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CLASSIFICATION OF TETRAVALENT 2-TRANSITIVE NONNORMAL CAYLEY GRAPHS OF FINITE SIMPLE GROUPS

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Abstract

A graph Γ is called (G, s)-arc-transitive if $G \leq \mathrm{Aut}(\Gamma)$ is transitive on the set of vertices of Γ and the set of s-arcs of Γ, where for an integer $s \geq 1$ an s-arc of Γ is a sequence of $s + 1$ vertices $(v_0, v_1, \ldots, v_s)$ of Γ such that v_{i-1} and v_i are adjacent for $1 \leq i \leq s$ and $v_{i-1} \neq v_{i+1}$ for $1 \leq i \leq s - 1$. A graph Γ is called 2-transitive if it is $(\mathrm{Aut}(\Gamma), 2)$-arc-transitive but not $(\mathrm{Aut}(\Gamma), 3)$-arc-transitive. A Cayley graph Γ of a group G is called normal if G is normal in $\mathrm{Aut}(\Gamma)$ and nonnormal otherwise. Fang *et al.* ['On edge transitive Cayley graphs of valency four', *European J. Combin.* **25** (2004), 1103–1116] proved that if Γ is a tetravalent 2-transitive Cayley graph of a finite simple group G, then either Γ is normal or G is one of the groups $\mathrm{PSL}_2(11)$, M_{11}, M_{23} and A_{11}. However, it was unknown whether Γ is normal when G is one of these four groups. We answer this question by proving that among these four groups only M_{11} produces connected tetravalent 2-transitive nonnormal Cayley graphs. We prove further that there are exactly two such graphs which are nonisomorphic and both are determined in the paper. As a consequence, the automorphism group of any connected tetravalent 2-transitive Cayley graph of any finite simple group is determined.

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1. Introduction

All groups considered in the paper are finite and all graphs considered are finite, simple and undirected. Given a group G and a subset S of G such that $1_G \notin S$ and $S = S^{-1} := \{x^{-1} \mid x \in S\}$, the *Cayley graph* of G relative to S is defined to be the graph $\Gamma = \mathrm{Cay}(G, S)$ with vertex set $V\Gamma = G$ and edge set $E\Gamma = \{\{x, y\} \mid yx^{-1} \in S\}$. It is readily seen that Γ has valency $|S|$. It is also easy to see that Γ is connected if and only if S is a generating set of G. In general, Γ has exactly $|G : \langle S\rangle|$ connected components, each of which is isomorphic to $\mathrm{Cay}(\langle S\rangle, S)$, where $\langle S\rangle$ is the subgroup of G generated by S. So we may focus on the connected case when dealing with Cayley graphs. Denote by G_R

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the right regular representation of G. Define

$$A(G, S) := \{ x \in \mathrm{Aut}(G) \mid S^x = S \}.$$

Then $A(G, S)$ is a subgroup of $\mathrm{Aut}(G)$ acting naturally on G. It is not difficult to see that $\Gamma = \mathrm{Cay}(G, S)$ admits $G_R A(G, S)$ as a subgroup of its full automorphism group $\mathrm{Aut}(\Gamma)$. It is well known (see [5, 12]) that $\mathrm{N}_{\mathrm{Aut}(\Gamma)}(G_R) = G_R A(G, S)$. Since $G_R \cong G$, we may use G in place of G_R, so that $G_R A(G, S)$ is written as $G.A(G, S)$. Γ is called a *normal* Cayley graph if G is normal in $\mathrm{Aut}(\Gamma)$, that is, $\mathrm{Aut}(\Gamma) = G.A(G, S)$.

A fundamental problem in studying the structure of a graph is to determine its full automorphism group. This is, in general, quite difficult. However, for a connected Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ of valency d, if Γ is normal, then we know that its automorphism group is given by $\mathrm{Aut}(\Gamma) = G.A(G, S)$. Moreover, the subgroup $A(G, S)$ of $\mathrm{Aut}(G)$ acts faithfully on the *neighbourhood* $\Gamma(\alpha)$ of $\alpha \in V\Gamma$, where $\Gamma(\alpha)$ is defined as the set of vertices of Γ adjacent to α in Γ. Hence $A(G, S)$ is isomorphic to a subgroup of the symmetric group S_d of degree d. In other words, if Γ is a normal Cayley graph, then the structure of $\mathrm{Aut}(\Gamma)$ is well understood. In contrast, it is more challenging to determine the automorphism groups of nonnormal Cayley graphs. For this reason, nonnormal Cayley graphs have attracted considerable attention in recent years.

