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Examples of exponentially many collisions in a hard ball system

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Abstract. Consider the system of n identical hard balls in $\mathbb{R}^3$ moving freely and colliding elastically. We show that there exist initial conditions such that the number of collisions is exponential in n.

Key words: hard ball gas model, number of collisions, examples with many collisions
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1. *Introduction*

Consider the system of n identical hard balls moving freely and colliding elastically. Since long ago the problem of counting the number of collisions that may occur between the balls has been extensively studied for both the system of balls confined to a box and in open space. The problem of estimating the number of collisions goes back to Boltzmann. Mathematically it had been proposed by Sinai; see [5]. It has been studied by many mathematicians.

Denote by $\mathrm{MaxCol}(n, d)$ the maximum number of collisions that may occur between n identical balls in $\mathbb{R}^d$, where simultaneous collisions are prohibited. This number is always finite. The fact that the number of collisions for any initial data is finite has been shown by Vaserstein [12] and Galperin [5]. The fact that $\mathrm{MaxCol}(n, d)$ is finite has been shown by Burago, Ferleger and Kononenko [2]; see also [1]. In fact, Theorem 1.3 in [2] provides a (rough) estimate $\mathrm{MaxCol}(n, d) \le (32n^{2/3})^{n^2}$ for all d.

Many authors studying hard ball systems used the following observation. Instead of studying the motion of balls, that is, their centers in $\mathbb{R}^d$, one can put all their coordinates together as a dn-tuple and study the motion of this point in $\mathbb{R}^{dn}$. Note that some points of

$\mathbb{R}^{dn}$ have to be removed. Namely, for each pair of balls there is a set of points which corresponds to configurations of balls where these two balls overlap. These sets are cylinders; in particular, they are convex. We denote by $\mathcal{B}_{d,n}$ the complement of the union of these cylinders; it is the configuration space of our system. It well known that the motion of the system of balls is represented by the billiard dynamics in $\mathcal{B}_{d,n}$. Namely, $\mathcal{B}_{d,n}$ is a billiard table whose walls are the boundaries of the cylinders and the usual billiard laws govern the motion exactly corresponding to the dynamics of the n balls in $\mathbb{R}^d$. The convexity of cylinders implies that this is a semi-hyperbolic billiard, though it is not hyperbolic [**10**] due to the presence of straight lines on the boundary. This makes its study more difficult. We forbid trajectories hitting singularities (intersections of two or more walls), since they correspond to simultaneous collisions in the ball system. The bounds obtained in [**2**] do not study the system of balls directly but rather by analyzing billiard trajectories in complements of unions of convex bodies. Earlier, Sinai [**9**] has shown that in a polyhedral cone there is a uniform upper bound (for all trajectories) for the number of collisions with walls.

Not much is known about the lower bounds on $\text{MaxCol}(n, d)$. It is easy to see that $\text{MaxCol}(n, 1) = n(n-1)/2$ and it is monotone in d. If one allows different masses of balls, even in $d = 1$ the situation becomes more complicated; see e.g. [**5**]. Beyond the trivial lower bound $\text{MaxCol}(n, d) \geq n(n-1)/2$, the first result we know of is by Thurston and Sandri [**11**] stating that $\text{MaxCol}(3, 2) \geq 4$ (which is not obvious). As a matter of fact, $\text{MaxCol}(3, d) = 4$ for all $d \geq 2$; see [**6**, **7**] and references therein. A cubic lower bound for $\text{MaxCol}(n, 2)$ was obtained in [**4**]. This seems to be all that is known so far.

The main result of this paper is the following theorem.

THEOREM 1.1. $\text{MaxCol}(n, 3) \geq 2^{\lfloor n/2 \rfloor}$ *for all* $n \geq 3$.

Note that the lower bound in Theorem 1.1 and the upper bound from [**2**] have a large gap between them but at least they are both poly-exponential. Making a better match after we went above polynomials seems not so interesting, there is little hope to make them match exactly, and the gap between $\exp(t)$ and $\exp(2t)$ is also huge. In fact, we prove a somewhat better lower bound which though is more cumbersome; see (3.10). To make the lower and upper bounds closer to each other, one now probably should rather concentrate on upper bounds; there obviously is some room for improvement.

In the proof of Theorem 1.1 we construct a trajectory with the desired number of collisions defined on a bounded time interval. The continuation of this trajectory may not be defined on the entire $\mathbb{R}$ due to a simultaneous collision. By a small perturbation of the initial data one can obtain a trajectory which is defined on the entire $\mathbb{R}$ and with at least the same number of collisions. Indeed, such initial data form a set of full measure in the phase space.

The collisions in our construction occur in a very small neighborhood of one singular point on the boundary of the configuration space (billiard table) $\mathcal{B}_{3,n} \subset \mathbb{R}^{3n}$. We find an appropriate singular point q on the boundary of $\mathcal{B}_{3,n}$ and consider the tangent cone to $\mathcal{B}_{3,n}$ at q. The point q is such that the billiard system in the cone has a trajectory with the number of collisions we need. By applying a homothety this trajectory can be moved arbitrarily close to the origin of the cone. Then it is easy to see that there is a nearby

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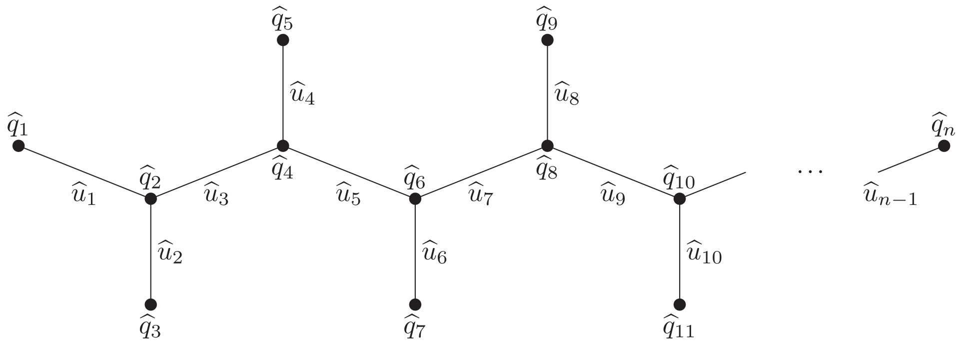

FIGURE 1. The configuration $\widehat{q}$ for $n = 4k$ and its graph of ball contacts. This is a projection of a three-dimensional configuration to the plane. The points $\widehat{q}_1, \widehat{q}_2, \widehat{q}_4, \widehat{q}_6, \widehat{q}_8, \widehat{q}_{10}, \ldots$ lie in the xy-plane. The points $\widehat{q}_5, \widehat{q}_9, \ldots$ lie above $\widehat{q}_4, \widehat{q}_8, \ldots$, respectively, and $\widehat{q}_3, \widehat{q}_7, \widehat{q}_{11}, \ldots$ are beneath $\widehat{q}_2, \widehat{q}_6, \widehat{q}_{10}, \ldots$. All the segments have unit lengths and meet at right angles.

trajectory in $\mathcal{B}_{3,n}$ with the same number of collisions; see Lemma 2.2. The point q must have very special properties.

