Proceedings of the Royal Society of Edinburgh, **152**, 102–127, *2022*
DOI:10.1017/prm.2020.100

Sharp Caffarelli–Kohn–Nirenberg inequalities on Riemannian manifolds: the influence of curvature

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(MS Received 5 January 2020; accepted 13 December 2020)

We first establish a family of sharp Caffarelli–Kohn–Nirenberg type inequalities (shortly, sharp CKN inequalities) on the Euclidean spaces and then extend them to the setting of Cartan–Hadamard manifolds with the same best constant. The quantitative version of these inequalities also is proved by adding a non-negative remainder term in terms of the sectional curvature of manifolds. We next prove several rigidity results for complete Riemannian manifolds supporting the Caffarelli–Kohn–Nirenberg type inequalities with the same sharp constant as in the Euclidean space of the same dimension. Our results illustrate the influence of curvature to the sharp CKN inequalities on the Riemannian manifolds. They extend recent results of Kristály (*J. Math. Pures Appl.* 119 (2018), 326–346) to a larger class of the sharp CKN inequalities.

Keywords: Caffarelli–Kohn–Nirenberg inequalities; sharp constant; Riemannian manifold; curvature

2010 *Mathematics subject classification:* 26D10; 31C12; 53C20; 53C21

1. Introduction

Let us start by recalling a celebrated family of the first-order interpolation inequalities due to Caffarelli, Kohn and Nirenberg [4] (nowadays, they are called Caffarelli–Kohn–Nirenberg (shortly, CKN) inequalities): let $n \geqslant 1$ and let $p, q, r, \alpha, \beta, \gamma, \delta$ and σ be real numbers such that

$$p, q \geqslant 1, \quad r > 0, \quad a \in [0, 1],$$

and

$$\frac{1}{p} + \frac{\alpha}{n} > 0, \quad \frac{1}{q} + \frac{\beta}{n} > 0, \quad \frac{1}{r} + \frac{\gamma}{n} > 0,$$

where

$$\gamma = \delta\sigma + (1 - \delta)\beta.$$

Then there exists a positive constant C such that the following inequality holds for any function $f \in C_0^\infty(\mathbb{R}^n)$

$$\left(\int_{\mathbb{R}^n} |f|^r |x|^{r\gamma}\, \mathrm{d}x\right)^{1/r} \leqslant C \left(\int_{\mathbb{R}^n} |f|^p |x|^{\alpha p}\, \mathrm{d}x\right)^{\delta/p} \left(\int_{\mathbb{R}^n} |f|^q |x|^{\beta q}\, \mathrm{d}x\right)^{(1-\delta)/q} \tag{1.1}$$

if and only if the following conditions hold

$$\frac{1}{r} + \frac{\gamma}{n} = \delta\left(\frac{1}{p} + \frac{\alpha-1}{n}\right) + (1-\delta)\left(\frac{1}{q} + \frac{\beta}{n}\right)$$

(this is dimensional balance),

$$\alpha - \sigma \geqslant 0 \quad \text{if} \quad \delta > 0,$$

and

$$\alpha - \sigma \leqslant 1 \quad \text{if} \quad \delta > 0 \quad \text{and} \quad \frac{1}{r} + \frac{\gamma}{n} = \frac{1}{p} + \frac{\alpha-1}{n}.$$

The CKN inequalities contain many well-known inequalities, for example, the Sobolev inequalities, the Hardy inequalities, the Hardy–Sobolev inequalities, the Gagliardo–Nirenberg inequalities, etc. They play an important role in theory of partial differential equations and have been intensively studied in many settings such as the stratified Lie groups, the homogeneous groups, the metric measure spaces, the Riemannian manifolds with negative curvature, Finsler manifolds, the fractional order derivatives, etc. We refer the readers to the papers [**6**, **10**–**13**, **15**, **16**, **18**, **22**, **30**–**37**, **42**] for more detailed discussions on this subject.

An interesting and non-trivial problem concerning to the CKN inequalities is looking for its sharp constant and its extremal functions (if they exists). Several results are well-known in this direction. For example, the sharp constants in the Sobolev inequalities was found independently by Aubin [**1**] and Talenti [**38**], the sharp Hardy–Sobolev inequalities was proved by Lieb [**28**], the sharp constants in the Gagliardo–Nirenberg inequalities was established by Del Pino and Dolbeault [**11**, **12**] (see also [**9**] for different proof by using mass transportation technique), etc. In [**42**], Xia found out the sharp constant in the following subclass of CKN inequalities: Let $r > p > 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$ be such that

$$\frac{1}{r} - \frac{\gamma}{n} > 0, \quad \frac{1}{p} - \frac{\alpha}{n} > 0, \quad 1 - \frac{\beta}{n} > 0 \tag{1.2}$$

and

$$\gamma = \frac{1+\alpha}{r} + \frac{p-1}{pr}\beta, \tag{1.3}$$

then the following inequality holds for any $f \in C_0^\infty(\mathbb{R}^n)$

$$\int_{\mathbb{R}^n} \frac{|f|^r}{|x|^{\gamma r}}\, \mathrm{d}x \leqslant \frac{r}{n-\gamma r}\left(\int_{\mathbb{R}^n} \frac{|\nabla f|^p}{|x|^{\alpha p}}\, \mathrm{d}x\right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{|f|^{p(r-1)/(p-1)}}{|x|^{\beta}}\, \mathrm{d}x\right)^{(p-1)/p}. \tag{1.4}$$

Furthermore, if

$$n - \beta < \left(1 + \alpha - \frac{\beta}{p}\right) \frac{p(r-1)}{r-p} \tag{1.5}$$

then inequality (1.4) is sharp, i.e. the constant $r/(n-\gamma r)$ is the best constant in (1.4), and a family of extremal functions is given by

$$f(x) = c(\lambda + |x|^{1+\alpha-\beta/p})^{(p-1)/(p-r)}, \quad c \in \mathbb{R},\ \lambda > 0.$$

In that paper, Xia also proved a rigidity result for complete Riemannian manifold with nonnegative Ricci curvature that supports a CKN inequality. More precisely, his result reads as follows: Let (M, g) be a complete Riemannian manifold with non-negative Ricci curvature, let $p, q, r, \alpha, \beta, \gamma$ satisfy $r > p > 1$ and conditions (1.2), (1.3) and (1.5) and let P be a fixed point in M; if the CKN inequality (1.4) holds on (M, g) with $|x|$ being replaced by $d(P, x)$ the geodesic distance from x to P on M, then (M, g) is isometric to $\mathbb{R}^n$. This rigidity result can be included into the best constant programme initiated by Aubin [**2**] and studied by Ledoux [**26**], Bakry, Concordet and Ledoux [**3**], Cheeger and Colding [**8**], Druet, Hebey and Vaugon [**14**], do Carmo and Xia [**5**], Minerbe [**29**], Li and Wang [**27**], Xia [**40**–**42**], Kristály [**23**, **24**], Kristály and Ohta [**25**], etc. In the aforementioned papers, the authors show that the complete Riemannian manifolds with non-negative Ricci curvature supporting some Sobolev-type inequalities should be close to Euclidean spaces whenever the constant is sufficiently close to the sharp constant in the corresponding inequality in Euclidean space. We refer the reader to the book by Hebey [**20**] for a thoroughgoing presentation of this subject.

Our origin motivation of this paper is to extend the CKN inequality (1.4) to a larger class of parameters $p, q, r, \alpha, \beta, \gamma$. Let $\partial_r = x/|x| \cdot \nabla$ denote the radial derivative of functions on $\mathbb{R}^n$. Our first main result in this paper reads as follows:

THEOREM 1.1. *Suppose that $n \geqslant 2$, $p > 1$, $r > 0$ and α, β, γ satisfy conditions* (1.2) *and* (1.3). *Then the following inequalities hold for any function $f \in C_0^\infty(\mathbb{R}^n)$:*
(a) *If $r > 1$, then we have*

$$\int_{\mathbb{R}^n} \frac{|f|^r}{|x|^{\gamma r}} \mathrm{d}x \leqslant \frac{r}{(n-\gamma r)} \left(\int_{\mathbb{R}^n} \frac{|\partial_r f|^p}{|x|^{\alpha p}} \mathrm{d}x\right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{|f|^{p(r-1)/(p-1)}}{|x|^\beta} \mathrm{d}x\right)^{(p-1)/p}. \tag{1.6}$$

(b) *If $r \in (0, 1)$, then we have*

$$\int_{\mathbb{R}^n} \frac{|f|^r}{|x|^{\gamma r}} \mathrm{d}x \leqslant \frac{r}{(n-\gamma r)} \left(\int_{\mathbb{R}^n} \frac{|\partial_r f|^p}{|x|^{\alpha p}} \mathrm{d}x\right)^{1/p} \left(\int_{suppf} \frac{|f|^{p(r-1)/(p-1)}}{|x|^\beta} \mathrm{d}x\right)^{(p-1)/p}, \tag{1.7}$$

where suppf denotes the support of function f.

Moreover, the constant $r/(n-\gamma r)$ is sharp if one of the following conditions holds:

(i) $1 < p < r$ *and* (1.5) *holds. A family of extremals is given by*

$$f(x) = c(\lambda(\omega) + |x|^{1+\alpha-\beta/p})^{(p-1)/(p-r)}, \quad \omega = \frac{x}{|x|},$$

where c *is constant and* $\lambda : S^{n-1} \to (0, \infty)$ *such that*

$$\int_{S^{n-1}} \lambda(\omega)^{\frac{p(r-1)}{p-r}+\frac{n-\beta}{1+\alpha-\beta/p}}\,\mathrm{d}\omega < \infty.$$

(ii) $0 < r < p$, $r \neq 1$ *and* $1 + \alpha - \beta/p > 0$. *A family of extremals is given by*

$$f(x) = c\left(\lambda(\omega) - |x|^{1+\alpha-\beta/p}\right)_+^{(p-1)/(p-r)}, \quad \omega = \frac{x}{|x|}$$

where $a_+ = \max\{a, 0\}$ *denotes the positive part of a real number* a, c *is a constant,* $\lambda : S^{n-1} \to (0, \infty)$ *such that*

$$\int_{S^{n-1}} \lambda(\omega)^{\frac{p(r-1)}{p-r}+\frac{n-\beta}{1+\alpha-\beta/p}}\,\mathrm{d}\omega < \infty.$$

(iii) $0 < r < p$, $r \neq 1$ *and* $1 + \alpha - \beta/p = 0$. *A family of extremals is given by*

$$f(x) = c\,(\lambda(\omega) - \ln|x|)_+^{(p-r)/(p-1)}, \quad \omega = \frac{x}{|x|},$$

where c *is constant,* $\lambda : S^{n-1} \to \mathbb{R}$ *such that* $\int_{S^{n-1}} \mathrm{e}^{(n-\beta)\lambda}\,\mathrm{d}\omega < \infty$

(iv) $0 < r < p$, $r \neq 1$, $1 + \alpha - (\beta/p) < 0$ *and* $n - \beta + (1 + \alpha - (\beta/p))((p(r-1))/(p-r)) > 0$. *A family of extremals is given by*

$$f(x) = c\left(|x|^{1+\alpha-\beta/p} - \lambda(\omega)\right)_+^{(p-1)/(p-r)}, \quad \omega = \frac{x}{|x|},$$

where c *is a constant and* $\lambda : S^{n-1} \to (0, \infty)$ *such that*

$$\int_{S^{n-1}} \lambda^{\frac{p(r-1)}{p-r}+\frac{n-\beta}{1+\alpha-\beta/p}}\,\mathrm{d}\omega < \infty.$$

(v) $r = p$ *and* $1 + \alpha - \beta/p > 0$. *A family of extremals is given by*

$$f(x) = \lambda(\omega)\mathrm{e}^{-c|x|^{1+\alpha-\beta/p}},$$

where $c > 0$ *is a constant and* $\lambda : S^{n-1} \to \mathbb{R}$ *such that* $\int_{S^{n-1}} |\lambda|^p\,\mathrm{d}\omega < \infty$.