Given an integer $s \geq 1$, an *s-arc* of a graph Γ is a sequence $(v_0, v_1, \ldots, v_s)$ of $s + 1$ vertices of Γ such that $\{v_{i-1}, v_i\} \in E\Gamma$ for $i = 1, 2, \ldots, s$ and $v_{i-1} \neq v_{i+1}$ for $i = 1, 2, \ldots, s - 1$. A graph Γ is called *(G, s)-arc-transitive* if G is a subgroup of $\mathrm{Aut}(\Gamma)$ that is transitive on $V\Gamma$ and transitive on the set of s-arcs of Γ. A (G, s)-arc-transitive graph is called *(G, s)-transitive* if it is not $(G, s + 1)$-arc-transitive. In particular, Γ is called *s-arc-transitive* if it is $(\mathrm{Aut}(\Gamma), s)$-arc-transitive, and *s-transitive* if it is $(\mathrm{Aut}(\Gamma), s)$-transitive. A 1-arc-transitive graph is also called an *arc-transitive* or *symmetric* graph.

For any integer $s \geq 1$, a complete classification of cubic s-transitive nonnormal Cayley graphs of finite simple groups was obtained by S. J. Xu, M. Y. Xu and the first two authors of the present paper (see [13, 14]). In the tetravalent case, C. H. Li, M. Y. Xu and the first author of the present paper proved [2, Theorem 1.1] that, if Γ is a tetravalent 2-transitive Cayley graph of a finite simple group G, then either Γ is normal or G is one of the following groups: $\mathrm{PSL}_2(11)$ (two-dimensional projective special linear group over $\mathbb{F}_{11}$), M_{11} (Mathieu group of degree 11), M_{23} (Mathieu group of degree 23), A_{11} (alternating group of degree 11). However, for a long time it was unknown whether Γ is normal when G is one of these four groups. In this paper we settle these unsolved cases and classify all connected tetravalent 2-transitive nonnormal Cayley graphs of finite simple groups. As a consequence, the automorphism group of any connected tetravalent 2-transitive Cayley graph of any finite simple group is determined.

The main result of this paper is as follows, where the graphs $\Gamma(\Delta_1)$ and $\Gamma(\Delta_2)$ will be defined in (3.3) in Section 3.

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THEOREM 1.1. *Let G be a finite nonabelian simple group and $\Gamma = \mathrm{Cay}(G, S)$ a connected tetravalent 2-transitive Cayley graph of G. Then one of the following occurs:*

(a) Γ *is normal and* $\mathrm{Aut}(\Gamma) = G.A_4$ *or* $G.S_4$;
(b) $G = \mathrm{M}_{11}$, $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathrm{M}_{12}) = \mathrm{M}_{12}{:}2$, $\mathrm{Aut}(\Gamma)_\alpha \cong S_4$ *for* $\alpha \in V\Gamma$, Γ *is nonnormal,* $\Gamma \cong \Gamma(\Delta_1)$ *or* $\Gamma(\Delta_2)$ *and* $\Gamma(\Delta_1)$ *and* $\Gamma(\Delta_2)$ *are not isomorphic.*

In the next section we introduce notation and give a few preliminary results. In Section 3 we determine all tetravalent 2-transitive nonnormal Cayley graphs of finite simple groups by analysing the four groups above. In Section 4 we settle the isomorphism problem and thus complete the proof of Theorem 1.1. As we will see shortly, even in the four innocent-looking cases above, considerable analysis and computation will be needed in order to establish Theorem 1.1. We will also use [2, Theorem 1.1] in our proof of Theorem 1.1.

2. Preliminaries

A permutation group G acting on a set Ω is said to be *quasiprimitive* if each of its nontrivial normal subgroups is transitive on Ω. The *socle* of a group G, denoted by $\mathrm{soc}(G)$, is the product of all minimal normal subgroups of G. In particular, G is said to be *almost simple* if $\mathrm{soc}(G)$ is a nonabelian simple group. Given a graph Γ and a group $K \le \mathrm{Aut}(\Gamma)$, the *quotient graph* Γ_K of Γ relative to K is defined as the graph with vertices the K-orbits on $V\Gamma$, such that two K-orbits, say, X and Y, are adjacent in Γ_K if and only if there is an edge of Γ with one end-vertex in X and the other end-vertex in Y.