One can see that the tangent cone to $\mathcal{B}_{d,n}$ at any point $q \in \partial\mathcal{B}_{d,n}$ is a polyhedral cone whose faces correspond to pairs of touching balls in the configuration represented by q. Furthermore, the angles between faces are bounded away from 0. In our examples the number of faces equals $m = n - 1$ and the angles between faces are very close to $\pi/2$. Note that, in a cone with m faces where all angles are equal to $\pi/2$, every billiard trajectory experiences no more than m collisions. Nonetheless, it turns out that an arbitrarily small change of angles can result in a cone admitting a billiard trajectory with exponentially many collisions; see Lemma 2.3. Using this fact, we first prove a model theorem (Theorem 2.4), which shows that $\mathrm{MaxCol}(n, n-1) \geq 2^{n-1} - 1$. Its proof already contains most of the principal ideas of the main construction.

A number of open questions are left.

(1) So far we were unable to prove an analog of Theorem 1.1 in dimension two. The reason is the lack of flexibility in constructing configurations with prescribed angles, like the one depicted in Figure 1.

(2) We do not know any interesting lower and upper bounds on the measure of the configurations in the phase space resulting in a large number of collisions. (For the sake of normalization, the energy and a cube to which the positions of balls are confined to must be fixed). The word ‘large’ is vague and could mean e.g. some polynomial or exponential bounds. An upper bound on the measure would be particularly interesting.As a matter of fact, analogous problems are more interesting not in the whole $\mathbb{R}^d$ but rather in a box where the density of balls is small enough. Of course, then the number of collisions is counted in unit time or by averaging $N(T)/T$. This allows one to think about dynamical characteristics like entropies (see [2, 3]).

(3) It seems that, if the number of collisions is ‘large’, then the overwhelming number of collisions are almost tangential. This problem had been posed in a preprint of this paper and was essentially answered in [8] using a completely different

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set of tools. A physicist would call such collisions 'inessential' in the sense that they result in almost zero exchange of momenta, energy, and directions of velocities of the balls. However, for a dynamical system person they may look very essential, for the analogs of Lyapunov exponents are huge. Thus, theoretically, such collisions could make a non-trivial contribution to metric entropy (which is rather unlikely) or to topological entropy (which is quite possible). Note that, under reasonable assumptions, the topological entropy is finite [3], though the proof uses a compactness argument in addition to Alexandrov geometry of $k \le 0$, and probably no reasonable formula for the upper bound is known or at least can be found in the literature. It seems that, to answer such questions, one needs to look at Question 2 above along with the above-mentioned estimate on the number of almost tangential collisions in [8].

Notation. Throughout the paper we denote by $\mathbb{N}$ the set of positive integers, by $\mathbb{R}_+$ the set of non-negative reals, and by $\mathbb{R}^m_+$ the set $(\mathbb{R}_+)^m \subset \mathbb{R}^m$. The symbol $\langle\, , \rangle$ denotes the Euclidean scalar product in $\mathbb{R}^m$. For a piecewise linear function f defined on an interval, we denote by $f'(t_+)$ and $f'(t_-)$ the right and left derivatives of f at t.

2. *Tangent cones*

Consider a hard ball gas system of n identical balls in $\mathbb{R}^d$. Without loss of generality we set the radii of the balls to be $\frac{1}{2}$. We denote the centers of the balls by $q_1, \ldots, q_n$. Recall that we regard a collection $(q_1, \ldots, q_n) \in (\mathbb{R}^d)^n$ as a point $q \in \mathbb{R}^{dn}$. Conversely, for a point $q \in \mathbb{R}^{dn}$ we denote by $q_1, \ldots, q_n$ its d-dimensional components. Denote by $\mathcal{B}_{d,n}$ the configuration space of the system, that is, $\mathcal{B}_{d,n} \subset \mathbb{R}^{dn}$ is defined by

$$\mathcal{B}_{d,n} = \{q \in \mathbb{R}^{dn} : |q_i - q_j| \ge 1 \text{ for all } i \ne j\}.$$

This set corresponds to configurations of balls with disjoint interiors. It is the complement of the union of round cylinders

$$C_{ij} = \{q \in \mathbb{R}^{dn} : |q_i - q_j| < 1\}, \quad 1 \le i < j \le n.$$

We refer to the boundaries ∂C_{ij} of these cylinders as walls. Recall that the evolution of a system of balls corresponds to the billiard dynamics in $\mathcal{B}_{d,n}$. We consider billiard trajectories defined on various intervals with no collisions at end points. Let a trajectory γ hit a wall at a moment t and let ν be the unit normal to the wall at $\gamma(t)$. Then the rule 'the angle of reflection equals the angle of incidence' takes the form

$$\gamma'(t_+) = \gamma'(t_-) - 2\langle \gamma'(t_-), \nu\rangle \nu. \tag{2.1}$$

Definition 2.1. Let $q \in \partial\mathcal{B}_{d,n}$. We denote by $\operatorname{Cone}(q)$ the tangent cone of $\mathcal{B}_{d,n}$ at q defined as follows. The point q belongs to several cylinders. They have unit outer normal vectors at q referred to as *normals* and denoted by $\nu_1, \ldots, \nu_m$. The tangent cone $\operatorname{Cone}(q)$ is the set of vectors $v \in \mathbb{R}^{dn}$ such that $\langle v, \nu_i\rangle \ge 0$ for all $i \in \{1, \ldots, m\}$.

According to this definition, $\operatorname{Cone}(q)$ is a convex polyhedral cone (with cone's origin at 0) whose faces are contained in hyperplanes orthogonal to $\nu_1, \ldots, \nu_m$. If $q \in \partial C_{ij}$, $i < j$, and $\nu \in \mathbb{R}^{dn}$ is the normal to C_{ij} at q, then

$$\nu = \frac{1}{\sqrt{2}}(0, \ldots, 0, q_i - q_j, 0, \ldots, 0, q_j - q_i, 0, \ldots, 0), \tag{2.2}$$

where the non-zero entries $q_i - q_j$ and $q_j - q_i$ are at the ith and jth positions, respectively. To avoid case chasing below, we use the notation C_{ij} for $i > j$ as well, that is, $C_{ij} = C_{ji}$. In the case when $i > j$ the formula for ν is similar to (2.2). In both cases the ith d-dimensional component of ν equals $q_i - q_j$, the jth one equals $q_j - q_i$, and all other components are zero.

The scalar products of the normals can be computed as follows. If $q \in \partial C_{ij} \cap \partial C_{lk}$ and ν_1 and ν_2 are the normals to C_{ij} and C_{lk} at q, then

$$\langle \nu_1, \nu_2 \rangle = 0 \quad \text{if } \{i, j\} \cap \{l, k\} = \emptyset. \tag{2.3}$$

If $i = l$, then

$$\langle \nu_1, \nu_2 \rangle = \tfrac{1}{2}\langle q_j - q_i, q_k - q_i \rangle. \tag{2.4}$$

The first case corresponds to configurations where two disjoint pairs of balls touch simultaneously and in the second case the ith ball touches the jth and kth ones. Recall that such configurations never occur in the dynamics we study. The cases when $i = k$, $j = l$, or $j = k$ reduce to (2.4) by swapping indices in C_{ij} and C_{kl}.

