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A MIXTURE OF EXPONENTIAL AND IFR GAMMA DISTRIBUTIONS HAVING AN UPSIDEDOWN BATHTUB-SHAPED FAILURE RATE

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We consider a mixture of one exponential distribution and one gamma distribution with increasing failure rate. For the right choice of parameters, it is shown that its failure rate has an upsidedown bathtub shape failure rate. We also consider a mixture of a family of exponentials and a family of gamma distributions and obtain a similar result.

1. INTRODUCTION

We consider mixtures of continuous distributions whose failure rate function turns out to have what we call an upside-down bathtub shape (UBT). We also say that distributions having this property are UBT. More specifically, the failure rate initially increases, then it eventually decreases. A more familiar shape for a failure rate in reliability is the bathtub (BT) shape which decreases at first, and then increases. Upside-down failure rates frequently appear in the literature, but sometimes under the names inverted bathtub or hump-shaped. Whereas there are few standard

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statistical distributions that have bathtub shapes, Marshall and Olkin [9] discuss many distributions that are upside-down bathtub. Some examples are the lognormal, log logistic and Pareto (IV). Distributions of this type have also shown up as first passage time distributions. They have also been encountered in survival analysis applications (see [7]). For extensive discussions concerning mixtures of survival functions and also BT and UBT failure rates, see the recent books of Lai and Xie [8] and Marshall and Olkin [9]. For a discussion of why mixtures are important in reliability see Block, Li and Savits [2]. Some other related papers on mixtures are Finkelstein [4], Navarro, Guillamon and Ruiz [10] and Navarro and Hernandez [11].

A simple mixture of a specific exponential distribution and a specific gamma distribution with an increasing failure rate (IFR) yields a distribution which is UBT, as given in Example 2.3 of Block et al. [2]. In this paper we show that this example is the prototype for a whole class of mixtures of a similar type. That is, we show that the mixture of an exponential distribution with rate λ and an IFR gamma distribution with scale parameter $\lambda_0 > \lambda$ and shape parameter $1 < \phi \leq 2$ is UBT. This is the content of the theorem in Section 2. In Section 3, we show a much more general result. That is, we take an arbitrary mixture of the exponentials and mix this with a mixture of the gammas. The gammas are mixed on the shape parameter. The result is still a UBT distribution. For this result there are restrictions on the range of the parameters.

We use the convention that if we say increasing or decreasing, this is not in the strict sense. For the strict case we would say strictly increasing or strictly decreasing.

2. ONE EXPONENTIAL AND ONE GAMMA

As mentioned in the Introduction, we are interested in determining the shape of the failure rate for a mixture of an exponential distribution and an IFR gamma distribution:

$$f(t) = p\lambda e^{-\lambda t} + q\lambda_0(\lambda_0 t)^{\phi-1} e^{-\lambda_0 t}/\Gamma(\phi), \tag{1}$$

$0 < p < 1, p + q = 1$. In a previous paper (Block et al. [1]), the authors showed that $f(t)$ had a BT-shaped failure rate when $0 < \lambda_0 < \lambda$ and $2 \leq \phi$. See also Block et al. [8] for extensions to continuous mixtures. One of the subcases considered by Gupta and Warren [6] was the case $\lambda = \lambda_0$ which gave that its failure rate was IFR for $1 < \phi \leq 2$ and BT for $\phi > 2$. Here we consider the case $0 < \lambda < \lambda_0$ and $1 < \phi \leq 2$.

THEOREM 1: *Consider the mixture given in (1):*

$$f(t) = p\lambda e^{-\lambda t} + q\lambda_0(\lambda_0 t)^{\phi-1} e^{-\lambda_0 t}/\Gamma(\phi), \quad 0 < p < 1, \quad p + q = 1.$$

Then for $0 < \lambda < \lambda_0$ and $1 < \phi \leq 2$, the failure rate of the mixture is UBT.

PROOF: Without loss of generality we may assume that $\lambda_0 = 1$. We then follow the usual paradym of Glaser's [5] technique, that is, we investigate the shape of

$$\eta(t) = -\frac{d}{dt}\log f(t).$$

See also Savits [12] for a more general version of the Glaser result. Some other extensions of Glaser's result were given in Gupta and Warren [6] and in Navarro and Hernandez [10]. We will show that $\eta(t)$ has an UBT. According to Glaser [8], it then follows that the failure rate $r(t) = f(t)/\bar{F}(t)$ of $f(t)$ is either UBT or DFR. We rule out the latter by showing that $r'(0+) > 0$.

