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Spatial segregation limit of a competition–diffusion system

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We consider a competition–diffusion system and study its singular limit as the interspecific competition rate tend to infinity. We prove the convergence to a Stefan problem with zero latent heat.

1 Introduction

The understanding of the interaction of biological species arising in ecological systems has recently developed as a central problem in population ecology. In particular, problems of coexistence and exclusion of competing species have been theoretically investigated using models based on partial and ordinary differential equations. Among many models proposed so far, reaction–diffusion equation models are used to study the spatial segregation of competing species which move by diffusion.

Consider a competing system which consists of n species living in a habitat $\Omega \subset \mathbf{R}^N$ ($N \geqslant 1$). We denote by $u_i(x,t)$ $(i = 1,2,\ldots,n)$ their population densities at position $x \in \Omega$ and time $t \geqslant 0$. The evolution of $u_i(x,t)$ $(i = 1,2,\ldots,n)$ is described by

$$u_{it} = d_i \Delta u_i + \left(r_i - a_i u_i - \sum_{j=1}^{n} b_{ij} u_j \right) u_i \quad (i = 1,2,\ldots,n) \quad x \in \Omega, \ t > 0, \tag{1.1}$$

where d_i is the diffusion rate, r_i the intrinsic growth rate, a_i the intraspecific competition rate, that is the competition between members of the *same* species u_i, and b_{ij} the interspecific competition rate, that is the competition between members of the *different* species u_i and u_j. All the rates are positive constants. We suppose that Ω is bounded and impose the no-flux boundary conditions on the boundary $\partial\Omega$,

$$\frac{\partial u_i}{\partial \nu} = 0, \quad (i = 1,2,\ldots,n) \quad x \in \partial\Omega, \ t > 0, \tag{1.2}$$

where ν is the outward normal unit vector to $\partial\Omega$. The initial conditions are given by

$$u_i(0,x) = u_{0i}(x) \geqslant 0 \quad (i = 1,2,\ldots,n) \quad x \in \Omega. \tag{1.3}$$

The long time behaviour of solutions of Problem (1.1)–(1.3) has been extensively analysed with the purpose of studying the spatio-temporal segregation of competing

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species. We first point out the special case when all the diffusion rates d_i are large with respect to the other parameters. In this situation, the diffusion processes are dominant, and therefore one easily finds that any (nonnegative) solution of Problem (1.1)–(1.3) tends to be spatially homogeneous as $t \to \infty$ [CHS]. In other words, the asymptotic behaviour of solutions of Problem (1.1)–(1.3) is qualitatively the same as that of the diffusionless system corresponding to (1.1),

$$\frac{dv_i}{dt} = \left(r_i - a_i v_i - \sum_{j=1}^{n} b_{ij} v_j \right) v_i \quad (i = 1, 2, \ldots, n) \quad t > 0. \tag{1.4}$$

Thus we know that in this case (1.1) exhibits no spatial segregation for competing species. We should note that (1.4) exhibits temporal segregation, depending on the values of the parameters r_i, a_i and b_{ij}. We shall not study this phenomenon here, but we refer for instance to an article by Mimura [M].

Our main interest in (1.1)–(1.3) involves the case when at least one of the diffusion coefficients d_i is not necessarily large from the viewpoint of spatial segregation of competing species. To analyse this case, we discuss the simplest case of (1.1) with $n = 2$, namely

$$\begin{cases} u_{1t} = d_1 \Delta u_1 + (r_1 - a_1 u_1 - b_1 u_2) u_1 & x \in \Omega, t > 0 \quad (1.5a) \\ u_{2t} = d_2 \Delta u_2 + (r_2 - a_2 u_2 - b_2 u_1) u_2 & x \in \Omega, t > 0. \quad (1.5b) \end{cases}$$

with the boundary conditions

$$\frac{\partial u_1}{\partial v} = 0, \quad \frac{\partial u_2}{\partial v} = 0, \quad x \in \partial\Omega, \quad t > 0. \tag{1.6}$$

We first note that the stable attractor of (1.5), (1.6) consists of equilibrium solutions only [H, MM]. Therefore, for the study of the asymptotic behaviour of solutions of (1.5), (1.6) we only need to focus our attention on the existence and stability of equilibrium solutions. Along this line, Kishimoto & Weinberger [KW] showed that if Ω is convex, then any spatially inhomogeneous equilibrium solution – whenever it exists – is unstable. If we suppose that two species are strongly competing, that is if the interspecific competition rate is stronger than the intraspecific one so that we require that

$$\frac{a_1}{b_2} < \frac{r_1}{r_2} < \frac{b_1}{a_2}, \tag{1.7}$$

then one finds that the only stable equilibrium solutions of (1.5), (1.6) are given by $(u_1, u_2) = (r_1/a_1, 0)$ and $(u_1, u_2) = (0, r_2/a_2)$. In ecological terms, this implies that the two competing species can never coexist under strong competition. This is called Gause's competitive exclusion.

On the other hand, if the domain Ω is not convex, the structure of equilibrium solutions is complicated, depending on the shape of Ω [EFM]. In fact, if Ω takes a suitable dumbbell shape in two dimensions, there exist stable spatially inhomogeneous equilibrium solutions which exhibit spatial segregation in the sense that u_1 and u_2 take values close to (r_1/a_1) in one subregion and close to $(0, r_2/a_2)$ in the other one. Thus, the results above give us information on the asymptotic behaviour of solutions. However, from the viewpoint of ecological applications, it is more interesting to know the transient behaviour of solutions. For this purpose, we consider the situation where the diffusion rates d_1 and

d_2 are sufficiently small or all of the other rates r_i, a_i and b_i are sufficiently large and satisfy (1.7). We rewrite (1.5) as

$$\begin{cases} u_{1t} = \varepsilon^2 \Delta u_1 + (r_1 - a_1 u_1 - b_1 u_2) u_1 & x \in \Omega, t > 0 \quad (1.8a) \\ u_{2t} = d\varepsilon^2 \Delta u_2 + (r_2 - a_2 u_2 - b_2 u_1) u_2 & x \in \Omega, t > 0 \quad (1.8b) \end{cases}$$

in which ε is a small parameter. If the competing species segregate according to (1.8) it is natural to define the subregions $\Omega_1(t) = \{x \in \Omega : (u_1, u_2)(x, t) \approx (r_1/a_1, 0)\}$ and $\Omega_2(t) = \{x \in \Omega : (u_1, u_2)(x, t) \approx (0, r_2/a_2)\}$.