The tangent cone has a non-empty interior. Indeed, if $q \in \partial C_{ij}$ and ν is the corresponding normal, then, by (2.2),

$$\langle q, \nu \rangle = \frac{1}{\sqrt{2}}|q_i - q_j|^2 = \frac{1}{\sqrt{2}} > 0.$$

Hence, in the notation of Definition 2.1, the vector q has positive scalar products with the normals $\nu_1, \ldots, \nu_m$ and thus belongs to the interior of $\mathrm{Cone}(q)$.

LEMMA 2.2. *Let $q \in \mathcal{B}_{d,n}$ and $N \in \mathbb{N}$ be such that there is a billiard trajectory in* $\mathrm{Cone}(q)$ *with N collisions. Then* $\mathrm{MaxCol}(n, d) \geq N$.

Proof. Let $W_1, \ldots, W_m$ be the walls of $\mathcal{B}_{d,n}$ (that is, boundaries of the cylinders) that contain q and $\nu_1, \ldots, \nu_m$ their normals at q. Let

$$\overline{W}_i = \{x \in \mathbb{R}^{dn} : \langle x, \nu_i \rangle = 0\}, \quad i = 1, \ldots, m,$$

be the respective walls of the cone $K := \mathrm{Cone}(q)$. Note that the walls W_i do not intersect the interior of K due to the convexity of the cylinders.

Let $\gamma : (a, b) \to K$ be a billiard trajectory in the cone with N collisions at moments $a < t_1 < \cdots < t_N < b$ with walls $\overline{W}_{i_1}, \ldots, \overline{W}_{i_N}$, respectively. For every $\lambda > 0$, consider a rescaled set $\mathcal{B}(\lambda) := \lambda(\mathcal{B}_{d,n} - q)$. It is bounded by the walls $W_i(\lambda) := \lambda(W_i - q)$. We send λ to infinity, fix $t_0 \in (a, t_1)$, and consider a billiard trajectory γ_λ in $\mathcal{B}(\lambda)$ with the initial conditions $\gamma_\lambda(t_0) = \gamma(t_0)$ and $\gamma'_\lambda(t_0) = \gamma'(t_0)$.

The walls $W_i(\lambda)$ converge to $\overline{W}_i$ as $\lambda \to \infty$ in C^1 topology on compact sets. To avoid lengthy discussion of general submanifold convergence, we use the following *ad hoc* definition in our special case. For every $\lambda > 0$, the rescaled wall $W_i(\lambda)$ is a

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codimension-one smooth submanifold of $\mathbb{R}^{dn}$, it contains 0, and its tangent hyperplane at 0 is $\overline{W}_i$. Hence, a part of $W_i(\lambda)$ near 0 is a graph of a smooth function $f_{i,\lambda}\colon U_{i,\lambda} \to (\overline{W}_i)^{\perp} \simeq \mathbb{R}$, where $U_{i,\lambda}$ is a neighborhood of 0 in $\overline{W}_i$, $f_{i,\lambda}(0) = 0$, and $df_{i,\lambda}(0) = 0$. Since $W_i(\lambda)$ is the λ-rescaled copy of $W_i(1)$, we can express $f_{i,\lambda}$ in terms of $f_{i,1}$ as follows:

$$U_{i,\lambda} = \lambda U_{i,1}$$

and

$$f_{i,\lambda}(x) = \lambda f_{i,1}(\lambda^{-1}x), \quad x \in U_{i,\lambda}.$$

These formulae imply that for any compact set $D \subset \overline{W}_i$, the domains $U_{i,\lambda}$ cover D for all sufficiently large λ and the restriction $f_{i,\lambda}|_D$ goes to zero in $C^1(D)$ as $\lambda \to \infty$. This is what we mean by convergence of $W_i(\lambda)$ to $\overline{W}_i$.

Fix a sequence $t_0 = \tau_0 < \tau_1 < \cdots < \tau_N < b$ such that $t_k \in (\tau_{k-1}, \tau_k)$ for all $k = 1, \ldots, N$. We claim that the trajectories γ_λ converge to γ in the following sense: for every k, one has $\gamma_\lambda(\tau_k) \to \gamma(\tau_k)$ and $\gamma'_\lambda(\tau_k) \to \gamma'(\tau_k)$ as $\lambda \to \infty$. We prove this by induction in k. The claim is trivial for $k = 0$. Assume that it holds for $k-1$ in place of k and consider the first moment $t_k^0(\lambda) > \tau_{k-1}$ when γ_λ hits the flat wall $\overline{W}_{i_k}$. The assumed convergence at τ_{k-1} implies that for all sufficiently large λ, the moment $t_k^0(\lambda)$ exists, the interval of γ_λ between τ_{k-1} and $t_k^0(\lambda)$ is a straight-line segment (not hitting any walls), and this segment converges to the segment between $\gamma(\tau_{k-1})$ and $\gamma(t_k)$. In particular, $t_k^0(\lambda) \to t_k$, $\gamma_\lambda(t_k^0(\lambda)) \to \gamma(t_k)$, and the left derivative of γ_λ at $t_k^0(\lambda)$ converges to that of γ at t_k as $\lambda \to \infty$.

Recall that $W_{i_k}(\lambda)$ is the graph of a smooth function $f_{i_k,\lambda}$ defined over a large region in $\overline{W}_{i_k}$, and the functions $f_{i_k,\lambda}$ tend to zero along with their derivatives as $\lambda \to \infty$. By an elementary analysis, it follows that γ_λ hits W_{i_k} at some moment $t_k(\lambda) > t_k^0(\lambda)$ such that $t_k(\lambda) - t_k^0(\lambda) \to 0$ as $\lambda \to \infty$. Thus, $t_k(\lambda) \to t_k$, $\gamma_\lambda(t_k(\lambda)) \to \gamma(t_k)$, and $\gamma'_\lambda(t_k(\lambda)_-) \to \gamma'(t_{k-})$ as $\lambda \to \infty$. The tangent direction of $W_{i_k}(\lambda)$ at $\gamma_\lambda(t_k(\lambda))$ converges to the direction of $\overline{W}_{i_k}$, since it is determined by the derivative of $f_{i_k,\lambda}$. Hence, the velocity of γ_λ after the collision also converges: $\gamma'_\lambda(t_k(\lambda)_+) \to \gamma'(t_{k+})$ as $\lambda \to \infty$. If λ is large enough, it follows that γ_λ does not hit any wall between $t_k(\lambda)$ and τ_{k-1}. The desired convergence of $\gamma_\lambda(\tau_k)$ and $\gamma'_\lambda(\tau_k)$ follows, completing the induction step and thus proving the claim that $\gamma_\lambda \to \gamma$.

Moreover, the argument implies that, for a sufficiently large λ, the trajectory γ_λ is well defined on an interval (τ_0, τ_N) and experiences N collisions with walls $W_{i_1}(\lambda), \ldots, W_{i_N}(\lambda)$ in this order.

Rescaling everything back, we obtain that there is a billiard trajectory $\widetilde{\gamma}$ in $\mathcal{B}_{d,n}$, namely the one defined by $\widetilde{\gamma}(t) = q + \lambda^{-1}\gamma_\lambda(t)$ for a sufficiently large λ, that experiences N collisions on the interval $(t_0, t_N + \varepsilon)$. □

Now we describe a simple example with exponentially many collisions in high dimensions. We do this mainly to facilitate understanding. This example is not used in the proof of the main theorem. We begin with the following lemma.