Obviously, the inequality $|\partial_r f| \leqslant |\nabla f|$ holds true. Consequently, inequality (1.6) is stronger than the one of Xia (1.4). In general, we have the following consequences.

Corollary 1.2. *Given* $n, p, q, r, \alpha, \beta$ *and* γ *as in the statements of theorem* 1.1. *Then the following inequalities hold true for any* $f \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$*:*

(a) *If $r > 1$, then we have*

$$\int_{\mathbb{R}^n} \frac{|f|^r}{|x|^{\gamma r}}\,\mathrm{d}x \leqslant \frac{r}{(n-\gamma r)} \left(\int_{\mathbb{R}^n} \frac{|\nabla f|^p}{|x|^{\alpha p}}\,\mathrm{d}x\right)^{1/p} \left(\int_{\mathbb{R}^n} \frac{|f|^{p(r-1)/(p-1)}}{|x|^{\beta}}\,\mathrm{d}x\right)^{(p-1)/p}. \tag{1.8}$$

(b) *If $r \in (0,1)$, then we have*

$$\int_{\mathbb{R}^n} \frac{|f|^r}{|x|^{\gamma r}}\,\mathrm{d}x \leqslant \frac{r}{(n-\gamma r)} \left(\int_{\mathbb{R}^n} \frac{|\nabla f|^p}{|x|^{\alpha p}}\,\mathrm{d}x\right)^{1/p} \left(\int_{supp f} \frac{|f|^{p(r-1)/(p-1)}}{|x|^{\beta}}\,\mathrm{d}x\right)^{(p-1)/p}. \tag{1.9}$$

Moreover, the constant $r/(n-\gamma r)$ is sharp if one conditions (i)–(v) in theorem 1.1 holds true, and all extremals are given by the corresponding family of extremals in theorem 1.1 with λ being identically constant.

The novelty in theorem 1.1 and corollary 1.2 is that the inequalities are established for only radial derivation ∂_r, a family of extremals is found out, and especially it extends the inequality of Xia (1.4) to the case $0 < r < p$. It is remarkable that if $1 < r < (2p-1)/p$ or $r \in (0,1)$ then $q := p(r-1)/(p-1) \in (0,1)$ or $q < 0$ respectively. Hence, theorem 1.1 and corollary 1.2 also provide the new type of CKN inequalities in comparing with the classical ones of Caffarelli, Kohn and Nirenberg (1.1). Also by $q < 0$ if $r \in (0,1)$, we then need a slight modification in inequalities (1.7) and (1.9) by taking the second integral in their right-hand side on supp$f$ to ensure these inequalities being sense. Note that the celebrated sharp Heisenberg–Pauli–Weyl uncertainty principle (see [**19**, **39**]) and its extremals (up to a constant) given by the family of Gaussian functions is a special case of corollary 1.2 corresponding to the case $p = r = 2$, $\alpha = 0$ and $\beta = -2$.

The next purpose of this paper is to describe a complete scenario concerning to the CKN inequalities (1.6) and (1.7) on the complete Riemannian manifolds. Our next results tell us that inequalities (1.6) and (1.7) still hold on the Cartan–Hadamard manifolds (M,g) (i.e. the n-dimensional complete simply connected Riemannian manifolds with the non-positive sectional curvature). For a Riemannian manifold (M,g) with Riemannian metric g, we denote by $\mathrm{d}V_g$ and $\nabla_g f$ the volume element on M, and the gradient of function f with respect to metric g respectively, and by $|\nabla_g f| = \sqrt{g(\nabla_g f, \nabla_g f)}$ the length of $\nabla_g f$, and by $d_P(x) = d(x,P), x \in M$ for a fixed point $P \in M$, where d is the geodesic distance on M. We also use $\partial_\rho f$ to denote the radial derivation of function f on M (the derivation along the geodesic curve starting from a fixed point $P \in M$). We then have the following results.

THEOREM 1.3. *Let $n, p, q, r, \alpha, \beta$ and γ be as in the statements of theorem 1.1. Let (M,g) be an n-dimensional Cartan–Hadamard manifold and $P \in M$ be a fixed point, and ∂_ρ denote the radial derivation along geodesic curve starting from P. Then the following inequalities hold true for any function $f \in C_0^\infty(M)$:*

(a) *If $r > 1$, then we have*

$$\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\,\mathrm{d}V_g \leqslant \frac{r}{(n-\gamma r)}\left(\int_M \frac{|\partial_\rho f|^p}{d_P(x)^{\alpha p}}\,\mathrm{d}V_g\right)^{1/p}\left(\int_M \frac{|f|^{p(r-1)/(p-1)}}{d_P(x)^{\beta}}\,\mathrm{d}V_g\right)^{(p-1)/p}. \tag{1.10}$$

(b) *If $r \in (0,1)$, then we have*

$$\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\,\mathrm{d}V_g \leqslant \frac{r}{(n-\gamma r)}\left(\int_M \frac{|\partial_\rho f|^p}{d_P(x)^{\alpha p}}\,\mathrm{d}V_g\right)^{1/p}\left(\int_{supp f} \frac{|f|^{p(r-1)/(p-1)}}{d_P(x)^{\beta}}\,\mathrm{d}V_g\right)^{(p-1)/p}. \tag{1.11}$$

Moreover, the constant $r/(n-\gamma r)$ is sharp if one of conditions (i)–(v) in theorem 1.1 holds.

Theorem 1.3 together with Gauss lemma which says that $|\partial_\rho f| \leqslant |\nabla_\rho f|$ implies the following extensions of corollary 1.2 to the Cartan–Hadamard manifolds.

Corollary 1.4. *Suppose the assumptions of theorem 1.3. Then the following inequalities hold true for any function $f \in C_0^\infty(M)$:*
(a) *If $r > 1$, then we have*

$$\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\,\mathrm{d}V_g \leqslant \frac{r}{(n-\gamma r)}\left(\int_M \frac{|\nabla_g f|^p}{d_P(x)^{\alpha p}}\,\mathrm{d}V_g\right)^{1/p}\left(\int_M \frac{|f|^{p(r-1)/(p-1)}}{d_P(x)^{\beta}}\,\mathrm{d}V_g\right)^{(p-1)/p}. \tag{1.12}$$

(b) *If $r \in (0,1)$, then we have*

$$\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\,\mathrm{d}V_g \leqslant \frac{r}{(n-\gamma r)}\left(\int_M \frac{|\nabla_g f|^p}{d_P(x)^{\alpha p}}\,\mathrm{d}V_g\right)^{1/p}\left(\int_{supp f} \frac{|f|^{p(r-1)/(p-1)}}{d_P(x)^{\beta}}\,\mathrm{d}V_g\right)^{(p-1)/p}. \tag{1.13}$$

Moreover, the constant $r/(n-\gamma r)$ is sharp if one of conditions (i)–(v) in theorem 1.1 holds.

Corollary 1.4 extends inequality (1.4) of Xia to the setting of Cartan–Hadamard manifolds and to a larger class of parameters $p, r, \alpha, \beta, \gamma$. In recent paper [**22**], Huang, Kristály and Zhao studied inequality (1.12) with $p=2$, $\alpha=0$, $\gamma=(2+\beta)/(2r)$ and $0<\beta<2<r$ on the general Finsler manifolds. In the setting of Carnot groups or Grushin type operator, the inequality of type (1.12) was recently established by Flynn in [**15**] and [**16**], respectively. It is worthy to note that if the sectional curvature of (M,g) is bounded from above by a strict negative constant then the CKN inequalities in theorem 1.3 and corollary 1.4 can be strengthened by adding a non-negative remainder term concerning to the upper bound of sectional curvature (see §3 for more details).

In the sequel, we characterize the complete Riemannian manifolds which support the sharp CKN inequalities in corollary 1.2 (i.e. inequalities (1.8) and (1.9)). Hereafter, in order to avoid the confusions, the sharpness is understood in the sense that the CKN inequalities (of the types (1.8) or (1.9)) hold on a Riemannian manifold (M,g) with the same best constant as in the Euclidean space. From now on, we always make the assumptions that $n\geqslant 2$, $p>1$, $r>0$ and α,β,γ satisfy conditions (1.2) and (1.3). Note that both inequalities (1.6) and (1.7) can be written in the form

$$\int_{\mathbb{R}^n}\frac{|f|^r}{|x|^{\gamma r}}\,\mathrm{d}x\leqslant\frac{r}{(n-\gamma r)}\left(\int_{\mathbb{R}^n}\frac{|\nabla f|^p}{|x|^{\alpha p}}\,\mathrm{d}x\right)^{1/p}\left(\int_{\operatorname{supp}f}\frac{|f|^{p(r-1)/(p-1)}}{|x|^{\beta}}\,\mathrm{d}x\right)^{(p-1)/p}.$$

Let (M,g) be a n-dimensional complete Riemannian manifold, $\mathrm{d}V_g$ denote its canonical volume element, and $d_P(x)=d(x,P)$ be the geodesic distance from a point $x\in M$ to a fixed point $P\in M$. For a fixed point $P\in M$, we consider the CKN inequalities on (M,g) at P (shortly, $(\mathbf{CKN})_P$) of the form: for all $f\in C_0^\infty(M)$

$$\int_M\frac{|f|^r}{d_P(x)^{\gamma r}}\,\mathrm{d}V_g\leqslant\frac{r}{(n-\gamma r)}\left(\int_M\frac{|\nabla_g f|^p}{d_P(x)^{\alpha p}}\,\mathrm{d}V_g\right)^{1/p}\left(\int_{\operatorname{supp}f}\frac{|f|^{p(r-1)/(p-1)}}{d_P(x)^{\beta}}\,\mathrm{d}V_g\right)^{(p-1)/p}. \tag{$(\mathbf{CKN})_P$}$$

corollary 1.4 says that $(\mathbf{CKN})_P$ holds true on the n-dimensional Cartan–Hadamard manifolds (M,g). Our next result characterizes the attainability of the sharp constant in $(\mathbf{CKN})_P$.

