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SUBCRITICAL SEVASTYANOV BRANCHING PROCESSES WITH NONHOMOGENEOUS POISSON IMMIGRATION

OLLIVIER HYRIEN,* *Fred Hutchinson Cancer Research Center*
KOSTO V. MITOV,** *Vasil Levski National Military University*
NIKOLAY M. YANEV,*** *Bulgarian Academy of Sciences*

Abstract

We consider a class of Sevastyanov branching processes with nonhomogeneous Poisson immigration. These processes relax the assumption required by the Bellman–Harris process which imposes the lifespan and offspring of each individual to be independent. They find applications in studies of the dynamics of cell populations. In this paper we focus on the subcritical case and examine asymptotic properties of the process. We establish limit theorems, which generalize classical results due to Sevastyanov and others. Our key findings include a novel law of large numbers and a central limit theorem which emerge from the nonhomogeneity of the immigration process.

Keywords: Branching process; immigration; Poisson process; limit theorem

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Secondary 60F05; 60J85; 62P10

1. Introduction

Age-dependent branching processes with immigration are well suited to describe the temporal evolution of populations in which individuals (for example, cells) appear randomly over time in accordance with two distinct mechanisms. One mechanism, called immigration, is the influx of new individuals in the population of which they are not natives. The other mechanism, referred to as branching, is the process by which individuals of the population generate new offspring. These models have attracted much attention since Sevastyanov's seminal work on continuous-time Markov branching processes with immigration [29]. Jagers [17] established asymptotic properties in the age-dependent (Bellman–Harris) case. Other properties, also for Bellman–Harris processes, were subsequently proven by Radcliffe [28], Pakes and Kaplan [27], and Kaplan and Pakes [19], among others. Olofsson considered a process with a more general branching mechanism [25]. These papers dealt with time-homogeneous immigration processes, and Mitov and Yanev [21]–[23] investigated a Bellman–Harris process with state-dependent immigration (see also [1, Chapter 3]). Age-dependent branching processes with immigration have been proposed to study the dynamics of cell populations developing *in vivo* [7], [8], [12], [15], [33], [34]. We refer the reader to [1]–[3], [6], [18], and [32] for general monographs on branching processes.

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* Postal address: Program in Biostatistics, Bioinformatics, and Epidemiology, Vaccine and Infectious Disease Division, Fred Hutchinson Cancer Research Center, 1100 Fairview Ave. N., Seattle, WA 98109, USA.
Email address: ohyrien@fredhutch.org
** Postal address: Faculty of Aviation, Vasil Levski National Military University, 5856 D. Mitropolia, Pleven, Bulgaria.
Email address: kmitov@yahoo.com
*** Postal address: Department of Operations Research, Probability and Statistics, Institute of Mathematics and Informatics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria. Email address: yanev@math.bas.bg

In this paper we consider a class of age-dependent branching processes with immigration, in which

- the branching mechanism obeys the assumptions of a Sevastyanov process [31]–[33], an extension of the Bellman–Harris process which allows lifespan and offspring to be dependent;
- the immigration process is time-inhomogeneous.

In Section 2 these two assumptions are motivated by using a cell biology example. Mitov and Yanev [24] and Hyrien *et al.* [10] studied critical processes for various intensities of the nonhomogeneous Poisson immigration process, while Hyrien *et al.* [11] investigated the supercritical case. Pakes [26] considered critical and subcritical Bellman–Harris processes with a nonhomogeneous Poisson immigration process that converges weakly to a homogeneous Poisson process. Yanev [35], [36] studied the Sevastyanov process when immigration is time-homogeneous. Here, we investigate the subcritical case which is adapted to model the dynamics of terminally differentiated cells.

The asymptotic properties are studied when the intensity of the Poisson immigration process belongs to the class of power and exponential functions.

The process is formulated in Section 3. The asymptotic behavior of its expectation and variance-covariance function is investigated in Section 4. All other limit theorems are stated in Section 5. The conditional limiting distributions of Theorems 5.1–5.3 are akin to those that have been established for continuous-time branching processes without immigration; see, e.g. [1]–[3], [6], [18], [32]. Theorems 5.4–5.7 uncover behaviors in the form of a law of large numbers (LLN) and the central limit theorem (CLT) that are novel for branching processes and which arise from the nonhomogeneity of the immigration process. Theorem 5.8 generalizes a classical result due to Sevastyanov [29] to the more general setting considered herein.

2. A biological motivation

The molecular events and pathways that control cell fate decision (e.g. division, death, differentiation) in multicellular systems are not well understood and are the subject of intensive basic science research. Experimental set-ups used in these studies often yield observations about the composition of the system at discrete time points only. A viable approach to inferring about cell fate on the basis of such data consists in modeling the dynamics of the cell population in order to relate experimental observations to (unobserved) cell cycle outcomes. Key to the success of this approach is an appropriate modeling framework in which the most prominent features of cell proliferation are properly captured. The Bellman–Harris process has been successfully used to develop stochastic models in this context. However, the rise of high-throughput, high-dimensional data produced by modern technologies (e.g. flow cytometry, imaging, sequencing) permits a wealth of cellular information to be produced at the single-cell resolution. This information allows teasing apart more subtle models.

An assumption that is central to the make-up of the Bellman–Harris process is that the duration of the lifespan, η, and the number of daughters, ξ, of any cell of the population are independent random variables. This assumption was found inadequate in several studies [4], [9], [13], [14], [16]. One of these studies investigated the generation of terminally differentiated oligodendrocytes from their progenitor cells (PCs), known as the oligodendrocytes type-2 astrocytes PCs (O-2A/OPCs) [16]. These cells of the central nervous system produce the myelin sheath that insulate axons that conduct electrical impulses to transmit information between

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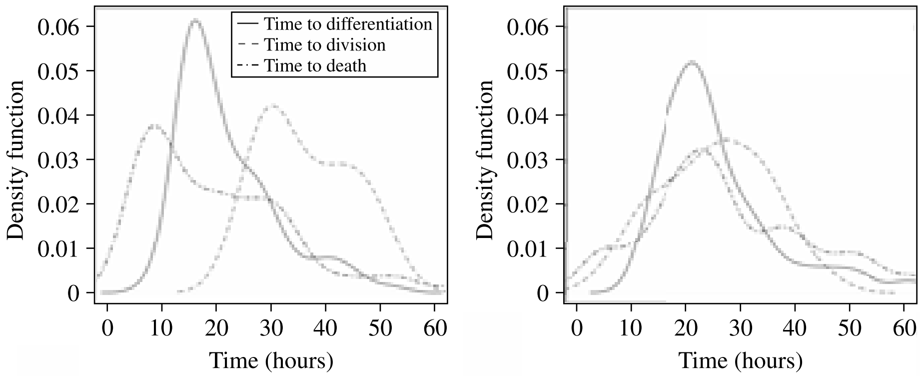

FIGURE 1: Kernel density estimates of the distributions of the time to differentiation, the time to division, and the time to death of O-2A/OPCs in the presence (*right*) and in the absence (*left*) of thyroid hormone observed in time-lapse experiments (Hyrien *et al.* [16]). The plots indicate that the distribution of the lifespan depended on the fate of the cell (here: differentiation, division, or death), and that these distributions were affected by thyroid hormone. For example, the presence of thyroid hormone appeared to shorten the time to differentiation and increase the time to death and the time to division. Unlike the Bellman–Harris process, the Sevastyanov process is adapted to describe these features of the cell cycle.

neurons. It was observed in this study that the time to division, the time to death, and the time to differentiation of these cells into oligodendrocytes had dissimilar distributions and that these distributions could be differentially affected by exposure to external signals (see Figure 1). This finding may be explained by the fact that fate-specific molecular events are triggered in order for cells to reach their ultimate transformation. It also reflects the fact that the time at which a specific fate is detected may be arbitrary and depends on the experimental set-up that is used. The time at which a cell divides may be unambiguously defined as the time at which it splits into two daughter cells at the end of cytokinesis. However, the definition of the time at which a cell dies is debatable because the event that leads to the ultimate disintegration of the cell is not observable, and the time of cell death is instead defined as the time at which an outcome of death (e.g. fragmentation of the cell membrane) becomes experimentally detectable. Relaxing the assumption of independence between η and ξ yields the class of Sevastyanov processes.