LEMMA 2.3. *For every $m \in \mathbb{N}$ and $\varepsilon > 0$, there exists a polyhedral cone $K \subset \mathbb{R}^m$ with m faces and such that:*

(1) *all pairwise angles between faces of K belong to* $(\pi/2 - \varepsilon, \pi/2 + \varepsilon)$*;*
(2) *there exists a billiard trajectory* $\gamma : \mathbb{R} \to K$ *with* $2^m - 1$ *collisions.*

Proof. We argue by induction in m. The base $m = 1$ is trivial. The induction step is from m to $m+1$. Let $K \subset \mathbb{R}^m$ be a cone from the induction hypothesis and $\gamma : \mathbb{R} \to K$ a billiard trajectory with $N := 2^m - 1$ collisions. Let $t_1 < \cdots < t_N$ be the moments of these collisions.

Consider the cone $K \times \mathbb{R} \subset \mathbb{R}^{m+1}$ and observe that for any two constants $C_0, C_1 \in \mathbb{R}$ the path $\overline{\gamma} : \mathbb{R} \to K \times \mathbb{R}$ defined by

$$\overline{\gamma}(t) = (\gamma(t), C_1 - C_0 t) \in K \times \mathbb{R} \tag{2.5}$$

is a billiard trajectory in $K \times \mathbb{R}$. We choose $C_0 > 0$ so large that the vector

$$v := -\frac{\overline{\gamma}'(t)}{|\overline{\gamma}'(t)|}, \quad t > t_N,$$

forms an angle smaller than ε with the last coordinate vector of $\mathbb{R}^m \times \mathbb{R}$.

Define a cone $\widehat{K} \subset \mathbb{R}^{m+1} = \mathbb{R}^m \times \mathbb{R}$ by

$$\widehat{K} = \{x \in K \times \mathbb{R} : \langle x, v \rangle \geq 0\}.$$

This is a polyhedral cone with $m+1$ faces forming pairwise angles between $\pi/2 - \varepsilon$ and $\pi/2 + \varepsilon$. Denote by W the newly added wall of this cone, that is,

$$W = \{x \in \widehat{K} : \langle x, v \rangle = 0\}.$$

We construct a billiard trajectory $\widehat{\gamma} : \mathbb{R} \to \widehat{K}$ with $2N + 1 = 2^{m+1} - 1$ collisions as follows. Choose $C_1 > 0$ in (2.5) so large that $\langle \overline{\gamma}(t_N + 1), v \rangle \geq 0$. This ensures that $\overline{\gamma}(t) \in \widehat{K}$ for all $t \in (-\infty, t_N + 1]$. Then $\overline{\gamma}$ hits W at some moment $t_{N+1} \geq t_N + 1$ and it hits W orthogonally. Then the path $\widehat{\gamma} : \mathbb{R} \to \widehat{K}$ defined by

$$\widehat{\gamma}(t) = \begin{cases} \overline{\gamma}(t), & t \leq t_{N+1}, \\ \overline{\gamma}(2t_{N+1} - t), & t \geq t_{N+1}, \end{cases}$$

is a billiard trajectory in $\widehat{K}$ with $2N + 1$ collisions. This completes the induction step. □

THEOREM 2.4. *For every* $n \geq 2$, $\operatorname{MaxCol}(n, n-1) \geq 2^{n-1} - 1$.

Proof. For $m = n - 1$ and a sufficiently small $\varepsilon > 0$ construct a cone $K \subset \mathbb{R}^{n-1}$ as in Lemma 2.3. Let $u_1, \ldots, u_{n-1}$ be the inner normals of faces of K. If ε is sufficiently small, then there exist unit vectors $q_1, \ldots, q_{n-1} \in \mathbb{R}^{n-1}$ such that $\langle q_i, q_j \rangle = 2\langle u_i, u_j \rangle$ and $|q_i - q_j| > 1$ for all $i \neq j$. (They form a basis of $\mathbb{R}^{n-1}$ close to an orthonormal one.)

Set $d = n - 1$ and consider the configuration of balls in $\mathbb{R}^{n-1}$ with centers at $q_1, \ldots, q_{n-1}$, and $q_n = 0$. In this configuration the nth ball touches all other balls while the other ones do not touch each other. Hence, the point $q \in \mathcal{B}_{n-1,n}$ belongs to the walls ∂C_{in}, $i = 1, \ldots, n-1$. Let $\nu_1, \ldots, \nu_{n-1}$ be the normals to these walls at q. Then, by

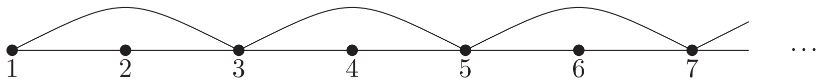

FIGURE 2. The set $\mathcal{E}$. For each $(i, j) \in \mathcal{E}$ the edge connecting i and j is depicted.

(2.4) and the construction of q,

$$\langle v_i, v_j \rangle = \tfrac{1}{2}\langle q_i, q_j \rangle = \langle u_i, u_j \rangle, \quad 1 \le i < j \le n-1.$$

Hence, the frame $(v_1, \ldots, v_{n-1})$ is isometric to the frame $(u_1, \ldots, u_{n-1})$. Therefore, the cone $\mathrm{Cone}(q)$ is isometric to $K \times \mathbb{R}^k$ for a suitable $k \in \mathbb{N}$. Since K admits a billiard trajectory with $2^{n-1} - 1$ collisions, so does $\mathrm{Cone}(q)$. This and Lemma 2.2 imply that there exists a billiard trajectory in $\mathcal{B}_{n-1,n}$ with at least $2^{n-1} - 1$ collisions. Theorem 2.4 follows. □

3. *An example in* $\mathbb{R}^3$

In this section we prove Theorem 1.1. Therefore, $d = 3$. We fix $n \ge 2$ for the rest of this section. Our goal is to construct a trajectory of a system of n identical balls in $\mathbb{R}^3$ with exponentially many collisions. All collisions in our construction occur near a special configuration $\widehat{q} = (\widehat{q}_1, \ldots, \widehat{q}_n) \in \mathbb{R}^{3n}$ defined as follows: we set $\widehat{q}_1 = (0, 0, 0) \in \mathbb{R}^3$ and, for $2 \le i \le n$,

$$\widehat{q}_i = \begin{cases} (k, k-1, 0) & \text{if } i = 4k-2, k \in \mathbb{Z}, \\ (k, k-1, -1) & \text{if } i = 4k-1, k \in \mathbb{Z}, \\ (k, k, 0) & \text{if } i = 4k, k \in \mathbb{Z}, \\ (k, k, 1) & \text{if } i = 4k+1, k \in \mathbb{Z}. \end{cases}$$

This configuration is illustrated in Figure 1. One sees that $\widehat{q} \in \mathcal{B}_{3,n}$ and $\widehat{q}$ has exactly $n-1$ pairs of contacting balls. We connect each pair of contacting balls by a segment and denote these segments by $\widehat{u}_1, \ldots, \widehat{u}_{n-1}$ as follows:

$$\begin{aligned} \widehat{u}_1 &= [\widehat{q}_1, \widehat{q}_2], \\ \widehat{u}_{2k} &= [\widehat{q}_{2k}, \widehat{q}_{2k+1}], \quad k = 1, 2, \ldots, \lfloor (n-1)/2 \rfloor, \\ \widehat{u}_{2k+1} &= [\widehat{q}_{2k}, \widehat{q}_{2k+2}], \quad k = 1, 2, \ldots, \lfloor (n-2)/2 \rfloor. \end{aligned}$$

This configuration is not the one whose tangent cone admits exponentially many collisions. Indeed, all angles between adjacent segments $\widehat{u}_i$ are equal to $\pi/2$. Hence, by (2.3) and (2.4), the tangent cone $\mathrm{Cone}(\widehat{q})$ is a right-angled cone. This implies that a billiard trajectory in $\mathrm{Cone}(\widehat{q})$ cannot experience more than $n-1$ collisions. Our plan is to construct a configuration $q \in \partial\mathcal{B}_{3,n}$ near $\widehat{q}$ whose cone does admit trajectories with exponentially many collisions and apply Lemma 2.2 to q. (Compare with Lemma 2.3 and Theorem 2.4.)