To show that $\eta(t)$ is UBT we consider the sign-change properties of $\eta'(t)$ as expressed by

$$K(t) = A_{22}(t)q^2 + A_{12}(t)pq,$$

where

$$A_{22}(t) = \frac{(\phi-1)t^{2(\phi-2)}e^{-2t}}{\Gamma^2(\phi)}$$

and

$$A_{12}(t) = \frac{\lambda e^{-t(\lambda+1)}}{\Gamma(\phi)}[(2-\phi)(\phi-1)t^{(\phi-3)} + 2(\phi-1)(1-\lambda)t^{(\phi-2)} - (1-\lambda)^2 t^{(\phi-1)}].$$

Since we are only interested in the sign-change properties of $K(t)$, we can renormalize as follows: let

$$\begin{aligned} W(t) &= \Gamma^2(\phi)e^{2t}K(t) \\ &= W_{22}(t)q^2 + W_{12}(t)pq, \end{aligned}$$

where

$$W_{22}(t) = (\phi-1)t^{2(\phi-2)}$$

and

$$\begin{aligned} W_{12}(t) &= \lambda\Gamma(\phi)e^{t(1-\lambda)}[(2-\phi)(\phi-1)t^{(\phi-3)} + 2(\phi-1)(1-\lambda)t^{(\phi-2)} \\ &\quad - (1-\lambda)^2 t^{(\phi-1)}] \\ &= \lambda\Gamma(\phi)e^{t(1-\lambda)}t^{(\phi-1)}\left[(2-\phi)(\phi-1)t^{-2} + 2(\phi-1)(1-\lambda)t^{-1} - (1-\lambda)^2\right] \\ &= \lambda\Gamma(\phi)e^{t(1-\lambda)}t^{(\phi-3)}\left[(2-\phi)(\phi-1) + 2(\phi-1)(1-\lambda)t - (1-\lambda)^2t^2\right]. \end{aligned}$$

Our goal is to show that $W(t)$ has the required sign-change properties, that is, first positive and then negative.

First we consider the special case $\phi = 2$. Then

$$W(t) = q^2 + \lambda(1-\lambda)e^{t(1-\lambda)}[2-(1-\lambda)t]pq$$

and

$$W'(t) = \lambda(1-\lambda)^2 e^{(1-\lambda)t}[1-(1-\lambda)t]pq.$$

Hence $W(t)$ is strictly increasing for $0 < t < (1-\lambda)^{-1}$ and strictly decreasing for $t > (1-\lambda)^{-1}$. Since $W(0+) > 0$ while $W(\infty) = -\infty$, $W(t)$ has one sign-change from $+$ to $-$.

Next we investigate the case when $1 < \phi < 2$. Here we examine the terms separately. The term $W_{22}(t)$ is strictly decreasing in $t > 0$ and satisfies $W_{22}(0+) = +\infty$ while $W_{22}(\infty) = 0$. For $W_{12}(t)$, we note that $W_{12}(0+) = +\infty$, $W_{12}(\infty) = -\infty$ (since $0 < \lambda < 1$) and $W_{12}(t) = 0$ has exactly one positive root given by

$$t_0 = \frac{\phi - 1 + \sqrt{\phi - 1}}{1-\lambda} \quad (1 < \phi < 2)\,.$$

The derivative of $W_{12}(t)$ is given by

$$W'_{12}(t) = \lambda\Gamma(\phi)e^{t(1-\lambda)}t^{\phi-4}P(t),$$

where

$$P(t) = (\phi-1)(\phi-3)(2-\phi) - (\phi-1)(2-\phi)(1-\lambda)t + (\phi-1)(1-\lambda)^2t^2 - (1-\lambda)^3t^3.$$

Note that $P(0) = (\phi-1)(\phi-3)(2-\phi) < 0$, $P(\infty) = -\infty$ and $P(-\infty) = +\infty$.