THEOREM 1.5. *Given $n\geqslant 2$, $r\geqslant p>1$ and $\alpha,\beta,\gamma\in\mathbb{R}$ satisfying conditions (1.2) and (1.3). Suppose an extra assumption that $n-\beta+(1+\alpha-\beta/p)((p(r-1))/(p-r))<0$ if $1<p<r$, or $1+\alpha-\beta/p>0$ if $r=p$. Let (M,g) be an n-dimensional Cartan–Hadamard manifold. Then the following statements are equivalent:*

(a) *$r/(n-\gamma r)$ is achieved by an extremal which is not identically zero in $(\boldsymbol{CKN})_P$ for some $P\in M$.*
(b) *$r/(n-\gamma r)$ is achieved by an extremal which is not identically zero in $(\boldsymbol{CKN})_P$ for all $P\in M$.*
(c) *M is isometric to $\mathbb{R}^n$.*

The case $p=2$ in theorem 1.5 was proved by Kristály (see, e.g. [**24**, theorems 1.1 and 1.3]) under an assumption that the extremal is positive in M. In our theorem, this condition is unnecessary. Theorem 1.5 gives a non-positively curved counterpart of the rigidity result of Xia (see [**42**, theorem 1.3]) which asserts that a complete Riemannian manifold (M,g) with non-negative Ricci curvature supporting the CKN

https://doi.org/10.1017/prm.2020.100 Published online by Cambridge University Press

inequality (1.6) with $r > p > 1$ and α, β, γ satisfying conditions (1.2), (1.3) and $n - \beta + (1 + \alpha - \beta/p)((p(r-1))/(p-r)) < 0$ must be isometric to $\mathbb{R}^n$. We refer readers to [**3**, **5**, **8**, **14**, **20**, **23**–**27**, **29**, **40**–**42**]) for another results in this subject. We next prove such a rigid result in the case $p = r$. It contains a recent rigidity result of Kristály [**24**, theorem 1.2] for the sharp Heisenberg–Pauli–Weyl principle (i.e. the case $p = r = 2$, $\alpha = 0$ and $\beta = -2$ of (1.8)) as a special case.

THEOREM 1.6. *Given $n \geqslant 2$, $p > 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$ such that conditions (1.2) and (1.3) hold true with $r = p$. Suppose, in addition, that $1 + \alpha - \beta/p > 0$. Let (M, g) be a n-dimensional complete Riemannian manifold with non-negative Ricci curvature. Then the following statements are equivalent:*

(a) $(\boldsymbol{CKN})_P$ *holds for some $P \in M$.*
(b) $(\boldsymbol{CKN})_P$ *holds for all $P \in M$.*
(c) *M is isometric to $\mathbb{R}^n$.*

We next consider the case $0 < r < p$, $r \neq 1$. As seen before (see theorem 1.1 and corollary 1.2), the extremals of (1.6) and (1.7) in the Euclidean space with $0 < r < p$, $r \neq 1$ are compactly supported functions. This is very different with the case $r \geqslant p > 1$ in which the extremals never vanish. Consequently, different with the result in theorem 1.5 in which a global result is proved, the attainability of the sharp constants in $(\mathbf{CKN})_P$ only characterizes locally the Riemannian manifold (M, g) around the point $P \in M$ as stated in the following theorem.

THEOREM 1.7. *Given $n \geqslant 2$, $p > 1$, $0 < r < p, r \neq 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$ satisfying conditions (1.2) and (1.3). Suppose that one of the following extra assumptions holds: $1 + \alpha - (\beta/p) \geqslant 0$ or $1 + \alpha - \beta/p < 0$ and $n - \beta + (1 + \alpha - \beta/p)((p(r-1))/(p-r)) > 0$. Let (M, g) be an n-dimensional Cartan–Hadamard manifold, and $P \in M$. Then the following statements are equivalent:*

(a) *$r/(n - \gamma r)$ is achieved by an extremal which is not identically zero in* $(\boldsymbol{CKN})_P$.
(b) *There exists $r_P > 0$ such that the geodesic ball $B(P, r_P)$ is isometric to $B_{r_P}(0)$, here $B_r(0)$ denotes the ball of radius $r > 0$ in $\mathbb{R}^n$ with centre at the origin.*

We next present a non-negatively curved counterpart of theorem 1.7, that is, an analogue of theorem 1.6 and theorem 1.3 of Xia [**42**] in the case $r \geqslant p > 1$. We will see that in the non-negatively curved case, the situation is even more rigid than in theorem 1.7.

THEOREM 1.8. *Given $n \geqslant 2$, $p > 1$, $0 < r < p$, $r \neq 1$ and $\alpha, \beta, \gamma \in \mathbb{R}$ satisfying conditions (1.2) and (1.3). Suppose that one of the following extra assumptions holds: $1 + \alpha - \beta/p \geqslant 0$ or $1 + \alpha - \beta/p < 0$ and $n - \beta + (1 + \alpha - \beta/p)((p(r-1))/(p-r)) > 0$. Let (M, g) be an n-dimensional complete Riemannian manifold with non-negative Ricci curvature. Then the following statements are equivalent:*

(a) $(\boldsymbol{CKN})_P$ *holds for some $P \in M$.*
(b) $(\boldsymbol{CKN})_P$ *holds for all $P \in M$.*
(c) *M is isometric to $\mathbb{R}^n$.*

The rest of this paper is organized as follows. In §2, we recall some notion and results from Riemannian geometry which are used throughout in our proofs. In §3, we first prove the sharp CKN inequalities in theorem 1.1 in the Euclidean spaces, and then extend them to the Cartan–Hadamard manifolds (i.e. prove theorem 1.3). We also prove in this section the quantitative CKN inequalities by adding the nonnegative remainder terms concerning to the upper bound of the sectional curvature of Riemannian manifolds. In §4, we prove the rigidity results for the Cartan–Hadamard manifolds whenever the inequality $(\mathbf{CKN})_P$ is attained, that is, we prove theorems 1.5 and 1.7. In §5, we prove theorems 1.6 and 1.8 on the rigidity results for the complete Riemannian manifolds with non-negative Ricci curvature which support the sharp CKN inequalities.

2. Preliminaries

In this section, we list some basic properties on Riemannian manifolds, especially the properties of the Cartan–Hadamard manifolds and the complete Riemannian manifolds with non-negative Ricci curvature. Let (M,g) be an n-dimensional complete Riemannian manifolds and let d be the geodesic distance associated to the Riemannian metric g on M. For each $P \in M$ and $\rho > 0$, let $B(P,\rho) = \{x \in M : d(x,P) < \rho\}$ denote the open geodesic ball with centre $P \in M$ and radius $\rho > 0$. Let $\mathrm{d}V_g$ denote the canonical volume element on (M,g), the volume of a bounded open set $\Omega \subset M$ is given by

$$\mathrm{Vol}_g(\Omega) = \int_\Omega \mathrm{d}V_g.$$

In general, we have for any $P \in M$ that

$$\lim_{\rho \to 0^+} \frac{\mathrm{Vol}_g(B(P,\rho))}{\omega_n \rho^n} = 1 \tag{2.1}$$

where ω_n denotes the volume of the unit ball in $\mathbb{R}^n$.

If $\{x^i\}_{i=1}^n$ is a local coordinate system, then we can write

$$g = \sum_{i,j=1}^n g_{ij}\, \mathrm{d}x^i \mathrm{d}x^j.$$

In such a local coordinate system, the Laplace–Beltrami operator Δ_g with respect to the metric $\mathrm{d}s^2$ has the form

$$\Delta_g = \sum_{i,j=1}^n \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_i}\left(\sqrt{|g|}\, g^{ij} \frac{\partial}{\partial x^j}\right),$$

where $|g| = \det(g_{ij})$ and $(g^{ij}) = (g_{ij})^{-1}$. Let us denote by ∇_g the corresponding gradient. Then

$$\langle \nabla_g u, \nabla_g v\rangle = \sum_{i,j=1}^n g^{ij} \frac{\partial u}{\partial x^i} \frac{\partial v}{\partial x^j}.$$

For simplicity, we shall use the notation $|\alpha| = \sqrt{\langle \alpha, \alpha\rangle}$ for any 1-form α.

Let K_M be the sectional curvature on M. A Riemannian manifold (M, g) is called a Cartan–Hadamard manifold if it is complete, simply connected and has nonnegative sectional curvature (i.e. $K_M \leqslant 0$ along each plane section at each point of M).

If (M, g) is a Cartan–Hadamard manifold, then for each point $P \in M$, M contains no points conjugate to P, and the exponential map $\mathrm{Exp}_P : T_P M \to M$ is a diffeomorphism, where $T_P M$ is the tangent space to M at P (see, e.g. [**21**, chapter I]). Fix a point $P \in M$ and denote $d_P(x) = d(x, P)$ for all $x \in M$. Note that $d_P(x)$ is smooth on $M \setminus \{P\}$ and satisfies

$$|\nabla_g d_P(x)| = \langle \nabla_g d_P(x), \nabla_g d_P(x)\rangle^{1/2} = 1, \qquad x \in M \setminus \{P\}.$$

Moreover, since Exp_P is a diffeomorphism, then the function

$$d_P(x)^2 = \|\mathrm{Exp}_P^{-1}(x)\|^2 \in C^\infty(M).$$

The radial derivation $\partial_\rho = \partial/\partial\rho$ is defined for any function f on M by

$$\partial_\rho f(x) = \frac{d(f \circ \mathrm{Exp}_P)}{dr}(\mathrm{Exp}_P^{-1}(x)),$$

where $\mathrm{d}/\mathrm{d}r$ denotes the radial derivation on $T_P M$, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}r} F(u) = \frac{\langle u, \nabla F(u)\rangle}{|u|}, \qquad u \in T_P M \setminus \{0\}.$$

Let (M, g) be a complete Riemannian manifold. We introduce the density function $J(u, t)$ of the volume form in normal coordinates as follows (see, e.g. [**17**, pp. 166–167]). Choose an orthonormal basis $\{u, e_2, \ldots, e_n\}$ on $T_P M$ and let $c(t) = \mathrm{Exp}_P(tu)$ be a geodesic curve. The Jacobian fields $\{Y_i(t)\}_{i=2}^n$ satisfy $Y_i(0) = 0$, $Y_i'(0) = e_i$, so that the density function can be given by

$$J(u, t) = t^{1-n}\sqrt{\det(\langle Y_i(t), Y_j(t)\rangle)}, \quad t > 0.$$

We note that $J(u, t)$ does not depend on $\{e_2, \ldots, e_n\}$ and $J(u, t) \in C^\infty(T_P M \setminus \{0\})$ by the definition of $J(u, t)$. Moreover, if we set $J(u, 0) \equiv 1$ then $J(u, t) \in C(T_P M)$ and has the following asymptotic expansion

$$J(u, t) = 1 + O(t^2) \tag{2.2}$$

as $t \to 0$ since $Y_i(t)$ has the asymptotic expansion (see, e.g. [**17**, p. 169])

$$Y_i(t) = te_i - \frac{t^3}{6} R(c'(t), e_i)c'(t) + o(t^3),$$

as $t \to 0$, where $R(\cdot, \cdot)$ is the curvature tensor on M.