We define a specific set $\mathcal{E} \subset \mathbb{N} \times \mathbb{N}$ by

$$\mathcal{E} = \{(i, j) \in \mathbb{N} \times \mathbb{N} : \text{either } j = i+1 \text{or } i \text{ is odd and } j = i+2\}.$$

This set is illustrated in Figure 2 as a set of edges of a graph with vertices in $\mathbb{N}$.

Let $m = n - 1$. Observe that for $1 \le i < j \le m$, $(i, j) \in \mathcal{E}$ if and only if the segments $\widehat{u}_i$ and $\widehat{u}_j$ meet at a common end point. We denote by $\mathcal{E}_m$ the set of pairs $(i, j) \in \mathcal{E}$ such that $i, j \le m$. We perturb our configuration by applying the following lemma.

LEMMA 3.1. *There exists $\theta = \theta(m) > 0$ such that the following holds. For any collection of numbers $\{\alpha_{ij}\}$ indexed by pairs $(i, j) \in \mathcal{E}_m$ and such that $|\alpha_{ij} - \pi/2| < \theta$ for all $(i, j) \in \mathcal{E}_m$, there exists a configuration $q \in \mathcal{B}_{3,m+1}$ of $m + 1$ balls such that:*

(1) *the combinatorics of ball contacts in q is the same as in $\widehat{q}$. That is, $|q_i - q_j| = 1$ if and only if $|\widehat{q}_i - \widehat{q}_j| = 1$;*
(2) *let $u_1, \ldots, u_m$ be the segments between the centers of pairs of touching balls of q enumerated in the same way as we have enumerated $\{\widehat{u}_i\}$. Then $\angle(u_i, u_j) = \alpha_{ij}$ for all $(i, j) \in \mathcal{E}_m$.*

Proof. This is an easy lemma. For completeness, we provide a proof. First consider the case when m is odd. Let $q_1 = \widehat{q}_1, q_2 = \widehat{q}_2$, and $u_1 = [q_1, q_2]$. Then, for $i = 3, 5, 7, \ldots, m$, let q_{i+1} be the unique point in the xy-plane such that $|q_{i-1} - q_{i+1}| = 1$, the segments u_{i-2} and $u_i := [q_{i-1}, q_{i+1}]$ satisfy $\angle(u_{i-2}, u_i) = \alpha_{i-2,i}$, and they form a triangle oriented in the same way as the one formed by $\widehat{u}_{i-2}$ and $\widehat{u}_i$.

Finally, for $i = 2, 4, 6, \ldots, m - 1$, let q_{i+1} be the unique point in $\mathbb{R}^3$ such that q_{i+1} lies in the same half-space as $\widehat{q}_{i+1}$ with respect to the xy-plane, $|q_i - q_{i+1}| = 1$, and the segments u_{i-1}, u_{i+1}, and $u_i := [q_i, q_{i+1}]$ satisfy $\angle(u_{i-1}, u_i) = \alpha_{i-1,i}$ and $\angle(u_i, u_{i+1}) = \alpha_{i,i+1}$. This is possible whenever $\theta < \pi/6$, since the three angles $\alpha_{i-1,i}$, $\alpha_{i,i+1}$, and $\alpha_{i-1,i+1}$ satisfy the triangle inequality and their sum is less than 2π.

The resulting configuration $q \in \mathbb{R}^{3n}$ tends to $\widehat{q}$ as $\alpha_{ij} \to \pi/2$. Thus, if θ is sufficiently small, then $|q_i - q_j| > 1$ for all i, j such that $|\widehat{q}_i - \widehat{q}_j| > 1$.

In the case when m is even, apply the above construction to $m + 1$ in place of m, assuming that $\alpha_{m,m+1} = \alpha_{m-1,m+1} = \pi/2$, and then remove the point q_{m+2}. □

Let q be a configuration constructed in Lemma 3.1 (for a sufficiently small θ and a collection of angles $\{\alpha_{ij}\}$ to be specified later). Define $K = \text{Cone}(q)$. Each wall of K corresponds to a pair of touching balls in q. We enumerate these walls in the same way as we have enumerated the segments $\{u_i\}$ and we denote by $\nu_1, \ldots, \nu_m$ their respective normals. By (2.4) and (2.3), for $1 \le i < j \le m$ we have

$$\langle \nu_i, \nu_j \rangle = \begin{cases} \frac{1}{2} \cos \alpha_{ij}, & (i, j) \in \mathcal{E}, \\ 0, & (i, j) \notin \mathcal{E}. \end{cases} \tag{3.1}$$

If ε is sufficiently small, then (3.1) and the assumption $|\alpha_{ij} - \pi/2| < \theta$ imply that the Gram matrix $(\langle \nu_i, \nu_j \rangle)$ is close to the identity one. Therefore, the vectors $\nu_1, \ldots, \nu_m$ are linearly independent. Hence, K is isometric to $K_0 \times \mathbb{R}^{3n-m}$, where K_0 is the intersection of K and the linear span of $\nu_1, \ldots, \nu_m$. The linear factor $\mathbb{R}^{3n-m}$ plays no role here and we construct a desired billiard trajectory in K_0.

Note that K_0 is an m-dimensional polyhedral cone with the same normals $\nu_1, \ldots, \nu_m$ to faces. Since the normals are linearly independent, for every m-tuple $(\xi_1, \ldots, \xi_m) \in \mathbb{R}^m_+$ there exists a unique point $x \in K_0$ such that $\langle x, \nu_i \rangle = \xi_i$ for all i.

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Using this fact, we represent a billiard trajectory $\gamma \colon I \to K_0$, where I is an interval, by the collection of functions $f_i \colon I \to \mathbb{R}_+$, $i = 1, \ldots, m$, given by $f_i(t) = \langle \gamma(t), \nu_i \rangle$. In other words, $f_i(t)$ is the distance from $\gamma(t)$ to the ith wall. These functions are piecewise linear, their break points (that is, discontinuity points of the derivative) occur only at moments where one of them vanishes, and the reflection rule (2.1) takes the following form: if $i \in \{1, \ldots, m\}$ and $t \in I$ are such that $f_i(t) = 0$, then

$$f_j'(t_+) = f_j'(t_-) - 2\langle \nu_i, \nu_j \rangle f_i'(t_-), \quad j = 1, \ldots, m. \tag{3.2}$$

Since γ never hits intersections of walls, at every moment $t \in I$ no more than one of the values $f_1(t), \ldots, f_m(t)$ can vanish.

