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Dynamics analysis of a novel 5-DoF parallel manipulator with couple-constrained wrench

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SUMMARY

A three-dimensional (3D) model of a novel 5-DoF type parallel manipulator with a couple-constrained wrench is constructed and its couple-constrained wrench is analyzed. First, the formulas are derived for solving the displacement, velocity, acceleration of the moving platform and moving links, and a workspace is constructed. Second, the formulas are derived for solving the inertial wrenches of the moving links. Third, a dynamics equation is established by considering the inertial wrenches and friction, and the formulas are derived for solving the dynamically active forces and the dynamically couple-constrained wrench. Finally, a numerical example is given to demonstrate the analytic solution of the kinematics and the dynamics, and the analytical solutions are verified by utilizing a simulation mechanism.

KEYWORDS: Parallel manipulator, Couple-constrained wrench, Kinematics, Dynamics, Workspace

Nomenclature

PM	parallel manipulator
DoF	degree of freedom
B, m	the base, moving platform
O, o	the center point of B, m
$\{B\}$	coordinate O-XYZ on B
$\{m\}$	coordinate o-xyz on m at o
P, R	prismatic joint, revolute joint
U, S	universal joint, spherical joint
B_i, b_i	the vertices of B and m
$\boldsymbol{v}_i$, $\boldsymbol{a}_i$	the velocity and acceleration vectors
r_i	active leg of PM
$\boldsymbol{\delta}_i$	the unit vectors of r_i $i = 1, 2, 3, 4, 5$
$\boldsymbol{R}_\alpha$, $\boldsymbol{R}_\beta$, $\boldsymbol{R}_\gamma$	the axis of Euler angles
$e_i = e$, E_i	the distance from b_i to o, B_i to O
θ_i	angles of every vertices about center
α, β, γ	Euler angles of m about (Y, X_1, Y_2)
v_{ri}	scalar velocity along r_i
$\boldsymbol{v}$, $\boldsymbol{\omega}$	linear and angular velocity of m at o
$\boldsymbol{a}$, $\boldsymbol{\varepsilon}$	linear and angular acceleration of m

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$\boldsymbol{V}, \boldsymbol{A}$	general velocity and acceleration of m
$\boldsymbol{c}_1, \boldsymbol{c}_2$	constrained force unit vectors of PM
$\boldsymbol{L}$	unit vector of the connection rod
$\boldsymbol{c}$	the unit vector of constrained force
$\boldsymbol{\tau}$	the unit vector of constrained torque
$x_l, x_m, x_n y_l, y_m, y_n z_l, z_m, z_n$	nine orientation parameters of m
$\boldsymbol{l}$	unit vector of the vector $\boldsymbol{b}_3\boldsymbol{b}_1$
X_o, Y_o, Z_o	the position components of o in B
$\boldsymbol{v}_r, \boldsymbol{a}_r$	general input velocity, general input acceleration
$\boldsymbol{V}_r, \boldsymbol{A}_r$	generalized velocity and acceleration of branch
$\boldsymbol{J}, \boldsymbol{H}$	Jacobian, Hessian matrix of PM with planar limb
$\Vert, \perp, \vert$	parallel, perpendicular, collinear constraint
$\boldsymbol{F}_o, \boldsymbol{T}_o$	the central force and torque applied on m at o
$\boldsymbol{F}_a$	the combination of driving and constrained force

1. Introduction

Among less mobility parallel machine tools, the 5-DoF parallel machine tools have been used for normal machining three-dimensional (3D) free-form surface of work pieces, such as the models of automobile windshield, the impeller blades of ships or airplanes, launches and turbine.[1–3] Currently, several 5-DoF parallel machine tools have been developed.[4–9] In this aspect, Gao *et al*. developed a 5-DoF parallel machine tool with four PSS-type limbs and a composite 3UU-type limb,[4,5] where, *P*, *U* and *S* represent the prismatic joint, the universal joint and the spherical joint, respectively. Liu,[6] Wang,[7] Wu[8] and Zhu[9] *et al*. developed different 5-DoF hybrid machine tools in which a 3-DoF parallel manipulator (PM) is combined with a 2-DoF tool head. Li *et al*. optimized the parameters of a 5-DoF gasbag polishing machine tool.[10] Generally, the 5-DoF parallel machine tools are be developed from the 5-DoF PMs. In this aspect, Qi *et al*. proposed a 5-DoF PM with four UPS-type limbs and a central UPU-type limb and analyzed its forward kinematics.[11] Kong *et al*. synthesized several 5-DoF PMs by utilizing the screw theory and the concept of virtual chains.[12] Shirazi *et al*. analyzed a 5-RPUR type PM and optimized its structure.[13] Borràs proposed a 5-DoF PM in which a rod (moving platform) is connected with five active limbs.[14] Fang *et al*. synthesized a class of 5-DoF over-constrained PMs with identical serial limbs.[15] Motevalli[16] and Piccin[17] studied the architecture synthesis of a 5-DoF PMs with three translational and two rotational movements. Li *et al*. synthesized the 5-DoF PMs with three rotational and two translational movements by utilizing the Lie group of displacements.[18] Sangveraphunsiri designed a hybrid 5-DoF manipulator based on an H-4 family PM with three rotational and one translational movements.[19] You *et al*. proposed a haptic device with pantograph parallel platform and they studied its kinematics.[20] Lu *et al*. proposed a 5-DoF 4SPS+1SPR parallel machine tool with two composite spherical joints[21] and analyzed its kinematics. In the aspect of dynamics and workspace of PMs, Wu *et al*. studied dynamics of a planar 3-DOF PM with actuation redundancy,[22] a PM in a spray-painting equipment[23] and a solar tracker with parallel mechanism.[24] Bonev and Gosselin determined the workspace of symmetrical spherical PMs.[25] Liu and Bonev studied two articulated tool heads with parallel kinematics.[26] In fact, it is difficult to manufacture a composite spherical joint. It is known based on the topology graph of the mechanisms[27] that the topology graph of the 5-DoF PM proposed by Gao[5] includes two pentagonal links, two quaternary links and groups of binary links. The topology graphs of the 5-DoF PMs in Refs. [10-19,21] include two pentagonal links and some groups of binary links. These studies have their merits and different focuses, and lay a theoretical foundation for this study.

Generally, many less mobility PMs include one or more constrained forces or constrained torques which are mutually independent. If the constrained forces or constrained torques in a less mobility PM are mutually dependent, this type less mobility PM is called as a PM with the couple-constrained wrench. We have constructed several novel PMs with couple-constrained wrench by utilizing CAD software, and found that they have a large position and orientation workspace, high rigidity, good property of isotropy horizontal motion and easy to be manufactured. One of them is authorized a patent with No. CN104369182B in China. Up to now, the dynamics of PMs with the couple-constrained wrench have not been studied. Therefore, this paper focuses on the dynamics analysis of a novel

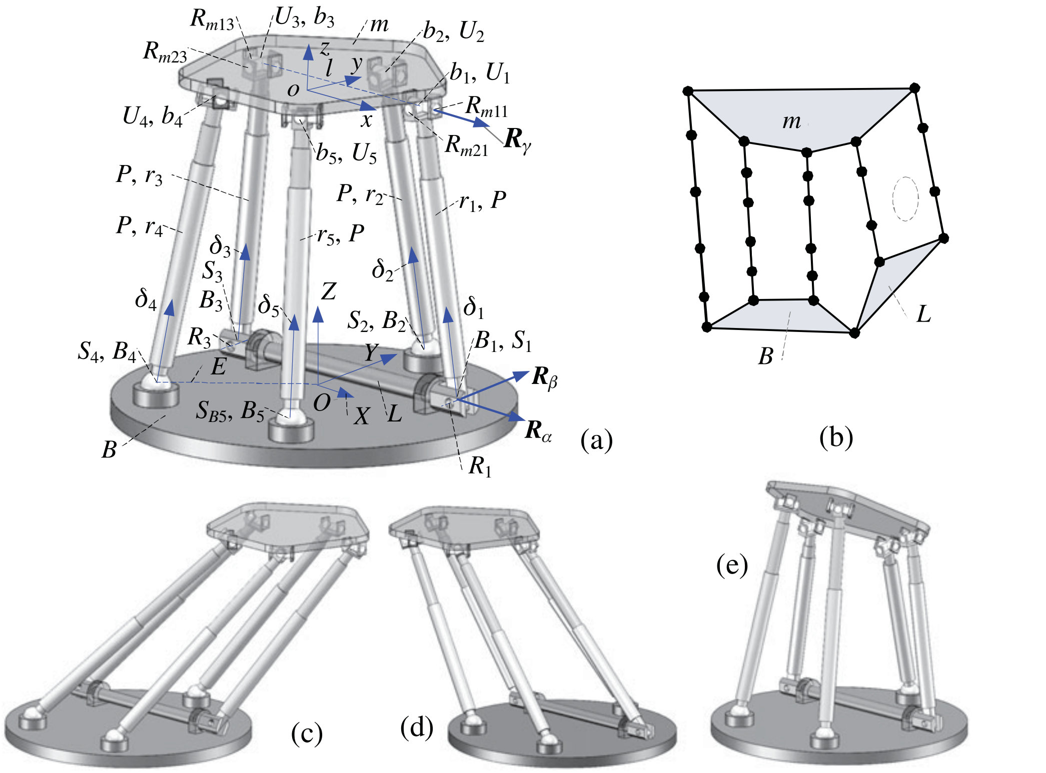

Fig. 1. (a) A 3D model of novel 3SPU+2RPU+R type PM with couple-constrained wrench and its coordinate system, (b) its revised topology graph, (c and d) two positions far from Z and (e) a position with large orientation.

3SPU+2RPU+R type PM with the couple-constrained wrench. Its kinematics of the moving links, inertial wrenches, structure characteristics, couple-constrained wrench, dynamically active forces and dynamically couple-constrained wrench are studied systematically.

2. Performances of Novel PM with Couple-Constrained Wrench and its DoFs

A 3D model of the novel 3SPU+2RPU+R type PM with the couple-constrained wrench is constructed, see Fig. 1a. Its other three position orientations are shown in Fig. 1c-e. This novel PM includes a moving platform m, a base B, a connection rod L, two RPU-type active legs r_i (i =1, 3) and three SPU-type active legs r_i (i =2, 4, 5). Here, B has a central point O and five joints B_i (i =1, 3) and B_i (i =2, 4, 5), which are located in the two circumferences on the same plane of B. m has a central point o and five joints b_i (i =1, ..., 5) which are uniformly located in the same circumference of m. L is connected with B by two revolute joints R_i (i =1, 3) at points B_i. Each of the RPU-type active legs r_i (i =1, 3) has a translational actuator, its upper end is connected with m at b_i (i =1, 3) by the universal joint U_i; its lower end is connected with the two ends of L at B_i (i =1, 3) by the revolute joint R_i. Each of U_i (i =1, 3) includes two crossed revolute joints R_{m1i} and R_{m2i}. Each of SPU-type active legs r_i (i =2, 4, 5) has a linear actuator, its upper end connects with m at b_i by the U_i joint; its lower end connects with B at B_i by spherical joint S. Let (‖, ⊥, |) be the parallel, perpendicular and collinear constraints, respectively. Let m:o-xyz be a coordinate system attached on m at its central point o; B:O-XYZ be a coordinate system attached on B at its central point O; l be a line from b_3 to b_1. The geometric constraints $R_1 \| R_3$, $z \perp m$, $x \| l$, $Z \perp B$, $X \| L$, $R_{m1i} \perp R_{m2i}$, $R_i \perp L$, $R_i \perp r_i$, $R_{m1i} | l$, $R_{m2i} \| R_i$, i =(1, 3) are satisfied.