The following lemma determines the vertex stabilisers for connected tetravalent 2-transitive graphs (see [9, Theorem 4] or [6, Proposition 2.2]).

LEMMA 2.1. *Let Γ be a connected tetravalent 2-transitive graph. Then the vertex stabiliser of Γ is A_4 or S_4.*

The next lemma describes possible structures of the full automorphism group of a connected Cayley graph of a finite simple group.

LEMMA 2.2 ([4], Theorem 1.1). *Let G be a finite nonabelian simple group and let $\Gamma = \mathrm{Cay}(G, S)$ be a connected Cayley graph of G. Let M be a subgroup of* $\mathrm{Aut}(\Gamma)$ *containing $G.A(G, S)$. Then either $M = G.A(G, S)$ or one of the following holds:*

(a) *M is almost simple and* $\mathrm{soc}(M)$ *contains G as a proper subgroup and is transitive on $V\Gamma$;*
(b) *$G \cdot \mathrm{Inn}(G) \le M = G \cdot A(G, S) \cdot 2$ and S is a self-inverse union of G-conjugacy classes;*
(c) *M is not quasiprimitive and there is a maximal intransitive normal subgroup H of M such that one of the following holds:*
    (i) *M/H is almost simple and* $\mathrm{soc}(M/H)$ *contains $GH/H \cong G$ and is transitive on $V\Gamma_H$;*

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TABLE 1. Groups G and T for Lemma 2.2(c)(iii).

	G	T	$\lvert V\Gamma_K \rvert$
1	A_6	G	36
2	M_{12}	G or A_m	144
3	$\mathrm{Sp}_4(q)$ ($q = 2^a > 2$)	G or A_m or $\mathrm{Sp}_{4r}(q_0)$ ($q = q_0^r$)	$\dfrac{q^4(q^2-1)^2}{4}$
4		$\mathrm{Sp}_{4r}(q_0)$ ($q = q_0^r$)	$\dfrac{q^4(q^2-1)^2}{2}$
5	$\mathrm{P}\Omega_8^+(q)$	G or A_m or $\mathrm{Sp}_8(2)$ (if $q = 2$)	$\dfrac{q^6(q^4-1)^2}{(2, q-1)^2}$

(ii) $M/H = \mathrm{AGL}_3(2)$, $G = \mathrm{L}_2(7)$ *and* $\Gamma_H \cong K_8$*;*
(iii) $\mathrm{soc}(M/H) \cong T \times T$ *and* $GH/H \cong G$ *is a diagonal subgroup of* $\mathrm{soc}(M/H)$*, where* T *and* G *are given in Table 1.*

Moreover, there are examples of connected Cayley graphs of finite simple groups in each of these cases.

A subgroup K of a group G is called *core-free* if $\bigcap_{g\in G} K^g = 1$. Given a core-free subgroup K of G and an element $g \in G \setminus \mathrm{N}_G(K)$ such that $g^2 \in K$ and $G = \langle K, g\rangle$, the *coset graph* $\Gamma^* = \Gamma(G, K, g)$ is defined by

$$V\Gamma^* = [G : K] = \{ Kx \mid x \in G \}, \quad E\Gamma^* = \{ \{Kx, Ky\} \mid xy^{-1} \in KgK \}.$$

A well-known result due to Sabidussi [10] and Lorimer [8] asserts that Γ^* is G-arc-transitive and up to isomorphism every G-arc-transitive graph can be constructed this way. The following lemma is a refinement of this result (see [3, Theorem 2.1]).

LEMMA 2.3. *Let* Γ *be a finite connected* $(G, 2)$*-arc-transitive graph of valency* d*. Then there exist a core-free subgroup* K *of* G *and an element* $g \in G$ *such that:*

(a) $g \notin \mathrm{N}_G(K)$, $g^2 \in G$, $\langle K, g\rangle = G$*;*
(b) *the action of* K *on* $[K : K \cap K^g]$ *by right multiplication is transitive, where* $|K : K \cap K^g| = d$*; and*
(c) $\Gamma \cong \Gamma(G, K, g)$.