We consider a more general problem where the scalar products $\langle \nu_i, \nu_j \rangle$ in (3.2) are replaced by entries of an $m \times m$ matrix $A = (a_{ij})$, which is not assumed to be positive definite or even symmetric.

Definition 3.2. We say that an $m \times m$ matrix $A = (a_{ij})$ is *admissible* if $a_{ii} = 1$ for all i. For an admissible matrix A, an *A-trajectory* is a piecewise linear function

$$f = (f_1, \ldots, f_m) \colon I \to \mathbb{R}_+^m$$

with finitely many break points, where $I \subset \mathbb{R}$ is an interval, such that:

(1) no two of the f_i vanish simultaneously. That is, if $f_i(t) = f_j(t) = 0$ for some i, j, and t, then $i = j$;

(2) f is linear on any interval where all the f_i are strictly positive;

(3) if i and t are such that $f_i(t) = 0$, then, for every $j \in \{1, \ldots, m\}$,

$$f_j'(t_+) = f_j'(t_-) - 2a_{ij} f_i'(t_-). \tag{3.3}$$

Such moments t are referred to as *collisions*;

(4) collisions do not occur at end points of I.

In particular, if $a_{ij} = \langle \nu_i, \nu_j \rangle$ for all i, j, then A-trajectories correspond exactly to billiard trajectories in K_0. Due to the condition $a_{ii} = 1$, the rule (3.3) for $j = i$ takes the form $f_i'(t_+) = -f_i'(t_-)$.

We describe two ways of modifying an admissible matrix A preserving the property that there is an A-trajectory with many collisions. The first one is a sufficiently small perturbation.

LEMMA 3.3. *Let $N \in \mathbb{N}$ and let A be an admissible matrix such that there is an A-trajectory with N collisions. Then there exists $\delta > 0$ such that for every admissible matrix $\widetilde{A}$ satisfying $\|\widetilde{A} - A\| < \delta$ there is an $\widetilde{A}$-trajectory with N collisions. (Here and below the matrix norm $\|\cdot\|$ is the maximum of the absolute values of the matrix entries.)*

Proof. This is yet another easy lemma. Let $f \colon (a, b) \to \mathbb{R}_+^n$ be an A-trajectory with N collisions at moments $t_1 < \cdots < t_N$. For $k = 1, \ldots, N$, let i_k be the index such that $f_{i_k}(t_k) = 0$. Fix $\tau_0 \in (a, t_1)$, $\tau_k \in (t_k, t_{k+1})$ for $k = 1, \ldots, N - 1$, and $\tau_N \in (t_N, b)$.

Clearly an $\widetilde{A}$-trajectory $\widetilde{f}$ is uniquely determined by the initial data $(\widetilde{f}(\tau_0), \widetilde{f}'(\tau_0))$. For convenience we consider the matrix $\widetilde{A}$ as a part of the initial data. Let $\widetilde{A} = (\widetilde{a}_{ij})$

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be an admissible matrix, $x = (x_1, \dots, x_m) \in \mathbb{R}^m_+$, and $v = (v_1, \dots, v_m) \in \mathbb{R}^m$. If $\widetilde{A}$ is sufficiently close to A, x to $f(\tau_0)$, and v to $f'(\tau_0)$, then there exists an $\widetilde{A}$-trajectory $\widetilde{f}\colon [\tau_0, \tau_1] \to \mathbb{R}^m_+$ with initial data $\widetilde{f}(\tau_0) = x$ and $\widetilde{f}'(\tau_0) = v$ and precisely one collision $f_{i_1}(\widetilde{t}_1) = 0$ at some moment $\widetilde{t}_1 \in (\tau_0, \tau_1)$. Moreover, the map $(\widetilde{A}, x, v) \mapsto (\widetilde{A}, f(\tau_1), f'(\tau_1))$ that sends the initial data to the terminal data is continuous. Indeed, $\widetilde{f}$ is given by the explicit formulae

$$\widetilde{f}_{i_1}(t) = |x_{i_1} + (t - \tau_0)v_{i_1}|$$

and

$$\widetilde{f}_j(t) = x_j + (t - \tau_0)v_j - v_{i_1}\widetilde{a}_{i_1 j}(t - \widetilde{t}_1 + |t - \widetilde{t}_1|), \quad j \neq i_1,$$

where $\widetilde{t}_1 = \tau_0 - x_{i_1}/v_{i_1}$.

Applying the same argument to intervals $[\tau_{k-1}, \tau_k]$, $k = 1, \dots, N$, and composing the resulting maps one sees that, if $\widetilde{A}$ is sufficiently close to A, then there is an $\widetilde{A}$-trajectory defined on $[\tau_1, \tau_N]$ with one collision on each of the intervals. □

The second modification of A is a rescaling described in the following lemma.

LEMMA 3.4. *Let $A = (a_{ij})$ be an admissible matrix and $\lambda = (\lambda_1, \dots, \lambda_m)$ an m-tuple of positive numbers. Define a matrix $A^\lambda = (a^\lambda_{ij})$ by*

$$a^\lambda_{ij} = \frac{\lambda_j}{\lambda_i} a_{ij}, \quad 1 \le i, j \le m.$$

Then, if A admits an A-trajectory with N collisions, then so does A^λ.

Proof. Note that $a^\lambda_{ij} = a_{ij} = 1$ and hence A^λ is an admissible matrix. Let $f\colon I \to \mathbb{R}^m_+$ be an A-trajectory with N collisions. Define $g\colon I \to \mathbb{R}^m_+$ by $g_i(t) = \lambda_i f_i(t)$ for $i = 1, \dots, m$. Multiplying (3.3) by λ_j yields

$$g'_j(t_+) = g'_j(t_-) - 2\frac{\lambda_j}{\lambda_i} a_{ij} g'_i(t_-) = g'_j(t_-) - 2a^\lambda_{ij} g'_i(t_-).$$

Thus, g is an A^λ-trajectory. The collisions of g are at the same moments as those of f. □

With there operations at hand, we reduce our goal to constructing an A-trajectory with many collisions for a concrete $m \times m$ matrix $A = A_m$ whose entries (a_{ij}) are given by

$$a_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } (i, j) \in \mathcal{E}_m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.4}$$

Recall that the set $\mathcal{E}_m$ is not symmetric; it includes only pairs (i, j) with $i < j$. Thus, the matrix A_m defined by (3.4) is upper triangular. Note that A_m is a submatrix of A_{m+1} in the sense that for $i, j \le m$, the (i, j)th entries of A_m and A_{m+1} are the same.

LEMMA 3.5. *Let A_m be the matrix defined by (3.4). Suppose that there is an A_m-trajectory with N collisions for some $N \in \mathbb{N}$. Then* $\mathrm{MaxCol}(m + 1, 3) \ge N$.

Proof. We choose a finite sequence $\lambda = (\lambda_1, \ldots, \lambda_m)$ of positive numbers that decay sufficiently fast. The precise requirements on λ are specified later.