Any cubic equation has exactly one or three real roots. In this case it has one real root, since $P(-\infty) = \infty$ and $P(0) < 0$, the root must be negative. Consequently $P(t) < 0$ on $(0, \infty)$. Thus $W_{12}(t)$ is positive, then negative, so $W_{12}(t)$ changes sign from + to $-$. For the case of three real roots, one of them must be negative as above. If all three are negative, then $P(t) < 0$ as above. This gives that $W_{12}(t)$ is decreasing on $(0, \infty)$. Since $W_{12}(0+) = \infty$, $W_{12}(\infty) = -\infty$ and $W_{12}(t)$ has one root, it follows that $W_{12}(t)$ changes sign once from + to $-$. The only other case is where one or more roots are positive. If only one is positive, then two others are negative and this is impossible since $P(0) < 0$. This leaves only the case where one root is negative and two are positive. Call these positive roots $z_1 \le z_2$. Then $W_{12}(t)$ is decreasing on $(0, z_1)$ and (z_2, ∞) and increasing on (z_1, z_2). Now consider the one root t_0 of $W_{12}(t)$. Either $t_0 \le z_1$, in which case $W_{12}(t) > 0$ on $(0, t_0)$ and $W_{12}(t) < 0$ on (t_0, ∞). In this case there is one sign change from $+$ to $-$. If $t_0 > z_1$, we cannot have $z_1 \le t_0 \le z_2$, so $t_0 > z_2$. Again $W_{12}(t) > 0$ on $(0, t_0)$ and $W_{12}(t) < 0$ on (t_0, ∞), so there is one sign change from $+$ to $-$. Finally, it is easy to see that there is one sign change of W from $+$ to $-$.

To conclude that the failure rate of the mixture is UBT and not DFR, we show that $r'(0+) > 0$. According to Block et al. [2], if

$$f(t) = pf_1(t) + qf_2(t),$$

then

$$r'(0+) = pf_1'(0+) + qf_2'(0+) + [pf_1(0+) + qf_2(0+)]^2.$$

It is then easy to show that for $0 < \lambda < 1,\ 1 < \phi \leq 2$,

$$r'(0+) = \begin{cases} \infty, & 1 < \phi < 2, \\ q(1-\lambda^2 p) > 0, & \phi = 2. \end{cases}$$

■

3. CONTINUOUS VERSION

Next, we consider a continuous mixture of exponential and gamma distributions:

$$f(t) = pE[\Lambda e^{-\Lambda t}] + qE[\lambda_0(\lambda_0 t)^{\Phi-1}e^{-\lambda_0 t}/\Gamma(\Phi)], \qquad \textbf{(2)}$$

where Λ and Φ are random variables. In a previous paper, Block et al. [1] investigated the BT case for $\Lambda > \lambda_0$ and $\Phi > 2$. Here we are interested in the UBT case.

THEOREM 2: *Consider the continuous mixture (2) above. Then for* $0 < \Lambda < \lambda_0$ *and* $1 < \Phi < \phi_1$*, where* $\phi_1 = 1.92431$ *is the smallest root of the quadratic equation* $16\phi^2 - 76\phi + 87 = 0$*, it follows that* $f(t)$ *has an UBT shaped failure rate.*

PROOF: As usual, we may assume that $\lambda_0 = 1$. We follow the ideas of Block et al. [3] and consider the shape properties of $\eta(t) = -f'(t)/f(t)$ which are determined by the sign-change properties $K(t)$ of $\eta'(t)$. According to Block et al. [3], we can write $K(t)$ as

$$K(t) = A_{11}(t)p^2 + A_{22}(t)q^2 + A_{12}(t)pq$$

with

$$A_{11}(t) = -(1/2)E[\Lambda_1\Lambda_2(\Lambda_2 - \Lambda_1)^2 e^{-(\Lambda_1+\Lambda_2)t}]),$$
$$A_{22}(t) = (1/2)e^{-2t}E[t^{\Phi_1+\Phi_2-4}\Delta(\Phi_1, \Phi_2)]$$

and

$$A_{12}(t) = -E[t^{2\Phi-4}\Lambda Q(\Lambda, \Phi, t)e^{-(\Lambda+1)t}],$$

where Λ_1 and Λ_2 are independent versions of Λ and similarly for Φ_i, $i = 1, 2$. Here

$$\begin{aligned}\Delta(a_1, a_2) &= \frac{(a_1 - 1)(a_2 - a_1 + 1) + (a_2 - 1)(a_1 - a_2 + 1)}{\Gamma(a_1)\Gamma(a_2)} \\ &= \frac{(a_1 + a_2 - 2) - (a_1 - a_2)^2}{\Gamma(a_1)\Gamma(a_2)}\end{aligned}$$

and

$$Q(\lambda, \phi, t) = \frac{(\lambda - 1)^2 t^{3-\phi} + 2(\lambda - 1)(\phi - 1)t^{2-\phi} + (\phi - 1)(\phi - 2)t^{1-\phi}}{\Gamma(\phi)}.$$