To study the dynamics of the segregation between u_1 and u_2, we take the limit $\varepsilon \downarrow 0$ in (1.8) so that the internal layers which exist for small values of $\varepsilon > 0$ become sharp interfaces, say $\Gamma(t)$, which is the boundary between two regions $\Omega_1(t)$ and $\Omega_2(t)$. Using singular limit analysis, Ei & Yanagida [EY] derived the following evolution equation to describe the motion of the interface $\Gamma(t)$:

$$V = \varepsilon L(d)(N-1)\kappa + c, \tag{1.9}$$

where V is the normal velocity of the interface, κ the mean curvature of the interface, $L(d)$ a positive constant depending on d such that $L(1) = 1$ and c the velocity of the travelling wave solution (u_1, u_2) of the one-dimensional system corresponding to (1.5) with $d_1 = 1$ and $d_2 = d$, namely

$$\begin{cases} u_{1t} = u_{1xx} + (r_1 - a_1 u_1 - b_1 u_2) u_1 & x \in \mathbf{R},\ t > 0, \quad (1.10a) \\ u_{2t} = d u_{2xx} + (r_2 - b_2 u_1 - a_2 u_2) u_2 & x \in \mathbf{R},\ t > 0, \quad (1.10b) \end{cases}$$

with the boundary conditions at infinity

$$(u_1, u_2)(-\infty, t) = \left(\frac{r_1}{a_1}, 0\right) \quad \text{and} \quad (u_1, u_2)(\infty, t) = \left(0, \frac{r_2}{a_2}\right). \tag{1.10c}$$

Kan-on [K] proved that the velocity of the travelling wave solution of Problem (1.10) is unique for fixed values of the rates r_i, a_i and b_i $(i = 1, 2)$. In particular, if a_1 is a free parameter and the other parameters are fixed and satisfy the inequalities (1.7), then there exists a unique constant $a^* > 0$ such that $c = 0$ if $a_1 = a^*$, $c > 0$ if $a_1 > a^*$, and $c < 0$ if $a_1 < a^*$. For the special case when $c = 0$, (1.9) becomes the equation of motion by mean curvature, which has been analytically and numerically investigated (see, for instance, [C]). The manifold $\Gamma(t)$ obtained from (1.9) provides information on the dynamics of the spatial segregation between the two competing species.

This result clearly shows the similarity between this class of problems and the Allen-Cahn equation first studied by Keller, Sternberg & Rubinstein [KSR], where the limiting interface moves according to its mean curvature.

In this paper, we consider a different situation from the one obtained above, namely the case that only the interspecific competition rates b_1 and b_2 are very large. To study this situation, it is convenient to rewrite (1.5) as

$$\begin{cases} u_{1t} = d_1 \Delta u_1 + r_1(1 - u_1) u_1 - b u_1 u_2 & x \in \Omega,\ t > 0 \quad (1.11a) \\ u_{2t} = d_2 \Delta u_2 + r_2(1 - u_2) u_2 - \alpha b u_1 u_2, & x \in \Omega,\ t > 0. \quad (1.11b) \end{cases}$$

where b and α are positive constants. We assume that b is the only parameter which is large and that all the other parameters are of order $O(1)$. The coefficient $\alpha > 0$ is the

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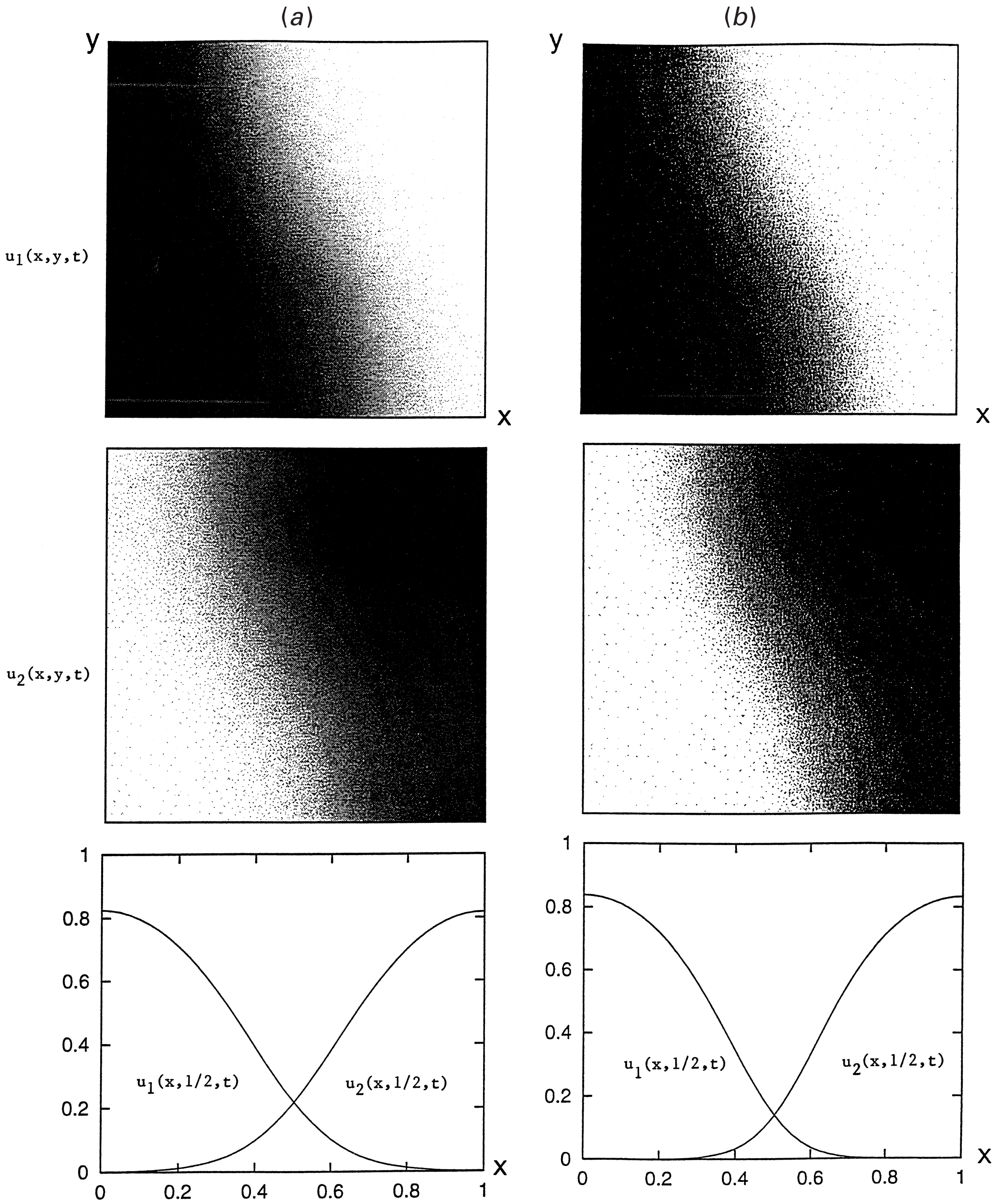

FIGURE 1a, b. For caption see facing page.

competition ratio between the two species u_1 and u_2. If $\alpha > 1$, then u_1 has a competitive advantage over u_2, while if $\alpha < 1$, the situation is reversed.