DoF M of the novel 3SPU+2RPU+R type PM with the couple-constrained wrench is calculated based on a revision Kutzbach Grubler equation in Ref. [2], as shown below:

$$\begin{aligned} M &= 6(n_0 - n_k - 1) + \sum M_i + \varsigma - M_0 \\ &= 6 \times (13 - 16 - 1) + 3 \times 3 + 5 \times 2 + 3 \times 1 + 5 \times 1 + 3 - 1 = 5 \end{aligned} \tag{1}$$

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here, n_0 =13 is the number of links for one m, one B, one L, five piston rods and five cylinder rods; n_k =16 is the number of kinematic pairs for three spherical joints, five universal joints, three revolute joints and five prismatic joints; ΣM_i =27 is the sum of local DoFs of kinematic pairs for three spherical joints, five universal joints, three revolute joints and five prismatic joints; ς =3 is for the redundant constraint of a sub-plane mechanism; M_0 =1 for one passive DoF due to $R_{m1i}|l$, (i =1, 3).

A revised topology graph of the novel 5-DoF PM with the couple-constrained wrench is constructed by considering M_0 and ς,[27,28] see Fig. 1b. It includes one pentagonal link for m, one quaternary link for B, one ternary link for link L and 18+M + M_0-ς =21 binary links for constructing two RPU-type legs and three SPU-type legs. It can be found that the complicated composite spherical joints in some PMs can be replaced by the revolute joints which have larger capability of pulling force bearing than that of S joint, and have larger rotation angle than the rotation cone angle of S joint before interference, and have higher rigidity.

3. Kinematics of Novel PM with Couple-Constrained Wrench

3.1. Inverse displacement

The novel PM includes a sub-planar mechanism formed from two RPU-type active legs r_i (i =1, 3), L and m. Let β_1 be an angle between L and r_1 and β_2 be an angle between l and r_1. Let (α, β, γ) be the three Euler angles of m in B and $(\boldsymbol{R}_\alpha, \boldsymbol{R}_\beta, \boldsymbol{R}_\gamma)$ be the unit vector of rotational axis of (α, β, γ), respectively. Here, α is a rotational angle of m about L (i.e. about X), β is an angle between l and L about Y and γ is a rotational angle of m about l. Thus, $(\beta = \beta_1 + \beta_2, \boldsymbol{R}_\alpha|L, \boldsymbol{R}_\beta|R_1, \boldsymbol{R}_\gamma|l)$ are satisfied, see Fig. 1a. Based on above geometric constraints, set ${}^B_m\boldsymbol{R}$ to be a rotation matrix from m to B in order XYX,[21] $(x_l, x_m, x_n, y_l, y_m, y_n, z_l, z_m, z_n)$ be the nine orientation parameters of m in B, (X_o, Y_o, Z_o) be the position components of o in B and φ be one of $(\alpha, \beta, \gamma, \theta_i)$. Set s_φ =sin φ and c_φ =cos φ. Let ${}^m\boldsymbol{b}_i$ (i =1,..., 5) $\boldsymbol{b}_i$ and $\boldsymbol{o}$ be the vectors of points b_i and $\boldsymbol{o}$ on m in m and B, respectively; $\boldsymbol{B}_i$ be the vectors of points B_i on B in B; (X_o, Y_o, Z_o), (X_{bi}, Y_{bi}, Z_{bi}) and (X_{Bi}, Y_{Bi}, Z_{Bi}) be the position components of $(\boldsymbol{o}, \boldsymbol{b}_i, \boldsymbol{B}_i)$ in B, respectively. $({}^B_m\boldsymbol{R}, \boldsymbol{o}, {}^m\boldsymbol{b}_i, \boldsymbol{b}_i, \boldsymbol{B}_i)$ are represented and derived as follows:

$$
\begin{aligned}
&{}^B_m\boldsymbol{R} = \begin{pmatrix} x_l & y_l & z_l \\ x_m & y_m & z_m \\ x_n & y_n & z_n \end{pmatrix} = \begin{pmatrix} c_\beta & s_\beta s_\gamma & s_\beta c_\gamma \\ s_\alpha s_\beta & c_\alpha c_\gamma - s_\alpha c_\beta s_\gamma & -c_\alpha s_\gamma - s_\alpha c_\beta c_\gamma \\ -c_\alpha s_\beta & s_\alpha c_\gamma + c_\alpha c_\beta s_\gamma & -s_\alpha s_\gamma + c_\alpha c_\beta c_\gamma \end{pmatrix}, \ \boldsymbol{o} = \begin{pmatrix} X_o \\ Y_o \\ Z_o \end{pmatrix}, \\
&{}^m\boldsymbol{b}_i = e_i \begin{pmatrix} c_{\theta i} \\ s_{\theta i} \\ 0 \end{pmatrix}, \ \boldsymbol{B}_i = \begin{pmatrix} X_{Bi} \\ Y_{Bi} \\ Z_{Bi} \end{pmatrix} = E_i \begin{pmatrix} c_{\theta i} \\ s_{\theta i} \\ 0 \end{pmatrix}, \ \boldsymbol{b}_i = \begin{pmatrix} X_{bi} \\ Y_{bi} \\ Z_{bi} \end{pmatrix} = {}^B_m\boldsymbol{R}\,{}^m\boldsymbol{b}_i + \boldsymbol{o} = \begin{pmatrix} e_i c_{\theta i} x_l + e_i s_{\theta i} y_l + X_o \\ e_i c_{\theta i} x_m + e_i s_{\theta i} y_m + Y_o \\ e_i c_{\theta i} x_n + e_i s_{\theta i} y_n + Z_o \end{pmatrix}
\end{aligned}
\tag{2}
$$

here, $e_i = e$ (i =1,..., 5) are the distances from o to b_i; E_i are the distances from O to B_i; θ_i are the angles between x and line from o to b_i, θ_i (i =1, ..., 5)=(18°, 90°, 162°, 234°, 306°).

The vector $\boldsymbol{r}_i$ (i =1,..., 5) of r_i and its unit vector $\boldsymbol{\delta}_i$ are derived from Eq. (2) as follows:

$$
\boldsymbol{r}_i = \boldsymbol{b}_i - \boldsymbol{B}_i = \begin{pmatrix} e_i c_{\theta i} x_l + e_i s_{\theta i} y_l + X_o - E_i c_{\theta i} \\ e_i c_{\theta i} x_m + e_i s_{\theta i} y_m + Y_o - E_i s_{\theta i} \\ e_i c_{\theta i} x_n + e_i s_{\theta i} y_n + Z_o \end{pmatrix}, \ \boldsymbol{\delta}_i = \frac{\boldsymbol{r}_i}{|\boldsymbol{r}_i|} \tag{3}
$$

Let $\boldsymbol{l}$ and $\boldsymbol{L}$ be the unit vectors of l and L, respectively. Since (l, L and r_1) locate in the same plane, and $(\boldsymbol{l} = \boldsymbol{x}, \boldsymbol{L} = \boldsymbol{X})$ are satisfied, a constrained equation of plane is derived from Eqs. (2) and (3) as follows:

$$
\begin{aligned}
&\boldsymbol{l} = \boldsymbol{x} = \begin{pmatrix} x_l \\ x_m \\ x_n \end{pmatrix} = \begin{pmatrix} c_\beta \\ s_\alpha s_\beta \\ -c_\alpha s_\beta \end{pmatrix}, \ \boldsymbol{r}_1 = \boldsymbol{b}_1 - \boldsymbol{B}_1 = \begin{pmatrix} X_{b1} - X_{B1} \\ Y_{b1} - Y_{B1} \\ Z_{b1} - Z_{B1} \end{pmatrix}, \ \boldsymbol{L} = \boldsymbol{X} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \\
&(\boldsymbol{l}\boldsymbol{r}_1\boldsymbol{L}) = 0 \ \Rightarrow \begin{vmatrix} x_l & x_m & x_n \\ X_{b1} - X_{B1} & Y_{b1} - Y_{B1} & Z_{b1} - Z_{B1} \\ 1 & 0 & 0 \end{vmatrix} = 0
\end{aligned}
\tag{4}
$$

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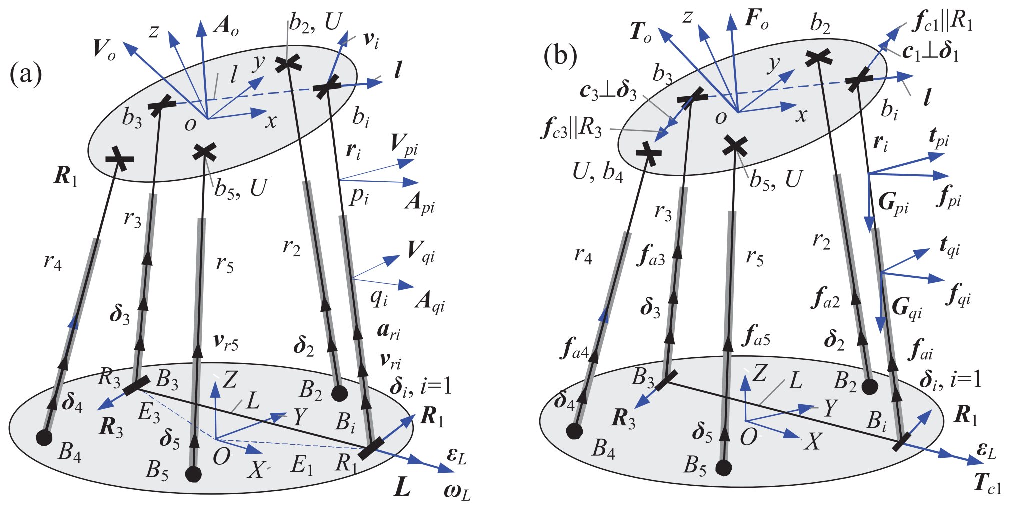

Fig. 2. The kinematics/dynamics model of the novel PM with couple-constrained wrench.

The relations among $(X_o, Y_o, Z_o, \alpha, \beta, \gamma)$ are derived from Eqs. (2) and (4) as follows:

$$x_m(Z_{b1} - Z_{B1}) - x_n(Y_{b1} - Y_{B1}) = 0 \Rightarrow e s_{\theta 1} c_\gamma + Z_o s_\alpha + Y_o c_\alpha - E_1 s_{\theta 1} c_\alpha = 0,$$
$$c_\gamma = (E_1 s_{\theta 1} c_\alpha - Z_o s_\alpha - Y_o c_\alpha)\big/(e s_{\theta 1}) \tag{5}$$

3.2. Analyses of velocity, couple-constrained wrench and singularity

Let $(\boldsymbol{v}, \boldsymbol{\omega}, \boldsymbol{V}_o)$ be the translational, angular and general velocities of m at o in B, respectively. Let v_{ri} (i =1,..., 5) be the input velocity along r_i of PM. Let $\boldsymbol{v}_i$ be the translational velocity of m at b_i in B. Let $\boldsymbol{e}_i$ be the vector from o to b_i. Let $\boldsymbol{T}_{ci}$ (i =1, 3) be the dynamic constrained torques exerted onto L. Let $\boldsymbol{f}_{ci}$ (i =1, 3) and $\boldsymbol{c}_i$ be the dynamic couple-constrained forces exerted onto m and their unit vectors. Let f_{ci} be scalar of $\boldsymbol{f}_{ci}$. A kinematics and dynamics model of the novel PM with the couple-constrained wrench is shown in Fig. 2.