From the definition of $J(u, t)$, we have the following polar coordinate formula on M

$$\int_M f \,\mathrm{d}V = \int_{S^{n-1}} \int_0^{\rho(u)} f(\mathrm{Exp}_P(tu)) J(u, t) t^{n-1} \,\mathrm{d}t \,\mathrm{d}u, \tag{2.3}$$

where $\mathrm{d}u$ denotes the canonical measure of the unit sphere of $T_P M$ and $\rho(u)$ denotes the distance to the cut-locus in the direction u (see [**17**, §2.C.7] for the definition of cut-locus). Moreover, the Laplacian of the distance function $d_P(x)$ has the following expansion via the function $J(u,t)$ (see, e.g. [17, §4.B.2])

$$\Delta_g d_P(x) = \frac{n-1}{d_P(x)} + \frac{J'(u_x, d_P(x))}{J(u_x, d_P(x))}, \quad \rho > 0,$$

for any point $x \neq p$ which is not on the cut-locus of P, where u_x is the unique direction in $S^{n-1} \subset T_P M$ such that $x = \mathrm{Exp}_P(d_P(x) u_x)$ and $J'(u,t) = \partial J(u,t)/\partial t$ with $t < \rho(u)$. Therefore, for any radial function $f(d_P)$ on M, we have

$$\Delta_g f(d_P(x)) = f''(d_P(x)) + \left(\frac{n-1}{d_P(x)} + \frac{J'(u_x, d_P(x))}{J(u_x, d_P(x))}\right) f'(d_P(x)),$$

for any point $x \neq P$ which is not on the cut-locus of P. Note that if the sectional curvature K_M is constant then $J(u,t)$ depends only on t. We shall denote by $J_b(t)$ the corresponding density function if $K_M \equiv -b$ for some $b \geqslant 0$. Hence, we have

$$J_b(t) = \begin{cases} 1 & \text{if } b = 0 \\ \left(\dfrac{\sinh(\sqrt{b}t)}{\sqrt{b}t}\right)^{n-1} & \text{if } b > 0. \end{cases}$$

For $b \geqslant 0$, we consider the function $\mathbf{ct}_b : (0,\infty) \to \mathbb{R}$ defined by

$$\mathbf{ct}_b(t) = \begin{cases} \dfrac{1}{t} & \text{if } b = 0 \\ \sqrt{b}\coth(\sqrt{b}t) & \text{if } b > 0, \end{cases}$$

and the function $\mathbf{D}_b : [0,\infty) \to \mathbb{R}$ defined by

$$\mathbf{D}_b(t) = \begin{cases} 0 & \text{if } t = 0 \\ t\mathbf{ct}_b(t) - 1 & \text{if } t > 0. \end{cases}$$

Clearly, we have $\mathbf{D}_b \geqslant 0$.

In our proofs below, we will need the following Bishop–Gunther comparison theorem (see, e.g. [17, p. 172] for its proof) which says that if the sectional curvature K_M on M satisfies $K_M \leqslant -b$ for some $b \geqslant 0$ then

$$\frac{J'(u,t)}{J(u,t)} \geqslant \frac{J_b'(t)}{J_b(t)} = \frac{n-1}{t}\mathbf{D}_b(t), \qquad t > 0. \tag{2.4}$$

In particular, the function $t \to J(u,t)$ is non-decreasing for any $u \in S^{n-1}$ which implies that the function $\rho \to \mathrm{Vol}_g(B(x,\rho))/\rho^n$ is non-decreasing. Combining this together with the limit (2.1), we obtain

$$\mathrm{Vol}_g(B(x,\rho)) \geqslant \omega_n \rho^n, \quad \forall x \in M,\ \rho > 0. \tag{2.5}$$

Furthermore, if the equality holds in (2.5) then $B(x,\rho)$ is isometric to $B_\rho(0)$ (see, e.g. [7, theorem III.4.2]).

If (M,g) has non-negative Ricci curvature, then the function $\rho \to \mathrm{Vol}_g(B(x,\rho))/\rho^n$ is non-increasing. Combining this together with (2.1), we obtain

$$\mathrm{Vol}_g(B(x,\rho)) \leqslant \omega_n \rho^n, \quad \forall x \in M,\ \rho > 0. \tag{2.6}$$

Furthermore, if the equality holds in (2.6) then $B(x,\rho)$ is isometric to $B_\rho(0)$ (see, e.g. [**7**, theorem III.4.4]).

3. The sharp CKN inequalities: proof of theorems 1.1 and 1.3

This section is devoted to prove the CKN inequalities in Cartan–Hadamard manifolds. In fact, we will prove a quantitative version of the CKN inequalities by adding the non-negative remainder terms concerning to the upper bound of the sectional curvature. Let (M,g) be an n-dimensional Cartan–Hadamard manifold with $n \geqslant 2$ and $K_M \leqslant -b$ for some $b \geqslant 0$. Let $P \in M$ be a fixed point, for $p > 1$ and $\xi, \eta \in T_P M$, we denote

$$R_p(\xi,\eta) = \frac{1}{p}|\eta|^p + \frac{p-1}{p}|\xi|^p - |\xi|^{p-2}\langle \xi, \eta\rangle.$$

By the convexity of $\xi \to |\xi|^p$ we see that $R_p(\xi,\eta) \geqslant 0$ with equality if and only if $\xi = \eta$. Furthermore, we can see that

$$R_p(\xi,\eta) = (p-1)\int_0^1 |t\xi + (1-t)\eta|^{p-2} t\,\mathrm{d}t |\xi - \eta|^2.$$

For $p > 1$, we always use p' to denote the conjugate exponent of p, i.e. $p' = p/(p-1)$. Let $g = (g_1,\ldots,g_m)$, $m \geqslant 1$ be such that each function g_i is in $L^{p'}(M)$, we define the new function $\mathcal{D}_p(g)$ on M by

$$\mathcal{D}_p(g)(x) = \begin{cases} 0 & \text{if } g(x) = 0 \\ \dfrac{g(x)}{|g(x)|}|g(x)|^{1/(p-1)} & \text{if } g(x) \neq 0 \end{cases}$$

Then $|\mathcal{D}_p(g)| \in L^p(M)$. With these notation, we have the following equality which improves Hölder inequality: let $f = (f_1,\ldots,f_m) \in L^p(M)$ and $g = (g_1,\ldots,g_m) \in L^{p'}(M)$ be non-identically zero functions, then it holds

$$\int_M f\cdot g\,\mathrm{d}V_g = \|f\|_p \|g\|_{p'}\left(1 - \int_M R_p\left(\frac{f}{\|f\|_p}, \frac{\mathcal{D}_p(g)}{\|g\|_{p'}^{1/(p-1)}}\right)\mathrm{d}V_g\right), \tag{3.1}$$

here we use the notation $\|f\|_p = (\int_M |f|^p \mathrm{d}V_g)^{1/p}$ and for a subset $\Omega \subset M$, we shall denote $\|f\|_{p,\Omega} = (\int_\Omega |f|^p \mathrm{d}V_g)^{1/p}$.

We now can state our first main result in this section as follows.

THEOREM 3.1. *Suppose $n \geqslant 2$ and $p, q, r, \alpha, \beta, \gamma$ satisfy conditions (1.2) and (1.3). Let (M, g) be an n-dimensional Cartan–Hadamard manifold. Then the following equality holds true for any function $f \in C_0^\infty(M)$*

$$\begin{aligned}\int_M \frac{|f|^r}{d_P^{\gamma r}}\, \mathrm{d}V_g &= \frac{r}{n-\gamma r}\left\|\frac{\partial_\rho f}{d_P^\alpha}\right\|_{p,suppf}\left\|\frac{|f|^{r-1}}{d_P^{\beta/p'}}\right\|_{p',suppf}\\ &\quad - \frac{r}{n-\gamma r}\left\|\frac{\partial_\rho f}{d_P^\alpha}\right\|_{p,suppf}\left\|\frac{|f|^{r-1}}{d_P^{\beta/p'}}\right\|_{p',suppf}\\ &\quad \times \int_{suppf} R_P\left(\frac{d_P^{-\alpha}\partial_\rho f}{\|d_P^{-\alpha}\partial_\rho f\|_{p,suppf}}, \frac{\mathcal{D}_p(-d_P^{-(\beta/p')} f|f|^{r-2})}{\|d_P^{-(\beta/p')}|f|^{r-1}\|_{p',suppf}^{1/(p-1)}}\right)\mathrm{d}V_g\\ &\quad - \frac{1}{n-\gamma r}\int_{suppf} \frac{|f|^r}{d_P(x)^{\gamma r}} \frac{d_P(x) J'(u_x, d_P(x))}{J(u_x, d_P(x))}\,\mathrm{d}V_g, \qquad (3.2)\end{aligned}$$

where u_x denotes the unique unit vector in $T_P M$ such that $x = \mathrm{Exp}_P(d_P(x) u_x)$.