Populations of lineage-committed PCs that develop *in vivo* are sustained by influxes of differentiated cells produced by multipotent stem cells or PCs. This mechanism ensures the maintenance of terminally differentiated tissues that have limited self-renewing capabilities. The influx may vary over time. For example, it may temporarily increase in order to accelerate the repair of damaged downstream cellular compartments. The dynamics of such systems may be described by subcritical branching processes with time-inhomogeneous immigration. Here we ask whether the distribution of the population size can be characterized. We find under mild assumptions that the process is asymptotically normal when immigration increases over time.

To avoid modeling the influx of new cells as a Poisson process, cell kinetics could be alternatively formulated as a two-type reducible Sevastyanov branching process in which the first type of cells would correspond to (unobservable) upstream stem and PCs, while the second type would describe the pool of observable cells (Figure 2; see also Kesten and Stigum [20] for a discrete-time version of the process). While conceptually simple, this formulation presents several drawbacks, including the following.

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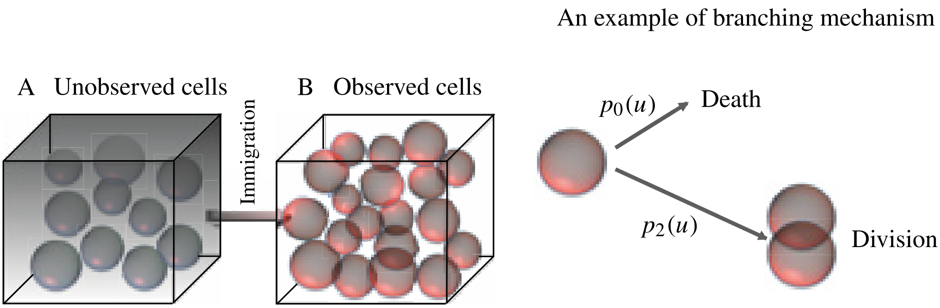

FIGURE 2: *Left:* schematic representation of a population consisting of two types of cells: those in box B are observed, and those in box A are unobservable but contribute to the observed population (box B) via immigration. Our process does not describe population dynamics within box A, but formulates the influx of cells from box A into box B as a nonhomogeneous Poisson process, and models the dynamics within box B as a Sevastyanov process. *Right:* an example of a branching mechanism allowed by the process in which a cell may either die or divide; the Sevastyanov process allows the probabilities of division and death to depend on the time elapsed since the cell was born. For instance, cell death may occur stochastically earlier than cell division.

- It assumes that type-1 cells form a population of homogeneous cells. This assumption may be too rigid in practice. For instance, the hematopoietic PCs that give rise to the various blood cell lineages (e.g. myeloid (erythrocytes, megakaryocytes, monocytes, neutrophils, basophils, or eosinophils) or lymphoid (T- and B-lymphocytes)) are known to exhibit varying degrees of commitment to these lineages, thereby presenting dissimilar probabilities of further specializing into any of them. By assuming that the offspring distribution is identical among all type-1 cells, the two-type process would be less realistic.

- It may lack the flexibility needed to capture the plasticity exhibited by stem and PCs in order to respond to the varying needs of the body. Feedback mechanisms between the two cell types could be included in the model to describe this plasticity. Such an extension, however, would define a model that is considerably more challenging to study, especially in the age-dependent case.

In contrast, our model restricts assumptions postulated on the unobservable cell population to its influx into the pool of observable cells. The rate of this influx may be time-dependent, as needed.

3. The process and its equations

3.1. The Sevastyanov process

We consider a branching process in which the joint distribution of the lifespan η and offspring ξ of any cell is specified as $\mathbb{P}\{\eta \in B,\ \xi = k\} = \int_B p_k(u)\,\mathrm{d}G(u)$ for every Borel set $B \subset \mathbb{R}$, where $G(u) = \mathbb{P}\{\eta \le u\}$ and $\sum_{k=0}^{\infty} p_k(u) = 1$ $(u \ge 0)$, and every cell evolves independently of all other cells. Put $h(u, s) = \sum_{k=0}^{\infty} p_k(u)s^k$, $|s| \le 1$, for the associated probability generating functions (PGF). These assumptions define a (G, h)-Sevastyanov process [32]. Write $\{Z(t)\}_{t\ge 0}$ for the population size at time t.

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Define the moments of the offspring and lifespan distributions $a(t) = h'_s(t,s)|_{s=1}$, $b(t) = h''_{ss}(t,s)|_{s=1}$, $a = \int_0^\infty a(t)\,\mathrm{d}G(t)$, $b = \int_0^\infty b(t)\,\mathrm{d}G(t)$, and $\mu = \int_0^\infty t\,\mathrm{d}G(t)$, all assumed finite. This paper is concerned with the subcritical case ($a < 1$), and we assume the existence of $\alpha < 0$, the Malthusian parameter, which solves $\int_0^\infty \mathrm{e}^{-\alpha x} a(x)\,\mathrm{d}G(x) = 1$. Let $G(t)$ and $G_a^{(\alpha)}(t) = \int_0^t a(u)\mathrm{e}^{-\alpha u}\,\mathrm{d}G(u)$ be nonlattice.

Throughout the paper, we will be assuming that

$$\int_0^\infty x\mathrm{e}^{-\alpha x}\,\mathrm{d}G(x) < \infty, \qquad \int_0^\infty x\,\mathrm{d}G_a^{(\alpha)}(x) < \infty, \tag{3.1}$$

and

$$\int_0^\infty b(u)\mathrm{e}^{-\alpha u}\,\mathrm{d}G(u) < \infty, \qquad \int_0^\infty ub(u)\mathrm{e}^{-\alpha u}\,\mathrm{d}G(u) < \infty. \tag{3.2}$$

Another key assumption is as follows.

Condition 3.1. *The cumulative distribution function (CDF) $G_a^\alpha(t) = \int_0^t a(u)\mathrm{e}^{-\alpha u}\,\mathrm{d}G(u)$ is of absolutely continuous type; that is, there exists $k \geq 1$ for which the k-fold convolution of $G_a^{(\alpha)}(\cdot)$ with itself has an absolutely continuous component (see [32, Definition 2, Chapter VIII.7]).*

Put $A(t) = \mathbb{E}[Z(t) \mid Z(0) = 1]$. When (3.1) holds, it is known that (see [32, Theorem 8.4])

$$A(t) = A\mathrm{e}^{\alpha t}(1 + o(1)), \qquad t \to \infty, \tag{3.3}$$

where

$$A = \frac{\int_0^\infty \mathrm{e}^{-\alpha t}(1 - G(t))\,\mathrm{d}t}{\int_0^\infty x\mathrm{e}^{-\alpha x}a(x)\,\mathrm{d}G(x)}.$$

Define $B(t,\tau) = \mathbb{E}[Z(t)Z(t+\tau) \mid Z(0) = 1]$, $B(t) = \mathbb{E}[Z(t)(Z(t) - 1) \mid Z(0) = 1]$, and $V(t) = \operatorname{var}[Z(t) \mid Z(0) = 1]$. Then $V(t) = B(t,0) - A^2(t) = B(t) + A(t) - A^2(t)$.

Lemma 3.1. *Assume that (3.1), (3.2), and Condition 3.1 hold. Then, for every fixed $\tau \geq 0$, there exists a constant $D_\tau > 0$ such that, as $t \to \infty$,*

$$B(t,\tau) = D_\tau \mathrm{e}^{\alpha(t+\tau)}(1 + o(1)). \tag{3.4}$$

Proof. Write $B_0(t,\tau) = B(t,\tau)\mathrm{e}^{-\alpha t}$ and $A_0(t) = A(t)\mathrm{e}^{-\alpha t}$. Then, we have

$$B_0(t,\tau) = \int_0^t B_0(t-u,\tau)\,\mathrm{d}G_a^{(\alpha)}(u) + I(t,\tau),$$

where

$$\begin{aligned} I(t,\tau) &= \mathrm{e}^{\alpha t}\int_0^t b(u)A_0(t-u)A_0(t-u+\tau)\mathrm{e}^{-2\alpha u}\,\mathrm{d}G(u) \\ &\quad + \int_t^{t+\tau} a(u)A_0(t-u+\tau)\mathrm{e}^{-\alpha u}\,\mathrm{d}G(u) + \mathrm{e}^{-\alpha(t+\tau)}(1 - G(t+\tau)) \\ &= I_1(t,\tau) + I_2(t,\tau) + I_3(t,\tau) \end{aligned}$$