Moreover, one can choose g *to be a* 2*-element.*

Conversely, if G *is a finite group with a core-free subgroup* K *and an element* g *satisfying (a) and (b) above, then* $\Gamma^* = \Gamma(G, K, g)$ *is a connected* $(G, 2)$*-arc-transitive graph and* G *acts faithfully on the vertex set* $[G : K]$ *of* Γ^* *by right multiplication.*

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3. Tetravalent 2-transitive nonnormal Cayley graphs

The purpose of this section is to prove the following proposition, which gives all tetravalent 2-transitive nonnormal Cayley graphs of finite simple groups. We postpone the definition of $\Gamma(\Delta_1)$ and $\Gamma(\Delta_2)$ to (3.3).

PROPOSITION 3.1. *Let G be a finite simple group and Γ a connected tetravalent 2-transitive nonnormal Cayley graph of G. Then $G = \mathrm{M}_{11}$, $\mathrm{Aut}(\Gamma) = \mathrm{Aut}(\mathrm{M}_{12}) = \mathrm{M}_{12}{:}2$, $\mathrm{Aut}(\Gamma)_\alpha = S_4$ and Γ is isomorphic to $\Gamma(\Delta_1)$ or $\Gamma(\Delta_2)$.*

PROOF. Suppose that G is a finite simple group and $\Gamma = \mathrm{Cay}(G, S)$ is a connected tetravalent 2-transitive nonnormal Cayley graph of G. Then, by [2, Theorem 1.1], G is one of the following groups:

$$\mathrm{PSL}_2(11),\ \mathrm{M}_{11},\ \mathrm{M}_{23},\ A_{11}. \tag{3.1}$$

Write $A = \mathrm{Aut}(\Gamma)$. Then $A = GA_\alpha$ with $G \cap A_\alpha = 1$ and $A_\alpha = A_4$ or S_4 by Lemma 2.1. We consider the following two situations separately.

Situation 1: A is quasiprimitive on $V\Gamma$. In this situation, A is almost simple by Lemma 2.2. Let $T = \mathrm{soc}(A)$. Note that $|S_4| = 24$ is divisible by $|A : G|$. It follows that (T, G) is one of the pairs:

$$(\mathrm{M}_{11},\ \mathrm{PSL}_2(11)),\ (\mathrm{M}_{12},\ \mathrm{M}_{11}),\ (\mathrm{M}_{24},\ \mathrm{M}_{23}),\ (A_{12},\ A_{11}). \tag{3.2}$$

Case 1. $(T, G) \in \{(\mathrm{M}_{11},\ \mathrm{PSL}_2(11)),\ (\mathrm{M}_{24},\ \mathrm{M}_{23}),\ (A_{12},\ A_{11})\}$.

First, we consider the case $(T, G) = (\mathrm{M}_{11},\ \mathrm{PSL}_2(11))$ and we suppose that $\mathrm{M}_{11} = \mathrm{PSL}_2(11)A_4$. It is well known that M_{11} has a faithful permutation representation of degree 12 acting on $\Omega = \{1, 2, \ldots, 12\}$. In this representation, $\mathrm{PSL}_2(11)$ is the point-stabiliser and the subgroup A_4 should be regular on Ω. However, according to the permutation character $\chi = \chi_1 + \chi_{11}$ taken from ATLAS [1, page 18], we have $\chi(1A) = 12$, $\chi(2A) = 4$ and $\chi(3A) = 3$. Therefore, the number of orbits of A_4 on Ω is

$$\frac{1}{|A_4|} \sum_{g \in A_4} \chi(g) = \frac{1}{12}(12 \cdot 1 + 4 \cdot 3 + 3 \cdot 8) = 4,$$

which contradicts the regularity of A_4.