To study how the segregation of two competing species depends on the value of b we present two-dimensional numerical simulations of System (1.11), together with the boundary conditions (1.6) in a rectangular domain (see Figure 1). We take b as a free parameter and keep the other parameters d_1, d_2, r_1, r_2 and α fixed. For values of b which

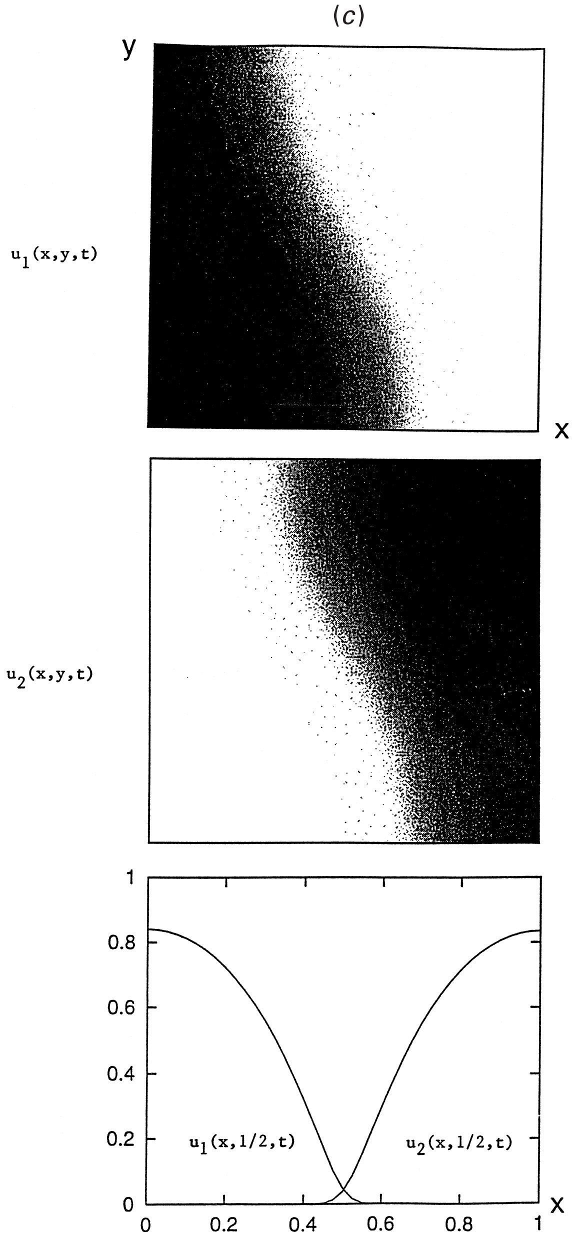

FIGURE 1. Two-dimensional patterns of (u_1, u_2) in the domain $\{(x, y), 0 < x < 1, 0 < y < 1\}$ with homogeneous Neumann boundary conditions in the case that $d_1 = d_2 = 0.5$, $r_1 = r_2 = 20.0$ and $\alpha = 1$. (a) b = 500; (b) b = 3000; (c) b = 15000.

are neither large nor small, it is shown that u_1 and u_2 exhibit spatial segregation with a rather wide zone of overlap. When the value of b increases, the zone of overlap becomes narrower. Thus, taking the limit $b \to \infty$, one can expect that u_1 and u_2 have disjoint supports (habitats) with only one common curve, which separates the habitats of the two competing species.

The purpose of this paper is to derive the limiting system as $b \to \infty$, which is called the *spatial segregation limit*, to describe the time evolution of the supports of u_1 and u_2. As it will be proved below, the limiting system can be described by a *free boundary problem* which is a two-phase Stefan-like problem with reaction terms.

Let $\Gamma(t)$ be the interface which separates the two subregions

$$\Omega_1(t) = \{x \in \Omega : u_1(x,t) > 0, \quad u_2(x,t) = 0\}$$

and

$$\Omega_2(t) = \{x \in \Omega : u_1(x,t) = 0, \quad u_2(x,t) > 0\}$$

in Ω (see Figure 1). Then u_1 and u_2 satisfy

$$\begin{cases} u_{1t} = d_1 \Delta u_1 + r_1(1-u_1)u_1 \quad x \in \Omega_1(t),\ t > 0 & (1.12a) \\ u_{2t} = d_2 \Delta u_2 + r_2(1-u_2)u_2 \quad x \in \Omega_2(t),\ t > 0 & (1.12b) \\ \dfrac{\partial u_1}{\partial \nu} = 0, \quad \dfrac{\partial u_2}{\partial \nu} = 0, \qquad x \in \partial\Omega, t > 0. & (1.12c) \end{cases}$$

On the interface,

$$u_1 = 0, \quad u_2 = 0 \quad x \in \Gamma(t) \quad \text{for} \quad t > 0, \tag{1.12 d}$$

and

$$0 = -\alpha d_1 \frac{\partial u_1}{\partial \nu}(x,t) - d_2 \frac{\partial u_2}{\partial \nu}(x,t) \quad x \in \Gamma(t) \quad \text{for} \quad t > 0, \tag{1.12 e}$$

where ν is a unit vector normal to $\Gamma(t)$. The initial conditions are given by

$$u_i(x,0) = u_{i0}(x), \quad x \in \Omega_1(0) \quad (i = 1, 2), \tag{1.13 a}$$

and are such that their support is separated by the line

$$\Gamma(0) = \Gamma_0. \tag{1.13 b}$$

The problem is to find functions $(u_1(x,t), u_2(x,t))$ and $\Gamma(t))$ which satisfy (1.12)–(1.13). If this problem can be solved, the interface $\Gamma(t)$ determines the segregating patterns between the two strongly competing species. One notices that the system (1.12)–(1.13) is quite similar to the standard two-phase Stefan problem except for the two following points: (i) the system (1.12 a)–(1.12 b) for u_1 and u_2 is not the heat equation, but the *logistic growth equation* which is well-known in theoretical ecology; (ii) the interface equation (1.12 e) is such that the latent heat is zero. The strength ratio α of the interspecific competition between u_1 and u_2 is contained in (1.12 e).