We note that R_p is always nonnegative. Moreover, $J' \geqslant 0$ since $K_M \leqslant 0$ and (2.4). Hence, inequality (3.2) implies the CKN inequalities in theorems 1.1 and 1.3 because

$$\int_{\mathrm{supp} f} \frac{|\partial_\rho f|^p}{d_P^{\alpha p}}\, \mathrm{d}V_g = \int_M \frac{|\partial_\rho f|^p}{d_P^{\alpha p}}\, \mathrm{d}V_g,$$

and if $r > 1$

$$\int_{\mathrm{supp} f} \frac{|f|^{p(r-1)/(p-1)}}{d_P(x)^\beta}\, \mathrm{d}V_g = \int_M \frac{|f|^{p(r-1)/(p-1)}}{d_P(x)^\beta}\, \mathrm{d}V_g.$$

Especially, if $K_M \leqslant -b$ for some $b \geqslant 0$, we obtain from (3.2) and Gauss lemma, the following quantitative CKN inequalities

COROLLARY 3.2. *Suppose the assumptions in theorem 3.1 and $K_M \leqslant -b$ for some $b \geqslant 0$. Then the following inequalities holds for any function $f \in C_0^\infty(M)$,*

$$\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\left(1 + \frac{n-1}{n-\gamma r}\boldsymbol{D}_b(d_P(x))\right)\mathrm{d}V_g \leqslant \frac{r}{n-\gamma r}\left\|\frac{\partial_\rho f}{d_P^\alpha}\right\|_p \left\|\frac{|f|^{r-1}}{d_P^{\beta/p'}}\right\|_{p',suppf}, \qquad (3.3)$$

and

$$\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\left(1 + \frac{n-1}{n-\gamma r}\boldsymbol{D}_b(d_P(x))\right)\mathrm{d}V_g \leqslant \frac{r}{n-\gamma r}\left\|\frac{\nabla_g f}{d_P^\alpha}\right\|_p \left\|\frac{|f|^{r-1}}{d_P^{\beta/p'}}\right\|_{p',suppf}, \qquad (3.4)$$

The case $p = r$, $\beta = p + \delta$ and $\alpha = \delta/p$ with $\delta < n - p$, corollary 3.2 implies the following quantitative weighted Hardy inequalities on Cartan–Hadamard manifolds which are recently prove by the author [**31**]

$$\int_M \frac{|f|^p}{d_P^{p+\delta}} \left(1 + \frac{p(n-1)}{n-p-\delta} \mathbf{D}_b(d_P(x))\right) \mathrm{d}V_g \leqslant \left(\frac{p}{n-p-\delta}\right)^p \int_M \frac{|\partial_\rho f|^p}{d_P^{\delta}} \mathrm{d}V_g,$$

and

$$\int_M \frac{|f|^p}{d_P^{p+\delta}} \left(1 + \frac{p(n-1)}{n-p-\delta} \mathbf{D}_b(d_P(x))\right) \mathrm{d}V_g \leqslant \left(\frac{p}{n-p-\delta}\right)^p \int_M \frac{|\nabla_g f|^p}{d_P^{\delta}} \mathrm{d}V_g. \quad (3.5)$$

These inequalities improve the weighted Hardy inequalities on Cartan–Hadamard manifolds due to Yang, Su and Kong [**43**]. The case $p = 2$ and $\delta = 0$, inequality (3.5) was proved by Kristály in [**24**]. We refer readers to [**31**] for more results about the critical Hardy, and Rellich type inequalities on Cartan–Hadamard inequality. The last comment in the case that (M, g) has constant sectional curvature, i.e. $K_M = -b$ for some $b \geqslant 0$ is that the extremal functions for (3.3) and (3.4) exist if one of conditions (i)–(v) in theorem 1.1 holds. Moreover, a family of extremal is given by the same family of extremal functions in the corresponding Euclidean case with $d_P(x)$ instead of $|x|$. The reason is that, in this case, we always have

$$R_P \left(\frac{d_P^{-\alpha} \partial_\rho f}{\|d_P^{-\alpha} \partial_\rho f\|_{p,\mathrm{supp} f}}, \frac{\mathcal{D}_p(-d_P^{-(\beta/p')} f|f|^{r-2})}{\|d_P^{-(\beta/p')} |f|^{r-1}\|_{p',\mathrm{supp} f}^{1/(p-1)}} \right) = 0$$

if f has such form, and $\rho J'(u, \rho)/J(u, \rho) = (n-1)\mathbf{D}_b(\rho)$. Therefore equality holds true in (3.2). Furthermore, the proof of theorems 1.5 and 1.7 can be applied to prove a rigidity results for Cartan–Hadamard manifolds (M, g) with $K_M \leqslant -b$ for some $b \geqslant 0$ such that extremal for inequality (3.4) exists. Such a manifold should be isometric to a manifold of constant sectional curvature $-b$. Evidently, the case $b = 0$ is considered in theorems 1.5 and 1.7.

Let us prove theorem 3.1.

Proof of theorem 3.1. The proof is simple by using integration by parts. Indeed, let $f \in \mathbb{C}_0^\infty(M)$, by abusing notation we still denote by $\mathrm{supp} f$ for the preimage of support of f in $T_P M$ and $f(tu)$ for $f(\mathrm{Exp}_P(tu))$. Using polar coordinate (2.3), we have

$$\begin{aligned} \int_M \frac{|f|^r}{d_P(x)^{\gamma r}} \mathrm{d}V_g &= \int_{S^{n-1}} \int_{\mathrm{supp} f \cap [0,\infty)} |f(tu)|^r t^{n-\gamma r-1} J(u,t) \,\mathrm{d}t \,\mathrm{d}u \\ &= \frac{1}{n-\gamma r} \int_{S^{n-1}} \int_{\mathrm{supp} f \cap [0,\infty)} |f(tu)|^r (t^{n-\gamma r})' J(u,t) \,\mathrm{d}t \,\mathrm{d}u. \end{aligned}$$

Using integration by parts and the assumption $n-\gamma r>0$, we get

$$\begin{aligned}
&\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\,\mathrm{d}V_g\\
&\quad= -\frac{r}{n-\gamma r}\int_{S^{n-1}}\int_{\mathrm{supp}\,f\cap[0,\infty)} |f(tu)|^{r-2} f(tu)\partial_\rho f(tu) t^{n-\gamma r} J(u,t)\,\mathrm{d}t\,\mathrm{d}u\\
&\quad= -\frac{1}{n-\gamma r}\int_{S^{n-1}}\int_{\mathrm{supp}\,f\cap[0,\infty)} |f(tu)|^{r} t^{n-\gamma r} J'(u,t)\,\mathrm{d}t\,\mathrm{d}u.
\end{aligned}$$

Using again polar coordinate (2.3) and condition (1.3), we arrive

$$\begin{aligned}
\int_M \frac{|f|^r}{d_P(x)^{\gamma r}}\,\mathrm{d}V_g = &-\frac{r}{n-\gamma r}\int_{\mathrm{supp} f} \frac{|f|^{r-2}f}{d_P^{\beta/p'}}\frac{\partial_\rho f}{d_P^{\alpha}}\,\mathrm{d}V_g\\
&-\frac{1}{n-\gamma r}\int_{\mathrm{supp} f}\frac{|f|^r}{d_P^{\gamma r}}\frac{d_P(x)J'(u_x,d_P(x))}{J(u_x,d_P(x))}\,\mathrm{d}V_g.
\end{aligned}$$

Now, using (3.1), we obtain our desired equality (3.2). □

We are now ready to prove theorems 1.1 and 1.3.

Proof of theorem 1.1. The inequalities are trivial by remarks after theorem 3.1. The sharpness of constant $r/(n-\gamma r)$ is immediately checked by functions given in theorem 1.1 corresponding to conditions (i)–(v). Indeed, in these cases, we have

$$R_P\left(\frac{d_P^{-\alpha}\partial_\rho f}{\|d_P^{-\alpha}\partial_\rho f\|_{p,\mathrm{supp} f}}, \frac{\mathcal{D}_p(-d_P^{-(\beta/p')}f|f|^{r-2})}{\|d_P^{-(\beta/p')}|f|^{r-1}\|_{p',\mathrm{supp} f}^{1/(p-1)}}\right)=0,$$

on supp$f$ and $J'\equiv 0$. This finishes the proof of theorem 1.1. □

Proof of theorem 1.3. The inequalities are trivial by remarks after theorem 3.1. Let us verify the sharpness of the constant $r/(n-\gamma r)$. We know from theorem 1.1 that if one of conditions (i)–(v) in theorem 1.1 holds true, then

$$\frac{r}{n-\gamma r} = \sup_{f\in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n}|x|^{-\gamma r}|f|^r\,\mathrm{d}x}{\left(\int_{\mathbb{R}^n}|x|^{-\alpha p}|\partial_r f|^p dx\right)^{1/p}\left(\int_{\mathrm{supp} f}|x|^{-\beta}|f|^{p'(r-1)}\,\mathrm{d}x\right)^{1/p'}}.$$

Furthermore, we can assume that the supremum is taken on non-negative radial functions. Indeed, from theorem 1.1, we see that the extremal of CKN inequalities contain non-negative radial functions by taking $\lambda\equiv$ const, and hence we can approximate these functions by non-negative radial functions in $C_0^\infty(\mathbb{R}^n)$. For any

https://doi.org/10.1017/prm.2020.100 Published online by Cambridge University Press

$\epsilon > 0$, we can chose a non-negative radial function $f_\epsilon \in C_0^\infty$ such that

$$\frac{r}{n-\gamma r} - \epsilon \leqslant \sup_{f_\epsilon \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |x|^{-\gamma r} |f_\epsilon|^r \,\mathrm{d}x}{\left(\int_{\mathbb{R}^n} |x|^{-\alpha p} |\partial_r f_\epsilon|^p dx\right)^{1/p} \left(\int_{\mathrm{supp} f_\epsilon} |x|^{-\beta} |f_\epsilon|^{p'(r-1)} \,\mathrm{d}x\right)^{1/p'}}.$$

For $\delta > 0$, denote $f_{\epsilon,\delta}(x) = f_\epsilon(x/\delta)$ then $\mathrm{supp} f_{\epsilon,\delta} = \delta \mathrm{supp} f_\epsilon$. The scaling invariant of CKN inequalities implies that

$$\frac{r}{n-\gamma r} - \epsilon \leqslant \sup_{f_\epsilon \in C_0^\infty(\mathbb{R}^n)} \frac{\int_{\mathbb{R}^n} |x|^{-\gamma r} |f_{\epsilon,\delta}|^r \,\mathrm{d}x}{\left(\int_{\mathbb{R}^n} |x|^{-\alpha p} |\partial_r f_{\epsilon,\delta}|^p \,\mathrm{d}x\right)^{1/p} \left(\int_{\mathrm{supp} f_{\epsilon,\delta}} |x|^{-\beta} |f_\epsilon|^{p'(r-1)} \,\mathrm{d}x\right)^{1/p'}} \tag{3.6}$$

for any $\delta > 0$.