Next assume $(T, G) = (\mathrm{M}_{24},\ \mathrm{M}_{23})$. In this case $\mathrm{M}_{24} = \mathrm{M}_{23}K$ for some subgroup $K \cong S_4$. Since K is regular on $\Omega = \{1, 2, \ldots, 24\}$, following the notation of [1, page 96], the involution of K must be in class $2B$ and the elements of order 3 in $3B$. There are two classes of regular elements of order 4, namely $4A$ and $4C$. However, the power map shows that $4A^2 = 2A$, which cannot be the case. So the elements of order 4 in K must be in $4C$. Now suppose that a 2-element $g \in \mathrm{M}_{24}$ satisfies (a) and (b) in Lemma 2.3. Since $8A^2 = 4B$, $4A^2 = 2A$ and $4B^2 = 2A$, we can only have $g \in 2A$, $2B$ or $4C$. However, an exhaustive search shows that, for such an element g, the subgroup $\langle K, g\rangle \lneqq \mathrm{M}_{24}$, a contradiction.

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Finally, we consider $(T,G) = (A_{12}, A_{11})$. If $A = A_{12}$, then $A = A_{11}K$ for some subgroup $K \cong A_4$. Since K is regular on $\Omega = \{1,2,\ldots,12\}$, the involution in K must be in conjugacy class $2B$ and the elements of order 3 in class $3C$, following the notation of [1, page 92]. Suppose that a 2-element $g \in A_{12}$ satisfies (a) and (b) in Lemma 2.3. It is evident that g has order 2 or 4. According to the power map of conjugacy classes of A_{12}, if g has order 4, then g^2 cannot be in $2B$. Thus g must have order 2. Furthermore, $|K \cap K^g| = 3$ implies that g normalises the element of $3C$. With the help of this information, an exhaustive search shows that $\langle K, g\rangle$ cannot be A. Similarly, when $A = S_{12}$, there is no 2-element satisfying (a) and (b) in Lemma 2.3.

The argument above shows that Case 1 does not occur.

Case 2. $(T,G) = (\mathrm{M}_{12}, \mathrm{M}_{11})$.

In this case, Γ is either $(\mathrm{Aut}(\mathrm{M}_{12}), 2)$-arc transitive or $(\mathrm{M}_{12}, 2)$-arc-transitive. Consider first $\mathrm{Aut}(\mathrm{M}_{12}) = \mathrm{M}_{12}{:}2$. This group contains a unique class of subgroups isomorphic to M_{11}. Since $|A : G| = 24$, we have $A_\alpha = S_4$ by Lemma 2.1. Computation using GAP [11] yields the following:

(a) $A = \mathrm{M}_{12}{:}2$ has a unique class of subgroups $K \cong S_4$ such that $K \cap \mathrm{M}_{11} = 1$;
(b) for a subgroup K in (a), there are in total sixteen 2-elements $g \in A$ such that K and g satisfy (a) and (b) in Lemma 2.3—denote the set of these 16 elements by Δ;
(c) $\mathrm{N}_A(K) = K \cong S_4$ and the conjugate action of K on Δ produces two orbits, denoted by Δ_1 and Δ_2, with $|\Delta_1| = 12$ and $|\Delta_2| = 4$.

Let $K = S_4$ be a subgroup obtained in (a). For any g satisfying (b), the coset graph $\Gamma(\mathrm{M}_{12}{:}2, S_4, g)$ must be a nonnormal 2-transitive tetravalent Cayley graph of M_{11}. For a coset graph $\Gamma(G, K, g)$, it is not difficult to verify that $\Gamma(G,K,g) \cong \Gamma(G, K^x, g^x)$ for any $x \in \mathrm{Aut}(G)$ (see [3, Fact 2.2]). It then follows that all coset graphs $\Gamma(\mathrm{M}_{12}{:}2, K, g)$ with $g \in \Delta_i$ are isomorphic, for $i = 1, 2$. Fix $g_i \in \Delta_i$ for $i = 1, 2$. Define

$$\Gamma(\Delta_i) := \Gamma(\mathrm{M}_{12}{:}2, S_4, g_i), \quad i = 1, 2. \tag{3.3}$$

These two graphs are, up to isomorphism, the only tetravalent 2-transitive nonnormal Cayley graphs of M_{11}, for $\mathrm{Aut}(\Gamma) = \mathrm{M}_{12}{:}2$.