In § 2, we precisely formulate the problem which we study and derive some *a priori* estimates. In § 3, we study the limiting behaviour of the solution of (1.11)–(1.6) as b tends to ∞ and prove that it converges to the solution of a free boundary problem. In § 4, we consider the one-dimensional limiting free boundary problem. We present a thorough analysis of the equilibrium solutions and discuss the large time behaviour. Finally, we present some remarks in § 5.

This paper extends a similar study due to Evans [E] in the case of a slightly simpler system without growth terms, which he considered with more restrictive hypotheses on the initial data. Also, let us mention results by Dancer & Du [DD] about the limiting behaviour of equilibrium solutions in higher space dimensions. For a study of the limiting free boundary problem without growth terms, we refer to Cannon & Hill [CH] and to a recent paper by Tonegawa [To], who proves regularity properties of the solution and of the interface.

In a forthcoming article, we will show how our method of analysing the spatial segregation limit can be applied to some three component competition–diffusion systems.

2 Formulation of the problem and some basic properties

In this section we formulate the reaction-diffusion system which we shall be studying and derive a number of basic properties of its solutions. As announced in the Introduction, we shall consider the problem

$$(\mathrm{P}_k)\begin{cases} u_t = d_1\Delta u + \{f(u) - kv\}u & \text{in} \quad Q = \Omega \times \mathbf{R}^+ & (2.1a)\\ v_t = d_2\Delta v + \{g(v) - \alpha k u\}v & \text{in} \quad Q = \Omega \times \mathbf{R}^+ & (2.1b)\\ \dfrac{\partial u}{\partial \nu} = 0, \quad \dfrac{\partial v}{\partial \nu} = 0 & \text{on} \quad S = \partial\Omega \times \mathbf{R}^+ & (2.1c)\\ u(x,0) = u_0^k(x), \quad v(x,0) = v_0^k(x) & \text{for} \quad x \in \Omega. & (2.1d)\end{cases}$$

We shall make the following hypotheses about the functions f and g and the initial functions u_0 and v_0.

H1. The functions f and g are continuously differentiable on $[0,\infty)$ such that

$$f(s) > 0,\ g(s) > 0 \quad \text{for} \quad s \in (0,1) \quad \text{and} \quad f(s) < 0,\ g(s) < 0 \quad \text{for} \quad s > 1.$$

We shall write

$$\ell_0 = \max\{f(s) : 0 \leqslant s \leqslant 1\} \quad \text{and} \quad \ell_1 = \max\{sf'(s) + f(s) : 0 \leqslant s \leqslant 1\},$$

$$m_0 = \max\{g(s) : 0 \leqslant s \leqslant 1\} \quad \text{and} \quad m_1 = \max\{sg'(s) + g(s) : 0 \leqslant s \leqslant 1\}.$$

H2.

$$u_0^k,\ v_0^k \in C(\overline{\Omega}), \quad 0 \leqslant u_0^k \leqslant 1, \quad 0 \leqslant v_0^k \leqslant 1,$$

$$u_0^k \rightharpoonup u_0, \quad v_0^k \rightharpoonup v_0, \quad \text{weakly in } L^2(\Omega) \text{ as } k \to \infty.$$

By a solution of Problem (P_k) we shall understand a pair of functions (u,v) such that $u,v \in C(\overline{Q}) \cap C^{2,1}(\overline{\Omega} \times [\delta, T])$ for any $\delta \in (0,T)$. We begin with *a priori* bounds for solutions of Problem (P_k).

Lemma 2.1 *Let (u_k, v_k) be a solution of Problem (P_k). Then*

$$0 \leqslant u_k \leqslant 1 \quad \text{and} \quad 0 \leqslant v_k \leqslant 1 \quad \text{in} \quad \overline{Q}.$$

Proof Define

$$\mathscr{L}_1(u) \stackrel{\text{def}}{=} u_t - d_1\,\Delta u - \{f(u) - kv\}u,$$

$$\mathscr{L}_2(v) \stackrel{\text{def}}{=} v_t - d_2\,\Delta v - \{g(v) - \alpha k u\}v.$$

Since

$$\mathscr{L}_i(0) = 0\ (i = 1,2) \quad \text{and} \quad \mathscr{L}_i(1) \geqslant 0\ (i = 1,2),$$

the assertion follows from the maximum principle. □

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The existence and uniqueness of the solution of Problem (P_k) follows from Lunardi [L, proposition 7.3.2].

In the next three lemmas we shall obtain *a priori* bounds for the solution (u_k, v_k) of Problem (P_k) which are *uniform* with respect to the parameter k in the equations. This will enable us to study the properties of the family of solutions (u_k, v_k) for large values of k.

Lemma 2.2 *We have*

$$\int_0^T \int_\Omega u_k\, v_k \leqslant \frac{|\Omega|}{k}\,(\ell_0 T + 1).$$

Proof Integration of the equation for u over Q_T yields

$$k\int_0^T \int_\Omega u_k\, v_k = d_1 \int_0^T \int_{\partial\Omega} \frac{\partial u_k}{\partial \nu} + \int_0^T \int_\Omega f(u_k)u_k - \int_\Omega u_k(T) + \int_\Omega u_{0k} \leqslant (\ell_0\, T + 1)|\Omega|.$$

Lemma 2.3 *There exists a positive constant $\mathscr{C}$, which does not depend on k, such that*

$$\int_0^T \int_\Omega |\nabla u_k|^2 \leqslant \mathscr{C} \quad \text{and} \quad \int_0^T \int_\Omega |\nabla v_k|^2 \leqslant \mathscr{C}.$$

Proof We multiply (2.1 a) by u and integrate over Ω. This yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u_k^2 + d_1 \int_\Omega |\nabla u_k|^2 + k\int_\Omega u_k^2\, v_k \leqslant \ell_0 |\Omega|,$$

where we have used Lemma 2.1. When we now integrate over $(0, T)$ we obtain the first estimate. The second estimate is proved similarly.