The formulas for solving $(v_{ri}, \boldsymbol{v}_i)$ can be derived as follows:

$$\boldsymbol{v}_i = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i,\ \boldsymbol{e}_i = \boldsymbol{b}_i - \boldsymbol{o},\ v_{ri} = \boldsymbol{v}_i \cdot \boldsymbol{\delta}_i = (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i) \cdot \boldsymbol{\delta}_i,$$
$$v_{ri} = \left(\boldsymbol{\delta}_i^T\ (\boldsymbol{e}_i \times \boldsymbol{\delta}_i)^T\right) \boldsymbol{V}_o,\ \boldsymbol{V}_o = \begin{pmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{pmatrix} \tag{6}$$

Generally, a 5-DoF PM has a general input velocity which includes five input velocity components and a general output velocity which includes six velocity components (i.e. three translational and three angular velocity components). Therefore, a 5×6 velocity mapped matrix $\boldsymbol{J}$ can be derived. However, it is a challenging issue to derive the formulas for solving the acceleration of the moving links and the dynamics of the 5-DoF PM by utilizing the 5×6 matrix $\boldsymbol{J}$. Since most 5-DoF PMs include a decoupled constrained force/torque which does not generate power, the 5×6 matrix $\boldsymbol{J}$ can be transformed into a 6×6 matrix $\boldsymbol{J}$ by utilizing the decoupled constrained force/torque based on the principle of the virtual power.[21] Thus, it is easy by utilizing the 6×6 matrix $\boldsymbol{J}$ to derive formulas for solving the acceleration and dynamics of the 5-DoF PMs.

However, the novel 5-DoF PM includes two couple-constrained forces. Hence, a relation equation of two couple-constrained forces must be derived. After that, a 6×6 matrix $\boldsymbol{J}$ can be derived based on the relation equation of two couple-constrained forces and the principle of the virtual power.

Based on the balancing condition of the dynamic constrained torques, ($\boldsymbol{T}_{c1} = \boldsymbol{T}_{c3}$ and $\boldsymbol{T}_{ci} | \boldsymbol{L}$) are satisfied, see Fig. 2b. Therefore, the dynamically couple-constrained forces must satisfy

$$(\boldsymbol{r}_1 \times f_{c1}\boldsymbol{c}_1 + \boldsymbol{r}_3 \times f_{c3}\boldsymbol{c}_3) \cdot \boldsymbol{L} = 0 \ \Rightarrow\ f_{c3} = k f_{c1},\ k = -(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}\big/[(\boldsymbol{r}_3 \times \boldsymbol{c}_3) \cdot \boldsymbol{L}] \tag{7}$$

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Since neither the couple-constrained forces $\boldsymbol{f}_{ci}$ (i =1, 3) nor the coupling constrained torques $\boldsymbol{e}_i \times \boldsymbol{f}_{ci}$ (i =1, 3) generate power, ($\boldsymbol{f}_{ci} \times \boldsymbol{r}_i = \boldsymbol{T}_{ci}$ and $\boldsymbol{f}_{c1} \| \boldsymbol{f}_{c3} \boldsymbol{c}_i \perp \boldsymbol{\delta}_i$ and $\boldsymbol{c}_i \| \boldsymbol{R}_i$) must be satisfied. Thus, from the principle of the virtual power, the couple-constrained wrench must satisfy

$$(f_{c1}\boldsymbol{c}_1 + f_{c3}\boldsymbol{c}_3) \cdot \boldsymbol{v} + (\boldsymbol{e}_1 \times f_{c1}\boldsymbol{c}_1 + \boldsymbol{e}_3 \times f_{c3}\boldsymbol{c}_3) \cdot \boldsymbol{\omega} = 0 \tag{8}$$

Substitute $f_{c3} = k f_{c1}$ in Eq. (7) into Eq. (8), it leads to

$$\begin{aligned}&\{[(\boldsymbol{r}_3 \times \boldsymbol{c}_3) \cdot \boldsymbol{L}] \cdot \boldsymbol{c}_1 - [(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_3\} \cdot \boldsymbol{v} + \\ &\{\boldsymbol{e}_1 \times [((\boldsymbol{r}_3 \times \boldsymbol{c}_3) \cdot \boldsymbol{L}) \cdot \boldsymbol{c}_1 - ((\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}) \cdot \boldsymbol{c}_3] + l\boldsymbol{l} \times ((\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}) \cdot \boldsymbol{c}_3\} \cdot \boldsymbol{\omega} = 0\end{aligned} \tag{9}$$

Equation (9) can be represented as follows:

$$\begin{aligned}&\boldsymbol{c}_l \cdot \boldsymbol{v} + \boldsymbol{\tau}_l \cdot \boldsymbol{\omega} = 0,\ \boldsymbol{c}_l = [(\boldsymbol{r}_3 \times \boldsymbol{c}_3) \cdot \boldsymbol{L}]\boldsymbol{c}_1 - [(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_3, \\ &\boldsymbol{\tau}_l = \boldsymbol{e}_1 \times \{[(\boldsymbol{r}_3 \times \boldsymbol{c}_3) \cdot \boldsymbol{L}]\boldsymbol{c}_1 - [(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_3\} + l\boldsymbol{l} \times [(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_3\end{aligned} \tag{10}$$

Since $\boldsymbol{R}_1 \| \boldsymbol{R}_3$, $\boldsymbol{c}_i \| \boldsymbol{R}_i$ (i =1, 3) and $\boldsymbol{R}_i \perp \boldsymbol{L}$ are satisfied, there are $\boldsymbol{c}_3 = -\boldsymbol{c}_1$, $\boldsymbol{c}_i \perp \boldsymbol{l}$, $(\boldsymbol{L} \times \boldsymbol{r}_i) \| \boldsymbol{c}_i$. Next, $\boldsymbol{c}_l$ and $\boldsymbol{\tau}_l$ are derived as follows:

$$\begin{aligned}\boldsymbol{c}_l &= [(\boldsymbol{r}_3 \times \boldsymbol{c}_3) \cdot \boldsymbol{L}]\boldsymbol{c}_1 - [(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_3 = -[(\boldsymbol{r}_3 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_1 + [(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_1 \\ &= -[(\boldsymbol{L} \times \boldsymbol{r}_3) \cdot \boldsymbol{c}_1]\boldsymbol{c}_1 + [(\boldsymbol{L} \times \boldsymbol{r}_1) \cdot \boldsymbol{c}_1]\boldsymbol{c}_1 = \boldsymbol{L} \times (\boldsymbol{r}_1 - \boldsymbol{r}_3) = \boldsymbol{L} \times (l\boldsymbol{l} - L\boldsymbol{L}) = l\boldsymbol{c}, \\ \boldsymbol{\tau}_l &= \boldsymbol{e}_1 \times \boldsymbol{c} + l\boldsymbol{l} \times (\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}\boldsymbol{c}_3 = \boldsymbol{e}_1 \times \boldsymbol{c} - \boldsymbol{l} \times [(\boldsymbol{r}_1 \times \boldsymbol{c}_1) \cdot \boldsymbol{L}]\boldsymbol{c}_1 \\ &= \boldsymbol{e}_1 \times \boldsymbol{c} - l\boldsymbol{l} \times [(\boldsymbol{L} \times \boldsymbol{r}_1) \cdot \boldsymbol{c}_1]\boldsymbol{c}_1 = \boldsymbol{e}_1 \times \boldsymbol{c} - l\boldsymbol{l} \times (\boldsymbol{L} \times \boldsymbol{r}_1) = l\boldsymbol{\tau}, \\ \boldsymbol{c} &= \boldsymbol{L} \times \boldsymbol{l},\ \boldsymbol{\tau} = \boldsymbol{e}_1 \times (\boldsymbol{L} \times \boldsymbol{l}) - \boldsymbol{l} \times (\boldsymbol{L} \times \boldsymbol{r}_1)\end{aligned} \tag{11}$$

From Eqs. (10) and (11), it leads to

$$\boldsymbol{c}_l \cdot \boldsymbol{v} + \boldsymbol{\tau}_l \cdot \boldsymbol{\omega} = 0 \ \Rightarrow\ \boldsymbol{c} \cdot \boldsymbol{v} + \boldsymbol{\tau} \cdot \boldsymbol{\omega} = 0 \tag{12}$$

Since $\boldsymbol{R}_1 \| (\boldsymbol{L} \times \boldsymbol{r}_1)$, $\boldsymbol{\tau} \perp \boldsymbol{L}$, $\boldsymbol{\tau} \perp \boldsymbol{R}_1$ are satisfied. Hence, when $l \| L$ is satisfied, the constrained wrench is the constrained torque $\boldsymbol{T}_c$, and $T_c = F_c L$, $\boldsymbol{T}_c \perp \boldsymbol{L}$, $\boldsymbol{T}_c \perp \boldsymbol{R}_1$ are satisfied.

Let $\boldsymbol{V}_r$ be a general input velocity of the novel PM. A formula of $\boldsymbol{V}_r$ is derived from Eq. (6) and Eq. (12) as follows:

$$\boldsymbol{V}_r = \boldsymbol{J}\boldsymbol{V}_o,\ \boldsymbol{V}_r = \begin{pmatrix} v_{r1} \\ v_{r2} \\ v_{r3} \\ v_{r4} \\ v_{r5} \\ 0 \end{pmatrix},\ \boldsymbol{J} = \begin{pmatrix} \boldsymbol{\delta}_1^T & (\boldsymbol{e}_1 \times \boldsymbol{\delta}_1)^T \\ \boldsymbol{\delta}_2^T & (\boldsymbol{e}_2 \times \boldsymbol{\delta}_2)^T \\ \boldsymbol{\delta}_3^T & (\boldsymbol{e}_3 \times \boldsymbol{\delta}_3)^T \\ \boldsymbol{\delta}_4^T & (\boldsymbol{e}_4 \times \boldsymbol{\delta}_4)^T \\ \boldsymbol{\delta}_5^T & (\boldsymbol{e}_5 \times \boldsymbol{\delta}_5)^T \\ \boldsymbol{c}^T & \boldsymbol{\tau}^T \end{pmatrix},\ \begin{aligned}&\boldsymbol{c} = \boldsymbol{L} \times \boldsymbol{l}, \\ &\boldsymbol{\tau} = \boldsymbol{e}_1 \times \boldsymbol{c} - \boldsymbol{l} \times (\boldsymbol{L} \times \boldsymbol{r}_1)\end{aligned} \tag{13}$$

here, $\boldsymbol{J}$ is a 6×6 Jacobian matrix of the novel PM.