First we require that $\lambda_j^2/\lambda_i^2 < \delta$ for all $i < j$, where δ is the number provided by Lemma 3.3 for A_m and N. Define an $m \times m$ matrix $\widetilde{A} = (\widetilde{a}_{ij})$ by

$$\widetilde{a}_{ij} = \begin{cases} 1 & \text{if } i = j, \\ -1 & \text{if } (i, j) \in \mathcal{E}_m, \\ -\lambda_i^2/\lambda_j^2 & \text{if } (j, i) \in \mathcal{E}_m, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

In the third case in (3.5) we have $i > j$ and therefore $|\widetilde{a}_{ij}| < \delta$. Since the other entries of $\widetilde{A}$ are the same as those of A_m, we have $|\widetilde{a}_{ij} - a_{ij}| < \delta$ for all i, j. Hence, by Lemma 3.3, there exists an $\widetilde{A}$-trajectory with at least N collisions.

Now rescale $\widetilde{A}$ using λ as in Lemma 3.4. Denote the resulting matrix $\widetilde{A}^\lambda$ by B. The entries (b_{ij}) of B are given by $b_{ii} = 1$, $b_{ij} = -\lambda_j/\lambda_i$ if $(i, j) \in \mathcal{E}_m$, $b_{ij} = -\lambda_i/\lambda_j$ if $(j, i) \in \mathcal{E}_m$, and 0 otherwise. Hence, B is symmetric.

Now we require that $\lambda_j/\lambda_i < \frac{1}{2} \sin\theta$, where θ is the number provided by Lemma 3.1. For each pair $(i, j) \in \mathcal{E}_m$, define $\alpha_{ij} \in (\pi/2 - \theta, \pi/2 + \theta)$ by

$$\cos\alpha_{ij} = -2\lambda_j/\lambda_i = 2b_{ij}.$$

Let $q \in \mathcal{B}_{3,m+1}$ be the configuration of balls constructed in Lemma 3.1 for this collection of angles $\{\alpha_{ij}\}$. Let $K = \mathrm{Cone}(q)$ and let $\nu_1, \ldots, \nu_m$ be the normals to faces of K as explained above. Then, by (3.1) and the definition of B, we have $\langle \nu_i, \nu_j \rangle = b_{ij}$ for all $1 \le i, j \le m$.

Therefore, as explained above, every B-trajectory corresponds to a billiard trajectory in K_0 (and hence in K) with the same number of collisions. Thus, K has a billiard trajectory with at least N collisions. Finally, we apply Lemma 2.2 and conclude that $\mathrm{MaxCol}(m + 1, 3) \ge N$. □

The rest of the paper is devoted to constructing an A-trajectory with exponentially many collisions for the matrix A_m defined by (3.4). Our plan is to first construct a generalized A_m-trajectory where simultaneous collision of certain type are allowed (see Definition 3.6), and then perturb the generalized A_m-trajectory to a obtain a genuine one (see Lemma 3.7).

Definition 3.6. Let A be an admissible $m \times m$ matrix. A *generalized A-trajectory* is a piecewise linear map

$$f = (f_1, \ldots, f_m) \colon I \to \mathbb{R}_+^m,$$

where $I \subset \mathbb{R}$ is an interval, such that the following holds.

(1) If $f_i(t) = f_j(t) = 0$ for some $i \ne j$ and $t \in I$, then $a_{ij} = a_{ji} = 0$.

(2) For every $t \in I$ and every $j \in \{1, \ldots, m\}$,

$$f_j'(t_+) = f_j'(t_-) - 2 \sum_{i : f_i(t) = 0} a_{ij} f_i'(t_-), \tag{3.6}$$

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where we sum over the set of all indices $i \in \{1, \dots, m\}$ such that $f_i(t) = 0$ for the given t. In particular, f is linear on any interval where all the f_i are positive.

(3) If t is an end point of I, then $f_i(t) > 0$ for all i.

By the *number of collisions* of a generalized A-trajectory f we mean the total number of roots of the f_i. That is, a moment t when exactly k of the values $f_i(t)$ have vanished contributes k to the total number of collisions.

LEMMA 3.7. *Let A be an admissible $m \times m$ matrix such that there exists a generalized A-trajectory with N collisions (see Definition 3.6). Then there exists an A-trajectory with N collisions.*

Proof. The argument is similar to that in the proof of Lemma 3.3. Let $f : (a, b) \to \mathbb{R}^m_+$ be a generalized A-trajectory and $t_1 < \cdots < t_M$ the moments of collisions. For $k = 1, \dots, M$, denote by n_k the number of collisions at the moment t_k. Then the total number of collisions N equals $\sum n_k$. Fix $\tau_0 \in (a, t_0)$, $\tau_M \in (t_M, b)$, and $\tau_k \in (t_k, t_{k+1})$ for $k = 1, \dots, M - 1$.

Just like A-trajectories, generalized A-trajectories are determined by their initial data. We claim that for every $k \in \{1, \dots, M\}$ and any (x, v) sufficiently close to $(f(\tau_{k-1}), f'(\tau_{k-1}))$ there exists a generalized A-trajectory $\widetilde{f} : [\tau_{k-1}, \tau_k] \to \mathbb{R}^m_+$ with initial data $\widetilde{f}(\tau_{k-1}) = x$, $\widetilde{f}'(\tau_{k-1}) = v$, and precisely n_k collisions. Moreover, the terminal data $(\widetilde{f}(\tau_k), \widetilde{f}'(\tau_k))$ depend smoothly on (x, v).

To prove the claim, fix k and define $J_k = \{i : f_i(t_k) = 0\}$. Note that $|J_k| = n_k$. For (x, v) sufficiently close to $(f(\tau_{k-1}), f'(\tau_{k-1}))$, define $\widetilde{f} : [\tau_{k-1}, \tau_k] \to \mathbb{R}^m_+$ by

$$\widetilde{f}_i(t) = |x_i + (t - \tau_{k-1})v_i|, \quad i \in J_k, \tag{3.7}$$

and

$$\widetilde{f}_j(t) = x_j + (t - \tau_{k-1})v_j - \sum_{i=1}^{n_k} v_i(t - \widetilde{t}_{k,i} + |t - \widetilde{t}_{k,i}|), \quad j \notin J_k, \tag{3.8}$$

where

$$\widetilde{t}_{k,i} = \tau_{k-1} - x_i/v_i, \quad i \in J_k, \tag{3.9}$$

are the roots of the f_i. Note that the roots $\widetilde{t}_{k,i}$ and the terminal data $(\widetilde{f}(\tau_k), \widetilde{f}'(\tau_k))$ defined by the above formulae depend smoothly on (x, v). In particular, $\widetilde{t}_{k,i} \in (\tau_{k-1}, \tau_k)$ if the initial data (x, v) are sufficiently close to $(\widetilde{f}(\tau_k), \widetilde{f}'(\tau_k))$.

The definition of a generalized A-trajectory implies that $a_{ij} = 0$ for $i, j \in J_k$. This ensures that $\widetilde{f}$ satisfies (3.6) as long as $\widetilde{f}_j(t) > 0$ for all $j \notin J_k$ and $t \in [\tau_{k-1}, \tau_k]$. The latter is true for $(x, v) = (f(\tau_{k-1}), f'(\tau_{k-1}))$, since in this case $\widetilde{f} = f$; hence, it is true for all (x, v) sufficiently close to $(f(\tau_{k-1}), f'(\tau_{k-1}))$. This finishes the proof of the claim. Also observe that the roots $\widetilde{t}_{k,i}$ defined by (3.9) are distinct for almost all pairs (x, v).