Next we consider the function

$$w_k = u_k - \frac{1}{\alpha}\, v_k,$$

which appears when we eliminate the terms involving k from (2.1 a) and (2.1 b). It satisfies

$$w_{kt} = d_1 \Delta u_k - \frac{d_2}{\alpha}\Delta v_k + u_k f(u_k) - \frac{1}{\alpha} v_k g(v_k) \quad \text{in} \quad Q_T \tag{2.2 a}$$

$$\frac{\partial w_k}{\partial \nu} = 0 \quad \text{on} \quad S_T \tag{2.2 b}$$

Lemma 2.4 *The family $\{w_{kt}\}$ is bounded in $L^2(0, T; (H^1(\Omega)')$, uniformly with respect to k.*

Proof We multiply (2.2 a) by $\zeta \in L^2(0,\, T; H^1(\Omega))$ and integrate over $Q_T = \Omega \times (0, T)$. Then we obtain, after integration by parts,

$$\int_0^T \langle w_{kt}, \zeta\rangle = -d_1 \int_0^T \int_\Omega \nabla u_k \cdot \nabla \zeta + \frac{d_2}{\alpha}\int_0^T \int_\Omega \nabla v_k \cdot \nabla\zeta + \int_0^T \int_\Omega \{u_k\, f(u_k) - \frac{1}{\alpha}\, v_k\, g(v_k)\}\zeta.$$

Hence, by Lemmas 2.1 and 2.3, we have

$$\left| \int_0^T \langle w_{kt}, \zeta\rangle \right| \leqslant M\, \|\zeta\|_{L^2(0,T;H^1(\Omega))}, \tag{2.3}$$

in which M is a positive constant, which does not depend on k or ζ. Thus, if we denote the duality product between the spaces $H^1(\Omega)$ and $(H^1(\Omega))'$ by $\langle\cdot,\cdot\rangle$, we have shown that

$$\left|\int_0^T \langle w_{kt}, \zeta\rangle\right| \leqslant M\,\|\zeta\|_{L^2(0,T;H^1(\Omega))} \quad \text{for all} \quad \zeta \in L^2(0,T;H^1(\Omega)).$$

This means that

$$\|w_{kt}\|_{L^2(0,T;(H^1(\Omega))')} \leqslant M. \tag{2.4}$$

and the proof is complete. □

3 The limit problem

We deduce from Lemmas 2.1, 2.3 and 2.4 that the families $\{u_k\}$ and $\{v_k\}$ are bounded in $L^2(0,T;H^1(\Omega))$ and that the family $\{u_k - \frac{1}{\alpha}v_k\}$ is precompact in $L^2(Q_T)$ [T, theorem 2.1]. Thus, there exist subsequences of $\{u_k\}$ and $\{v_k\}$, which we denote again by $\{u_k\}$ and $\{v_k\}$, and functions $\overline{u},\overline{v} \in L^2(0,T;H^1(\Omega))$ such that $0 \leqslant \overline{u},\overline{v} \leqslant 1$ and

$$u_k \rightharpoonup \overline{u} \quad \text{and} \quad v_k \rightharpoonup \overline{v} \quad \text{weakly in} \quad L^2(0,\,T;H^1(\Omega)) \tag{3.1}$$

and

$$w_k = u_k - \frac{v_k}{\alpha} \to w \quad \text{in} \quad L^2(Q_T) \quad \text{and} \quad \text{a.e. in} \quad Q_T \quad \text{as} \quad k \to \infty. \tag{3.2}$$

Furthermore, it follows from Lemma 2.2 that the product

$$u_k v_k \to 0 \quad \text{as} \quad k \to \infty \quad \text{in} \quad L^1(Q_T) \quad \text{and} \quad \text{a.e. in} \quad Q_T. \tag{3.3}$$

Next we relate the functions $\overline{u}$, $\overline{v}$ and w.

Lemma 3.1 (a) *The subsequences u_k and v_k are such that*

$$u_k \to w^+ \quad \text{and} \quad v_k \to \alpha w^- \quad \text{as} \quad k \to \infty$$

in $L^1(Q_T)$ and a.e. in Q_T.

(b) $$\overline{u} = w^+ \quad \text{and} \quad \overline{v} = \alpha w^- \quad \text{and so} \quad w = \overline{u} - \frac{\overline{v}}{\alpha}.$$

Proof (a) Let $(x,t) \in Q_T$ be such that

$$w_k(x,t) = \left(u_k - \frac{v_k}{\alpha}\right)(x,t) \to w(x,t) \quad \text{and} \quad (u_k v_k)(x,t) \to 0 \quad \text{as} \quad k \to \infty.$$

(i) We first consider the case that $w(x,t) > 0$. Then there exists a positive constant k_0 such that

$$u_k(x,t) \geqslant \frac{w(x,t)}{2} > 0 \quad \text{for all} \quad k \geqslant k_0,$$

which implies that

$$v_k(x,t) \to 0 \quad \text{and} \quad u_k(x,t) \to w(x,t) = w^+(x,t) \quad \text{as} \quad k \to \infty.$$

(ii) Next we consider the case that $w(x,t) < 0$. Then there exists a positive constant k_1 such that

$$v_k(x,t) \geqslant -\frac{\alpha}{2}\,w(x,t) > 0 \quad \text{for all} \quad k \geqslant k_1,$$

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so that

$$u_k(x,t) \to 0 \quad \text{and} \quad v_k(x,t) \to -\alpha w(x,t) = \alpha w^-(x,t) \quad \text{as} \quad k \to \infty.$$

(iii) Finally, we consider the case that $w(x,t) = 0$. If a subsequence of $u_k(x,t)$, which we denote again by $u_k(x,t)$, is such that $u_k(x,t) \to \lambda > 0$, then $v_k(x,t) \to 0$, so that $u_k(x,t) - \frac{1}{\alpha} v_k(x,t) \to \lambda$ which contradicts the fact that $w(x,t) = 0$. Similarly, it is impossible to have that $v_k(x,t) \to \mu > 0$. Hence

$$u_k(x,t) \to 0 \quad \text{and} \quad v_k(x,t) \to 0 \quad \text{as} \quad k \to \infty.$$

The convergence in $L^1(Q_T)$ follows from the boundedness of u_k and v_k.

Finally, Part (b) is an immediate corollary of Part (a). □

Lemma 3.2 *Let T be an arbitrary positive number. The pair of functions $(\overline{u}, \overline{v})$ defined in (3.1) is such that*

$$\int_0^T \int_\Omega \left\{ \left(\overline{u} - \frac{1}{\alpha} \overline{v} \right) \varphi_t - \nabla \left(d_1 \overline{u} - \frac{d_2}{\alpha} \overline{v} \right) \nabla \varphi + \left(\overline{u} f(\overline{u}) - \frac{1}{\alpha} \overline{v} g(\overline{v}) \right) \varphi \right\} = - \int_\Omega \left(u_0 - \frac{v_0}{\alpha} \right) \varphi(0), \quad (3.4)$$

for all functions $\varphi \in C^\infty(Q_T)$ such that $\varphi(T) = 0$.