Several singularities are determined as follows:

1. When $r_i | L$ (i =1, 3) and $l \| L$ are satisfied, there are $\boldsymbol{c} = \boldsymbol{\tau} = \boldsymbol{0}$, $|\boldsymbol{J}| = 0$ in Eq. (13). A singularity occurs.
2. When l =0 is satisfied, there are $\boldsymbol{c} = \boldsymbol{\tau} = \boldsymbol{0}$, $|\boldsymbol{J}| = \boldsymbol{0}$ in Eq. (13). A singularity occurs.
3. When L =0 and is satisfied, there are $\boldsymbol{c} = \boldsymbol{\tau} = \boldsymbol{0}$, $|\boldsymbol{J}|$=0 in Eq. (13). A singularity occurs.
4. When r_i =0, i is one of (1, …, 5) is satisfied, there are $\boldsymbol{\delta}_i = \boldsymbol{e}_i \times \boldsymbol{\delta}_i = \boldsymbol{0}$, $|\boldsymbol{J}| = 0$. A singularity occurs.

3.3. Acceleration model and Hessian matrix

The acceleration analysis is a pre-condition for deriving dynamics formulas. Let $\boldsymbol{u}$, $s(\boldsymbol{u})$ be a vector and its skew-symmetric matrix and $\boldsymbol{I}$ be a 3×3 unit matrix. They satisfy[1,2]

$$\boldsymbol{u} \times = s(\boldsymbol{u}) = \hat{\boldsymbol{u}},\ s(\boldsymbol{u})^T = -s(\boldsymbol{u}),\ -s^2(\boldsymbol{u}) = -\hat{\boldsymbol{u}}^2 = \boldsymbol{I} - \boldsymbol{u}\boldsymbol{u}^T \tag{14}$$

here, $\boldsymbol{u}$ may be one of $\boldsymbol{L}, \boldsymbol{l}, \boldsymbol{e}_i, \boldsymbol{\delta}_i, \boldsymbol{r}_1, \boldsymbol{L} \times \boldsymbol{r}_1, \boldsymbol{L} \times \boldsymbol{l}$, ($i$ =1, …, 5).

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Let ($\boldsymbol{a}$, $\boldsymbol{\epsilon}$, $\boldsymbol{A}_o$) be the translational, angular and general accelerations of m at o in B, respectively. Let a_{ri} (i =1, 2, 3, 4, 5) be the input acceleration along r_i of novel PM. Differentiate Eq. (6), a_{ri} (i =1, 2, 3, 4, 5) can be derived based on Eqs. (6) and (14) in Ref. [21] as follows:

$$\begin{array}{l} a_{ri} = \begin{pmatrix} \boldsymbol{\delta}_i^T & (\boldsymbol{e_i} \times \boldsymbol{\delta}_i)^T \end{pmatrix} \boldsymbol{A}_o + \boldsymbol{V}_o{}^T \boldsymbol{h}_{ri} \boldsymbol{V}_o, \\ \underset{6\times 6}{\boldsymbol{h}_{ri}} = \frac{1}{r_i}\begin{pmatrix} \boldsymbol{h}_{11} & \boldsymbol{h}_{12} \\ \boldsymbol{h}_{21} & \boldsymbol{h}_{22} \end{pmatrix}, \begin{array}{l} \boldsymbol{h}_{11} = -s^2(\boldsymbol{\delta}_i),\ \boldsymbol{h}_{12} = s^2(\boldsymbol{\delta}_i)s(\boldsymbol{e}_i),\ \boldsymbol{h}_{21} = -s(\boldsymbol{e}_i)s^2(\boldsymbol{\delta}_i), \\ \boldsymbol{h}_{22} = r_i s(\boldsymbol{e}_i)s(\boldsymbol{\delta}_i) + s(\boldsymbol{e}_i)s^2(\boldsymbol{\delta}_i)s(\boldsymbol{e}_i) \end{array} \end{array} \tag{15}$$

here, $\boldsymbol{h}_{ri}$ (i =1, 2, 3, 4, 5) is a 6×6 sub-Hessian matrix corresponding to active force along r_i.

Differentiate Eq. (12), there are

$$\begin{array}{l} \boldsymbol{c}' = (\boldsymbol{L} \times \boldsymbol{l})' = \boldsymbol{L} \times (\boldsymbol{\omega} \times \boldsymbol{l}) = \begin{pmatrix} \boldsymbol{0} & -\boldsymbol{Q} \end{pmatrix} \boldsymbol{V}_o,\ \boldsymbol{Q} = s(\boldsymbol{L})s(\boldsymbol{l}), \\ \boldsymbol{\tau}' = [\boldsymbol{e}_1 \times (\boldsymbol{L} \times \boldsymbol{l}) - \boldsymbol{l} \times (\boldsymbol{L} \times \boldsymbol{r}_1)]' \\ \quad = \boldsymbol{\omega} \times \boldsymbol{e}_1 \times (\boldsymbol{L} \times \boldsymbol{l}) + \boldsymbol{e}_1 \times (\boldsymbol{L} \times (\boldsymbol{\omega} \times \boldsymbol{l})) - (\boldsymbol{\omega} \times \boldsymbol{l}) \times (\boldsymbol{L} \times \boldsymbol{r}_1) - \boldsymbol{l} \times (\boldsymbol{L} \times \boldsymbol{v}_1). \\ \boldsymbol{\tau}' = (\boldsymbol{L} \times \boldsymbol{l}) \times (\boldsymbol{e}_1 \times \boldsymbol{\omega}) - \boldsymbol{e}_1 \times [\boldsymbol{L} \times (\boldsymbol{l} \times \boldsymbol{\omega})] - (\boldsymbol{L} \times \boldsymbol{r}_1) \times (\boldsymbol{l} \times \boldsymbol{\omega}) - \boldsymbol{l} \times [\boldsymbol{L} \times \boldsymbol{v} + \boldsymbol{L} \times (\boldsymbol{\omega} \times \boldsymbol{e}_1)] \\ \quad = s(\boldsymbol{L} \times \boldsymbol{l})s(\boldsymbol{e}_1)\boldsymbol{\omega} - s(\boldsymbol{e}_1)s(\boldsymbol{L})s(\boldsymbol{l})\boldsymbol{\omega} - s(\boldsymbol{L} \times \boldsymbol{r}_1)s(\boldsymbol{l})\boldsymbol{\omega} - s(\boldsymbol{l})s(\boldsymbol{L})\boldsymbol{v} + s(\boldsymbol{l})s(\boldsymbol{L})s(\boldsymbol{e}_1)\boldsymbol{\omega} \\ \quad = \begin{pmatrix} -\boldsymbol{Q}^T & \boldsymbol{Q}_1 \end{pmatrix} \boldsymbol{V}_o, \\ \boldsymbol{Q}_1 = s(\boldsymbol{L} \times \boldsymbol{l})s(\boldsymbol{e}_1) - s(\boldsymbol{e}_1)s(\boldsymbol{L})s(\boldsymbol{l}) - s(\boldsymbol{L} \times \boldsymbol{r}_1)s(\boldsymbol{l}) + s(\boldsymbol{l})s(\boldsymbol{L})s(\boldsymbol{e}_1) \end{array} \tag{16}$$

From Eq. (16), it leads to

$$\begin{pmatrix} (\boldsymbol{c}^T)' & (\boldsymbol{\tau}^T)' \end{pmatrix} = \boldsymbol{V}^T \boldsymbol{h}_c,\ \boldsymbol{h}_c = \begin{pmatrix} \boldsymbol{0}_{3\times 3} & -\boldsymbol{Q} \\ -\boldsymbol{Q}^T & \boldsymbol{Q}_1^T \end{pmatrix} \tag{17}$$

here, $\boldsymbol{h}_c$ is a 6×6 sub-Hessian matrix corresponding to constrained wrench.

Differentiate equation $\boldsymbol{c} \cdot \boldsymbol{v} + \boldsymbol{\tau} \cdot \boldsymbol{\omega} = 0$ in Eq. (12), it leads to

$$\begin{array}{l} 0' = [\begin{pmatrix} \boldsymbol{c}^T & \boldsymbol{\tau}^T \end{pmatrix} \boldsymbol{V}_o]' \Rightarrow \\ 0 = \begin{pmatrix} \boldsymbol{c}^T & \boldsymbol{\tau}^T \end{pmatrix} \boldsymbol{A}_o + \begin{pmatrix} (\boldsymbol{c}^T)' & (\boldsymbol{\tau}^T)' \end{pmatrix} \boldsymbol{V}_o \end{array} \tag{18}$$

A formula for solving general acceleration is derived from Eqs. (15), (17) and (18) as follows:

$$\begin{array}{l} \boldsymbol{A}_{ri} = \boldsymbol{J}\boldsymbol{A}_o + \boldsymbol{V}_o{}^T \boldsymbol{H} \boldsymbol{V}_o, \\ \boldsymbol{A}_o = \boldsymbol{J}^{-1}[\boldsymbol{A}_{ri} - \boldsymbol{V}_{ri}^T(\boldsymbol{J}^{-1})^T \boldsymbol{H} \boldsymbol{J}^{-1} \boldsymbol{V}_{ri}], \\ \boldsymbol{A}_{ri} = \begin{pmatrix} a_{r1} & a_{r2} & a_{r3} & a_{r4} & a_{r5} & 0 \end{pmatrix}^T, \\ \boldsymbol{H} = \begin{pmatrix} \boldsymbol{h}_{r1} & \boldsymbol{h}_{r2} & \boldsymbol{h}_{r3} & \boldsymbol{h}_{r4} & \boldsymbol{h}_{r5} & \boldsymbol{h}_c \end{pmatrix}^T \end{array} \tag{19}$$

here, $\boldsymbol{H}$ is a 6 layer 6×6 Hessian matrix of the novel PM.

4. Workspace

The reachable workspace and the orientation workspace of PMs are two important indices to evaluate their performance and dexterity.[25] In this section, a reachable workspace W of the novel PM is constructed by utilizing CAD variation geometry.[29] Since this PM has a symmetry structure in OYZ plane, its W is also symmetry in OXZ plane and is formed by an upper surface S_u and a lower surface S_l. Let $r_{i\max}$, $r_{i\min}$ and Δr_i (i =1, 2, 3, 4, 5) be the maximum extension of r_i, the maximum extension of r_i and the variation increment of r_i at each step. When set the basic parameters e =150, E_i =270 (i =1, 3), E_i =265 (i =2, 4, 5), $r_{i\max}$ =850, $r_{i\min}$ =650, Δr_i =10 mm, the reachable workspace W of the novel 5-DoF is constructed by utilizing Matlab and is transformed into Solidwork by utilizing CAD variation geometry, see Fig. 3. The volume dimensions of W in (X, Y, Z) directions are (1350, 1371, 511) mm, respectively.

Let (ϕ_{zX}, ϕ_{zY}, ϕ_{zZ}) be the angles between z and (X, Y, Z), respectively. The (ϕ_{zX}, ϕ_{zY}, ϕ_{zZ}) of the novel PM are solved by utilizing CAD variation geometry and the simulation mechanism, see Table I. It is known that the novel PM has better property of isotropy horizontal motion.