Suppose that $f_\epsilon(x) = \varphi(|x|)$ for some function φ with $\varphi \equiv 0$ on $[a,\infty)$ for some $a > 0$. Define the functions F_δ on M by $F(x) = \varphi(d_P(x)/\delta)$. Evidently, $\mathrm{supp} F_\delta \subset B(P, a\delta)$. Using polar coordinate (2.3), we have

$$\begin{aligned}
\int_M \frac{|F_\delta|^r}{d_P^{\gamma r}} \,\mathrm{d}V_g &= \int_{S^{n-1}} \int_{\delta \mathrm{supp}\varphi} \varphi(t/\delta)^r t^{n-\gamma r-1} J(u,t) \,\mathrm{d}t \,\mathrm{d}u \\
&= (1+O(\delta^2)) \int_{S^{n-1}} \int_{\delta \mathrm{supp}\varphi} \varphi(t/\delta)^r t^{n-\gamma r-1} \,\mathrm{d}t \,\mathrm{d}u \\
&= (1+O(\delta^2)) \int_{\mathbb{R}^n} \frac{|f_{\epsilon,\delta}|^r}{|x|^{\gamma r}} \,\mathrm{d}x,
\end{aligned}$$

here we use (2.2). Similarly, we get

$$\int_M \frac{|\partial_\rho F_\delta|^p}{d_P^{\alpha p}} \,\mathrm{d}V_g = (1+O(\delta^2)) \int_{\mathbb{R}^n} \frac{|\partial_r f_{\epsilon,\delta}|^p}{|x|^{\alpha p}} \,\mathrm{d}x,$$

and

$$\int_{\mathrm{supp} F_\delta} \frac{|F_\delta|^{p(r-1)/(p-1)}}{d_P^\beta} \mathrm{d}V_g = (1+O(\delta^2)) \int_{\mathrm{supp} f_{\epsilon,\delta}} \frac{|f_{\epsilon,\delta}|^{p(r-1)/(p-1)}}{|x|^\beta} \,\mathrm{d}x,$$

here we use the fact f is radial. Combining these three equalities together with (3.6), we obtain

$$\liminf_{\delta \to 0} \frac{\int_M \frac{|F_\delta|^r}{d_P^{\gamma r}} \,\mathrm{d}V_g}{\left(\int_M \frac{|\partial_\rho F_\delta|^p}{d_P^{\alpha p}} \,\mathrm{d}V_g\right)^{1/p} \left(\int_{\mathrm{supp} F_\delta} \frac{|F_\delta|^{p(r-1)/(p-1)}}{d_P^\beta} \,\mathrm{d}V_g\right)^{(p-1)/p}} \geqslant \frac{r}{n-\gamma r} - \epsilon$$

for any $\epsilon > 0$. This implies the sharpness of $r/(n-\gamma r)$. □

4. Rigidity results on Cartan–Hadamard manifolds: proof of theorems 1.5 and 1.7

In this section, we give the proof of theorems 1.5 and 1.7. The main ingredients in our proofs are theorem 3.1, the Gauss lemma and the explicit solutions of several ordinary differential equations related to the Euler–Lagrange equations of the extremal for the CKN inequalities. We first prove theorem 1.5.

Proof of theorem 1.5. Obviously, from corollary 1.2, we always have $(c) \Rightarrow (b) \Rightarrow (a)$. It remains to prove $(a) \Rightarrow (c)$. Let f be an extremal which is not identically zero for $(\mathbf{CKN})_P$. Evidently, $|f|$ also is an extremal for $(\mathbf{CKN})_P$. Hence we can assume that f is non-negative. Moreover, by theorem 1.3 (more precisely, inequality (1.10)), we must have

$$\int_M \frac{|\partial_\rho f|^p}{d_P(x)^{\alpha p}}\,\mathrm{d}V_g \geqslant \int_M \frac{|\nabla_g f|^p}{d_P(x)^{\alpha p}}\,\mathrm{d}V_g,$$

which implies $|\partial_\rho f| = |\nabla_g f|$ by Gauss lemma. Therefore, f is radial function, that is, f depends only on d_P or $f = \varphi(d_P)$ with $\varphi : [0,\infty) \to [0,\infty)$. By (3.2), we must have

$$R_P\left(\frac{d_P^{-\alpha}\partial_\rho f}{\|d_P^{-\alpha}\partial_\rho f\|_{p,\mathrm{supp} f}}, \frac{\mathcal{D}_p(-d_P^{-(\beta/p')} f|f|^{r-2})}{\|d_P^{-(\beta/p')}|f|^{r-1}\|_{p',\mathrm{supp} f}^{1/(p-1)}}\right) = 0$$

and $J'(u_x, d_P(x)) = 0$ on supp$f$. The first condition is equivalent to

$$\partial_\rho f = -c f^{(r-1)/(p-1)} d_P^{\alpha-\beta/p}$$

on supp$f$ for some $c > 0$. Writing f as $\varphi(d_P(x))$, the previous equation is equivalent to

$$\varphi'(t) = -c\varphi(t)^{(r-1)/(p-1)} t^{\alpha-\beta/p}, \tag{4.1}$$

on $\{\varphi > 0\}$. Since f is not identically zero, then $\varphi(0) > 0$. Equation (4.1) has unique solution

$$\varphi(t) = \left(\varphi(0)^{(p-r)/(p-1)} + c\frac{r-p}{p-1}\frac{t^{1+\alpha-\beta/p}}{1+\alpha-\beta/p}\right)^{(p-1)/(p-r)}$$

if $r > p > 1$ and

$$\varphi(t) = \varphi(0)\exp\left(-\frac{c}{1+\alpha-\beta/p} t^{1+\alpha-\beta/p}\right)$$

if $r = p$. Hence supp$f = M$, and the condition $J'(u_x, d_P(x)) = 0$ on supp$f$ translates to $J'(u, t) = 0$ for any $t > 0$ and for each fixed $u \in S^{n-1}$. This implies $J(u, t) \equiv 1$ for any $t > 0$ and $u \in S^{n-1}$. Hence M is isometric to $\mathbb{R}^n$. □

Theorem 1.7 is proved by the same way.

Proof of theorem 1.7. Obviously, as in the proof of theorem 1.5, we can assume that the extremal f is non-negative and f is radial function, that is, f depends only on d_P or $f = \varphi(d_P)$ with $\varphi : [0,\infty) \to [0,\infty)$. Moreover, we must have $J'(u_x, d_P(x)) = 0$

on supp$f$ and

$$\partial_\rho f = -c|f|^{(r-p)/(p-1)} f d_P^{\alpha-\beta/p}$$

on supp$f$ for some $c > 0$. Consequently, φ satisfies

$$\varphi'(t) = -c\varphi(t)^{(r-1)/(p-1)} t^{\alpha-\beta/p} \tag{4.2}$$

on $\{\varphi > 0\}$. Hence, φ is strict increasing on $\{\varphi > 0\}$. If $0 < r < p, r \neq 1$ and $1 + \alpha - \beta/p > 0$ then (4.2) has only solutions of the form

$$\varphi(t) = \left(\lambda - c\frac{p-r}{p-1}\frac{t^{1+\alpha-\beta/p}}{1+\alpha-\beta/p}\right)_+^{(p-1)/(p-r)}, \quad \lambda > 0.$$

If $0 < r < p, r \neq 1$ and $1 + \alpha - \beta/p = 0$ then (4.2) has only solutions of the form

$$\varphi(t) = \left(\lambda - c\frac{p-r}{p-1}\ln t\right)_+^{(p-1)/(p-r)}, \quad \lambda \in \mathbb{R}.$$

If $0 < r < p, r \neq 1, 1 + \alpha - \beta/p < 0$ and $n - \beta + (1 + \alpha - \beta/p)((p(r-1))/(p-r) > 0$ then (4.2) has only solutions of the form

$$\varphi(t) = \left(-c\frac{p-r}{p-1}\frac{t^{1+\alpha-\beta/p}}{1+\alpha-\beta/p} - \lambda\right)_+^{(p-1)/(p-r)}, \qquad \lambda \in \mathbb{R}.$$

To ensure $\int_{\text{supp}f} d_P^{-\beta} f^{((p(r-1))/(p-1))}\, \mathrm{d}V_g < \infty$, we must have $\lambda > 0$.

The form of function φ above shows that f has compact support and there is $r_P > 0$ such that $\{f > 0\} = B(P, r_P)$. Therefore, the condition $J'(u_x, d_P(x)) = 0$ on supp$f$ is equivalent to $J'(u, t) = 0$ for any $t < r_P$ and $u \in S^{n-1}$. Thus $J(u, t) \equiv 1$ for any $t < r_P$ and $u \in S^{n-1}$ which implies $\text{Vol}_g(B(P, r_P)) = \omega_n r_P^n$ by polar coordinate (2.3). Hence $B(P, r_P)$ is isometric to $B_{r_P}(0)$ (see, e.g. [**7**, theorem III.4.2]). □

An immediate consequence of theorem 1.7 is that if the constant $r/(n - \gamma r)$ is attained by an extremal which is not identically zero in $(\mathbf{CKN})_P$ for a point $P \in M$, then the sectional curvature at P vanishes. Therefore, if the constant $r/(n - \gamma r)$ is attained by an extremal which is not identically zero in $(\mathbf{CKN})_P$ for any point $P \in M$, then M is flat, i.e. the sectional curvature vanishes at any point of M. This implies that the Ricci curvature of M vanishes. By Bishop–Gromov volume comparison theorem (2.6), we get $\text{Vol}_g(B(P, r)) \leqslant \omega_n r^n$ for any $P \in M$ and $r > 0$. The reversed inequality holds since M is Cartan–Hadamard manifold (by (2.5)). Hence $\text{Vol}_g(B(P, r)) = \omega_n r^n$ for any $P \in M$ and $r > 0$ which implies M is isometric to $\mathbb{R}^n$. Thus, we have proved the following result.

Corollary 4.1. *Suppose the assumptions in theorem 1.7. Then the following statements are equivalent:*

(a) $r/(n - \gamma r)$ *is achieved by an extremal which is not identically zero in* $(\boldsymbol{CKN})_P$ *for any point* $P \in M$.
(b) M *is isometric to* $\mathbb{R}^n$.

https://doi.org/10.1017/prm.2020.100 Published online by Cambridge University Press

5. Rigidity results on manifolds with non-negative Ricci curvature: proof of theorems 1.6 and 1.8

This section is devoted to prove the rigidity results in theorems 1.6 and 1.8. The main idea in our proof goes back to the study of Ledoux on the sharp Sobolev inequality [26] and then developed by several studies [3, 5, 8, 14, 23–25, 27, 29, 40–42]. The crucial ingredient is the explicit form of extremal in the Euclidean spaces. Exploiting this form of extremal, we define a new function in (M, g) which depends only on d_P and then applying the CKN inequality to obtain a differential inequality. Using this differential inequality, we obtain the equality in the Bishop–Gromov volume comparison theorem, and hence obtain the desired result. It is worthy to emphasize here that the extremal of CKN inequalities considered in theorem 1.8 are compactly supported functions. This arises several difficulties in the proof of theorem 1.8. In the knowledge of author, this is the first time such a rigidity result is established for the Sobolev type inequalities with compactly supported extremal (comparing with [3, 5, 8, 14, 23–27, 29, 40–42]).