Proof When we multiply (3.2 a) by a test function $\varphi \in C_0^\infty(Q_T)$ such that $\varphi(T) = 0$, and integrate by parts, we obtain the identity

$$\int_0^T \int_\Omega \left\{ \left(u_k - \frac{1}{\alpha} v_k \right) \varphi_t - \nabla \left(d_1 u_k - \frac{d_2}{\alpha} v_k \right) \nabla \varphi + \left(u_k f(u_k) - \frac{1}{\alpha} v_k g(v_k) \right) \varphi \right\} = - \int_\Omega \left(u_0^k - \frac{v_0^k}{\alpha} \right) \varphi(0). \quad (3.5)$$

We now let $k \to \infty$ along the sequence for which (3.1) holds. Then, because

$$u_k \to \overline{u}, \quad \text{and} \quad v_k \to \overline{v} \quad \text{as} \quad k \to \infty \quad \text{a.e. in} \quad Q_T,$$

and $|u_k|, |v_k| \leqslant 1$ for all $k \geqslant 1$, it follows by the dominated convergence theorem that

$$\int_0^T \int_\Omega u_k f(u_k) \to \int_0^T \int_\Omega \overline{u} f(\overline{u}) \quad \text{as} \quad k \to \infty.$$

A similar result holds for the sequence $\{v_k\, g(v_k)\}$. Passing to the limit in (3.5), we obtain (3.4). This completes the proof. □

Next we show that the limit function w defined in Lemma 3.1 is a weak solution of the problem

$$(\text{P}) \begin{cases} w_t = \operatorname{div}(d(w) \nabla w) + h(w) & \text{in} \quad Q \\ \dfrac{\partial w}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times \mathbf{R}^+ \\ w(x,0) = w_0(x) \stackrel{\text{def}}{=} u_0(x) - \dfrac{v_0(x)}{\alpha} & \text{for} \quad x \in \Omega, \end{cases}$$

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where

$$d(s) = \begin{cases} d_1 & \text{if} \quad s > 0 \\ d_2 & \text{if} \quad s < 0, \end{cases}$$

and

$$h(s) = \begin{cases} f(s)s & \text{if} \quad s > 0 \\ g(-\alpha s)s & \text{if} \quad s < 0. \end{cases}$$

Definition 3.3 A function w is a weak solution of Problem (P) if

(i) $$w \in L^\infty(\Omega \times \mathbf{R}^+) \cap L^2(0, T, H^1(\Omega)) \cap C([0, \infty); L^2(\Omega));$$

(ii) $$\int_\Omega w(T)\varphi(T) - \int\int_{Q_T} \{w\varphi_t - d(w)\nabla w \nabla \varphi + h(w)\varphi\} = \int_\Omega w_0 \varphi(0)$$

for all $\varphi \in C^1(\overline{\Omega} \times \mathbf{R}^+)$.

Lemma 3.4 *The function w is a weak solution of Problem (P).*

Proof It follows from Lemma 2.1 that $w \in L^\infty(\Omega \times \mathbf{R}^+)$, and from Lemma 2.3 that $w \in L^2(0, T; H^1(\Omega))$. Since $w_t \in L^2(0, T; (H^1(\Omega))')$ as well, it follows from a standard regularity result that $w \in C([0, T]); L^2(\Omega))$ (e.g. see [T, lemma 1.2, p. 260]).

We consider (3.4), and observe that

$$d_1 \nabla \overline{u} - \frac{d_2}{\alpha} \nabla \overline{v} = d(w) \nabla w$$

and that

$$\overline{u} f(\overline{u}) - \frac{\overline{v}}{\alpha} g(\overline{v}) = h(w).$$

Therefore w satisfies the integral equality

$$\int_0^T \int_\Omega \{w\varphi_t - d(w)\nabla w \nabla \varphi + h(w)\varphi\} = \int_\Omega \left(u_0 - \frac{v_0}{\alpha}\right) \varphi(0) \tag{3.6}$$

for all functions $\varphi \in C^\infty(Q_T)$ such that $\varphi(T) = 0$ and for all $T > 0$. As a consequence, w satisfies the differential equation in Problem (P) as well as the homogeneous Neumann boundary condition in the sense of distributions, and the initial condition

$$w(x, 0) = u_0(x) - \frac{v_0(x)}{\alpha} \quad \text{for all} \quad x \in \Omega.$$

The fact that w is a weak solution of Problem (P) easily follows. □

Lemma 3.5 *Problem (P) has exactly one weak solution w, and $w \in C^{\alpha,\alpha/2}(\overline{\Omega} \times [0, \infty))$ for all $\alpha \in (0, 1)$.*

Proof The proof of uniqueness is similar to that of Aronson, Crandall & Peletier [ACP] (we remark that a weak solution of Problem (P) is a solution of Problem (P) in the sense of [ACP] as well). The regularity of w follows from DiBenedetto [Di, theorems 1.1 and 1.3, pp. 41 and 43].

Finally, we rewrite Problem P as an explicit free boundary problem.

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Theorem 3.6 *Let w be a weak solution of Problem P such that there exists a family of closed hypersurfaces $\Gamma := \{\cup \Gamma(t),\ t \in [0,T]\}$ such that $\Gamma(t) \subset\subset \Omega$ for all $t \in [0,T], w(t) > 0$ inside $\Gamma(t)$, say in Ω_t^{int} and $w(t) < 0$ outside $\Gamma(t)$, say in Ω_t^{ext} for each $t \in [0,T]$. Then, if Γ is smooth enough, and if the functions*

$$\overline{u} = w^+ \quad \text{and} \quad \overline{v} = \alpha w^-$$

are smooth up to $\Gamma(t)$, then $\overline{u}$ and $\overline{v}$ satisfy

$$\begin{cases} \overline{u}_t = d_1 \Delta \overline{u} + f(\overline{u})\overline{u} & \text{in} \quad \cup\{\Omega_t^{\mathrm{int}},\ t \in (0,T]\} \\ \overline{v}_t = d_2 \Delta \overline{v} + g(\overline{v})\overline{v} & \text{in} \quad \cup\{\Omega_t^{\mathrm{ext}}, t \in (0,T]\} \\ \overline{u} = 0, \quad \text{and} \quad \overline{v} = 0 & \text{on} \quad \Gamma \\ d_1 \dfrac{\partial \overline{u}}{\partial \nu} = -\dfrac{d_2}{\alpha} \dfrac{\partial \overline{v}}{\partial \nu} & \text{on} \quad \Gamma \\ \dfrac{\partial \overline{v}}{\partial \nu} = 0 & \text{on} \quad \partial\Omega \times (0,T] \\ \overline{u}(x,0) = u_0(x), \quad \overline{v}(x,0) = v_0(x) & \text{for} \quad x \in \Omega, \end{cases}$$

where we suppose that $u_0 > 0, v_0 = 0$ in Ω_0^{int} and $u_0 = 0, v_0 > 0$ in Ω_0^{ext}.