Table I. Orientation of the novel PM.

r_1, mm	r_2, mm	r_3, mm	r_4, mm	r_5, mm	ϕ_{zX} ,°	ϕ_{zY} ,°	ϕ_{zZ} ,°
708.14	650	715.46	850	850	90	27.17	62.86
769.10	850	769.10	650	650	90	136.94	46.94
650	719.17	850	850	734.64	56.40	93.10	33.78
850	720.70	650	732.91	850	122.19	88.85	32.21
850	850	732.44	650	720.05	113.51	114	34.72
732.59	850	850	719.93	650	66.25	113.89	34.83

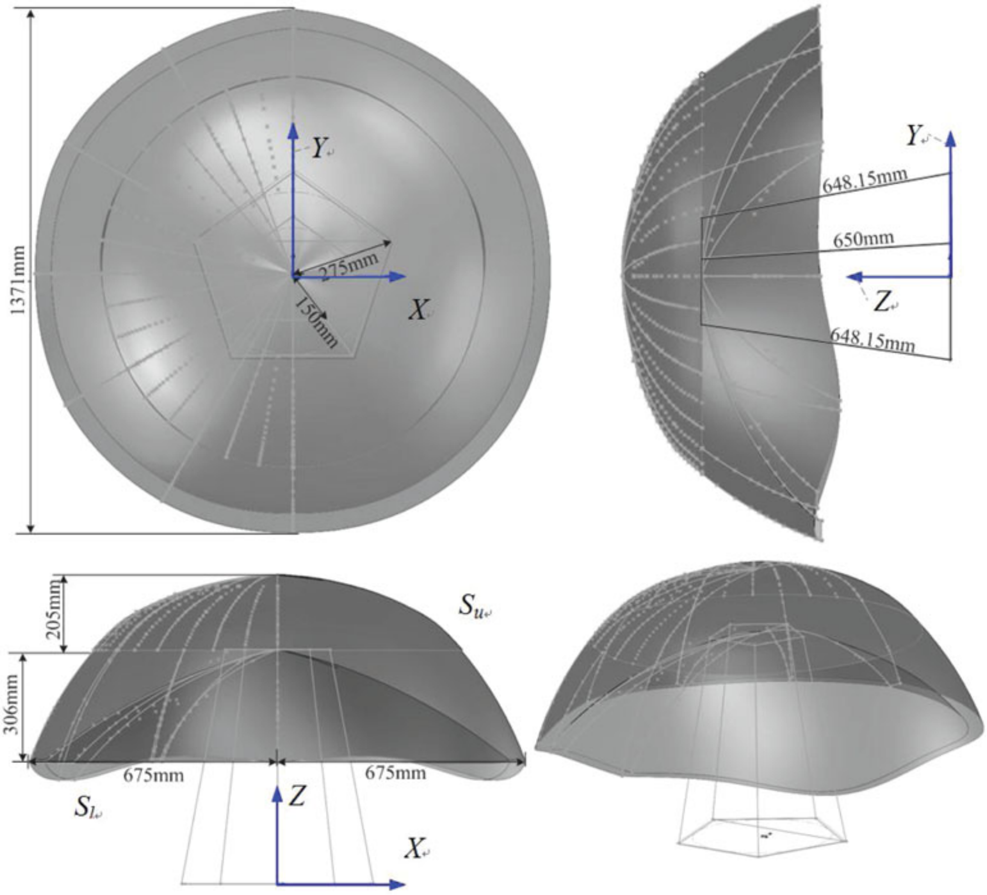

Fig. 3. A reachable workspace of novel PM. A top view (a), a side view (b), front view (c) and isometric view of 3D (d).

5. Kinematics of Active Legs r_i and Connected Rod L

Let $\boldsymbol{r}_i$ (i =1, ..., 5) and $\boldsymbol{\delta}_i$ be the vector of the active leg r_i and its unit vector, respectively. Let $\boldsymbol{e}_i$ be the vector from o to b_i. Let $\boldsymbol{v}_i$ be the translational velocity of r_i at b_i. Let $\boldsymbol{\omega}_{ri}$ and $\boldsymbol{\epsilon}_{ri}$ be the angular velocity and angular acceleration of r_i. Let v_{ri} be the scalar velocity along r_i.

The relative formulas among ($\boldsymbol{r}_i$, $\boldsymbol{e}_i$, $\boldsymbol{v}$, $\boldsymbol{\omega}$, $\boldsymbol{v}_i$, $\boldsymbol{\omega}_{ri}$ and v_{ri}) can be represented as follows:

$$\boldsymbol{v}_i = v_{ri}\boldsymbol{\delta}_i + \boldsymbol{\omega}_{ri} \times \boldsymbol{r}_i,\ \boldsymbol{v}_i = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i,\ \boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i = \boldsymbol{\omega}_{ri} \times \boldsymbol{r}_i \tag{20}$$

Cross-multiply both sides of the first formula in Eq. (20) in the right by $\boldsymbol{r}_i$, it leads to

$$\boldsymbol{\delta}_i \times \boldsymbol{v}_i = \boldsymbol{\delta}_i \times v_{ri}\boldsymbol{\delta}_i + r_i\boldsymbol{\delta}_i \times (\boldsymbol{\omega}_{ri} \times \boldsymbol{\delta}_i) = r_i\boldsymbol{\delta}_i \times (\boldsymbol{\omega}_{ri} \times \boldsymbol{\delta}_i) = r_i\boldsymbol{\omega}_{ri} - r_i\boldsymbol{\delta}_i(\boldsymbol{\omega}_{ri} \cdot \boldsymbol{\delta}_i) \tag{21}$$

5.1. Kinematics of RPU type active legs r_i *(i=1, 3)*
Let $\boldsymbol{r}_i$ (i =1, 3) and $\boldsymbol{\delta}_i$ be the vector of the RPU-type active leg r_i. Let ω_{i1} and $\boldsymbol{R}_{i1}$ be the scalar angular velocities of r_i about L and its unit vector. Let ω_{i2} and $\boldsymbol{R}_{i2}$ be the scalar angular velocities of r_i about R_{Bi} and its unit vector. Since r_i (i =1, 3) are connected with B by L, $\boldsymbol{\omega}_{ri}$ (i =1, 3) are represented as follows:

$$\boldsymbol{\omega}_{ri} = \omega_{i1}\boldsymbol{R}_{i1} + \omega_{i2}\boldsymbol{R}_{i2},\ \boldsymbol{R}_{i1} = \boldsymbol{L}/|\boldsymbol{L}|,\ \boldsymbol{R}_{i2} = (\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)/|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i| \tag{22}$$

Cross-multiply both sides of Eq. (22) in the right by $\boldsymbol{r}_i$, from Eqs. (14), (20) and (22), it leads to

$$\begin{aligned}&\omega_{i1}\boldsymbol{R}_{i1} \times \boldsymbol{r}_i + \omega_{i2}\boldsymbol{R}_{i2} \times \boldsymbol{r}_i = \boldsymbol{\omega}_{ri} \times \boldsymbol{r}_i = \boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i = \boldsymbol{v}_i - (\boldsymbol{v}_i \cdot \boldsymbol{\delta}_i)\boldsymbol{\delta}_i \\ &= (\boldsymbol{I} - \boldsymbol{\delta}_i\boldsymbol{\delta}_i^T)\boldsymbol{v}_i = -s^2(\boldsymbol{\delta}_i)\boldsymbol{v}_i = -s^2(\boldsymbol{\delta}_i)(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i) = -s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}]\end{aligned} \tag{23}$$

Dot multiply both sides of Eq. (23) in the right by $\boldsymbol{R}_{i1}$ and $\boldsymbol{R}_{i2}$, respectively, it leads to

$$\begin{aligned}&\omega_{i2}(\boldsymbol{R}_{i2} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i1} = \omega_{i2}D_{i1} = -\boldsymbol{R}_{i1}^T s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}], \\ &\omega_{i1}(\boldsymbol{R}_{i1} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i2} = -\omega_{i1}D_{i1} = -\boldsymbol{R}_{i2}^T s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}],\ D_{i1} = (\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{r}_i\end{aligned} \tag{24}$$

From Eq. (24), it leads to

$$\omega_{i1} = \boldsymbol{R}_{i2}^T s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}]\big/D_{i1}, \omega_{i2} = -\boldsymbol{R}_{i1}^T s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}]\big/D_{i1} \tag{25}$$

Substitute Eq. (25) into Eq. (22), a formula for solving $\boldsymbol{\omega}_{ri}$ is derived as follows:

$$\begin{aligned}&\boldsymbol{\omega}_{ri} = \omega_{i1}\boldsymbol{R}_{i1} + \omega_{i2}\boldsymbol{R}_{i2} = (\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T)s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}]\big/D_{i1}, \\ &\boldsymbol{\omega}_{ri} = \boldsymbol{J}_{\omega i}\boldsymbol{V},\ \boldsymbol{J}_{\omega i} = \boldsymbol{D}_{i2}s^2(\boldsymbol{\delta}_i)\left(\boldsymbol{I} - s(\boldsymbol{e}_i)\right)/D_{i1},\ \boldsymbol{D}_{i2} = \boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T\end{aligned} \tag{26}$$

In order to solve $\boldsymbol{\epsilon}_{ri}$, Eq. (26) is represented as follows:

$$\begin{aligned}&\boldsymbol{\omega}_{ri} = (\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T)s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}]\big/D_{i1} = D_{i3}/D_{i1}, \\ &D_{i3} = (\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T)s^2(\boldsymbol{\delta}_i)[\boldsymbol{v} - s(\boldsymbol{e}_i)\boldsymbol{\omega}] = -\boldsymbol{D}_{i2}(\boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i)\end{aligned} \tag{27}$$

Differentiate Eq. (27), a unified formula for solving $\boldsymbol{\varepsilon}_{ri}$ of r_i (i =1, 3) is derived as follows:

$$\begin{aligned}&\boldsymbol{\varepsilon}_{ri} = (D'_{i3}D_{i1} - D'_{i1}D_{i3})\big/D_{i1}^2,\ \boldsymbol{\delta}'_i = (\boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i)\big/r_i, \\ &D'_{i3} = (\boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T - \boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T)'(\boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i) + (\boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T - \boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T)(\boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i)' \\ &\quad = [(\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})\boldsymbol{R}_{i1}^T - \boldsymbol{R}_{i1}(\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})^T](\boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i) + (\boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T - \boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T)(\boldsymbol{a}_i - a_{ri}\boldsymbol{\delta}_i - v_{ri}\boldsymbol{\delta}'_i), \\ &D'_{i1} = [(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2})\boldsymbol{r}_i]' = (\boldsymbol{R}_{i1} \times \boldsymbol{R}'_{i2})\boldsymbol{r}_i + (\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2})\boldsymbol{r}'_i = [\boldsymbol{R}_{i1} \times (\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})]\boldsymbol{r}_i + (\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2})\boldsymbol{v}_i\end{aligned} \tag{28}$$

5.2. Kinematics of SPU-type active legs r_i *(i=2, 4, 5)*
Let R_{ij} (j =1, 2; i =2, 4, 5) be the two crossed revolute joints of the universal joint U_i of r_i at b_i, and $R_{i1} \perp R_{i2}$ is satisfied. Let $\boldsymbol{R}_{ij}$ be the unit vectors of R_{ij} joints. Let ϕ'_{ij} (j =1, 2) be the scalar angular velocities about R_{ij}. A kinematic equation among ($\boldsymbol{\omega}$, $\boldsymbol{\omega}_{ri}$, ϕ'_{ij}) is represented as follows:

$$\boldsymbol{\omega}_{ri} + \phi'_{i1}\boldsymbol{R}_{i1} + \phi'_{i2}\boldsymbol{R}_{i2} = \boldsymbol{\omega},\ \boldsymbol{\omega}_{ri} = \boldsymbol{\omega} - \phi'_{i1}\boldsymbol{R}_{i1} - \phi'_{i2}\boldsymbol{R}_{i2} \tag{29}$$