(see [32, Equation (23), Chapter VIII.8]). Since $A_0(t)$ is bounded on $[0,\infty)$, there exists $K > 0$ such that

$$I_1(t,\tau) = \mathrm{e}^{\alpha t}\int_0^t b(u)A_0(t-u)A_0(t-u+\tau)\mathrm{e}^{-2\alpha u}\,\mathrm{d}G(u) \leq 2K\mathrm{e}^{\alpha t}\int_0^t b(u)\mathrm{e}^{-2\alpha u}\,\mathrm{d}G(u).$$

Applying the line of argument used in the proof of Theorem 8.10 in [32] gives $I_1(t,\tau) \in L_1[0,\infty)$ and $I_1(t,\tau) \to 0$ as $t \to \infty$ for any fixed τ. Similarly, there exists $K > 0$ such that

$$I_2(t,\tau) = \int_t^{t+\tau} a(u)A_0(t-u+\tau)\mathrm{e}^{-\alpha u}\,\mathrm{d}G(u) \le K[1 - G_a^{(\alpha)}(t)].$$

The function $K[1 - G_a^{(\alpha)}(t)] \in L_1[0,\infty)$ because of (3.2), hence so does $I_2(t,\tau)$. We deduce from (3.1) that $I_3(t,\tau) = \mathrm{e}^{-\alpha(t+\tau)}[1 - G(t+\tau)] \in L_1[0,\infty)$. Therefore, $I(t,\tau) \in L_1[0,\infty)$ and $I(t,\tau) \to 0$ as $t \to \infty$. Applying Theorem 7.9 from [32] completes the proof. □

An immediate consequence of Lemma 3.1 is that the variance satisfies

$$V(t) = D_0\mathrm{e}^{\alpha t}(1 + o(1)), \qquad t \to \infty. \tag{3.5}$$

3.2. Sevastyanov process with immigration

Let $\{S_k\}_{k=1}^{\infty}$ be a sequence of increasing time points with $S_0 = 0$ that arises from a non-homogeneous Poisson process $\{\Pi(t)\}_{t\ge 0}$, where $\mathbb{P}\{\Pi(t) = k\} = \mathrm{e}^{-R(t)}R^k(t)/k!$, $k \ge 0$, with instantaneous and cumulative rates $r(t)$ and $R(t) = \int_0^t r(u)\,\mathrm{d}u$, with $r(t) \ge 0$. Let, for every $k = 1, 2, \ldots$, I_k denote the number of cells immigrating at time S_k, assumed mutually independent. Put $g(s) = \mathbb{E}[s^{I_k}]$, $|s| \le 1$ for its PGF, and $\gamma = \mathbb{E}[I_k] = g'(1)$ and $\gamma_2 = g''(1) = \mathbb{E}[I_k(I_k - 1)]$ for its mean and second factorial moment. Let $Y(t)$ denote the number of cells in the population at time t described by a branching process with immigration in which the branching mechanism obeys a (G, h)-Sevastyanov process. We decompose $Y(t)$ as

$$Y(t) = \begin{cases} \displaystyle\sum_{k=1}^{\Pi(t)}\sum_{i=1}^{I_k} Z^{(k,i)}(t - S_k) & \text{if } \Pi(t) > 0, \\ 0 & \text{if } \Pi(t) = 0, \end{cases} \tag{3.6}$$

where $\{Z^{(k,i)}(t)\}_{t\ge 0}$, $k, i = 1, 2, \ldots$, are independent and identically distributed (i.i.d.) copies of $\{Z(t)\}_{t\ge 0}$.

Define the PGF $\Psi(t; s) = \mathbb{E}[s^{Y(t)} \mid Y(0) = 0]$. Proceeding as in Yakovlev and Yanev ([34, Theorem 1]), we obtain

$$\Psi(t; s) = \exp\biggl\{-\int_0^t r(t-u)[1 - g(F(u; s))]\,\mathrm{d}u\biggr\}, \qquad \Psi(0, s) = 1, \tag{3.7}$$

where the PGF $F(t; s) = \mathbb{E}[s^{Z(t)} \mid Z(0) = 1]$ satisfies the integral equation

$$F(t, s) = s(1 - G(t)) + \int_0^t h(u, F(t-u, s))\,\mathrm{d}G(u)$$

with the initial condition $F(0, s) = s$. Under mild regularity conditions, $F(t, s)$ is the only solution of this equation that belongs to the class of PGFs (see [32]).

We note that $\{Y(t)\}_{t\ge 0}$ is a time-inhomogeneous and non-Markov process. If $\{U_k = S_k - S_{k-1}\}_{k=1}^{\infty}$ are i.i.d. random variables with CDF $G_0(x) = \mathbb{P}\{U_k \le x\} = 1 - \mathrm{e}^{-x/r_0}$ $(x \ge 0)$, the immigration process $\Pi(t)$ reduces to an ordinary Poisson process with instantaneous and cumulative rates $r(t) \equiv r_0$ and $R(t) = r_0 t$, respectively. Then we obtain the Sevastyanov age-dependent branching process with homogeneous Poisson immigration proposed and investigated by Yanev [35].

Define $M(t) = \mathbb{E}[Y(t) \mid Y(0) = 0]$. Differentiating both sides of (3.7), we obtain

$$M(t) = \Psi'_s(t; s)|_{s=1} = \gamma \int_0^t r(t-u)A(u)\,\mathrm{d}u. \tag{3.8}$$

Put $\Psi(s_1, s_2; t, \tau) = \mathbb{E}[s_1^{Y(t)} s_2^{Y(t+\tau)} \mid Y(0) = 0]$ for $t, \tau \ge 0$. Using a line of argument similar to that used to prove (3.7) yields

$$\begin{aligned}\Psi(s_1, s_2; t, \tau) = \exp\bigg\{ &-\int_0^t r(u)[1 - g(F(s_1, s_2; t-u, \tau))]\,\mathrm{d}u \\ &- \int_t^{t+\tau} r(v)[1 - g(F(1, s_2; t, \tau - v))]\,\mathrm{d}v \bigg\},\end{aligned} \tag{3.9}$$

where $F(s_1, s_2; t, \tau) = \mathbb{E}[s_1^{Z(t)} s_2^{Z(t+\tau)}]$ is given in [32, Equation (22), Chapter VIII.8].

The proof also uses (3.6) and the line of argument used to prove Theorem 1 of Yakovlev and Yanev [34]. Equation (3.9) implies that

$$\begin{aligned} C(t, \tau) &= \operatorname{cov}[Y(t), Y(t+\tau)] \\ &= \log \Psi(s_1, s_2; t, \tau)''_{s_1 s_2}|_{s_1 = s_2 = 1} \\ &= \int_0^t r(u)[\gamma B(t-u, \tau) + \gamma_2 A(t-u)A(t+\tau-u)]\,\mathrm{d}u, \end{aligned} \tag{3.10}$$

with initial conditions $B(0, \tau) = A(\tau)$ and $C(0, \tau) = 0$. Setting $\tau = 0$ in (3.10) yields

$$W(t) = \operatorname{var}[Y(t)] = \int_0^t r(t-u)[\gamma V(u) + (\gamma + \gamma_2)A^2(u)]\,\mathrm{d}u. \tag{3.11}$$

4. Asymptotic formulas for the moments

This section is concerned with the expectation, variance, and covariance of $Y(t)$ as $t \to \infty$. We consider three cases distinguished based on the immigration rate.

(i) $\int_0^\infty r(u)\mathrm{e}^{-\alpha u}\,\mathrm{d}u < \infty$ (Proposition 4.1).

(ii) $r(t) = r_0 \mathrm{e}^{\rho t}$, where $r_0 > 0$, and $\rho \in \mathbb{R}$ (Propositions 4.2 and 4.3). In this case the asymptotics depend on how the immigration parameter ρ compares to the Maltusian parameter α.

(iii) $r(t) = r_0 \times t^\theta$ or $r(t) = r_0 \times (t+1)^\theta$, where $r_0 > 0$ and $\theta \in \mathbb{R}$. In this case the moments either converge to 0 (if $\theta < 0$) or diverge to ∞ (if $\theta > 0$) (Proposition 4.4).

We will use the lemma and corollary below to derive asymptotics for the moments.

