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HOW MANY MATRICES HAVE ROOTS?

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0. Introduction. In many basic linear algebra texts it is shown that various classes of square matrices (normal, positive, invertible) possess square roots. In this note we characterize those $n \times n$ matrices with complex entries which possess at least one square root without any restriction on the class of root or matrix involved. We then use this characterization to obtain asymptotic estimates for the relative profusion of such matrices.

In Section 1 we characterize those $n \times n$ matrices with entries in $\mathbf{C}$ (or any algebraically complete field) which have square roots over $\mathbf{C}$. This characterization is in terms of similarity classes. In Section 2 we give asymptotic estimates for the number of Jordan forms of nilpotent $n \times n$ matrices which are squares. Section 3 is given over to numerical results concerning the actual and asymptotic frequency of such forms.

Since the results involve both elementary algebraic notions and less elementary analytic number theoretic notions, we have made some attempt to keep our development self-contained. The essence of our algebraic criterion can be found on pages 234 to 239 [**3**], but is nowhere explicitly given, indeed no criterion is given which can be immediately verified given the Jordan normal form. Furthermore in [**5**, p. 96] several papers are referenced in which necessary and sufficient conditions on the matrix A are given for the solvability of $p(X) = A$ where $p(X)$ is a polynomial in X. The conditions most similar to ours seem to be those of [**4**] for the case $p(X) = X^n$. As we shall see when A is nonsingular there is always a solution and it is sufficient to consider the case when the only eigenvalue is zero. Given the multiplicities of the elementary divisors Kreis defines a set of numbers $\{d_2\}$ by means of congruences mod n and gives necessary and sufficient conditions in terms of the d_i's for A to have an n-th root. These conditions are not as simple as ours although of course they and the others referenced by MacDuffee must be equivalent. No doubt the reason our conditions were not stated before is that most of the earlier work is directed towards finding all solutions of a more general problem and so involved more complicated conditions.

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The reaction of many people that we showed these results to is invariably that they must be well-known and later that they are rather different than anything previously stated. We are indebted to these people and especially the referee for their helpful comments and references.

1. Square roots of matrices. Throughout our discussion all matrices are presumed to lie in $M_n(\mathbf{C})$, the $n \times n$ matrices with complex entries. Any other algebraically complete field would suffice. The first question we wish to answer is the following.

Given a matrix A in $M_n(\mathbf{C})$, when can we solve

(1) $\quad A = B^2$

for some B in $M_n(\mathbf{C})$?

Recall that two matrices A_1 and A_2 are *similar* (over $M_n(\mathbf{C})$) if there exists an invertible matrix S with $A_2 = S^{-1}A_1{}^{-1}S$. Now similarity is an equivalence relation and

(2) $\quad A = B^2 \Leftrightarrow S^{-1}AS = (S^{-1}BS)^2.$

We will write $A_1 \sim_S A_2$ when A_1 and A_2 are similar. Thus one may solve (1) for a matrix A if and only if one may solve (1) for some matrix similar to A. Since each matrix is similar to its Jordan (canonical) form we may rephrase our question by asking when a Jordan form is a square.

Let I_m, S_m denote respectively diagonal and super diagonal unit matrices of dimension m. Let a be any complex number and set

(3) $\quad J_m(a) = aI_m + S_m.$

Then $J_m(a)$ is a Jordan block and every matrix A in $M_n(\mathbf{C})$ is similar to a matrix in Jordan normal form which we write

(4) $$J_A = \bigoplus_k J_{n_k}(a_k),$$

where $\sum_k n_k = n$ and J_A is the direct sum of the $J_{n_k}(a_k)$ for suitable numbers a_k (the eigenvalues of A). This representation is of course unique up to rearrangement of the Jordan blocks. In this notation one has

(5) $$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 5 & 7 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 2 \end{bmatrix} \sim_S \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} = J_2(1) \oplus J_2(2).$$

This very standard material may be found in **[3]**, **[6]**, **[7]** or elsewhere.

We begin by analysing the Jordan forms of the squares of Jordan blocks. Denote the greatest integer less than k by $[k]$.