Cross-multiply both sides of Eq. (29) in the right by $\boldsymbol{r}_i$ (i = 2, 4, 5), from Eq. (4), it leads to

$$\boldsymbol{\omega} \times \boldsymbol{r}_i - \phi'_{i1}\boldsymbol{R}_{i1} \times \boldsymbol{r}_i - \phi'_{i2}\boldsymbol{R}_{i2} \times \boldsymbol{r}_i = \boldsymbol{\omega}_{ri} \times \boldsymbol{r}_i \tag{30}$$

Dot multiply both sides of Eq. (30) in the right by $\boldsymbol{R}_{ij}$ (j =1, 2), respectively, it leads to

$$\begin{aligned}&(\boldsymbol{\omega} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i1} - (\phi'_{i2}\boldsymbol{R}_{i2} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i1} = \hat{\boldsymbol{\delta}}_i^2(-\boldsymbol{v} + \hat{\boldsymbol{e}}_i\boldsymbol{\omega})\boldsymbol{R}_{i1}, \\ &(\boldsymbol{\omega} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i2} - (\phi'_{i1}\boldsymbol{R}_{i1} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i2} = \hat{\boldsymbol{\delta}}_i^2(-\boldsymbol{v} + \hat{\boldsymbol{e}}_i\boldsymbol{\omega})\boldsymbol{R}_{i2}\end{aligned} \tag{31}$$

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From Eq. (31), the formulas for solving ϕ'_{ij} (i =2, 4, 5; j =1, 2) are derived as follows:

$$\phi'_{i1} = [-(\hat{\boldsymbol{r}}_i\boldsymbol{R}_{i2})\boldsymbol{\omega} - \hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega})\boldsymbol{R}_{i2}]/d_{i1},\ d_{i1}=(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2})\boldsymbol{r}_i,\ \phi'_{i2}= [(\hat{\boldsymbol{r}}_i\boldsymbol{R}_{i1})\boldsymbol{\omega} + \hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega})\boldsymbol{R}_{i1}]/d_{i1} \tag{32}$$

From Eqs. (30) and (32), $\boldsymbol{\omega}_{ri}$ (i =2, 4, 5) are derived as follows:

$$\begin{aligned}
\boldsymbol{\omega}_{ri} &= \boldsymbol{\omega} - \phi'_{i1}\boldsymbol{R}_{i1} - \phi'_{i2}\boldsymbol{R}_{i2}\\
&= \boldsymbol{\omega} - \{[-(\hat{\boldsymbol{r}}_i\boldsymbol{R}_{i2})\boldsymbol{\omega} - \hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega})\boldsymbol{R}_{i2}]\boldsymbol{R}_{i1} - [(\hat{\boldsymbol{r}}_i\boldsymbol{R}_{i1})\boldsymbol{\omega} + \hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega})\boldsymbol{R}_{i1}]\boldsymbol{R}_{i2}\}/d_{i1}\\
&= \boldsymbol{\omega} - \tfrac{1}{d_{i1}}\{(\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T)\hat{\boldsymbol{r}}_i\boldsymbol{\omega} + (\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T)\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\boldsymbol{\omega} - (\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T)\hat{\boldsymbol{\delta}}_i^2\boldsymbol{v}\}\\
&= \boldsymbol{\omega} + \boldsymbol{d}_{i2}(\hat{\boldsymbol{\delta}}_i^2\boldsymbol{v} - \hat{\boldsymbol{r}}_i\boldsymbol{\omega} - \hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\boldsymbol{\omega})/d_{i1}= [\boldsymbol{d}_{i2}\hat{\boldsymbol{\delta}}_i^2\boldsymbol{v} + (d_{i1}\boldsymbol{I} - \boldsymbol{d}_{i2}\hat{\boldsymbol{r}}_i - \boldsymbol{d}_{i2}\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i)\boldsymbol{\omega}]/d_{i1},\\
\boldsymbol{\omega}_{ri} &= \boldsymbol{J}_{\omega ri}\boldsymbol{V}_o,\ \boldsymbol{J}_{\omega ri}=\tfrac{1}{d_{i1}}\left(\boldsymbol{d}_{i2}\hat{\boldsymbol{\delta}}_i^2\ \ d_{i1}\boldsymbol{I} - \boldsymbol{d}_{i2}\hat{\boldsymbol{\delta}}_i(r_i + \hat{\boldsymbol{\delta}}_i\hat{\boldsymbol{e}}_i)\right),\\
d_{i1} &= (\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{r}_i,\ \boldsymbol{d}_{i2}=\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T,\ \boldsymbol{R}_{i1} = \frac{-\boldsymbol{e}_i}{|\boldsymbol{e}_i|},\ \boldsymbol{R}_{i2} = \frac{\boldsymbol{R}_{i1} \times \boldsymbol{r}_i}{|\boldsymbol{R}_{i1} \times \boldsymbol{r}_i|},\ i= 2, 4, 5
\end{aligned} \tag{33}$$

In order to solve the angular acceleration of SPU-type active legs r_i (i =3, 4, 5), Eq. (33) and its relative items are written as follows:

$$\begin{aligned}
&\boldsymbol{\omega}_{ri}=\boldsymbol{\omega} + \boldsymbol{d}_{i2}\boldsymbol{D}_i\big/d_{i1},\ \boldsymbol{d}_{i2}\boldsymbol{D}_i\big/d_{i1}=\boldsymbol{\omega}_{ri} - \boldsymbol{\omega},\ \boldsymbol{D}_i=\hat{\boldsymbol{\delta}}_i^2\boldsymbol{v} - \hat{\boldsymbol{r}}_i\boldsymbol{\omega} - \hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\boldsymbol{\omega},\\
&v_{ri}=\boldsymbol{v}_i\boldsymbol{\delta}_i,\ a_{ri}=\boldsymbol{a}_i \cdot \boldsymbol{\delta}_i + \boldsymbol{v}_i \cdot \boldsymbol{\delta}'_i,\ \boldsymbol{\delta}'_i= (\boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i)/r_i,\ \boldsymbol{a}_i=\boldsymbol{a}+\boldsymbol{\varepsilon} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{e}_i),\\
&d_{i1}=(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2})^T\boldsymbol{r}_i,\ \boldsymbol{d}_{i2}=\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T
\end{aligned} \tag{34a}$$

Here, d'_{i1} and d'_{i2} are derived as follows:

$$\begin{aligned}
d'_{i1} &= (\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2})' \cdot \boldsymbol{r}_i + (\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{r}'_i\\
&= (\boldsymbol{R}'_{i1} \times \boldsymbol{R}_{i2} + \boldsymbol{R}_{i1} \times \boldsymbol{R}'_{i2}) \cdot \boldsymbol{r}_1+(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{v}_i\\
&= [(\boldsymbol{\omega} \times \boldsymbol{R}_{i1}) \times \boldsymbol{R}_{i2} + \boldsymbol{R}_{i1} \times (\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})] \cdot \boldsymbol{r}_1+(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{v}_i\\
&= (\boldsymbol{\omega} \times \boldsymbol{R}_{i1}) \cdot (\boldsymbol{R}_{i2} \times \boldsymbol{r}_1) + (\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2}) \cdot (\boldsymbol{r}_1 \times \boldsymbol{R}_{i1})+(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{v}_i\\
&= (\boldsymbol{\omega} \cdot \boldsymbol{R}_{i2})(\boldsymbol{R}_{i1} \cdot \boldsymbol{r}_1) - (\boldsymbol{\omega} \cdot \boldsymbol{r}_1)(\boldsymbol{R}_{i1} \cdot \boldsymbol{R}_{i2})\\
&\quad +(\boldsymbol{\omega}_{ri} \cdot \boldsymbol{r}_1)(\boldsymbol{R}_{i2} \cdot \boldsymbol{R}_{i1}) - (\boldsymbol{\omega}_{ri} \cdot \boldsymbol{R}_{i1})(\boldsymbol{R}_{i2} \cdot \boldsymbol{r}_1)+(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{v}_i\\
&= (\boldsymbol{\omega} \cdot \boldsymbol{R}_{i2})(\boldsymbol{R}_{i1} \cdot \boldsymbol{r}_1)+(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{v}_i,\\
\boldsymbol{d}'_{i2}&=\boldsymbol{R}'_{i1}\boldsymbol{R}_{i2}^T + \boldsymbol{R}_{i1}\boldsymbol{R}'^T_{i2} - \boldsymbol{R}'_{i2}\boldsymbol{R}_{i1}^T - \boldsymbol{R}_{i2}\boldsymbol{R}'^T_{i1}\\
&= (\boldsymbol{\omega} \times \boldsymbol{R}_{i1})\boldsymbol{R}_{i2}^T + \boldsymbol{R}_{i1}(\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})^T - (\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})\boldsymbol{R}_{i1}^T - \boldsymbol{R}_{i2}(\boldsymbol{\omega} \times \boldsymbol{R}_{i1})^T
\end{aligned} \tag{34b}$$

Differentiate Eq. (34a), a unified formula for solving the angular accelerations $\boldsymbol{\epsilon}_{ri}$ of r_i (i =2, 4, 5) is derived based on Eq. (34b) as follows:

$$\begin{aligned}
&\boldsymbol{\varepsilon}_{ri}=\boldsymbol{\varepsilon} + [(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri})d'_{i1} + \boldsymbol{d}'_{i2}\boldsymbol{D}_i+\boldsymbol{d}_{i2}\boldsymbol{D}'_i]\big/d_{i1},\ (i= 2\ 4,\ 5),\\
&\boldsymbol{\varepsilon}_{ri}=\boldsymbol{\varepsilon}+\{(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri})[(\boldsymbol{\omega} \cdot \boldsymbol{R}_{i2})(\boldsymbol{R}_{i1} \cdot \boldsymbol{r}_1)+(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) \cdot \boldsymbol{v}_i]\\
&\quad +[(\boldsymbol{\omega} \times \boldsymbol{R}_{i1})\boldsymbol{R}_{i2}^T+\boldsymbol{R}_{i1}(\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})^T - (\boldsymbol{\omega}_{ri} \times \boldsymbol{R}_{i2})\boldsymbol{R}_{i1}^T - \boldsymbol{R}_{i2}(\boldsymbol{\omega} \times \boldsymbol{R}_{i1})^T](v_{ri}\boldsymbol{\delta}_i - \boldsymbol{v}_i - \boldsymbol{r}_i \times \boldsymbol{\omega})\\
&\quad +(\boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^T - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^T)[a_{ri}\boldsymbol{\delta}_i - \boldsymbol{a}_i - \boldsymbol{r}_i \times \boldsymbol{\varepsilon} - \boldsymbol{v}_i \times \boldsymbol{\omega}+v_{ri}(\boldsymbol{v}_i - v_{ri}\boldsymbol{\delta}_i)/r_i]\}/[(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2})^T\boldsymbol{r}_i]
\end{aligned} \tag{35}$$