Lemma 4.1. *Let $f(t) \sim Ct^\theta$ as $t \to \infty$, where $C > 0$ and $\theta \in \mathbb{R}$. Assume that*

$$\sup_{0 \le x \le t} f(x) \le Dt^{\max(\theta, 0)} \quad \textit{for some } D < \infty.$$

Let $y(\cdot)$ be any function such that $y(t) \ge 0$ $(t \ge 0)$, $\overline{y} = \int_0^\infty y(u)\,\mathrm{d}u < \infty$, and $y(t) = o(t^{\theta-1})$ if $\theta < 0$. Then, as $t \to \infty$,

$$I(t) = \int_0^t f(t-u)y(u)\,\mathrm{d}u \sim \overline{y} f(t).$$

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Proof. For every $0 < \delta < 1$, we have $I(t) = \int_0^t f(t-u)y(u)\,\mathrm{d}u = \int_0^{\delta t} + \int_{\delta t}^t = I_1(t) + I_2(t)$. When the assumptions of the lemma hold, we have, for every $\varepsilon > 0$ and when t is large enough, that $(C-\varepsilon)t^\theta \le f(t) \le (C+\varepsilon)t^\theta$ and

$$(C-\varepsilon)t^\theta \int_0^{\delta t}\Big(1-\frac{u}{t}\Big)^\theta y(u)\,\mathrm{d}u \le I_1(t) \le (C+\varepsilon)t^\theta \int_0^{\delta t}\Big(1-\frac{u}{t}\Big)^\theta y(u)\,\mathrm{d}u.$$

To determine the limit of $I_1(t)$ as $t \to \infty$, assume first that $\theta \ge 0$. Then

$$(C-\varepsilon)t^\theta(1-\delta)^\theta \int_0^{\delta t} y(u)\,\mathrm{d}u \le I_1(t) \le (C+\varepsilon)t^\theta \int_0^{\delta t} y(u)\,\mathrm{d}u,$$

$$\Big(1-\frac{\varepsilon}{C}\Big)(1-\delta)^\theta\overline{y} \le \liminf_{t\to\infty}\frac{I_1(t)}{f(t)} \le \limsup_{t\to\infty}\frac{I_1(t)}{f(t)} \le \Big(1+\frac{\varepsilon}{C}\Big)\overline{y}.$$

Therefore, $\lim_{t\to\infty}[I_1(t)/f(t)] = \overline{y}$. Assume next that $\theta \le 0$. Then

$$(C-\varepsilon)t^\theta \int_0^{\delta t} y(u)\,\mathrm{d}u \le I_1(t) \le (C+\varepsilon)t^\theta(1-\delta)^\theta \int_0^{\delta t} y(u)\,\mathrm{d}u,$$

$$\Big(1-\frac{\varepsilon}{C}\Big)\overline{y} \le \liminf_{t\to\infty}\frac{I_1(t)}{f(t)} \le \limsup_{t\to\infty}\frac{I_1(t)}{f(t)} \le \Big(1+\frac{\varepsilon}{C}\Big)(1-\delta)^\theta\overline{y}.$$

Hence $\lim_{t\to\infty}[I_1(t)/f(t)] = \overline{y}$.

Likewise, to study $I_2(t)$ as $t \to \infty$, assume first that $\theta \ge 0$. Then

$$I_2(t) \le \sup_{0\le x\le t(1-\delta)} f(x) \int_{\delta t}^t y(u)\,\mathrm{d}u \le Dt^\theta(1-\delta)^\theta \int_{\delta t}^t y(u)\,\mathrm{d}u,$$

which establishes that $\limsup_{t\to\infty}[I_2(t)/f(t)] = 0$. Assume next that $\theta < 0$. Then $I_2(t) \le \sup_{0\le x\le t(1-\delta)} f(x)\int_{\delta t}^t y(u)\,\mathrm{d}u = o(t^\theta)$, and $\limsup_{t\to\infty}[I_2(t)/f(t)] = 0$.

We finally deduce that $\lim_{t\to\infty}[I(t)/f(t)] = \overline{y}$, which completes the proof. □

Corollary 4.1. *Let $f(t)$ and $y(t)$ be nonnegative functions. Assume that $f(t)$ is bounded in $\mathbb{R}^+$, and that there exists $f^* < \infty$ and $\overline{y} < \infty$ such that $\lim_{t\to\infty} f(t) = f^*$ and $\int_0^\infty y(t)\,\mathrm{d}t = \overline{y}$. Then*

$$\lim_{t\to\infty}\int_0^t f(u)y(t-u)\,\mathrm{d}u = f^*\overline{y}.$$

Define, for every $\alpha < 0$, $\widehat{r}_t(\alpha) = \int_0^t r(u)\mathrm{e}^{-\alpha u}\,\mathrm{d}u$, and assume that

$$\lim_{t\to\infty}\widehat{r}_t(\alpha) = \widehat{r}(\alpha) < \infty. \tag{4.1}$$

Inequality (4.1) holds if, for example, the intensity of the immigration assumes the form $r(t) = O(\mathrm{e}^{\rho t})$ with $\rho < \alpha$.

Proposition 4.1. *Assume that (3.1), (4.1), and $\gamma_2 < \infty$ hold. Then, as $t \to \infty$,*

$$M(t) = A\gamma\hat{r}(\alpha)\mathrm{e}^{\alpha t}(1+o(1)). \tag{4.2}$$

Moreover, if (3.2) and Condition 3.1 hold, then, as $t \to \infty$,

$$C(t,\tau) = \gamma\hat{r}(\alpha)D_\tau\mathrm{e}^{\alpha(t+\tau)}(1+o(1)). \tag{4.3}$$

Proof. Equation (3.8) entails that

$$M(t) = \gamma e^{\alpha t} \int_0^t r(t-u) e^{-\alpha(t-u)} A(u) e^{-\alpha u} \, du.$$

Using (3.3) and (4.1) and applying Corollary 4.1 yields

$$\int_0^t r(t-u) e^{-\alpha(t-u)} A(u) e^{-\alpha u} \, du \to A\hat{r}(\alpha),$$

from which (4.2) follows. The proof of (4.3) relies on (3.3), (3.4), and (3.10), but remains otherwise similar to that of (4.2). □

Therefore, when the assumptions of Proposition 4.1 hold, the variance $W(t)$ satisfies

$$W(t) = D_0 \gamma \hat{r}(\alpha) e^{\alpha t}(1 + o(1)), \qquad t \to \infty.$$

Proposition 4.2. *Assume that (3.1), $r(t) = r_0 e^{\rho t}$ for some constant $r_0 > 0$, and $t \to \infty$ hold.*

(i) *If $\rho > \alpha$ then $M(t) = \gamma r_0 \hat{A}(\rho) e^{\rho t}(1 + o(1))$, $\hat{A}(\rho) = \int_0^\infty e^{-\rho u} A(u) \, du < \infty$.*

(ii) *If $\rho = \alpha$ then $M(t) = \gamma r_0 A t e^{\alpha t}(1 + o(1))$.*

(iii) *If $\rho < \alpha$ then $M(t) = (r_0 \gamma A/(\alpha - \rho)) e^{\alpha t}(1 + o(1))$.*

Proof. (i) From (3.8), we have

$$M(t) = \gamma r_0 e^{\rho t} \int_0^t e^{-(\rho-\alpha)u} A(u) e^{-\alpha u} \, du.$$

Since $A(u)e^{-\alpha u} \to A$, $u \to \infty$ (by (3.3)), and $\rho - \alpha > 0$, the integral converges to $\hat{A}(\rho)$, which completes the proof.

(ii) In this case $M(t) = \gamma r_0 e^{\alpha t} \int_0^t A(u) e^{-\alpha u} \, du$ and $\int_0^t A(u) e^{-\alpha u} \, du \sim At$, $t \to \infty$, which completes the proof.