Proof of theorem 1.6. The implications $(c) \Rightarrow (b) \Rightarrow (a)$ are trivial by corollary 1.2. It remains to verify $(a) \Rightarrow (c)$. For $\lambda > 0$, we define a function T on $(0, \infty)$ by

$$T(\lambda) = \int_{\mathbb{R}^n} \mathrm{e}^{-p\lambda |x|^{1+\alpha-\beta/p}} |x|^{-\gamma p}\, \mathrm{d}x.$$

We can check that

$$T(\lambda) = \lambda^{-\frac{n-\gamma p}{1+\alpha-\beta/p}} p^{-\frac{n-\gamma p}{1+\alpha-\beta/p}} \frac{n\omega_n}{1+\alpha-\beta/p} \Gamma\left(\frac{n-\gamma p}{1+\alpha-\beta/p}\right),$$

and hence T satisfies the equation

$$-\lambda T'(\lambda) = \frac{n-\gamma p}{1+\alpha-\beta/p} T(\lambda), \quad \lambda > 0. \tag{5.1}$$

Let $P \in M$ be fixed. Since $(\mathbf{CKN})_P$ holds, then (M, g) cannot be compact. For $\lambda > 0$, we define

$$u_\lambda(x) = \mathrm{e}^{-\lambda d_P(x)^{1+\alpha-\beta/p}}, \quad \lambda > 0.$$

By a simple approximation procedure, we can apply $(\mathbf{CKN})_P$ to function u_λ and then obtain the following inequality (note that $|\nabla d_P| = 1$)

$$\int_M \mathrm{e}^{-p\lambda d_P^{1+\alpha-\beta/p}} d_P^{-\gamma p}\, \mathrm{d}V_g \leqslant \lambda \frac{p(1+\alpha-\beta/p)}{n-\gamma p} \int_M \mathrm{e}^{-p\lambda d_P^{1+\alpha-\beta/p}} d_P^{-\beta}\, \mathrm{d}V_g, \quad \lambda > 0.$$

Define

$$F(\lambda) = \int_M \mathrm{e}^{-p\lambda d_P^{1+\alpha-\beta/p}} d_P^{-\gamma p}\, \mathrm{d}V_g.$$

Using Bishop–Gromov comparison theorem, we can easily check that $0 < F(\lambda) < \infty$ for any $\lambda > 0$ and F is differentiable on $(0, \infty)$. Moreover, we can compute that

$$F'(\lambda) = -p \int_M \mathrm{e}^{-p\lambda d_P^{1+\alpha-\beta/p}} d_P^{-\beta}\, \mathrm{d}V_g,$$

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and hence

$$-\lambda F'(\lambda) \geqslant \frac{n-\gamma p}{1+\alpha-\beta/p} F(\lambda), \quad \lambda > 0. \tag{5.2}$$

Combining (5.1) and (5.2), we get $(F/T)' \leqslant 0$ hence the function $\lambda \to F(\lambda)/T(\lambda)$ is non-increasing. In particular, for any $\lambda > 0$,

$$\frac{F(\lambda)}{T(\lambda)} \geqslant \lim_{\lambda\to\infty} \frac{F(\lambda)}{T(\lambda)}.$$

We next make an estimate of $F(\lambda)$ for $\lambda > 0$ large enough. A traditional way is to use the layer cake representation

$$F(\lambda) = \int_0^\infty \mathrm{Vol}_g \left(\{x \in M \,:\, \mathrm{e}^{-p\lambda d_P^{1+\alpha-\beta/p}} d_P^{-\gamma p} > t\} \right) \mathrm{d}t,$$

and then making the change of variable $t = \mathrm{e}^{-p\lambda s^{1+\alpha-\beta/p}} s^{-\gamma p}$. It seems that this argument does not work for $\gamma < 0$ since the function $s \to \mathrm{e}^{-p\lambda s^{1+\alpha-\beta/p}} s^{-\gamma p}$ is not decreasing monotone on $[0, \infty)$. The same situation also appeared in the proof of theorem 1.3 of Xia [**42**]. Instead of using the layer cake representation, we will use the polar coordinate formula (2.3). For $0 < \epsilon < \inf\{\rho(u) \,:\, u \in S^{n-1}\}$ we have

$$\begin{aligned}
F(\lambda) &= \int_{S^{n-1}} \int_0^{\rho(u)} \mathrm{e}^{-p\lambda t^{1+\alpha-\beta/p}} J(u,t) t^{n-\gamma p-1}\,\mathrm{d}t\,\mathrm{d}u \\
&\geqslant \int_0^{\epsilon} \int_{S^{n-1}} \mathrm{e}^{-p\lambda t^{1+\alpha-\beta/p}} J(u,t) t^{n-\gamma p-1}\,\mathrm{d}t\,\mathrm{d}u \\
&= (1+O(\epsilon)^2) n\omega_n \int_0^{\epsilon} \mathrm{e}^{-p\lambda t^{1+\alpha-\beta/p}} t^{n-\gamma p-1}\,\mathrm{d}t\,\mathrm{d}u \\
&= (1+O(\epsilon)^2) \lambda^{-\frac{n-\gamma p}{1+\alpha-\beta/p}} p^{-\frac{n-\gamma p}{1+\alpha-\beta/p}} \frac{n\omega_n}{1+\alpha-\beta/p} \int_0^{(p\lambda)^{1/(1+\alpha-\beta/p)}\epsilon} \mathrm{e}^{-t} t^{n-\gamma r-1}\,\mathrm{d}t.
\end{aligned}$$

Consequently, we obtain

$$\lim_{\lambda\to\infty} \frac{F(\lambda)}{T(\lambda)} \geqslant 1 + O(\epsilon^2).$$

Letting $\epsilon \to 0$, we get

$$\frac{F(\lambda)}{T(\lambda)} \geqslant \lim_{\lambda\to\infty} \frac{F(\lambda)}{T(\lambda)} \geqslant 1.$$

Hence $F(\lambda) \geqslant T(\lambda)$ for any $\lambda > 0$. On the contrary, for any $u \in S^{n-1}$, we have $J(u,t) \leqslant 1$ for any $t < \rho(u)$ (see, e.g. [**17**, p. 172, line 5]). Hence, using again the polar coordinate (2.3), we obtain

$$\begin{aligned} T(\lambda) \leqslant F(\lambda) &= \int_{S^{n-1}} \int_0^{\rho(u)} \mathrm{e}^{-p\lambda t^{1+\alpha-\beta/p}} t^{n-\gamma p-1} J(u,t)\,\mathrm{d}t\,\mathrm{d}u \\ &\leqslant \int_{S^{n-1}} \int_0^{\rho(u)} \mathrm{e}^{-p\lambda t^{1+\alpha-\beta/p}} t^{n-\gamma p-1}\,\mathrm{d}t\,\mathrm{d}u \\ &\leqslant \int_{S^{n-1}} \int_0^{\infty} \mathrm{e}^{-p\lambda t^{1+\alpha-\beta/p}} t^{n-\gamma p-1}\,\mathrm{d}t\,\mathrm{d}u \\ &= T(\lambda) \end{aligned}$$

which yields $F(\lambda) = T(\lambda)$ for any $\lambda > 0$. Consequently, all inequalities in the preceding estimates must be equalities. This fact implies that $\rho(u) = \infty$ for almost $u \in S^{n-1}$ and $J(u,t) \equiv 1$ for any $t < \rho(u)$. This together with the polar coordinate formula (2.3) gives $\mathrm{Vol}_g(B(P,r)) = \omega_n r^n$ for any $r > 0$. Hence, M is isometric to $\mathbb{R}^n$ by the equality condition in Bishop–Gromov volume comparison theorem (see [**7**, theorem III.4.4]). □

We continue by proving theorem 1.8. We will need the following simple result.

PROPOSITION 5.1. *Let (Ω,μ) be a measure space and $f:\Omega\to[0,\infty)$ be a measurable function. Given $q\in(0,1)\cup(1,\infty)$ and suppose that*

$$\int_{\{f<\lambda\}} (\lambda - f)^{q-1}\mathrm{d}\mu < \infty \tag{5.3}$$

for any $\lambda > 0$. Suppose, in addition, that

$$\mu(\{\lambda \leqslant f < \lambda + h\}) = \begin{cases} O(h) & \text{if } q \in (0,1) \\ O(1) & \text{if } q > 1 \end{cases} \tag{5.4}$$

for $\lambda, h > 0$. Then the function $G : \lambda \to \int_\Omega (\lambda - f)_+^q\,\mathrm{d}\mu$ is differentiable on $(0,\infty)$ and

$$G'(\lambda) = q \int_{\{f<\lambda\}} (\lambda - f)^{q-1}\,\mathrm{d}\mu.$$

Proof. Denote $A_\lambda = \{f < \lambda\}$. For $h > 0$, we have

$$\frac{G(\lambda+h) - G(\lambda)}{h} = \int_{A_\lambda} \frac{(\lambda+h-f)^q - (\lambda-f)^q}{h}\,\mathrm{d}\mu + \frac{1}{h}\int_{\{\lambda\leqslant f<\lambda+h\}} (\lambda+h-f)^q\,\mathrm{d}\mu.$$

It is easy to check that

$$(a+b)^q - a^q \leqslant C a^{q-1} b \quad a, b > 0,$$

for some constant $C > 0$ depending only on q. Combining the previous inequality with (5.3) and the Lebesgue dominated convergence theorem, we get

$$\lim_{h\to 0^+} \int_{A_\lambda} \frac{(\lambda+h-f)^q - (\lambda-f)^q}{h}\,\mathrm{d}\mu = q \int_{\{f<\lambda\}} (\lambda-f)^{q-1}\,\mathrm{d}\mu.$$

On the contrary, we have $(1/h)\int_{\{\lambda\leqslant f<\lambda+h\}}(\lambda+h-f)^q\,\mathrm{d}\mu\leqslant h^{q-1}\mu(\{\lambda\leqslant f<\lambda+h\})$. The assumption (5.4) implies

$$\lim_{h\to 0^+}\frac{1}{h}\int_{\{\lambda\leqslant f<\lambda+h\}}(\lambda+h-f)^q\,\mathrm{d}\mu=0.$$