5.3. Kinematics of connected rod L

Let $(\boldsymbol{v}_L, \boldsymbol{a}_L, \boldsymbol{\omega}_L, \boldsymbol{\epsilon}_L, \boldsymbol{V}_L)$ be the translational velocity, translational acceleration, angular velocity, angular acceleration and general velocity of L. Since L can be moved in translation, there is $\boldsymbol{v}_L = \boldsymbol{a}_L = \boldsymbol{0}$. A formula for solving $\boldsymbol{\omega}_L$ and $\boldsymbol{V}_L$ is derived from the first formula of Eq. (25) as follows:

$$\begin{aligned}
&\boldsymbol{\omega}_L=\omega_{11}\boldsymbol{R}_{11}=\frac{\boldsymbol{R}_{11}\boldsymbol{R}_{12}^T(\hat{\boldsymbol{\delta}}_1^2\boldsymbol{v} - \hat{\boldsymbol{\delta}}_1^2\hat{\boldsymbol{e}}_1\boldsymbol{\omega})}{(\boldsymbol{R}_{11} \times \boldsymbol{R}_{12}) \cdot \boldsymbol{r}_1} = \boldsymbol{J}_{\omega L}\boldsymbol{V}_o,\ \boldsymbol{J}_{\omega L} = \frac{\boldsymbol{R}_{11}\boldsymbol{R}_{12}^T}{D_{11}}\hat{\boldsymbol{\delta}}_1^2\left(\boldsymbol{I}\ -\hat{\boldsymbol{e}}_1\right),\\
&\boldsymbol{\omega}_L=D_L\big/D_{11},\ \ D_L = \boldsymbol{R}_{11}\boldsymbol{R}_{12}^T(v_{r1}\boldsymbol{\delta}_1 - \boldsymbol{v}_1),\ D_{11} = (\boldsymbol{R}_{11} \times \boldsymbol{R}_{12}) \cdot \boldsymbol{r}_1,\\
&\boldsymbol{V}_L = \begin{pmatrix}\boldsymbol{v}_L\\ \boldsymbol{\omega}_L\end{pmatrix} = \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{\omega}_L\end{pmatrix} =\boldsymbol{J}_L\boldsymbol{V}_o,\ \ \boldsymbol{J}_L = \begin{pmatrix}\boldsymbol{0}\\ \boldsymbol{J}_{\omega L}\end{pmatrix}
\end{aligned} \tag{36}$$

Differentiate Eq. (36), a formula for solving $\boldsymbol{\varepsilon}_L$ is derived as follows:

$$\begin{aligned}&\boldsymbol{\varepsilon}_L=\frac{D'_L D_{11}-D'_{11}D_L}{D_{11}^2},\quad \boldsymbol{\delta}'_1=(\boldsymbol{v}_1-v_{r1}\boldsymbol{\delta}_1)/r_1,\\&D'_L=\boldsymbol{R}_{11}(\boldsymbol{\omega}_{r1}\times\boldsymbol{R}_{12})^T(v_{r1}\boldsymbol{\delta}_1-\boldsymbol{v}_1)+\boldsymbol{R}_{11}\boldsymbol{R}_{12}^T(a_{r1}\boldsymbol{\delta}_1+v_{r1}\boldsymbol{\delta}'_1-\boldsymbol{a}_1)\end{aligned}\tag{37}$$

5.4. Velocity and acceleration of piston and cylinder of active legs r_i *(i=1, ..., 5)*

Each of RPU-type active legs r_i (i =1, 3) and SPU-type active legs r_i (i =2, 4, 5) is composed of a piston rod and a cylinder. The piston rod of r_i is connected with m at b_i. The cylinder of r_i is connected with B at B_i. Let p_i be the mass center of the piston rod in r_i, l_{pi} be the distance from b_i to p_i, q_i be the mass center of the cylinder in r_i and l_{qi} be the distance from B_i to q_i, see Fig. 2b. Let ($\boldsymbol{v}_g$, $\boldsymbol{\omega}_g$, $\boldsymbol{V}_g$, $\boldsymbol{a}_g$, $\boldsymbol{\varepsilon}_g$) be the translational velocity, the angular velocity, the general velocity, the translational acceleration and the angular acceleration of g ($g=p_i, q_i$), respectively. $\boldsymbol{v}_g$, $\boldsymbol{\omega}_g$, $\boldsymbol{V}_g$ ($g=p_i, q_i$) in B are derived as follows:

$$\begin{aligned}&\boldsymbol{v}_{pi}=v_{ri}\boldsymbol{\delta}_{ri}+\boldsymbol{\omega}_{ri}\times(r_i-l_{pi})\boldsymbol{\delta}_{ri}=\boldsymbol{J}_{vpi}\boldsymbol{V}_o,\quad \boldsymbol{\omega}_{pi}=\boldsymbol{\omega}_{ri}=\boldsymbol{J}_{\omega ri}\boldsymbol{V}_o,\ (i=1,\cdots,5),\\&\boldsymbol{J}_{vpi}=\boldsymbol{\delta}_{ri}[\,\boldsymbol{\delta}_{ri}^T\ (\boldsymbol{e}_i\times\boldsymbol{\delta}_{ri})^T\,]-(r_i-l_{pi})\hat{\boldsymbol{\delta}}_{ri}\boldsymbol{J}_{\omega ri},\ \ \boldsymbol{v}_{qi}=\boldsymbol{\omega}_{ri}\times l_{qi}\boldsymbol{\delta}_{ri}=-\,l_{qi}\hat{\boldsymbol{\delta}}_{ri}\boldsymbol{J}_{\omega ri}\boldsymbol{V}_o,\ \boldsymbol{\omega}_{qi}=\boldsymbol{J}_{\omega ri}\boldsymbol{V}_o,\\&\boldsymbol{V}_{pi}=\begin{pmatrix}\boldsymbol{v}_{pi}\\\boldsymbol{\omega}_{pi}\end{pmatrix}=\boldsymbol{J}_{pi}\boldsymbol{V}_o,\ \boldsymbol{V}_{qi}=\begin{pmatrix}\boldsymbol{v}_{qi}\\\boldsymbol{\omega}_{qi}\end{pmatrix}=\boldsymbol{J}_{qi}\boldsymbol{V}_o,\ \ \boldsymbol{J}_{pi}=\begin{pmatrix}\boldsymbol{J}_{vpi}\\\boldsymbol{J}_{\omega ri}\end{pmatrix},\ \boldsymbol{J}_{qi}=\begin{pmatrix}-l_{qi}\hat{\boldsymbol{\delta}}_{ri}\boldsymbol{J}_{\omega ri}\\\boldsymbol{J}_{\omega ri}\end{pmatrix}\end{aligned}\tag{38}$$

Differentiate $\boldsymbol{v}_g$ and $\boldsymbol{\omega}_g$ ($g=p_i, q_i$), respectively, in Eq. (38) with respect to time, $\boldsymbol{a}_g$, $\boldsymbol{\varepsilon}_g$ ($g=p_i, q_i$) are derived and represented as follows:

$$\begin{aligned}&\boldsymbol{a}_{pi}=a_{ri}\boldsymbol{\delta}_{ri}-(r_i-l_{pi})\hat{\boldsymbol{\delta}}_{ri}\boldsymbol{\varepsilon}_{ri}-2v_{ri}\hat{\boldsymbol{\delta}}_{ri}\boldsymbol{\omega}_{ri}+(r_i-l_{pi})\boldsymbol{\omega}_{ri}\times(\boldsymbol{\omega}_{ri}\times\boldsymbol{\delta}_{ri}),\\&\boldsymbol{a}_{qi}=-\,l_{qi}\hat{\boldsymbol{\delta}}_{ri}\boldsymbol{\varepsilon}_{ri}+l_{qi}\boldsymbol{\omega}_{ri}\times(\boldsymbol{\omega}_{ri}\times\boldsymbol{\delta}_{ri}),\ \ \boldsymbol{\varepsilon}_{pi}=\boldsymbol{\varepsilon}_{qi}=\boldsymbol{\varepsilon}_{ri},\ (i=1,\ \cdots,5)\end{aligned}\tag{39}$$

6. Dynamics of Novel PM with Couple-Constrained Wrench

Let $\boldsymbol{f}_\tau, \boldsymbol{t}_\tau, m_\tau, \boldsymbol{G}_\tau, \boldsymbol{I}_\tau$ and $\tau=o, L, p_i, q_i$, (i =1, ..., 5) be the inertial force, the inertial torque, the mass, the gravity, the inertial moment and the general inertial wrench of the moving links at their masse centers in B, respectively. Let ($\boldsymbol{F}_o, \boldsymbol{T}_o$) be a working-load wrench applied on m at o. ($\boldsymbol{f}_d, \boldsymbol{t}_d$) be the damping force and torque applied on m at o. Let μ be a damping coefficient. ($\boldsymbol{f}_\tau, \boldsymbol{t}_\tau, \boldsymbol{G}_\tau, \boldsymbol{I}_\tau, \boldsymbol{f}_d, \boldsymbol{t}_d$) are represented and solved as follows:

$$\begin{aligned}&\boldsymbol{G}_\tau=m_\tau\boldsymbol{g},\ \ \boldsymbol{f}_\tau=-m_\tau\boldsymbol{a}_\tau,\ \boldsymbol{t}_\tau=-\boldsymbol{I}_\tau\boldsymbol{\varepsilon}_\tau,\\&\boldsymbol{f}_d=-\mu\boldsymbol{v},\ \ \boldsymbol{t}_d=-\mu\boldsymbol{\omega},\ (\tau=o,\ L,\ p_i,\ q_i;\ i=1,\ \cdots,5)\end{aligned}\tag{40}$$

Let $\boldsymbol{F}_r$, be the general dynamic input forces and constrained wrench. Let $\boldsymbol{V}_r$ be the general input velocity. They are represented as follows:

$$\begin{aligned}&\boldsymbol{F}_r=\left(f_{a1}\ f_{a2}\ f_{a3}\ f_{a4}\ f_{a5}\ f_c\right)^T\\&\boldsymbol{V}_r=\left(v_{r1}\ v_{r2}\ v_{r3}\ v_{r4}\ v_{r5}\ 0\right)^T=\boldsymbol{J}\boldsymbol{V}_o\end{aligned},\quad \boldsymbol{F}=\begin{pmatrix}F_x\\F_y\\F_z\end{pmatrix},\quad \boldsymbol{T}=\begin{pmatrix}T_x\\T_y\\T_z\end{pmatrix}\tag{41}$$

When ignoring the friction of all the joints in the novel mechanism, based on the principle of virtual work, a power equation is derived as follows:

$$\boldsymbol{F}_r^T\boldsymbol{V}_r+\begin{pmatrix}\boldsymbol{F}_o+\boldsymbol{G}_o+\boldsymbol{f}_o+\boldsymbol{f}_d\\\boldsymbol{T}_o+\boldsymbol{t}_o+\boldsymbol{t}_d\end{pmatrix}^T\boldsymbol{V}_o+\begin{pmatrix}\boldsymbol{G}_L+\boldsymbol{f}_L\\\boldsymbol{t}_L\end{pmatrix}^T\boldsymbol{V}_L+\sum_{i=1}^{n=5}\left[\begin{pmatrix}\boldsymbol{G}_{pi}+\boldsymbol{f}_{pi}\\\boldsymbol{t}_{pi}\end{pmatrix}^T\boldsymbol{V}_{pi}+\begin{pmatrix}\boldsymbol{G}_{qi}+\boldsymbol{f}_{qi}\\\boldsymbol{t}_{qi}\end{pmatrix}^T\boldsymbol{V}_{qi}\right]=0\tag{42}$$

When considering the friction of all the joints in the mechanism, a coefficient ($\eta\le$1) of this novel PM can be added here. Thus, substitute Eqs. (25) and (38) into Eq. (42), a formula for solving the

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Table II. Given geometric parameters of m and B, mass and inertial moment of moving links, and workloads of novel PM.