(iii) We have $M(t) = \gamma r_0 e^{\alpha t} \int_0^t e^{(\rho-\alpha)(t-u)} A(u) e^{-\alpha u} \, du$. Since $A(u)e^{-\alpha u} \to A$, $u \to \infty$, (by (3.3)) and $\int_0^\infty e^{(\rho-\alpha)t} \, dt = 1/(\alpha - \rho) > 0$, the proof follows from Corollary 4.1. □

Proposition 4.3. *Assume that (3.1), (3.2), Condition 3.1, $r(t) = r_0 e^{\rho t}$, $r_0 > 0$, and $t \to \infty$ hold.*

(i) *If $\rho > \alpha$ then $C(t, \tau) = C(\tau) r_0 e^{\rho t + \alpha \tau}(1 + o(1))$, where*

$$C(\tau) = \int_0^\infty e^{-\rho u - \alpha \tau} [\gamma B(u, \tau) + \gamma_2 A(u) A(u + \tau)] \, du < \infty.$$

(ii) *If $\rho = \alpha$ then $C(t, \tau) = r_0 \gamma D_\tau t e^{\alpha(t+\tau)}(1 + o(1))$.*

(iii) *If $\rho < \alpha$ then $C(t, \tau) = (r_0 \gamma D_\tau/(\alpha - \rho)) e^{\alpha(t+\tau)}(1 + o(1))$.*

Proof. We deduce from (3.10) that

$$C(t, \tau) = r_0 e^{\rho t + \alpha \tau} \int_0^t e^{-(\rho-\alpha)u} [\gamma B(u, \tau) + \gamma_2 A(u) A(u + \tau)] e^{-\alpha(u+\tau)} \, du.$$

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In case (i), (3.3)–(3.5) entail that

$$[\gamma B(u,\tau) + \gamma_2 A(u)A(u+\tau)]\mathrm{e}^{-\alpha(u+\tau)} \to \gamma D_\tau, \qquad t \to \infty. \tag{4.4}$$

Since $\rho > \alpha$, the integral

$$C(\tau) = \int_0^\infty \mathrm{e}^{-(\rho-\alpha)u}[\gamma B(u,\tau) + \gamma_2 A(u)A(u+\tau)]\mathrm{e}^{-\alpha(u+\tau)}\,\mathrm{d}u$$

converges. This completes the proof of (i).

In case (ii), it follows from (4.4) that

$$\int_0^t [\gamma B(u,\tau) + \gamma_2 A(u)A(u+\tau)]\mathrm{e}^{-\alpha u}\,\mathrm{d}u \sim \gamma D_\tau t, \qquad t \to \infty,$$

which completes the proof of (ii).

In case (iii), we have

$$C = \int_0^\infty \mathrm{e}^{-(\alpha-\rho)u}\,\mathrm{d}u = \frac{1}{\alpha-\rho} \in (0,\infty)$$

and the statement follows from (4.4) and Corollary 4.1. □

Corollary 4.2. *Suppose that the assumptions of Proposition 4.3 hold and $t \to \infty$.*

(i) *If $\rho > \alpha$ then $W(t) = W\mathrm{e}^{\rho t}(1+o(1))$, where*

$$W = r_0 \int_0^\infty \mathrm{e}^{-\rho u}[\gamma V(u) + (\gamma+\gamma_2)A^2(u)]\,\mathrm{d}u < \infty.$$

(ii) *If $\rho = \alpha$ then $W(t) = r_0\gamma D_0 t\mathrm{e}^{\alpha t}(1+o(1))$.*

(iii) *If $\rho < \alpha$ then $W(t) = (r_0\gamma D_0/(\alpha-\rho))\mathrm{e}^{\alpha t}(1+o(1))$.*

Proposition 4.4. *Suppose that (3.1), (3.2), and Condition 3.1 hold, and assume further that $r(t) = r_0 \times t^\theta$, $0 < \theta < \infty$, or $r(t) = r_0 \times (t+1)^\theta$, $-\infty < \theta < 0$, where $r_0 > 0$ is a constant. Then, as $t \to \infty$,*

$$M(t) = Mt^\theta(1+o(1)), \qquad M = \gamma r_0 \int_0^\infty A(u)\,\mathrm{d}u, \tag{4.5}$$

$$W(t) = Wt^\theta(1+o(1)), \qquad W = r_0 \int_0^\infty [\gamma V(u) + (\gamma+\gamma_2)A^2(u)]\,\mathrm{d}u, \tag{4.6}$$

and, for any $\tau \ge 0$,

$$C(t,\tau) = C(\tau)t^\theta(1+o(1)), \tag{4.7}$$

where $C(\tau) = r_0\int_0^\infty(\gamma B(u,\tau) + \gamma_2 A(u)A(u+\tau))\,\mathrm{d}u$.

Proof. Equation (3.3) implies that $\int_0^\infty A(u)\,\mathrm{d}u < \infty$. Then (4.5) follows from (3.8) and Lemma 4.1. Equation (4.6) is a consequence of (3.3), (3.5), (3.11), and Lemma 4.1. The proof of (4.7) proceeds similarly using (3.3), (3.4), and (3.10). □

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5. Limit theorems

This section presents four classes of limit theorems. The first one is a conditional limit theorem of the form $\lim_{t\to\infty}\mathbb{P}\{Y(t)=k \mid Y(0)>0\}=q_k$, with $\sum_{k=1}^{\infty}q_k=1$, based on the fact that $Y(t)\xrightarrow{\mathbb{P}}0$ as $t\to\infty$ when $\lim_{t\to\infty}M(t)=0$, where '$\xrightarrow{\mathbb{P}}$' denotes convergence in probability. This result is akin to a limit theorem for subcritical processes without immigration (see [32, Section IX.3, Theorem 2]). The second one is an LLN that shows that $Y(t)/M(t)$ converges to 1 in some appropriate sense as $M(t)\to\infty$. The third one is a CLT satisfied by $Y(t)$ when properly normalized. The fourth class of theorems generalizes a classical result on the asymptotic distribution of $Y(t)$ for Markov branching processes with immigration when $\lim_{t\to\infty}r(t)=r_0>0$ that is due to Sevastyanov [29].

One of the byproducts of these limit theorems is a classification of the process into three subclasses:

(i) the subcritical-subcritical case for which conditional limiting distributions are obtained (see Theorems 5.1–5.3);

(ii) the subcritical-supercritical case for which an LLN and a CLT are established (see Theorems 5.4–5.7);

(iii) the pure subcritical case for which a stationary distribution exists and is characterized in Theorem 5.8.

It is interesting to point out that in case (i) the probability of nonextinction converges exponentially to 0 at rate α, the Malthusian parameter (see Theorem 5.1). In Theorem 5.2, the convergence rate is ρ, the immigration rate, and in Theorem 5.3 it is regular varying at ∞ with exponent $\theta<0$ (the parameter of the immigration intensity).

In case (ii), we show that the asymptotic variance of the normalized process (σ^2) depends on the parameter ρ of the immigration intensity in Theorem 5.5, whereas in Theorem 5.7 it is independent of the corresponding parameter θ. In both cases we show that $0<\sigma^2<1$.

Theorem 5.1. *Assume that (3.1), (3.2), and* $\lim_{t\to\infty}\hat{r}_t(\alpha)=\hat{r}(\alpha)<\infty$ *hold. Then we have the following.*

(i) $\mathbb{P}\{Y(t)>0\}=C\mathrm{e}^{\alpha t}(1+o(1))$, $C>0$, *as* $t\to\infty$.

(ii) *There exists a conditional stationary distribution*

$$\lim_{t\to\infty}\mathbb{P}\{Y(t)=k \mid Y(t)>0\}=q_k>0, \qquad k=1,2,\ldots.$$

Proof. Under the assumptions of the theorem we obtain from [32, Section IX.3, Theorems 1 and 2], that, as $t\to\infty$,

$$\mathbb{P}\{Z(t)>0\}=1-F(t,0)=Q\mathrm{e}^{\alpha t}(1+o(1)), \tag{5.1}$$

$$1-F(t,s)=Q(s)\mathrm{e}^{\alpha t}(1+o(1)) \quad \text{for every } s\in[0,1]. \tag{5.2}$$

Since $1-g(s)=\gamma(1-s)(1+o(1))$ it follows that, as $t\to\infty$,

$$1-g(F(t,0))=\gamma Q\mathrm{e}^{\alpha t}(1+o(1)), \tag{5.3}$$

and, for every $s\in[0,1]$,

$$1-g(F(t,s))=\gamma Q(s)\mathrm{e}^{\alpha t}(1+o(1)). \tag{5.4}$$

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Define the conditional PGF

$$\Psi^*(t;s) = \mathbb{E}[s^{Y(t)} \mid Y(t) > 0] = 1 - \frac{1-\Psi(t;s)}{1-\Psi(t;0)}. \tag{5.5}$$

Note first that $\Psi(t;0) = \mathrm{e}^{-J(t)}$, where $J(t) = \int_0^t r(t-u)[1-g(F(u;0))]\,\mathrm{d}u$. Then, (5.3) and Lemma 4.1 entail that, as $t \to \infty$,

$$J(t) = \mathrm{e}^{\alpha t} \int_0^t r(t-u)\mathrm{e}^{-\alpha(t-u)}\mathrm{e}^{-\alpha u}(1-g(F(u,0))\,\mathrm{d}u \sim \mathrm{e}^{\alpha t}\gamma Q\hat{r}(\alpha).$$