PROPOSITION 1.

(6) (a) $\quad J_m^2(a) \sim_s J_m(a^2) \quad (a \neq 0),$

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(7) (b) $J_m^2(0) \sim_s J_{\left[\frac{m+1}{2}\right]}(0) \oplus J_{\left[\frac{m}{2}\right]}(0).$

Proof. (a) Jordan's Theorem [7, page 163] says that $J_m^2(a)$ is the direct sum of blocks each of whose only eigenvalue is a^2. Now the null space of

$$[J_m^2(a) - a^2 I_m] = L(a)$$

is easily seen to be at least one-dimensional since $L(a)$ is an upper triangular matrix with zero entries on the diagonal and entries of $2a$ on the super diagonal. As each block in the Jordan form of a matrix contributes exactly one eigenvector (up to multiples), the Jordan form of $J_m^2(a)$ has only one block and must be $J_m(a^2)$.

(b) A similar argument shows that

$$[J_m^2(0) - 0I_m] = L(0)$$

has a two-dimensional null space, so that $J_m^2(0)$ is similar to the direct sum of two blocks $J_{m_1}(0)$ and $J_{m_2}(0)$ with $m_1 + m_2 = m$. In addition, however,

$$(J_{m_2}^2(0))^{\left[\frac{m+1}{2}\right]} = (S_m^2)^{\left[\frac{m+1}{2}\right]} = 0,$$

since $2\left[\frac{m+1}{2}\right] \geqq m$ and $S_m^m = 0$. Thus the minimal polynomial of $J_m^2(0)$ has degree no greater than $\left[\frac{m+1}{2}\right]$. This, in turn, implies that the largest block has dimension $\left[\frac{m+1}{2}\right]$ which uniquely determines the size of the other block as $\left[\frac{m}{2}\right]$.

THEOREM 1. *A matrix A is a square if and only if when the dimensions $\bar{n}_i$, of its zero eigenvalue Jordan blocks, are listed in decreasing order they satisfy*

(8) $\bar{n}_{2i-1} - \bar{n}_{2i} \leqq 1,\ i = 1, 2, \dots, k,$

where $\sum_{i=1}^{2k} n_i$ is the total dimension of the zero blocks and we assume (by adding a zero dimensional block if need be) that there are exactly $2k$ blocks.

Proof. Suppose B is an arbitrary matrix in $M_n(\mathbf{C})$ with

(9) $B \sim_s \oplus J_{n_i}(b_i) \quad (\sum n_i = n).$

Then, on applying Proposition 1,

$$B^2 \sim_s \oplus J^2_{n_i}(b_i)$$

$$(10) \quad \sim_s \bigoplus_{(b_i \neq 0)} J_{n_i}(b_i^2) \bigoplus_{(b_i \neq 0)} (J_{\left[\frac{n_i+1}{2}\right]}(0) \oplus J_{\left[\frac{n_i}{2}\right]}(0) \oplus J_{\left[\frac{n}{2}\right]}(0)).$$

Since (10) is a Jordan form, any matrix A for which (1) has solution must have Jordan form as in (10). Thus we have:

Suppose A is B^2. Then by (10), A has the desired property except that the $\bar{n}_i$ need not be decreasing. We can, however, always rearrange things so that the $\bar{n}_{2i-1}$ are decreasing and then that the $\bar{n}_{2i}$ decrease. The only way that things go wrong is if we now have numbers such as

$$d, d - 1, d, d - 1$$

in a sequence. These may be replaced by $d, d, d - 1, d - 1$ and we will still have a square. Conversely, if (8) holds, each pair of zero blocks has

$$J_{\bar{n}_{2i-1}}(0) \oplus J_{\bar{n}_{2i}}(0) \sim_s J^2_{(\bar{n}_{2i}+\bar{n}_{2i-1})}(0),$$

and for a_i non-zero

$$J_{m(a_i)} \sim_s J^2_m(\sqrt{a_i})$$

so that A is a square.