Symbol	Value, unit	Symbol	Value, unit
e_i (i =1, 2, 3, 4, 5)	150 mm	$\boldsymbol{F}_o$	$(10\ 10\ 500)^T$ N
E_i (i =1, 3)	270 mm	$\boldsymbol{T}_o$	$(0\ 0\ 1000)^T$ N·m
E_i (i =2, 4, 5)	265 mm	$\theta_1, \theta_2, \theta_3, \theta_4, \theta_5$	18°, 90°, 162°, 234°, 306°
η, μ	1, 0	$\boldsymbol{I}_o, \boldsymbol{I}_l, \boldsymbol{I}_{gi}, \boldsymbol{I}_{qi}$	5$\boldsymbol{I}$, 2$\boldsymbol{I}$, 3$\boldsymbol{I}$, 2$\boldsymbol{I}$ kg·m2
$l_{p1} = l_{p3}, l_{p2} = l_{p4} = l_{p5}, l_{qi}$	100, 98, 50 mm	m_o, m_l, m_{gi}, m_{qi}	3, 2, 3, 2 kg

general dynamic input forces and constrained wrench is derived as follows:

$$\boldsymbol{F}_r = -(\boldsymbol{J}^{-1})^T \left\{ \frac{1}{\eta} \begin{pmatrix} \boldsymbol{F}_o + \boldsymbol{G}_o + \boldsymbol{f}_o + \boldsymbol{f}_d \\ \boldsymbol{T}_o + \boldsymbol{t}_o + \boldsymbol{t}_d \end{pmatrix} + \boldsymbol{J}_l^T \begin{pmatrix} \boldsymbol{G}_l + \boldsymbol{f}_l \\ \boldsymbol{t}_l \end{pmatrix} + \sum_{i=1}^{n=5} \left[\boldsymbol{J}_{pi}^T \begin{pmatrix} \boldsymbol{G}_{pi} + \boldsymbol{f}_{pi} \\ \boldsymbol{t}_{pi} \end{pmatrix} + \boldsymbol{J}_{qi}^T \begin{pmatrix} \boldsymbol{G}_{qi} + \boldsymbol{f}_{qi} \\ \boldsymbol{t}_{qi} \end{pmatrix} \right] \right\} \tag{43}$$

The general dynamic constrained force f_c can be solved from Eqs. (41) and (43). After that, the formulas for solving the dynamic couple-constrained forces f_{c1} and f_{c3} and the dynamic constrained torque $\boldsymbol{T}_c$ exerted on to L are derived as follows:

$$f_c = f_{c1} + f_{c3},\ f_{c1} = \frac{f_c}{1+k},\ f_{c3} = f_c - f_{c1},\ \boldsymbol{T}_{c1} = \boldsymbol{r}_1 \times f_{c1}\boldsymbol{c}_1,\ k = \frac{(\boldsymbol{r}_1 \times \boldsymbol{c}_1)\cdot \boldsymbol{L}}{(\boldsymbol{r}_3 \times \boldsymbol{c}_1)\cdot \boldsymbol{L}} \tag{44}$$

7. Numerical Example of Kinematics/Dynamics and Analysis

The geometric parameters of the moving platform m, the base B and the workloads ($\boldsymbol{F}_o, \boldsymbol{T}_o$) exerted on m are given in Table II. A program is compiled in Matlab based on the derived relative analytic formulas and the parameters in Table II.

The solving processes of the analytical solutions are explained as follows:

1. Give the five displacement components ($X_o, Y_o, Z_o, \alpha, \beta$) of m at o in B of the novel PM, see Fig. 4a and f. Solve γ, and solve the velocities and accelerations of m at o in B of the proposed PM. The solved results are shown in Fig. 4b-f.
2. Solve r_i (i =1,..., 5) using Eq. 4. The solved results are shown in Fig. 4g.
3. Solve v_{ri} and a_{ri} of r_i of the novel 5-DoF PM. The solved results are shown in Fig. 4h and i.
4. Given the workloads ($\boldsymbol{F}_o, \boldsymbol{T}_o$) applied onto m in Table II, based on the solved r_i (i =1,..., 5) of active legs, solve the five dynamic active forces f_{ai} (i =1,..., 5) of the novel 5-DoF PM and the dynamic couple-constrained forces f_{c1}, f_{c3}. The solved results are shown in Fig. 4j and k.
5. Solve the dynamic constrained torque T_c exerted onto the connection rod, see Fig. 4l.

The analytic solutions are verified by the simulation solutions of a simulation mechanism in Matlab/Simulink/Mechanics. The characteristics of the novel PM are found from the solutions and are analyzed as follows:

1. When the displacement, translational velocity and translational acceleration of m are varied smoothly in a large range, the displacement, velocity and acceleration of r_i, the dynamic active forces, the dynamic couple-constrained force and dynamic couple-constrained torque are varied smoothly. It implies that the novel PM has good characteristics of the kinematics and dynamics.
2. Comparing with the dynamic active forces, the dynamic couple-constrained forces are quite small.

Errors between analytic solutions and simulation solutions are given in Table III. It is known from Table III that all derived analytical formulas of kinematics/dynamics are correct because the errors between the analytic solutions and the simulation solutions are very small.

Table III. Errors between analytic solutions and simulation solutions.

Absolute maximum errors	Unit	r_1	r_2	r_3	r_4	r_5
Displacement error $\Delta r_{i\max}$	mm	−1.7E−09	1.0E−09	2.5E−09	−9.7E−09	−1.4E−09
Velocity error $\Delta v_{ri\max}$	mm/s	−5.0E−08	3.8E−08	8.1E−08	1.1E−09	−5.0E−08
Acceleration error $\Delta a_{ri\max}$	mm/s^2	1.5E−09	1.3E−09	1.4E−09	1.3E−09	1.3E−09
Dynamic active force error $\Delta F_{ai\max}$	N	−0.017	0.019	−0.013	−0.003	0.009

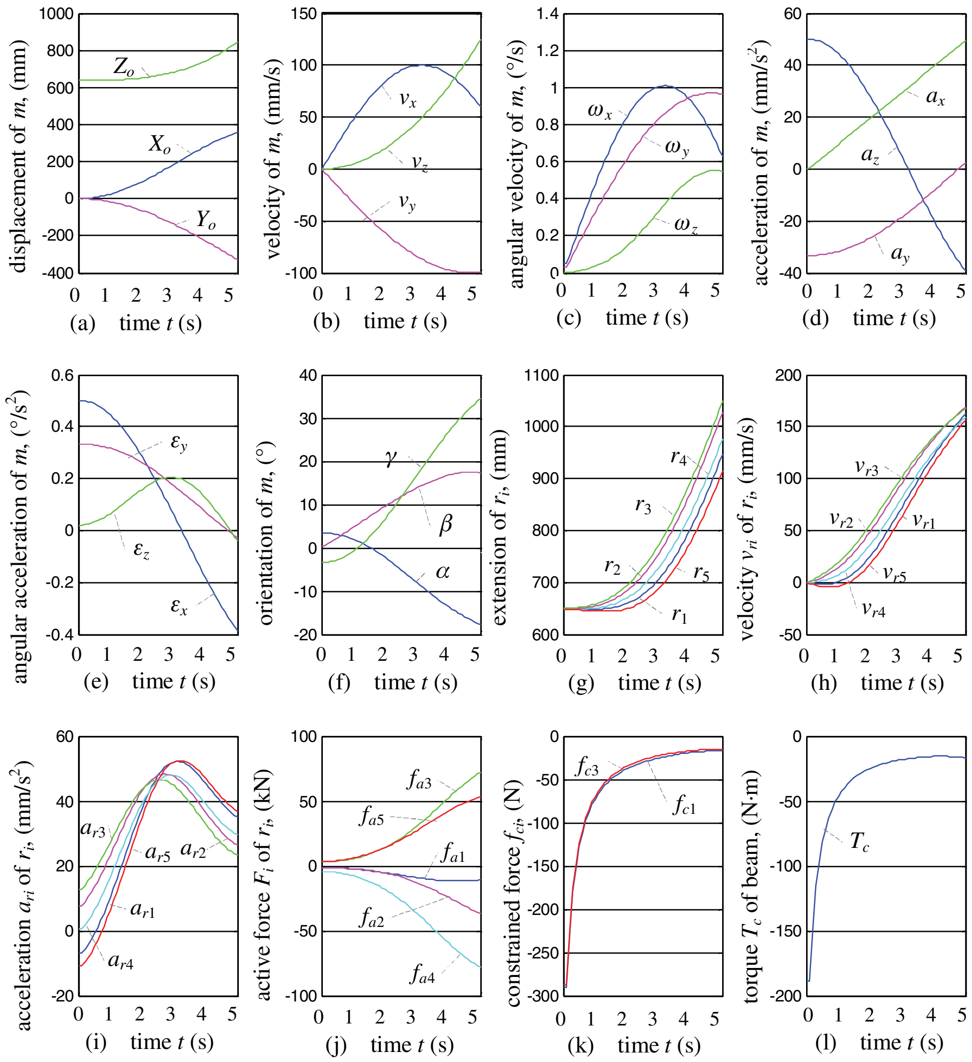

Fig. 4. Analytical solutions of kinematics and dynamics of novel PM.

8. Conclusions

A 3D model of the novel 3SPU+2RPU+R type parallel manipulator with the couple-constrained wrench is constructed. It has five DoFs and is composed of a quaternary link for the fixed base, a pentagonal link for the moving platform, a ternary link for the connection rod, two RPU-type active limbs and three SPU-type active limbs. The novel parallel manipulator is simple in structure and its capability of the load bearing and the rigidity are increased.

The kinematics formulas are derived for solving the displacement of the five active legs, Jacobian matrix, Hessian matrix, and the velocities/accelerations of the moving links for the novel parallel manipulator. Its dynamics formulas are derived for solving the dynamically active forces and the dynamically couple-constrained torque/force.

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All derived analytical formulas of kinematics/dynamics are verified to be correct by utilizing a simulation mechanism, and provide a theoretical foundation for the structure optimization, control, manufacturing and applications of the different 5-DoF parallel manipulators with the couple-constrained wrench.

Its position/orientation workspace is quite large and it has a good property of isotropy horizontal motion. It implies that the novel parallel manipulator with the couple-constrained wrench has a large moving workspace in any direction.

The novel parallel manipulator has potential applications for the hybrid hand, the surgical manipulator, the 5-DoF parallel machine tool, the tunnel borer, the barbette of warship, the human health robot and the satellite surveillance platform.

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