Next, we consider $\Gamma(\mathrm{M}_{12}, K, g)$. Computation shows that M_{12} has a unique class of subgroups $K \cong A_4$ satisfying $K \cap \mathrm{M}_{11} = 1$. So we may choose $K = A_4$ such that A_4 is a subgroup of S_4 given in the previous case. In addition, there are in total twelve 2-elements g such that K and g satisfy (a) and (b) in Lemma 2.3. Moreover, these 2-elements are all in Δ_1 above and K is transitive on Δ_1 by conjugate action. Thus, up to isomorphism, we obtain a unique tetravalent 2-transitive nonnormal Cayley graph of M_{11}, which is isomorphic to

$$\Gamma^*(\Delta_1) := \Gamma(\mathrm{M}_{12}, A_4, g_1)$$

for $g_1 \in \Delta_1$.

We claim that $\Gamma^*(\Delta_1)$ and $\Gamma(\Delta_1)$ are isomorphic. Note that A_4 is contained in S_4 and M_{12} is transitive on both $V\Gamma^*(\Delta_1)$ and $V\Gamma(\Delta_1)$. Define

$$\sigma: \quad A_4x \mapsto S_4x, \quad x \in M_{12}.$$

It is straightforward to verify that σ is an isomorphism from $\Gamma^*(\Delta_1)$ to $\Gamma(\Delta_1)$.

Therefore, any quasiprimitive tetravalent 2-transitive nonnormal Cayley graph of a finite simple group is isomorphic to $\Gamma(\Delta_1)$ or $\Gamma(\Delta_2)$.

Situation 2: A is not quasiprimitive on $V\Gamma$. In this case, let H be a maximal intransitive normal subgroup of A. Recall that $A = GA_\alpha$ with $G \cap A_\alpha = 1$, where $A_\alpha \cong A_4$ or S_4. By Lemma 2.2, we see that only (c)(i) in Lemma 2.2 occurs. This means that A/H is an almost simple group and $\mathrm{soc}(A/H)$ contains $GH/H \cong G$ and is transitive on $V\Gamma_H$, where Γ_H is the quotient graph of Γ relative to H. Set $T = \mathrm{soc}(A/H)$.

Case 1. $T \cong G$.

Since G is simple and $H \lhd A$, we have $H \cap G = 1$, which implies that $|H|$ is a divisor of $|S_4| = 24$. If G acts on H nontrivially by conjugation, then G is isomorphic to a subgroup of $\mathrm{Aut}(H)$. On the other hand, it is not hard to verify that this is not the case for $G = \mathrm{PSL}_2(11)$, M_{11}, M_{23} or A_{11}. So we assume that $GH = G \times H$. Now $T \cong G$. It follows that $|\mathrm{Out}(T)| = 1$ for $G = M_{11}$ and $G = M_{23}$, while $|\mathrm{Out}(T)| = 2$ for $G = \mathrm{PSL}_2(11)$ and $G = A_{11}$. In the former case we have $A = G \times H$ and hence $G \lhd A$, which is impossible. In the latter case we have $|A : G \times H| = 1$ or 2, which implies that $G \lhd A$, a contradiction. So Case 1 does not occur.

Case 2. $T \ncong G$.

Clearly, $G \cap H = 1$ and $|H|$ divides $|A_\alpha|$. So 24 is divisible by $|H|$. If 3 divides $|H|$, then $|T : GH/H|$ is a divisor of $8 = 2^3$, which is impossible by [7]. So H is a 2-group with $|H|$ dividing 8. Further, if $H_\alpha \neq 1$, then $d(\Gamma_H) = 2$, and hence $\mathrm{Aut}(\Gamma_H)$ is a dihedral group, a contradiction. Hence H is semiregular on $V\Gamma$ and $d(\Gamma_H) = d(\Gamma) = 4$.

For $\alpha = 1 \in G = V\Gamma$, set $\bar{\alpha} = \alpha^H$. Since Γ is A-arc-transitive, Γ_H is A/H-arc transitive. Since $(A/H)_{\bar{\alpha}} = \{Hx \mid x \in A_\alpha\}$ and $H \cap A_\alpha = 1$, it follows that $(A/H)_{\bar{\alpha}} \cong A_\alpha$. From this, Γ_H is $(A/H, 2)$-arc transitive.

Next, we determine all pairs (T, G). Note that $|A : G|$ divides 24. So $|A/H : GH/H|$ divides $24/|H|$. Since H is a 2-group, $24/|H|$ is 6 or 12. Hence, by [1], (T, G) must be one of the pairs

$$(M_{11}, \mathrm{PSL}_2(11)),\ (M_{12}, M_{11}),\ (A_{12}, A_{11}). \tag{3.4}$$

From this, $|H| = 2$ and $\mathrm{soc}(A/H) = A/H = T$. Thus Γ_H is $(T, 2)$-arc-transitive with $|V\Gamma_H| = |G|/2$ and (T, G) given in (3.4).