Notice that in general

$$A = B^2 = C^2 \not\Rightarrow B \sim_s C$$

as is shown by

$$J_2(0) \oplus J_2(0) \oplus J_1(0) \sim_s (J_4(0) \oplus J_2(0))^2$$
$$\sim_s (J_3(0) \oplus J_3(0))^2.$$

As an immediate corollary we observe that normal and invertible matrices have square roots. In the normal case each zero block is one-dimensional while in the invertible case there are none. We don't, of course, deduce that the root of a normal matrix may be assumed normal.

In the same fashion we may prove:

THEOREM 2. *A is the k^{th}-power of some matrix B if and only if the dimensions of the zero blocks in the Jordan form of A ordered decreasingly satisfy*

$$(11) \quad n_{rk+1} - n_{(r+1)k} \leqq 1, \; 0 \leqq r \leqq s - 1,$$

where $\sum_{t=1}^{sk} n_t$ is the total dimension of the zero blocks and we assume exactly sk such blocks.

Proof. This is much as before except that we now show

$$(12)\quad J_{nk+i}^{k}(0) \sim_s \left(\bigoplus_{m=1}^{i} J_{k+1}(0)\right) \oplus \left(\bigoplus_{m=1}^{n-i} J_k(0)\right).$$

Example.

$$A = J_3(0) \oplus J_2(0) \oplus J_2(0) \sim_s J_7^3(0)$$

so that A is a cube, but A is not a square.

Example. (1) In $M_2(\mathbf{C})$ the only non-square Jordan form is

$$J_2(0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Thus a 2×2 matrix A is a square unless it is of the form

$$(13)\quad A = \frac{1}{s_1s_4 - s_2s_3} \begin{bmatrix} s_1 & s_2 \\ s_3 & s_4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s_4 & -s_2 \\ -s_3 & s_1 \end{bmatrix}, \quad (s_1s_4 \neq s_2s_3)$$

$$= \frac{1}{s_1s_4 - s_2s_3} \begin{bmatrix} -s_1s_3 & s_1^2 \\ -s_3^2 & s_1s_3 \end{bmatrix}.$$

Equivalently

$$(14)\quad A = \begin{bmatrix} -xy & y^2 \\ -x^2 & xy \end{bmatrix}, \quad (x, y) \neq (0, 0).$$

(ii) In $M_3(\mathbf{C})$ there are only two non-square forms

$$\begin{bmatrix} \alpha & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = J_1(\alpha) \oplus J_2(0), \quad (\alpha \neq 0)$$

and

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = J_3(0).$$

One could explicitly write down the matrices which possess these forms but it does not yield anything particularly pleasant such as (14).

The problem of telling whether an arbitrary matrix is a square or k^{th}-power is thus reduced to combinatoric calculations concerning the size of the zero blocks in its Jordan form. These will in general be complicated and not easily computable. We can, however, use Theorem 1 to estimate the relative "density" of such square matrices. This is the subject of the next section in which we count the relative profusion of "squarable" Jordan forms of nilpotent matrices.

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2. How many squares are there? There are of course many ways of measuring the relative frequency of square matrices of dimension n inside $M_n(\mathbf{C})$. Some such estimates, especially probabilistic ones, would show squares to be in the preponderance, since a necessary condition for a matrix to be non-square is that its determinant vanish. As our characterization is in terms of Jordan forms, and since every invertible matrix is square, it seems reasonable to ask initially for an asymptotic estimate of the number, $N(n)$, of Jordan forms of n-dimensional nilpotent matrices (matrices which have only zero eigenvalues) which are squares, and then to compare that number to the total number of Jordan forms of such matrices, $M(n)$.

It is immediate from Theorem 1 that $N(n)$ is the number of partitions of n which satisfies:

$$\text{(a) } n = n_1 + n_2 + \ldots + n_{2m} \quad (n_{2m} = 0 \text{ if need be}),$$

$$(15) \quad \text{(b) } n_1 \geqq n_2 \geqq n_3 \geqq \ldots \geqq n_{2m} \quad n_{2m-1} \neq 0,$$

$$(c)\ n_{2i-1} - n_{2i} \leqq 1 \quad i = 1, 2, \ldots, m.$$

We will call such partitions *squarable.*

It is also immediate that $M(n)$ is just the total number of partitions of n.