Thus, we get

$$\lim_{h\to 0^+}\frac{G(\lambda+h)-G(\lambda)}{h}=q\int_{\{f<\lambda\}}(\lambda-f)^{q-1}\,\mathrm{d}\mu. \tag{5.5}$$

For $h<0$, we have

$$\frac{G(\lambda+h)-G(\lambda)}{h}=\int_{A_\lambda}\frac{(\lambda+h-f)_+^q-(\lambda-f)^q}{h}\,\mathrm{d}\mu.$$

We claim that

$$a^q-(a-b)_+^q\leqslant Ca^{q-1}b,\quad a,b>0.$$

for some constant $C>0$. Indeed, if $b\geqslant a/2$ then

$$a^q-(a-b)_+^q\leqslant a^q\leqslant 2a^{q-1}h.$$

If $0<b<a/2$, denote $t=b/a\in(0,1/2)$, we have

$$a^q-(a-b)^q=a^q(1-(1-t)^q)\leqslant Ca^qt=Ca^{q-1}b,$$

for some constant $C>0$, here we use $\lim_{t\to 0^+}(1-(1-t)^q)/t=q$ and $t\in(0,1/2)$. Hence our claim has been proved. Our claim together with (5.3) and the Lebesgue dominated convergence theorem, we get

$$\lim_{h\to 0^-}\int_{A_\lambda}\frac{(\lambda+h-f)^q-(\lambda-f)^q}{h}\,\mathrm{d}\mu=q\int_{\{f<\lambda\}}(\lambda-f)^{q-1}\,\mathrm{d}\mu. \tag{5.6}$$

Combining (5.5) and (5.6) finishes the proof of this proposition. □

Proof of theorem 1.8. The implications $(c)\Rightarrow(b)\Rightarrow(a)$ is trivial by corollary 1.2. It remains to prove $(a)\Rightarrow(c)$. In the sequel, we prove only for the case $0<r<p$ and $1+\alpha-\beta/p>0$. The proof in the other cases is completely similar.

For $\lambda>0$, define

$$T(\lambda)=\int_{\mathbb{R}^n}\left(\lambda-|x|^{1+\alpha-\beta/p}\right)_+^{r((p-1)/(p-r))}|x|^{-r\gamma}\,\mathrm{d}x.$$

A straightforward computation shows that

$$T(\lambda)=\lambda^{\frac{n-r\gamma}{1+\alpha-\beta/p}+\frac{r(p-1)}{p-r}}n\omega_n\frac{1}{1+\alpha-\beta/p}B\left(\frac{r(p-1)}{p-r}+1,\frac{n-\gamma r}{1+\alpha-\beta/p}\right)$$

where B denotes the usual beta function. Denote

$$\delta=\frac{n-r\gamma}{1+\alpha-\beta/p}+\frac{r(p-1)}{p-r}$$

for short, then it is evident that

$$\lambda T'(\lambda) = \delta T(\lambda), \quad \lambda > 0. \tag{5.7}$$

Let $P \in M$ be fixed. Since $(\mathbf{CKN})_P$ holds at P, then M cannot be compact. For $\lambda > 0$, define the function u_λ on M by

$$u_\lambda(x) = \left(\lambda - d_P(x)^{1+\alpha-\beta/p}\right)_+^{(p-1)/(p-r)},$$

and denote

$$F(\lambda) = \int_M \frac{u_\lambda^r}{d_P^{\gamma r}}\,\mathrm{d}V_g = \int_M \left(\lambda - d_P(x)^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} d_P(x)^{-\gamma r}\,\mathrm{d}V_g.$$

By a simple approximation argument, we can apply $(\mathbf{CKN})_P$ for this function u_λ and obtain the following inequality for any $\lambda > 0$,

$$F(\lambda) \leqslant \frac{r}{n-\gamma r}\left(1+\alpha-\frac{\beta}{p}\right)\frac{p-1}{p-r}\int_{\{d_P<\lambda^{\frac{1}{1+\alpha-\beta/p}}\}}\left(\lambda - d_P^{1+\alpha-\beta/p}\right)^{p(r-1)/(p-r)} d_P^{-\beta}\,\mathrm{d}V_g, \tag{5.8}$$

here we use $|\nabla d_P| = 1$. Since $J(u,t) \leqslant 1$ for any $t < \rho(u)$ then the assumptions in proposition 5.1 satisfy for $f = d_P^{1+\alpha-\beta/p}$, $(\Omega,\mu) = (M, V_g)$ and $q = r(p-1)/(p-r) \in (0,1)\cup(1,\infty)$. By proposition 5.1, the function F is differentiable, and F' is given by

$$F'(\lambda) = \frac{r(p-1)}{p-r}\int_{\{d_P<\lambda^{\frac{1}{1+\alpha-\beta/p}}\}}\left(\lambda - d_P^{1+\alpha-\beta/p}\right)^{p(r-1)/(p-r)} d_P^{-\gamma r}\,\mathrm{d}V_g.$$

Recall that $\gamma r = 1+\alpha-\beta/p+\beta$. By an easy computation, we get

$$\begin{aligned}
\lambda F'(\lambda) &= \frac{r(p-1)}{p-r}\int_{\{d_P<\lambda^{\frac{1}{1+\alpha-\beta/p}}\}}\lambda\left(\lambda - d_P^{1+\alpha-\beta/p}\right)^{p(r-1)/(p-r)} d_P^{-\gamma r}\,\mathrm{d}V_g\\
&= \frac{r(p-1)}{p-r}\int_M\left(\lambda - d_P^{1+\alpha-\beta/p}\right)^{r(p-1)/(p-r)} d_P^{-\gamma r}\,\mathrm{d}V_g\\
&\quad + \frac{r(p-1)}{p-r}\int_{\{d_P<\lambda^{\frac{1}{1+\alpha-\beta/p}}\}}\left(\lambda - d_P^{1+\alpha-\beta/p}\right)^{p(r-1)/(p-r)} d_P^{-\beta}\,\mathrm{d}V_g.
\end{aligned}$$

This together with (5.8) yields

$$\lambda F'(\lambda) \geqslant \delta F(\lambda), \quad \lambda > 0,\ \delta = \frac{n-r\gamma}{1+\alpha-\beta/p} + \frac{r(p-1)}{p-r}. \tag{5.9}$$

From (5.7) and (5.9), we get $F'T - T'F \geqslant 0$ or the function $\lambda \to F(\lambda)/T(\lambda)$ is non-decreasing on $(0,\infty)$. On the contrary, for $0 < \lambda^{1/(1+\alpha-\beta/p)} < \inf\{\rho(u) : u \in$

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$S^{n-1}\}$, we have by using polar coordinate (2.3)

$$\begin{aligned}
F(\lambda) &= \int_{S^{n-1}} \int_0^{\rho(u)} \left(\lambda - t^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} t^{n-\gamma r-1} J(u,t)\, \mathrm{d}t\, \mathrm{d}u \\
&= \int_{S^{n-1}} \int_0^{\lambda^{\frac{1}{1+\alpha-\beta/p}}} \left(\lambda - t^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} t^{n-\gamma r-1} J(u,t)\, \mathrm{d}t\, \mathrm{d}u \\
&= (1 + O(\lambda^{\frac{2}{1+\alpha-\beta/p}}))n\omega_n \int_0^{\lambda^{\frac{1}{1+\alpha-\beta/p}}} \left(\lambda - t^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} t^{n-\gamma r-1}\, \mathrm{d}t \\
&= (1 + O(\lambda^{\frac{2}{1+\alpha-\beta/p}}))T(\lambda).
\end{aligned}$$

Thus, we obtain

$$\lim_{\lambda\to 0^+} \frac{F(\lambda)}{T(\lambda)} = 1,$$

which together with the non-decreasing monotonicity of F/T immediately implies

$$F(\lambda) \geqslant T(\lambda), \qquad \lambda > 0.$$

On the contrary, we have by using polar coordinate formula (2.3) and $J(u,t) \leqslant 1$ for $t < \rho(u)$ that

$$\begin{aligned}
&T(\lambda) \leqslant F(\lambda) \\
&= \int_{S^{n-1}} \int_0^{\rho(u)} \left(\lambda - t^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} t^{n-\gamma r-1} J(u,t)\, \mathrm{d}t\, \mathrm{d}u \\
&= \int_{S^{n-1}} \int_0^{\min\{\rho(u),\lambda^{\frac{2}{1+\alpha-\beta/p}}\}} \left(\lambda - t^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} t^{n-\gamma r-1} J(u,t)\, \mathrm{d}t\, \mathrm{d}u \\
&\leqslant \int_{S^{n-1}} \int_0^{\min\{\rho(u),\lambda^{\frac{2}{1+\alpha-\beta/p}}\}} \left(\lambda - t^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} t^{n-\gamma r-1}\, \mathrm{d}t\, \mathrm{d}u \\
&\leqslant \int_{S^{n-1}} \int_0^{\lambda^{\frac{2}{1+\alpha-\beta/p}}} \left(\lambda - t^{1+\alpha-\beta/p}\right)_+^{r(p-1)/(p-r)} t^{n-\gamma r-1}\, \mathrm{d}t\, \mathrm{d}u \\
&= T(\lambda)
\end{aligned}$$

which implies $F(\lambda) = T(\lambda)$ for any $\lambda > 0$. So, all inequalities in the preceding estimates must be equalities. Consequently, for almost $u \in S^{n-1}$ we have $\rho(u) \geqslant \lambda^{2/(1+\alpha-\beta/p)}$ for any $\lambda > 0$. Hence, for almost $u \in S^{n-1}$, we have $\rho(u) = \infty$. Moreover, for such a $u \in S^{n-1}$ we have $J(u,t) = 1$ for almost $t > 0$ (which ensures the equality in the first inequality in the preceding estimates). By the continuity, we have $J(u,t) = 1$ for any $t > 0$. Using again the polar coordinate formula (2.3), we get, for any $r > 0$, $\mathrm{Vol}_g(B(P,r)) = \omega_n r^n$. By the equality condition in Bishop–Gromov volume comparison theorem (see [7, theorem III.4.4]), M is isometric to $\mathbb{R}^n$. □

Acknowledgements

The author is very graceful to thank the anonymous referee for the useful comments which improved the presentation of this paper and for introducing him the paper of Huang, Kristály and Zhao [22] on the sharp Caffarelli–Kohn–Nirenberg inequalities on the general Finsler manifolds. He also would like to thank Professor Alexandru Kristály for drawing our attentions to his studies [23–25]. This study was partially supported by the CIMI's postdoctoral research fellowship.

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