Similarly, one shows that the initial data $(\widetilde{f}(\tau_{k-1}), \widetilde{f}'(\tau_{k-1}))$ depend smoothly on the terminal data $(\widetilde{f}(\tau_k), \widetilde{f}'(\tau_k))$. Thus, the map that sends the initial data to the terminal data is a diffeomorphism from a neighborhood of $(f(\tau_{k-1}), f'(\tau_{k-1}))$ to a neighborhood of $(f(\tau_k), f'(\tau_k))$. Composing such diffeomorphisms for all k, we obtain that any initial data

(x, v) sufficiently close to $(f(\tau_0), f'(\tau_0))$ determine a generalized A-trajectory defined on $[\tau_0, \tau_M]$ with N collisions. Then by the routine of smooth topology one sees that for almost all initial data the roots $\widetilde{t}_{k,i}$ are distinct for all k and i.

Thus, a suitable perturbation of the initial data $(f(\tau_0), f'(\tau_0))$ gives us a generalized A-trajectory with N collisions occurring at N distinct moments. Such a generalized A-trajectory is a genuine A-trajectory. □

It remains to construct a generalized A_m-trajectory, for A_m given by (3.4), with exponentially many collisions. This is achieved by the following lemma.

LEMMA 3.8. *For A defined by (3.4), there exists a generalized A-trajectory $f : \mathbb{R} \to \mathbb{R}^m_+$ satisfying the following conditions.*

(1) $|f_i'(t)| = 1$ *for all* $i \in \{1, \ldots, m\}$ *and all* $t \in \mathbb{R}$ *except the break points of* f_i.

(2) *Denote by* T_i *the set of all* $t \in \mathbb{R}$ *such that* $f_i(t) = 0$. *Then* T_i *is a finite arithmetic progression for every* i.

(3) *For all even* $i = 2k \le m$, *one has* $|T_i| = 2^k + 2^{k-1} - 1$.

(4) *For all odd* $i = 2k + 1 \le m$, *one has* $|T_i| = 2^{k+1} + 2^k - 2$.

Proof. We argue by induction in m. For the induction base $m = 1$ we set $f_1(t) = |t|$. Then $T_1 = \{0\}$ and $|T_1| = 1$. We regard T_1 as an arithmetic progression with common difference 1.

For the induction step, we assume that $(f_1, \ldots, f_{2k-1})$ is a generalized A_m-trajectory satisfying (1)–(4) for $m = 2k - 1$ and prove the assertion for $m = 2k$ and $m = 2k + 1$. We do not change the existing f_i for $i \le 2k - 1$ and just add new functions f_{2k} and f_{2k+1}.

By the induction hypothesis, the set T_{2k-1} is a finite arithmetic progression. We denote its elements by $x_1 < x_2 < \cdots < x_M$, where $M = 2^k + 2^{k-1} - 2$, and its common difference is denoted by β. We first define the set T_{2k} by $T_{2k} = \{y_1, \ldots, y_{M+1}\}$, where

$$y_s = x_1 + (s - \tfrac{3}{2})\beta, \quad s = 1, \ldots, M + 1.$$

Note that

$$y_1 < x_1 < y_2 < x_2 < \cdots < y_M < x_M < y_{M+1},$$

T_{2k} is an arithmetic progression with common difference β, and the union $T_{2k-1} \cup T_{2k}$ is an arithmetic progression with common difference $\beta/2$. Now define

$$f_{2k}(t) = \operatorname{dist}(t, T_{2k}) = \min\{|t - y_s| : 1 \le s \le M + 1\}$$

for all $t \in \mathbb{R}$. The requirements (1) and (2) for f_{2k} follow from the construction. For (3), observe that $|T_{2k}| = M + 1 = 2^k + 2^{k-1} - 1$. It remains to verify that $(f_1, \ldots, f_{2k})$ is a generalized A_{2k}-trajectory. Since A_{2k} is upper triangular and contains A_{2k-1} as a submatrix, the requirements of the definition of the generalized A-trajectory for the components $f_1, \ldots, f_{2k-1}$ persist. The indices i such that $a_{i,2k} \ne 0$ are only $i = 2k - 1$ and $i = 2k$. Since $T_{2k} \cap T_{2k-1} = \emptyset$, simultaneous collisions $f_i(t) = f_{2k}(t) = 0$ can occur only if $a_{i,2k} = 0$ or $i = 2k$.

Let us verify (3.6) for $j = 2k$ and all $t \in \mathbb{R}$. If $t \in T_{2k}$, then $f_{2k}'(t_-) = -1$ and $f_{2k}'(t_+) = 1$. This agrees with (3.6), since $a_{2k,2k} = 1$. For $t = x_s \in T_{2k-1}$, observe that t is

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the mid point between y_s and y_{s+1} and hence it is a break point of f_{2k} with $f'_{2k}(t_-) = 1$ and $f'_{2k}(t_+) = -1$. The requirement (1) for $i = 2k-1$ implies that $f'_{2k-1}(t_-) = -1$. Since $a_{2k-1,2k} = -1$ and $a_{i,2k} = 0$ for all $i < 2k-1$, these values agree with (3.6). Finally, if $t \notin T_{2k} \cup T_{2k-1}$, then it is not a break point of f_{2k} and no f_i with $a_{i,2k} \neq 0$ vanishes at t. Thus, (3.6) is satisfied for $j = 2k$ in all cases and we have shown that $(f_1, \ldots, f_{2k})$ is a generalized A_{2k}-trajectory satisfying (1)–(4).

Now we construct f_{2k+1}. Recall that $T_{2k-1} \cup T_{2k}$ is an arithmetic progression of $2M+1$ elements starting at y_1 with common difference $\beta/2$. We construct f_{2k+1} from $T_{2k-1} \cup T_{2k}$ in the same way as f_{2k} is constructed from T_{2k-1}. Namely, define $T_{2k+1} = \{z_1, \ldots, z_{2M+2}\}$, where

$$z_s = y_1 + \left(s - \frac{3}{2}\right)\frac{\beta}{2}$$

and

$$f_{2k+1}(t) = \operatorname{dist}(t, T_{2k+1}) = \min\{|t - z_s| : 1 \leq s \leq 2M+2\}.$$

Note that $|T_{2k+1}| = 2M + 2 = 2^{k+1} + 2^k - 2$, verifying the induction step for (4). Using the facts that $a_{2k-1,2k+1} = a_{2k,2k+1} = -1$ and $a_{i,2k+1} = 0$ for all $i < 2k-1$, we prove that $(f_1, \ldots, f_{2k+1})$ is a generalized trajectory satisfying all requirements by the same argument as in the above proof for $(f_1, \ldots, f_{2k})$. $\square$

Now Lemmas 3.8, 3.7, and 3.5 imply that $\operatorname{MaxCol}(m+1, 3) \geq N$, where

$$N = \sum_{i=1}^{m} |T_i| = \begin{cases} 2^{k+2} + 2^{k-1} - 3k - 5, & m = 2k-1, \\ 2^{k+2} + 2^{k+1} - 3k - 6, & m = 2k. \end{cases} \tag{3.10}$$

One easily checks that $N \geq 2^k$ for all $m \geq 2$. Since $k = \lfloor n/2 \rfloor$ for $n = m+1$, it follows that

$$\operatorname{MaxCol}(n, 3) \geq 2^{\lfloor n/2 \rfloor}$$

for all $n \geq 3$.

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