemma 2.3], $\Phi(d) \subseteq P$, where $\Phi(d)$ is the separating space of d. Let $\pi_{\bar{P}} \colon \mathcal{A} \to \mathcal{A}/\bar{P}$ be the canonical epimorphism defined by $\pi_{\bar{P}}(a) = a + \bar{P}$ for all $a \in \mathcal{A}$. Since $\pi_{\bar{P}}(\Phi(d)) = 0$, by [**35**, Lemma 1.3], $\pi_{\bar{P}} \circ d \colon \mathcal{A} \to \mathcal{A}/\bar{P}$ is continuous. By a standard argument [**34**, Theorem 2.2], $\pi_{\bar{P}} \circ d(\bar{P}) = 0 + \bar{P}$. This implies that $d(\bar{P}) \subseteq \bar{P}$. So d naturally induces a derivation $d_{\bar{P}}$ of $\mathcal{A}/\bar{P}$ defined by $d_{\bar{P}}(a+\bar{P}) = d(a) + \bar{P}$ for all $a \in \mathcal{A}$. Note that $d_{\bar{P}}$ is continuous by [**35**, Lemma 1.4]. Thus, $\delta_{\bar{P}} = (u+\bar{P})_\ell d_{\bar{P}}$ is a continuous $\sigma_{\bar{P}}$-derivation of $\mathcal{A}/\bar{P}$, where $\sigma_{\bar{P}}$ is an inner automorphism of $\mathcal{A}/\bar{P}$ defined by $\sigma_{\bar{P}}(a+\bar{P}) = (u+\bar{P})\cdot(a+\bar{P})\cdot(u+\bar{P})^{-1}$ for all $a \in \mathcal{A}$. Recall that $[\delta(a), a]$ is quasinilpotent for every $a \in \mathcal{A}$. So $[\delta_{\bar{P}}(a+\bar{P}), a+\bar{P}]$ is quasinilpotent for every $a \in \mathcal{A}$. By Theorem 4.3, $\delta_{\bar{P}}(\mathcal{A}/\bar{P}) \subseteq \mathrm{rad}(\mathcal{A}/\bar{P})$. Clearly, $\bar{P} \subseteq Q$ as Q is closed in $\mathcal{A}$. Using $\mathrm{rad}(\mathcal{A}/\bar{P}) \subseteq Q/\bar{P}$, we obtain $\delta(\mathcal{A}) \subseteq Q$. Consequently, $\delta(\mathcal{A}) \subseteq \mathrm{rad}(\mathcal{A})$. The proof is now complete. □

In general, a continuous skew derivation satisfying the assumptions in Theorem 1.2, Theorem 1.5 or Corollary 1.4 does not necessarily map into the radical. See the example below.

Example. Let $\mathcal{A} = \mathbb{C}\oplus\mathbb{C}$ and let σ be the automorphism of $\mathcal{A}$ defined by $\sigma((a_1, a_2)) = (a_2, a_1)$ for all $a_1, a_2 \in \mathbb{C}$. Then, $\mathrm{rad}(\mathcal{A}) = 0$ and $\delta = \sigma - 1_{\mathcal{A}}$ is a non-zero continuous σ-derivation of $\mathcal{A}$ satisfying $[\delta(a), a] = 0$ for all $a \in \mathcal{A}$.

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