Proof In the set $\{w > 0\}$ the solution w satisfies the equation

$$w_t = d_1 \Delta w + h(w)$$

which yields

$$\overline{u}_t = d_1 \Delta \overline{u} + \overline{u} f(\overline{u}),$$

whereas in $\{w < 0\}$ we find that w satisfies

$$w_t = d_2 \Delta w + g(-\alpha w) w,$$

so that $\overline{v} = -\alpha w$ satisfies

$$\overline{v}_t = d_2 \Delta \overline{v} + \overline{v} g(\overline{v})$$

and $\overline{u} = \overline{v} = \overline{w} = 0$ on Γ.

Taking $\varphi \in C_0^\infty(Q_T)$ in Definition 3.3 (ii) and using that the partial differential equation is satisfied in the sense of distributions, we finally deduce the free boundary condition:

$$d_1 \frac{\partial \overline{u}}{\partial \nu} = -\frac{d_2}{\alpha} \frac{\partial \overline{v}}{\partial \nu} \quad \text{on} \quad \Gamma.$$

4 Steady state solutions

In this section we shall give a description of the set of equilibrium solutions of the limit problem (P), when Ω is a one-dimensional domain. Thus, we consider the problem

$$(\mathrm{E}) \begin{cases} (\mathscr{D}(w))'' + h(w) = 0 \quad \text{for} \quad 0 < x < L & (4.1a) \\ w'(0) = 0 \quad \text{and} \quad w'(L) = 0, & (4.1b) \end{cases}$$

in which

$$\mathscr{D}(s) = \begin{cases} d_1 s & \text{if} \quad s > 0 \\ d_2 s & \text{if} \quad s < 0 \end{cases} \quad \text{and} \quad h(s) = \begin{cases} s(1-s) & \text{if} \quad s > 0, \\ s(1+\alpha s) & \text{if} \quad s < 0, \end{cases} \tag{4.2}$$

and d_1, d_2 and α are positive constants.

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We shall prove the following theorem, in which the length

$$L_0 \stackrel{\text{def}}{=} (\sqrt{d_1} + \sqrt{d_2})\frac{\pi}{2} \tag{4.3}$$

will play a critical role.

Theorem 4.1 (a) *If $L \leqslant L_0$, then Problem (E) has no nonconstant solutions.*
(b) *If $L \in (nL_0, (n+1)L_0)$ for some integer $n \geqslant 1$, then Problem (E) has precisely n nontrivial solutions (modulo reflection with respect to $L/2$), $\{w_k : k = 1, \ldots, n\}$ where w_k has precisely k zeros.*

Before proving this theorem, we introduce the following auxiliary problem:

$$\begin{cases} y'' + h(y) = 0 \quad \text{for} \quad x > 0, & (4.4a) \\ y(0) = \gamma > 0 \quad \text{and} \quad y'(0) = 0. & (4.4b) \end{cases}$$

For each $\gamma \in (0, 1)$ this problem has a unique solution $y(x, \gamma)$ in a neighbourhood of the origin, which, since $h(s) > 0$ when $s \in (0, 1)$, has the properties

$$y' < 0, \quad y'' < 0 \quad \text{as long as} \quad y > 0, \tag{4.5}$$

and can be continued at least until it vanishes. Let

$$\ell(\gamma) \stackrel{\text{def}}{=} \sup\{x > 0 : y(\cdot, \gamma) > 0 \quad \text{on} \quad [0, x)\}.$$

Then (4.5) guarantees that $\ell(\gamma) < \infty$ if $\gamma \in (0, 1)$.

The following properties of $\ell(\gamma)$ are well-known and easily established.

Lemma 4.2 *We have $\ell \in C^1(0, 1)$, and*

(a) $$\ell'(\gamma) > 0 \quad \text{for} \quad 0 < \gamma < 1,$$

(b) $$\ell(\gamma) \to \frac{\pi}{2} \quad \text{as} \quad \gamma \to 0^+,$$

(c) $$\ell(\gamma) \to \infty \quad \text{as} \quad \gamma \to 1^-.$$

In what follows we shall also need an expression for the slope of y at its first zero, and so we introduce the function

$$\varphi(\gamma) \stackrel{\text{def}}{=} -y'(\ell(\gamma), \gamma) \quad \text{for} \quad 0 < \gamma < 1. \tag{4.6}$$

If we multiply (4.4 a) by $2y'$ and integrate over $(0, \ell(\gamma))$, we obtain

$$\varphi^2(\gamma) = 2H(\gamma) \stackrel{\text{def}}{=} 2\int_0^\gamma h(s)\, ds \tag{4.7}$$

From this expression we can deduce the following properties of φ:

Lemma 4.3 *We have $\varphi \in C^1(0, 1)$, and*

(a) $$\varphi'(\gamma) > 0 \quad \text{for} \quad 0 < \gamma < 1,$$

(b) $$\varphi(\gamma) \sim \gamma \quad \text{as} \quad \gamma \to 0^+,$$

(c) $$\varphi(\gamma) \to \frac{1}{\sqrt{3}} \quad \text{as} \quad \gamma \to 1^-.$$

We are now ready to give the proof of Theorem 4.1. Let w be a solution of Problem (E). Then, by the strong maximum principle $-1/\alpha < w(x) < 1$ for $0 \leqslant x \leqslant L$, and by uniquenss, $w(0) \neq 0$. Without loss of generality, we may choose

$$\gamma \stackrel{\text{def}}{=} w(0) > 0.$$

Thus, w is then the solution of the problem

$$\begin{cases} d_1 w'' + h(w) = 0 \quad \text{on} \quad \{w > 0\} & (4.8a) \\ w(0) = \gamma \in (0, 1) \quad \text{and} \quad w'(0) = 0, & (4.8b) \end{cases}$$

and hence, by an easy transformation, we see that

$$w(x) = y\left(\frac{x}{\sqrt{d_1}}, \gamma\right) \quad \text{for} \quad 0 \leqslant x \leqslant \sqrt{d_1}\,\ell(\gamma), \tag{4.9}$$

and

$$w'(\sqrt{d_1}\,\ell(\gamma)) = \frac{1}{\sqrt{d_1}}\, y'(\ell(\gamma), \gamma) = -\frac{1}{\sqrt{d_1}}\,\varphi(\gamma). \tag{4.10}$$

At the zeros of w, the function $(\mathscr{D}(w))'$ must be continuous. Thus we must have

$$d_1 w'(x_\gamma^-) = d_2 w'(x_\gamma^+), \quad x_\gamma = \sqrt{d_1}\,\ell(\gamma). \tag{4.11}$$

Therefore $w < 0$ and $w' < 0$ in a right-neighbourhood of x_γ.