Finally, we construct all $(T, 2)$-arc-transitive graphs for (T, G) as given in (3.4). Note that $|T|/|G| = 12$, $|V\Gamma_H| = |G|/2$ and $|T_{\bar{\alpha}}| = 24$. So $T_{\bar{\alpha}} \cong S_4$.

Consider $T = M_{11}$ first. There is only one class of subgroups isomorphic to S_4. Let K be such a subgroup. Computation using GAP [11] shows that there is no 2-element g in T satisfying (a) and (b) in Lemma 2.3, which is a contradiction.

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Consider $T = \mathrm{M}_{12}$. Computation shows that there are four classes of subgroups isomorphic to S_4. Using GAP [11], we see that there is no 2-element $g \in T$, together with $K \cong S_4$, satisfying (a) and (b) in Lemma 2.3, which is a contradiction.

Finally, consider $T = A_{12}$. There are 24 conjugate classes of subgroups $K \cong S_4$. A systematic search using GAP [11] shows that there is no 2-element $g \in T$ such that K and g satisfy (a) and (b) in Lemma 2.3. This completes the proof of Proposition 3.1. □

4. Proof of Theorem 1.1

By Proposition 3.1, a connected tetravalent 2-transitive nonnormal Cayley graph of a finite simple group is isomorphic to $\Gamma(\Delta_1)$ or $\Gamma(\Delta_2)$. In this section we prove that these two graphs are nonisomorphic and thus complete the proof of Theorem 1.1.

PROPOSITION 4.1. *The graphs* $\Gamma(\Delta_1)$ *and* $\Gamma(\Delta_2)$ *defined in (3.3) are not isomorphic.*

PROOF. Write $\Gamma_i = \Gamma(\Delta_i)$ and $X_i = \mathrm{Aut}(\Gamma)_i$, for $i = 1, 2$. It follows from Proposition 3.1 that Γ_1 and Γ_2 have the same vertex set and full automorphism group. Denote $V = V\Gamma_1 = V\Gamma_2$ and $X = X_1 = X_2 = \mathrm{Aut}(\mathrm{M}_{12}) = \mathrm{M}_{12}{:}2$. Now $X_\alpha = S_4$. Suppose by way of contradiction that $\Gamma_1 \cong \Gamma_2$. Let ϕ be an isomorphism from Γ_1 to Γ_2. Then $\phi \in \mathrm{N}_{\mathrm{Sym}(V)}(X)$ by [3, Fact 2.3]. Write $N = \mathrm{N}_{\mathrm{Sym}(V)}(X)$ and $C = \mathrm{C}_{\mathrm{Sym}(V)}(X)$. Then N/C is isomorphic to a subgroup of $\mathrm{Aut}(X)$. Moreover, since the vertex stabiliser $X_\alpha \cong S_4$ is self-normalised in X (see Case 2, result (c) of Situation 1 in Section 3), $C = 1$ by [3, Proposition 2.4] and hence N is a subgroup of $\mathrm{Aut}(X)$. Note that $X = \mathrm{Aut}(\mathrm{M}_{12})$ and $\mathrm{Out}(X) = 1$. Thus $N = X$. It follows that $\phi \in X$ is an automorphism of Γ_1, which implies that $\Gamma_1 = \Gamma_2$. On the other hand, for $\alpha = S_4 \in V$, the neighbourhood $\Gamma_i(\alpha)$ of α in Γ_i is given by

$$\Gamma_i(\alpha) = \{S_4 g_i x \mid x \in S_4\}, \quad \text{for } i = 1, 2.$$

However, computation shows that $\Gamma_1(\alpha) \neq \Gamma_2(\alpha)$, which contradicts the statement that $\Gamma_1 = \Gamma_2$. Therefore, Γ_1 and Γ_2 are not isomorphic. □

PROOF OF THEOREM 1.1. By [2, Theorem 1.1], Lemma 2.1, Proposition 3.1 and Proposition 4.1, we obtain Theorem 1.1 immediately. □

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