Let $q(n)$ denote the number of partitions of n whose odd parts occur at most once. As an example, notice that $13 = 5 + 5 + 2 + 1$ will be counted in $N(13)$ but not $q(13)$, while $13 = 6 + 4 + 3$ will be counted in $q(13)$ but not $N(13)$. The reader is referred to **[1]**, **[2]** for details of the relevant partition theory.

THEOREM 3. *With $N(n)$, $M(n)$ and $q(n)$ defined as above we have:*

(a) $q(n - 1) \leq N(n) \leq q(n)$.

$$\text{(b) } N(n) = \frac{1}{4\sqrt{2}n} e^{(\sqrt{n}\pi/\sqrt{2})} (1 + O(n^{-1/4+\delta}))$$

where δ is any positive constant.

$$(c)\ M(n) = \frac{1}{4\sqrt{3}n} e^{(\sqrt{n}\pi/\sqrt{3/2})} (1 + O(n^{-1/2})).$$

Proof. Consider the *Ferrer's* graph of a partition

$$\pi = (n_1, n_2, \ldots, n_{2m}) \quad \text{where} \quad \sum_{i=1}^{2m} n_i = n.$$

This is the diagram with n_1 dots in the first row, n_2 dots in the second row and so on. Let $\tilde{\pi}$ denote the *conjugate partition* of π. This is the partition whose Ferrer graph is obtained from the Ferrer graph of π by interchanging the rows and the columns. The condition (15) (c) in a squarable partition π that $n_1 - n_2 \leq 1$, states that at most one part equal to 1 occurs in $\tilde{\pi}$. Similarly the condition $n_{2i-1} - n_{2i} \leq 1$ states that at most one part equal to $2i - 1$ occurs in π. When π has an odd number of parts we have, from (15) (b), the additional restriction that the largest part occurs exactly once. Omitting this last restriction yields

(16) $N(n) \leq q(n).$

Imposing this last restriction, we see that we have at least $q(n - 1)$ such $\tilde{\pi}$. Indeed, we may add one to the largest summand in a partition of $n - 1$, counted in $q(n - 1)$, and obtain a partition of n with unique largest part whose conjugate is squarable. Thus

(17) $q(n - 1) \leq N(n).$

We now show

(18) $q(n - 1) = q(n)(1 + O(n^{-1/4+\delta}))$

and that

(19) $$q(n) = \frac{1}{4\sqrt{2n}} e^{\sqrt{n}\pi/\sqrt{2}}(1 + O(n^{-1/4+\delta}))$$

by using a result of [**6**] (see also [**1**, Chapter 6]). The *generating function* for $q(n)$ (with $q(0) = 1$) is

$$\sum_{n=0}^{\infty} q(n)x^n = \prod_{i=1}^{\infty} (1 + x^{2i+1}) \prod_{i=1}^{\infty} (1 - x^{2i})^{-1}$$

as each odd part may occur only once. Thus

(20) $$\sum_{n=0}^{\infty} q(n)x^n = \prod_{i \not\equiv 2(\text{mod } 4)} (1 - x^i)^{-1} = \prod_{n=1}^{\infty} (1 - x^n)^{-a_n}$$

where

$$a_n = \begin{cases} 0 \text{ if } n \equiv 2(\text{mod } 4) \\ 1 \text{ if } n \not\equiv 2(\text{mod } 4). \end{cases}$$

Before applying Meinardus' theorem we state it in its full generality.