Furthermore, $\Psi(t;s) = \exp\{-J(t;s)\}$, where, from (5.4) and Lemma 4.1,

$$\begin{aligned} J(t;s) &= \int_0^t r(t-u)(1-g(F(u;s))\,\mathrm{d}u \\ &= \mathrm{e}^{\alpha t}\int_0^t r(t-u)\mathrm{e}^{-\alpha(t-u)}\mathrm{e}^{-\alpha u}(1-g(F(u;s))\,\mathrm{d}u \\ &\sim \mathrm{e}^{\alpha t}\gamma Q(s)\hat{r}(\alpha), \qquad t\to\infty. \end{aligned}$$

Since $\alpha < 0$, $J(t) \to 0$ and $J(t;s) \to 0$, uniformly in $s \in [0,1]$. Therefore, as $t \to \infty$,

$$\begin{aligned} 1-\Psi(t;s) &= 1-\mathrm{e}^{-J(t;s)} = J(t;s)(1+o(1)) = \mathrm{e}^{\alpha t}\gamma Q(s)\hat{r}(\alpha)(1+o(1)), \\ 1-\Psi(t;0) &= 1-\mathrm{e}^{-J(t)} = J(t)(1+o(1)) = \mathrm{e}^{\alpha t}\gamma Q\hat{r}(\alpha)(1+o(1)). \end{aligned}$$

The last two relationships show that, uniformly in $s \in [0,1]$,

$$\lim_{t\to\infty}\Psi^*(t;s) = \Psi^*(s) = \sum_{k=1}^{\infty} q_k s^k = \frac{1-Q(s)}{Q},$$

which completes the proof of the theorem by invoking the continuity theorem for PGFs. □

Remark 5.1. The limiting PGF $\Psi^*(s) = \sum_{k=1}^{\infty} q_k s^k$, $0 \le s \le 1$, in Theorem 5.1 is similar to that holding for the standard Sevastyanov process without immigration.

Theorem 5.2. *Assume that (3.1), (3.2), and* $r(t) = r_0\mathrm{e}^{\rho t}$, $r_0 > 0$, $\alpha < \rho < 0$ *hold. Then we have the following.*

(i) $\mathbb{P}\{Y(t) > 0\} = K\mathrm{e}^{\rho t}(1+o(1))$, $K > 0$, *as* $t \to \infty$.

(ii) *There exists a conditional stationary distribution*

$$\lim_{t\to\infty}\mathbb{P}\{Y(t) = k \mid Y(t) > 0\} = q_k > 0, \qquad k = 1,2,\ldots,$$

with limiting conditional PGF

$$\Psi^*(s) = 1 - \frac{\int_0^\infty \mathrm{e}^{-\rho u}[1-g(F(u;s))]\,\mathrm{d}u}{\int_0^\infty \mathrm{e}^{-\rho u}[1-g(F(u;0))]\,\mathrm{d}u}, \qquad \Psi^*(1) = 1.$$

Proof. Following the proof of the previous theorem, we obtain

$$J(t) = r_0\mathrm{e}^{\rho t}\int_0^t \mathrm{e}^{-\rho u}[1-g(F(u;0))]\,\mathrm{d}u \sim r_0 K\mathrm{e}^{\rho t}, \qquad t\to\infty,$$

where $K = \int_0^\infty \mathrm{e}^{-\rho u}[1 - g(F(u;0))]\,\mathrm{d}u < \infty$. Similarly, for every $s \in [0,1)$,

$$J(t;s) = r_0 \mathrm{e}^{\rho t} \int_0^t \mathrm{e}^{-\rho u}[1 - g(F(u;s))]\,\mathrm{d}u \sim r_0 K(s)\mathrm{e}^{\rho t}, \qquad t \to \infty,$$

where $K(s) = \int_0^\infty \mathrm{e}^{-\rho u}[1 - g(F(u;s))]\,\mathrm{d}u < \infty$. Therefore, following the proof of Theorem 5.1, we obtain, as $t \to \infty$,

$$\mathbb{P}\{Y(t) > 0\} = 1 - \Psi(t;0) \sim r_0 K \mathrm{e}^{\rho t}, \qquad 1 - \Psi(t;s) \sim r_0 K(s)\mathrm{e}^{\rho t}.$$

Hence, by (5.5), there exists $\Psi^*(s) = \lim_{t\to\infty} \Psi^*(t;s) = 1 - K(s)/K$, which proves the theorem. □

Theorem 5.3. *Assume that (3.1), (3.2), and* $r(t) = r_0 \times (t+1)^\theta, \theta < 0, r_0 > 0$ *hold. Then we have the following.*

(i) $\mathbb{P}\{Y(t) > 0\} \sim -(r_0 \gamma Q/\alpha) t^\theta$ *as* $t \to \infty$.

(ii) *There exists a conditional stationary distribution* $\{q_k\}_{k=1}^\infty$ *such that*

$$\lim_{t\to\infty} \mathbb{P}\{Y(t) = k \mid Y(t) > 0\} = q_k \qquad (k = 1, 2, \ldots),$$

where $\Psi^*(s) = 1 - Q(s)/Q$, $\Psi^*(1) = 1$, *and* Q *and* $Q(s)$ *as defined in (5.1) and (5.2).*

Proof. Consider the conditional PGF $\Psi^*(t;s)$ as defined in (5.5) and note that $\Psi(t;s) = \exp\{-r_0 \times (t+1)^\theta J_1(t;s)\}$, where

$$\begin{aligned} J_1(t;s) &= \int_0^t \left(1 - \frac{u}{t+1}\right)^\theta [1 - g(F(u;s))]\,\mathrm{d}u \\ &= (t+1) \int_0^{1-1/(t+1)} (1-x)^\theta [1 - g(F(x(t+1);s))]\,\mathrm{d}x. \end{aligned}$$

Setting $s = 0$, we deduce from (5.3) that

$$1 - g(F(x(t+1);0)) \sim \gamma[1 - F(x(t+1);0)] \sim \gamma Q \mathrm{e}^{\alpha x(t+1)}, \qquad t \to \infty.$$

Therefore,

$$J_1(t;0) \sim \gamma Q \times (t+1) \int_0^{1-1/(t+1)} (1-x)^\theta \mathrm{e}^{\alpha x(t+1)}\,\mathrm{d}x, \qquad t \to \infty.$$

Furthermore, as $t \to \infty$,

$$\begin{aligned} &(t+1) \int_0^{1-1/(t+1)} (1-x)^\theta \mathrm{e}^{\alpha x(t+1)}\,\mathrm{d}x \\ &\quad = \alpha^{-1}\left[\mathrm{e}^{\alpha t}(t+1)^{-\theta} - 1 + \theta \int_0^{1-1/(t+1)} (1-x)^{\theta-1} \mathrm{e}^{\alpha x(t+1)}\,\mathrm{d}x\right] \\ &\quad \to -\alpha^{-1}, \end{aligned}$$

since Lemma 4.1 ensures that

$$I(t) = \int_0^{1-1/(t+1)} (1-x)^{\theta-1} \mathrm{e}^{\alpha x(t+1)}\,\mathrm{d}x = (t+1)^{-\theta} \int_0^t (t+1-u)^{\theta-1} \mathrm{e}^{\alpha u}\,\mathrm{d}u \sim -(\alpha t)^{-1}.$$

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Hence, $\lim_{t\to\infty} J_1(t;0) = \gamma Q/(-\alpha)$ such that

$$1-\Psi(t;0)\sim 1-\exp\biggl\{\biggl(\frac{r_0\gamma Q}{\alpha}\biggr)t^{\theta}\biggr\}\sim -\biggl(\frac{r_0\gamma Q}{\alpha}\biggr)t^{\theta}, \qquad t\to\infty,$$

which completes the proof of (i).