Since the sign-change properties of $K(t)$ and $W(t) = e^{2t}K(t)$ are the same, we instead study

$$W(t) = W_{11}(t)p^2 + W_{22}(t)q^2 + W_{12}(t)pq$$

with

$$\begin{aligned}W_{11}(t) &= -(1/2)E[\Lambda_1\Lambda_2(\Lambda_2 - \Lambda_1)^2 e^{(2-\Lambda_1-\Lambda_2)t}], \\ W_{22}(t) &= (1/2)E[t^{\Phi_1+\Phi_2-4}\Delta(\Phi_1, \Phi_2)]\end{aligned}$$

and

$$\begin{aligned}W_{12}(t) = E[\Lambda e^{(1-\Lambda)t}t^{(\Phi-3)}\{&(2 - \Phi)(\Phi - 1) + 2(1 - \Lambda)(\Phi - 1)t \\ &- (1 - \Lambda)^2 t^2\}/\Gamma(\Phi)].\end{aligned}$$

We now investigate the individual terms. If Λ is degenerate, then $W_{11}(t) = 0$; otherwise, since $0 < \Lambda < 1$ with probability one, $W_{11}(t)$ is decreasing and satisfies $W_{11}(0+) = -\frac{1}{2}E[\Lambda_1\Lambda_2(\Lambda_1 - \Lambda_2)^2] < 0$ and $W_{11}(\infty) = -\infty$. For $W_{22}(t)$, we need to make use of the following fact established in Block et al. (2008): $\Delta(x, y) > 0$ for (x, y) belonging to the set

$$C = \{(x, y) : x > 7/8,\ x + 0.5 - 0.5\sqrt{8x - 7} < y < x + 0.5 + 0.5\sqrt{8x - 7}.$$

Since $(\Phi_1, \Phi_2) \in (1, 2) \times (1, 2) \subset C$, it follows that $W_{22}(t)$ is decreasing. Also, $W_{22}(0+) = +\infty$ and $W_{22}(\infty) = 0$ for our restrictions on Φ.

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Next, we note that $W_{12}(0+) = +\infty$ while $W_{12}(\infty) = -\infty$. To conclude our proof we need to show that $W_{12}(t)$ is decreasing. Toward this end, we calculate its derivative:

$$W'_{12}(t) = E[\frac{\Lambda}{\Gamma(\Phi)} t^{\Phi-4} P(t, \Lambda, \Phi)],$$

where

$$P(t, \lambda, \phi) = (\phi - 1)(\phi - 3)(2 - \phi) - (\phi - 1)(2 - \phi)(1 - \lambda)t + (\phi - 1)(1 - \lambda)^2 t^2 - (1 - \lambda)^3 t^3.$$

The discriminant of the cubic equation

$$f(x) = ax^3 + bx^2 + cx + d$$

is given by

$$\Delta = abcd - 4b^3 d + b^2 c^2 - 4ac^3 - 27a^2 d^2.$$

and it is well known that if this is negative, the cubic equation has only one real root. It can be verified that the discriminant for the above cubic equation for $P(t, \lambda, \phi)$ is

$$\Delta = -4(1 - \lambda)^6 (2 - \phi)(\phi - 1)^2 (16\phi^2 - 76\phi + 87)$$

which is negative for $1 < \phi < \phi_1$ where ϕ_1 is the smallest root of the quadratic equation obtained by equating the second-order polynomal to zero in the previous display. Hence $P(t, \lambda, \phi) = 0$ has only one real root which must be negative since $P(-\infty, \lambda, \phi) = +\infty$ while $P(0, \lambda, \phi) < 0$. Again we conclude that $P(t, \lambda, \phi) < 0$ for all $t > 0, 0 < \lambda < 1$ and $1 < \phi < \phi_1$.

Thus we can now conclude that $W_{12}(t)$ is strictly decreasing and hence $W(t)$ is strictly decreasing with $W(0+) = +\infty$ and $W(\infty) = -\infty$. Consequently, $W(t)$ has exactly one sign-change and it goes from + to $-$; that is, $W(t)$ has an UBT shape. Since the initial derivative of the gamma distribution with shape parameter $1 < \phi < 2$ is $+\infty$, it follows using the arguments of Section 2 that $r'(0+) = +\infty$ and hence cannot be DFR. ■

Note. We have not been able to prove it but we conjecture that the result continues to hold for $\phi_1 \leq \phi \leq 2$.

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