THEOREM 4 (Meinardus). *Let*

(21) $$\prod_{i=1}^{\infty} (1 - e^{-\tau n})^{-a_n} = 1 + \sum_{n=1}^{\infty} q(n)e^{-n\tau} \quad (\text{Re}\,\tau > 0).$$

Consider the Dirichlet series

(22) $D(s) = \sum_{n=1}^{\infty} a_n n^{-s} \quad (s = \sigma + it)$

and assume that (i) *for* $\sigma > \alpha$, *a positive real number*, $D(s)$ *converges. Assume also that* (ii) $D(s)$ *may be analytically continued to the region* $\sigma \geq -C_0 \quad (0 < C_0 < 1)$ *and that in this region* $D(s)$ *is analytic except at* $s = \alpha$ *where it has a simple pole with residue A. Assume that* (iii) $D(s) = O(|t|^{C_1})$ *uniformly in* $\sigma \geq -C_0$ *as* $|t|$ *tends to infinity, where* C_1 *is a fixed positive number. Consider the function*

(23) $g(\tau) = \sum_{n=1}^{\infty} a_n e^{-n\tau},$

and assume that (iv) *if* $\tau = y + 2\pi ix$ *(x, y real) one has for* $|\arg \tau| > \pi/4$, $|x| \leqq 1/2$

(24) $\operatorname{Re}[g(\tau) - g(y)] \leqq -C_2 y^{-t_0},$

for y sufficiently small where t_0 *is an arbitrary positive constant and* C_2 *is a suitable positive constant depending on* t_0. *Then as n tends to infinity*

(25) $$q(n) = Cn^K \exp\left\{ n^{\frac{\alpha}{1+a}} \left(1 + \frac{1}{\alpha}\right) [A\Gamma(\alpha + 1)\zeta(\alpha + 1)]^{\frac{1}{\alpha+1}} \right\} \times (1 + O(n^{-K_1}))$$

where ζ *is the Riemann zeta function,* Γ *is the gamma function, and*

(26) $$C = e^{D'(0)}[2\pi(1 + \alpha)]^{-1/2}[A\Gamma(\alpha + 1)\zeta(\alpha + 1)]^{\frac{1-2D(0)}{(2+2\alpha)}},$$

(27) $$K = \frac{D(0) - 1 - \alpha/2}{(1+\alpha)},$$

(28) $$K_1 = \frac{\alpha}{\alpha+1} \min\left(\frac{C_0}{\alpha} - \frac{\delta}{4}, \frac{1}{2} - \delta\right),$$

and δ *is an arbitrary real number.*

In our case

(29) $$D(s) = \sum_{n=1}^{\infty} n^{-s} - \sum_{n=0}^{\infty} (4n+2)^{-s} = (1 - 2^{-s} + 4^{-s})\zeta(s).$$

Thus, by well known properties of ζ, $D(s)$ is analytically continuable to the entire complex plane and is analytic everywhere except at $s = 1$ where it has a simple pole with residue 3/4. Furthermore, since $\zeta(s)$ is a function of finite order in the sense of Dirichlet series in any half plane $\sigma \geq C_0$,

$$D(\sigma + it) = O(|t|^k), \quad k = k(C_0).$$

Finally, in our case

$$(30)\quad g(\tau) = \sum_{n=1}^{\infty} a_n\tau^n = \sum_{n=1}^{\infty} e^{-n\tau} - 2\sum_{n=1}^{\infty} ne^{-2n\tau} + 4\sum_{n=1}^{\infty} ne^{-4n\tau}$$

$$= \frac{1}{e^{\tau}-1} + \frac{2e^{2\tau}}{(e^{2\tau}-1)^2} - \frac{4e^{4\tau}}{(e^{4\tau}-1)^2}.$$

Hence if $\tau = y + 2\pi ix$

$$(31)\quad \mathrm{Re}(g(\tau) - g(y)) \leq C_2 y^{-1}$$

and the hypotheses of Meinardus Theorem are satisfied. Moreover

$$K = \frac{D(0)-1-1/2}{1+1} = \frac{\zeta(0)-3/2}{2} = -1,$$

$$(32)\quad K_1 = \frac{1}{2}\min\left(Co - \frac{\delta}{4}, \frac{1}{2} - \delta\right) = \frac{1}{4} - \delta,$$

$$A\Gamma(2)\zeta(2) = -2/8,$$

and

$$D'(s) = \zeta'(s)\left(1 - \frac{1}{2^s} + \frac{1}{4^s}\right) + \left(\frac{\log 2}{2^s} - \frac{\log 4}{4^s}\right)\zeta(s),$$

so

$$(33)\quad D'(0) = \zeta'(0) + \frac{\log 2}{2} = -\frac{1}{2}(\log 2\pi - \log 2).$$