Similarly, (5.4) implies that, as $t\to\infty$,

$$1-g(F(xt;s))\sim \gamma[1-F(xt;s)]\sim \gamma Q(s)\mathrm{e}^{\alpha xt}.$$

Hence,

$$J_1(t;s)\sim \gamma Q(s)(t+1)\int_0^{1-1/(t+1)}(1-x)^{\theta}\mathrm{e}^{\alpha x(t+1)}\,\mathrm{d}x, \qquad t\to\infty.$$

Therefore, $\lim_{t\to\infty} J_1(t;s)=\gamma Q(s)/(-\alpha)$, and $1-\Psi(t;s)\sim 1-\exp\{(r_0\gamma Q(s)/\alpha)t^{\theta}\}\sim -(r_0\gamma Q(s)/\alpha)t^{\theta}$. Hence, $\lim_{t\to\infty}\Psi^*(t;s)=\Psi^*(s)=1-Q(s)/Q$. □

Corollary 5.1. (Markov case.) *Assume that* $G(x)=1-\mathrm{e}^{-x/\mu}$ $(x\ge 0)$ *for some* $\mu>0$, $h(\cdot;s)\equiv h(s)$ *for every* $|s|\le 1$, *and*

$$0<-\log Q=\int_0^1\biggl\{\frac{\alpha x+f(1-x)}{xf(1-x)}\biggr\}\,\mathrm{d}x<\infty,$$

where $f(s)=[h(s)-s]/\mu$ *is the infinitesimal generating function. Then*

$$\Psi^*(s)=1-\exp\biggl\{\alpha\int_0^s\frac{\mathrm{d}x}{f(x)}\biggr\}\quad \textit{with } \Psi^*(1)=1.$$

Proof. We deduce from the assumptions and from [32, Chapter II.2, Theorem 1 and Chapter II.4, Theorem 1], that

$$1-F(t,0)\sim Q\mathrm{e}^{\alpha t}, \quad Q>0, \qquad \frac{1-F(t,s)}{1-F(t,0)}\to \exp\biggl\{\alpha\int_0^s\biggl(\frac{1}{f(x)}\biggr)\mathrm{d}x\biggr\}.$$

Therefore, we deduce from the proof of Theorem 5.3 that $Q(s)=Q\exp\{\alpha\int_0^s \mathrm{d}x/f(x)\}$. Hence,

$$\Psi^*(s)=1-\frac{Q(s)}{Q}=1-\exp\biggl\{\alpha\int_0^s\biggl(\frac{1}{f(x)}\biggr)\mathrm{d}x\biggr\}.$$ □

Theorem 5.4. *Assume that (3.1) and (3.2), and Condition 3.1 hold. Assume further that* $r(t)=r_0\mathrm{e}^{\rho t}$, $r_0>0$, $\rho>0$, $\gamma_2<\infty$. *Then, as* $t\to\infty$,

$$\zeta(t)=\frac{Y(t)}{M(t)}\to 1 \quad \textit{a.s. and in } L_2.$$

Proof. To establish the convergence in L_2, it is sufficient to show that, as $t\to\infty$,

$$\Delta(t,\tau)=\mathbb{E}[\zeta(t+\tau)-\zeta(t)]^2\to 0,$$

uniformly for $\tau\ge 0$. Note that $\mathbb{E}[\zeta(t)]\equiv 1$, and

$$\Delta(t,\tau)=\operatorname{var}(\zeta(t+\tau))+\operatorname{var}(\zeta(t))-2\operatorname{cov}\{\zeta(t),\zeta(t+\tau)\},$$

$$\operatorname{var}(\zeta(t))=W(t)M(t)^{-2}, \qquad \operatorname{cov}\{\zeta(t),\zeta(t+\tau)\}=\frac{C(t,\tau)}{M(t)M(t+\tau)}.$$

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Since $\rho > \alpha$, we deduce from Proposition 4.2(i), Corollary 4.2(i), and Proposition 4.3(i) that, as $t \to \infty$,

$$W(t)M(t)^{-2} = W\mathrm{e}^{\rho t}(\gamma r_0 \hat{A}(\rho))^{-2}\mathrm{e}^{-2\rho t} \to 0,$$
$$W(t+\tau)M(t+\tau)^{-2} = W\mathrm{e}^{\rho(t+\tau)}(\gamma r_0 \hat{A}(\rho))^{-2}\mathrm{e}^{-2\rho(t+\tau)} \to 0,$$
$$C(t,\tau)(M(t)M(t+\tau))^{-1} = C(\tau)r_0\mathrm{e}^{\alpha\tau+\rho t}(\gamma r_0 \hat{A}(\rho))^{-2}\mathrm{e}^{-\rho(2t+\tau)} \to 0,$$

where the convergence in the last two relationships are uniform in $\tau \geq 0$. Therefore, we have $\lim_{t\to\infty} \Delta(t,\tau) = 0$ uniformly in $\tau \geq 0$, which proves the convergence in L_2, and

$$\Delta(t) = \lim_{\tau\to\infty} \Delta(t,\tau) = \mathbb{E}[\zeta(t) - 1]^2 = \frac{W(t)}{M^2(t)} \sim K_1 \mathrm{e}^{-\rho t},$$

where $K_1 = W/(\gamma r_0 \hat{A}(\rho))^2$. Therefore, $\int_0^\infty \Delta(t)\,\mathrm{d}t < \infty$ and by Theorem 21.1 of [6], we deduce that $\zeta(t)$ converges a.s. to 1. □

Theorem 5.5. *Assume that (3.1), (3.2), Condition 3.1, $r(t) = r_0\mathrm{e}^{\rho t}$, $r_0 > 0$, $\rho > 0$, and $\gamma_2 < \infty$ hold. Then*

$$X(t) = \frac{Y(t) - M(t)}{\sqrt{W(t)}} \xrightarrow{\text{D}} N(0,\sigma^2), \qquad t \to \infty,$$

where

$$\sigma^2 = \frac{\int_0^\infty \mathrm{e}^{-\rho u}(\gamma B(u) + \gamma_2 A^2(u))\,\mathrm{d}u}{\int_0^\infty \mathrm{e}^{-\rho u}(\gamma B(u) + \gamma A(u) + \gamma_2 A^2(u))\,\mathrm{d}u} \in (0,1) \tag{5.6}$$

and '$\xrightarrow{\text{D}}$' denotes convergence in distribution.

Proof. Let $\varphi_t(z) = \mathbb{E}[\mathrm{e}^{\mathrm{i}zX(t)}]$ denote the characteristic function of $X(t)$. Then

$$\varphi_t(z) = \exp\biggl\{-\frac{\mathrm{i}zM(t)}{\sqrt{W(t)}}\biggr\}\mathbb{E}\biggl[\exp\biggl\{\frac{\mathrm{i}zY(t)}{\sqrt{W(t)}}\biggr\}\biggr] = \exp\biggl\{-\frac{\mathrm{i}zM(t)}{\sqrt{W(t)}}\biggr\}\Psi\biggl(t;\exp\biggl\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\biggr\}\biggr).$$

We deduce from (3.7) that

$$\log\varphi_t(z) = -\frac{\mathrm{i}zM(t)}{\sqrt{W(t)}} - \int_0^t r(t-u)\biggl[1 - g\biggl(F\biggl(u;\exp\biggl\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\biggr\}\biggr)\biggr)\biggr]\,\mathrm{d}u.$$

The following asymptotic expansions hold as $s \to 1$ (see [32]):

$$1 - g(s) \sim \gamma(1-s) - \tfrac{1}{2}\gamma_2(1-s)^2, \qquad 1 - F(u;s) \sim A(u)(1-s) - \tfrac{1}{2}B(u)(1-s)^2.$$

Moreover, $1 - \mathrm{e}^x = -x(1 + o(1))$ as $x \to 0$. Hence, as $t \to \infty$,

$$\log\varphi_t(z) \sim -\frac{\mathrm{i}zM(t)}{\sqrt{W(t)}} - \int_0^t r(t-u)\biggl\{\gamma\biggl[1 - F\biggl(u;\exp\biggl\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\biggr\}\biggr)\biggr] - \frac{\gamma_2}{2}\biggl[1 - F\biggl(u;\exp\biggl\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\biggr\}\biggr)\biggr]^2\biggr\}\,\mathrm{d}u, \tag{5.7}$$

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and

$$1 - F\left(u; \exp\left\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\right\}\right) \sim A(u)\left(1 - \exp\left\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\right\}\right) - \frac{B(u)}{2}\left(1 - \exp\left\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\right\}\right)^2$$
$$\sim -\frac{\mathrm{i}zA(u)}{\sqrt{W(t)}} + \frac{z^2B(u)}{2W(t)}.$$

Therefore,

$$\int_0^t r(t-u)\left\{\gamma\left[1 - F\left(u; \exp\left\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\right\}\right)\right] - \frac{\gamma_2}{2}\left[1 - F\left(u; \exp\left\{\frac{\mathrm{i}z}{\sqrt{W(t)}}\right\}\right)\right]^2\right\}\mathrm{d}u$$
$$\sim -\frac{\mathrm{i}z\gamma}{\sqrt{W(t)}}\int_0^t r(t-u)A(u)\,\mathrm{d}u + \frac{z^2\gamma}{2W(t)}\int_0^t r(t-u)B(u)\,\mathrm{d}u$$
$$+ \frac{\gamma_2 z^2}{2W(t)}\int_0^t r(t-u)A^2(u)\,\mathrm{d}u$$
$$\sim -\frac{\mathrm{i}zM(t)}{\sqrt{W(t)}} + \frac{z^2}{2}\left[1 - \frac{M(t)}{W(t)}\right].$$