It follows that for our particular $q(n)$,

$$q(n) = \frac{1}{4n\sqrt{2}}\, e^{(\sqrt{n}\pi/\sqrt{2})}\,(1 + O(n^{-1/4+\delta}))$$

and that

$$q(n-1) = q(n)(1 + O(n^{-1/4+\delta})).$$

Our proof of Theorem 3 (a) (b) is complete. Part (c) is more standard but could be deduced in the same fashion. Our estimate is due to Siegel [**2**].

In particular, it follows that the ratio, $R(n)$, of square nilpotent forms to all nilpotent forms is

$$(34)\quad R(n) = \frac{N(n)}{M(n)} \sim \sqrt{3/2}\,\exp\left[\pi\sqrt{n}\left(\frac{\sqrt{18}-\sqrt{24}}{6}\right)\right].$$

(Recall that $f(n) \sim g(n)$ if $\lim_{n\to\infty} f(n)/g(n) = 1$.)

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Thus, while for small n many nilpotent Jordan forms are squares, asymptotically very few forms are.

Similar calculations could be undertaken for the density of the Jordan forms of nilpotents which are k^{th}-powers.

Another calculation which in some senses is more representative of the density of squares is to look at

$$(35) \quad T(n) = \sum_{k=0}^{n} N(k)M(n-k).$$

$T(n)$ counts the number of Jordan forms of dimension n which are squares if one identifies all non-zero eigenvalues. In this case it would make sense to compare $T(n)$ to

$$(36) \quad S(n) = \sum_{k=0}^{n} M(k)M(n-k),$$

since $S(n)$ will count all distinct Jordan forms of matrices with eigenvalues zero and one.

Since $S(n)$ is the coefficient of x^n in

$$\prod_{n=1}^{\infty} (1 - x^n)^{-a'_n} \quad \text{where } a'_n = 2$$

and

$$U(n) = \sum_{k=0}^{n} q(n)M(n-k)$$

is the coefficient of x^n in

$$\prod_{n=1}^{\infty} (1 - x^n)^{-a''_n}$$

where

$$a''_n = \begin{cases} 1 \text{ if } n \equiv 2 \pmod 4 \\ 2 \text{ if } n \not\equiv 2 \pmod 4 \end{cases}$$

we may use Meinardus' theorem to determine the asymptotic behaviour of $S(n)$ and $U(n)$. Arguments completely analogous to those in the proof of Theorem 3 apply and we deduce

$$(37) \quad U(n) = \frac{(7/6)^{3/4}}{8n^{5/4}} \exp(\pi\sqrt{n}/\sqrt{6/7})(1 + O(n^{-1/4+\delta})),$$

(38) $$S(n) = \frac{1}{4\ 3^{3/4}\ n^{5/4}} \exp(2\pi\sqrt{n}/\sqrt{3})(1 + O(n^{-1/4+\delta})).$$

Since $N(k) \sim q(n)$ it is easily shown that $T(n) \sim U(n)$ and hence we have

THEOREM 5. *As* $n \to \infty$

a) $S(n) \sim \dfrac{1}{3^{3/4}4n^{5/4}} \exp(2\pi\sqrt{n}/\sqrt{3})$

b) $T(n) \sim \dfrac{(7/6)^{3/4}}{8n^{5/4}} \exp(\pi\sqrt{n}/\sqrt{6/7})$.

Thus $T(n)/S(n)$ becomes very small however it is much larger than the ratio of $N(n)/M(n)$.

Finally the error terms in equations (6) of Theorem 3 and equations a) and b) of Theorem 5 may be shown to be $O(n^{-1/2})$ of the principal terms. This requires an analysis which is more complicated than the one we have given using Meinardus' theorem and so in the interests of simplicity we have chosen the above course.

3. Numerical calculations. The following tables show the actual and asymptotic estimates of $N(n)$, $M(n)$ and $R(n)$ with errors.

Specifically, N, M and R are, as above, the actual partition numbers while

Asymptotic Estimates

$$N_A(n) = \frac{1}{4\sqrt{2}n} \exp(\sqrt{n}\ \pi/\sqrt{2}),$$

(39) $$M_A(n) = \frac{1}{4\sqrt{3}n} \exp(\sqrt{n}\ \pi/\sqrt{3/2}),$$

$$R_A(n) = N_A(n)/M_A(n),$$

Error Estimates

$$EN(n) = (N(n) - N_A(n))/(N(n)),$$

(40) $$EM(n) = (M(n) - M_A(n))/M(n),$$

$$ER(n) = (R(n) - R_A(n))/R(n).$$

We have included the actual numbers for n less than twenty-one; all three sets of numbers, incremented by twenties, for n less than one-hundred and fifty; and the asymptotic numbers, incremented by hundreds, between one and two thousand.

The actual values of $M(n)$ were calculated directly from the generating function while $N(n)$ is calculable from a relatively straightforward two variable, two term recurrence relationship.

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TABLE I Actual Values of N, M and R

n	$N(n)$	$M(n)$	$R(n)$
1	1	1	1.000000
2	1	2	.500000
3	2	3	.666667
4	3	5	.600000
5	4	7	.571429
6	5	11	.454545
7	7	15	.466667
8	10	22	.454545
9	13	30	.433333
10	16	42	.380952
11	21	56	.375000
12	28	77	.363636
13	35	101	.346535
14	43	135	.318519
15	55	176	.312500
16	70	231	.303030
17	86	297	.289562
18	105	385	.272727
19	130	490	.265306
20	161	627	.256778

TABLE 2 Asymptotic Values of N_A, M_A and R_A

n	$N_A(n)$	$M_A(n)$	$R_A(n)$
1000	.5699119345E + 27	.2440199631E + 32	.00002335514
1100	.1597695778E + 29	.1162714315E + 34	.00001374109
1200	.3877060741E + 30	.4683699364E + 35	.00000827777
1300	.8286858422E + 31	.1627752327E + 37	.00000509098
1400	.1582348882E + 33	.4961779550E + 38	.00000318908
1500	.2730983322E + 34	.1344797089E + 40	.00000203078
1600	.4302252581E + 35	.3277966561E + 41	.00000131248
1700	.6237930303E + 36	.7255759369E + 42	.00000085972
1800	.8383842692E + 37	.1470585184E + 44	.00000057010
1900	.1050932722E + 39	.2748755413E + 45	.00000038233
2000	.1235275003E + 40	.4767929011E + 46	.00000025908

TABLE III Comparitive Values and Errors

(For each n, the first line gives $EN(n)$, $EM(n)$, $ER(n)$, the second $N(n)$, $M(n)$, $R(n)$ and the third gives $N_A(n)$, $M_A(n)$, $R_A(n)$.)

n	$N(n)$	$M(n)$	$R(n)$
10	−.24206208	−.14534069	−.084448
10	16	42	.380952
10	19.873	48.104	.413123
30	−.11587859	−.08501701	−.028443
30	1016	5604	.181299
30	1133.733	6080.435	.186456
50	−.08708927	−.06543975	−.020320
50	21581	204226	.105672
50	23460.474	217590.499	.107819
70	−.07281424	−.05496666	−.016918
70	277691	4087968	.069078
70	297910.859	4312669.963	.069078
90	−.06388875	−.04826745	−.014902
90	2623017	56634173	.046315
90	2790598.289	59367760.238	.047005
110	−.05760878	−.04352276	−.013499
110	19961498	607163746	.032877
110	21111455.534	633589185.901	.033320
130	−.05287423	−.03993928	−.012438
130	129139468	5371315400	.024042
130	.13596E+09	.558584E+10	.024341
150	−.04913933	−.03711092	−.011598
150	735099980	40853235313	.017994
150	.77122E+09	.423693E+11	.018202

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