Returning to (5.7), and letting $t \to \infty$, we find that

$$\log \varphi_t(z) \sim -\frac{z^2}{2}\left[1 - \frac{M(t)}{W(t)}\right].$$

We deduce from Proposition 4.2(i) and Corollary 4.2(i) that

$$\lim_{t\to\infty} \frac{M(t)}{W(t)} = \frac{\int_0^\infty \mathrm{e}^{-\rho u}\gamma A(u)\,\mathrm{d}u}{\int_0^\infty \mathrm{e}^{-\rho u}(\gamma B(u) + \gamma A(u) + \gamma_2 A^2(u))\,\mathrm{d}u},$$

from which the expression for σ^2 given in (5.6) follows. Finally, $\lim_{t\to\infty} \varphi_t(z) = \mathrm{e}^{-z^2\sigma^2/2}$, which is the characteristic function of a normal distribution with mean 0 and variance σ^2, and the assertion follows from the continuity theorem [5]. □

Corollary 5.2. *Theorem 5.5, Proposition 4.2(i), and Corollary 4.2(i) entail the asymptotic normality*

$$Y(t)\mathrm{e}^{-\rho t} \sim N(r_0\gamma \hat{A}(\rho), \sigma^2 W \mathrm{e}^{-\rho t}), \qquad t \to \infty.$$

Theorem 5.6. *Assume that (3.1), (3.2), $\gamma_2 < \infty$, Condition 3.1, and $r(t) = r_0 t^\theta$ with $\theta > 0$ and $r_0 > 0$ hold. Then $\zeta(t) = Y(t)/M(t) \to 1$ in L_2 as $t \to \infty$. The convergence is almost sure if $\theta > 1$.*

Proof. We first deduce from (4.5)–(4.7) that, as $t \to \infty$,

$$\operatorname{var}(\zeta(t)) = W(t)M^{-2}(t) \sim WM^{-2}t^{-\theta}, \tag{5.8}$$

and

$$\operatorname{cov}\{\zeta(t), \zeta(t+\tau)\} = C(t,\tau)(M(t)M(t+\tau))^{-1} \sim C(\tau)M^{-2}(t+\tau)^{-\theta}. \tag{5.9}$$

Then (5.8) and (5.9) entail that $\Delta(t,\tau) \to 0$ uniformly in $\tau \geq 0$, and the convergence in L_2 follows.

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Assume now that $\theta > 1$. Equations (5.8) and (5.9) entail that $\mathrm{var}(\zeta(t+\tau)) \sim WM^{-2}(t+\tau)^{-\theta} \to 0$ and $\mathrm{cov}\{\zeta(t), \zeta(t+\tau)\} \sim C(\tau)M^{-2}(t+\tau)^{-\theta} \to 0$, $\tau \to \infty$. Hence, $\Delta(t) \sim WM^{-2}t^{-\theta}$, $t \to \infty$, and $\int_0^\infty \Delta(t)\,\mathrm{d}t < \infty$. We deduce from [6, Theorem 21.1] that $\zeta(t)$ converges to 1 a.s. □

Theorem 5.7. *Assume that (3.1), (3.2), $\gamma_2 < \infty$, Condition 3.1, and $r(t) = r_0 t^\theta$ with $\theta > 0$ and $r_0 > 0$ hold. Then $X(t) = [Y(t) - M(t)]/\sqrt{W(t)} \xrightarrow{\mathrm{D}} N(0, \sigma^2)$ as $t \to \infty$, where*

$$\sigma^2 = \frac{\int_0^\infty [\gamma B(u) + \gamma_2 A^2(u)]\,\mathrm{d}u}{\int_0^\infty [\gamma B(u) + \gamma_2 A^2(u) + \gamma A(u)]\,\mathrm{d}u} \in (0, 1).$$

Proof. Following the line of argument used in the proof of Theorem 5.5 yields

$$\log \varphi_t(z) \sim -\frac{z^2}{2}\biggl[1 - \frac{M(t)}{W(t)}\biggr], \qquad t \to \infty. \tag{5.10}$$

We deduce from (4.5) and (4.6) in Proposition 4.4 that $\lim_{t\to\infty} M(t)/W(t) = M/W$. Finally, we obtain from (5.10) that $\lim_{t\to\infty} \varphi_t(z) = \mathrm{e}^{-z^2\sigma^2/2}$, which is the characteristic function of the normal distribution with mean 0 and variance σ^2. The assertion follows from the continuity theorem (see, e.g. [5]). □

Corollary 5.3. *Theorem 5.7 and Proposition 4.4 entail the asymptotic normality $Y(t)t^{-\theta} \sim N(M, \sigma^2 W t^{-\theta})$, $t \to \infty$.*

Theorem 5.8. *Assume that (3.1), $\gamma < \infty$, and that $\lim_{t\to\infty} r(t) = r_0 > 0$ hold. Then there exists a limiting distribution $Q_k = \lim_{t\to\infty} \mathbb{P}\{Y(t) = k\} > 0$ ($k = 0, 1, 2, \dots$), such that*

$$\Psi^*(s) = \sum_{k=0}^\infty Q_k s^k = \exp\biggl\{-r_0 \int_0^\infty [1 - g(F(u, s))]\,\mathrm{d}u\biggr\}, \qquad |s| \le 1.$$

Proof. Under the assumptions of the theorem, $|1 - g(s)| \le \gamma|1 - s|$ and $|1 - F(u; s)| \le A(u)|1 - s|$. Therefore,

$$\biggl|\int_0^t r(t-u)[1 - g(F(u, s))]\,\mathrm{d}u\biggr| \le \gamma|1 - s| \int_0^t r(t-u)A(u)\,\mathrm{d}u.$$

Corollary 4.1 implies that

$$\lim_{t\to\infty} \int_0^t r(t-u)A(u)\,\mathrm{d}u = r_0 \int_0^\infty A(u)\,\mathrm{d}u < \infty.$$

Hence,

$$\lim_{t\to\infty} \Psi(t; s) = \Psi^*(s) = \exp\biggl\{-r_0 \int_0^\infty [1 - g(F(u, s))]\,\mathrm{d}u\biggr\}$$

uniformly in $|s| \le 1$. □

Pakes [26] obtained a similar limiting distribution for the Bellman–Harris process.

Corollary 5.4. *If $G(x) = 1 - \mathrm{e}^{-x/\mu}$ ($x \ge 0$), for some $\mu > 0$, and $h(u; s) \equiv h(s)$ then*

$$\Psi^*(t; s) = \exp\biggl\{-r_0 \int_s^1 \frac{1 - g(x)}{f(x)}\,\mathrm{d}x\biggr\}.$$

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Proof. It follows from the assumptions that $\{Z(t)\}_{t\geq 0}$ is a Markov branching process, and it is characterized by the Kolmogorov differential equations

$$\frac{\partial}{\partial t}F(t;s)=f(F(t;s)),\qquad \frac{\partial}{\partial t}F(t;s)=f(s)\frac{\partial}{\partial s}F(t;s),\qquad F(0;s)=s,$$

where $f(s)=(h(s)-s)/\mu$ (see, e.g. [6]). Therefore,

$$\begin{aligned}\frac{\partial}{\partial s}\int_0^\infty[1-g(F(u,s))]\,\mathrm{d}u&=-\int_0^\infty g'(F(u;s))\frac{\partial F(u;s)}{\partial s}\,\mathrm{d}u\\&=-\frac{1}{f(s)}\int_0^\infty g'(F(u;s))\frac{\partial F(u;s)}{\partial u}\,\mathrm{d}u\\&=-\frac{1-g(s)}{f(s)},\end{aligned}$$

using the fact that $F(\infty;s)=1$ and $F(0;s)=s$. Hence,

$$\int_0^\infty[1-g(F(u,s))]\,\mathrm{d}u=\int_s^1\frac{1-g(x)}{f(x)}\,\mathrm{d}x,$$

which completes the proof. □

Sevastyanov [29] obtained the same probability density function when $r(\cdot)\equiv r_0>0$.

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