Let $L > x_\gamma$ be the first zero of w'. Then w satisfies

$$\begin{cases} d_2 w'' + h(w) = 0, \quad w' < 0 \quad \text{on} \quad (x_\gamma, L) & (4.12a) \\ w(x_\gamma) = 0 \quad \text{and} \quad w'(L) = 0 & (4.12b) \end{cases}$$

and can be written as

$$w(x) = -\frac{1}{\alpha}\, y\left(\frac{L - x}{\sqrt{d_2}}, \beta\right) \quad \text{for} \quad x_\gamma \leqslant x \leqslant L, \quad \beta = -\alpha w(L). \tag{4.13}$$

So that this solution matches up at the zero x_γ of w, we require that

$$\frac{L - x_\gamma}{\sqrt{d_2}} = \ell(\beta), \tag{4.14}$$

or

$$\sqrt{d_1}\,\ell(\gamma) + \sqrt{d_2}\,\ell(\beta) = L, \tag{4.15}$$

and, in view of condition (4.11) on the derivatives,

$$\sqrt{d_1}\,\varphi(\gamma) = \frac{1}{\alpha}\sqrt{d_2}\,\varphi(\beta). \tag{4.16}$$

Since by Lemma 4.3, $\varphi' > 0$ on $(0, 1)$, we can solve equation (4.16) for β:

$$\beta = \beta(\gamma) \stackrel{\text{def}}{=} \varphi^{-1}(k\,\varphi(\gamma)), \tag{4.17}$$

where $k = \alpha\sqrt{d_1/d_2}$, and the domain $D(\beta)$ and range $R(\beta)$ of β are given by

$$D(\beta) = (0, 1) \quad \text{and} \quad R(\beta) = (0, \beta_k) \quad \text{if} \quad k \leqslant 1 \tag{4.18 a}$$

$$D(\beta) = (0, \gamma_k) \quad \text{and} \quad R(\beta) = (0, 1) \quad \text{if} \quad k > 1, \tag{4.18 b}$$

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where

$$\beta_k = \varphi^{-1}(k\,\varphi(1)) \quad \text{and} \quad \gamma_k = \varphi^{-1}(k^{-1}\varphi(1)). \tag{4.18 c}$$

Finally, we conclude from (4.16) and (4.17) that

$$\beta'(\gamma) > 0 \quad \text{for} \quad \gamma \in D(\beta). \tag{4.18 d}$$

If we substitute $\beta(\gamma)$ in (4.15), we obtain the equation

$$\psi(\gamma) = L, \tag{4.19 a}$$

where

$$\psi(\gamma) \stackrel{\text{def}}{=} \sqrt{d_1}\,\ell(\gamma) + \sqrt{d_2}\,\ell(\beta(\gamma)). \tag{4.19 b}$$

From Lemmas 4.2 and 4.3, and the properties (4.18) of $\beta(\gamma)$, we see that $\psi \in C^1(D(\beta))$, and

$$\psi' > 0 \quad \text{on} \quad D(\beta) \tag{4.20 a}$$

$$\psi(\gamma) \to L_0 \quad \text{as} \quad \gamma \to 0^+, \tag{4.20 b}$$

$$\psi(\gamma) \to \infty \quad \text{as} \quad \gamma \to \gamma_k^- \quad \text{if} \quad k > 1 \quad \text{and} \quad \gamma \to 1^- \quad \text{if} \quad k \leqslant 1. \tag{4.20 c}$$

The properties (4.20) of ψ ensure that if $L \leqslant L_0$, then equation (4.19 a) has no nontrivial solution, and if $L > L_0$, it has precisely one strictly decreasing solution, with one zero.

If $L > 2L_0$, then $L/2 > L_0$, and we can find a decreasing solution on $[0, L/2]$ and extend this solution symmetrically into $(L/2, L]$, and thus construct a solution with two zeros.

Continuing in this manner we find that if $L \in (nL_0, (n+1)L_0]$, where n is a positive integer, there exist n solutions $\{w_k : k = 1, \ldots, n\}$, where w_k has precisely k zeros.

This completes the proof of Theorem 4.1. □

4.1 Large time behaviour

Let $w = w(t, w_0)$ be the solution of the problem

$$(P_1)\begin{cases} w_t = (d(w)w_x)_x + h(w) & \text{in} \quad \Omega \times \mathbf{R}^+ \\ w_x(0,t) = w_x(L,t) = 0 & t > 0 \\ w(x,0) = w_0(x) & x \in \Omega \end{cases}$$

where $\Omega := (0, L)$ and $w_0 \in L^2(\Omega)$ is such that $-1/\alpha \leqslant w_0 \leqslant 1$.

First we remark that since $w \in L^2(0, T; H^1(\Omega))$ for all $T > 0$, this implies that $w(t, w_0) \in H^1(\Omega)$ for a.e. $t > 0$. Regularizing Problem (P_1) and then passing to the limit, one can deduce the following result:

Lemma 4.4 *The following inequality holds:*

$$\int_s^t \int_\Omega d(w)w_t^2 + \int_\Omega \{\tfrac{1}{2}(d(w)w_x)^2\,(t) - \mathscr{H}(w(t))\}$$
$$\leqslant \int_\Omega \{\tfrac{1}{2}(d(w)w_x)^2\,(s) - \mathscr{H}(w(s))\}$$

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for all $t > s > 0$, *where*

$$\mathscr{H}(r) \stackrel{\text{def}}{=} \int_0^r d(\tau)h(\tau)d\tau.$$

Let $\delta > 0$ be arbitrary. We deduce from Lemma 4.4 that

$$w_t \in L^2(\delta, \infty; L^2(\Omega)), \tag{4.21 a}$$

$$\|w_x(t)\|_{L^2(\Omega)} \leqslant C_1(\delta) \quad \text{for all} \quad t \geqslant \delta > 0, \tag{4.21 b}$$

$$\|\mathscr{D}(w)_x(t)\|_{L^2(\Omega)} \leqslant C_2(\delta) \quad \text{for all} \quad t \geqslant \delta > 0, \tag{4.21 c}$$

where we recall that $\mathscr{D}(s) = \int_0^s d(\tau)\,d\tau$. Thus, since also $-1/\alpha \leqslant w(t) \leqslant 1$ for all $t \geqslant 0$,

$$\{w(t, w_0), t \geqslant 1\} \quad \text{is precompact in} \quad L^2(\Omega). \tag{4.22}$$

Define $\omega(w_0) = \{v \in L^2(\Omega)$ such that there exists $\{t_n\}$ such that

$$w(t_n, w_0) \to v \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad t_n \to \infty.$$

Then one can easily deduce that

(i) $\{w(t, w_0), t \geqslant 1\}$ is a precompact set of $L^2(\Omega)$;

(ii) $\omega(w_0)$ is nonempty and connected in $L^2(\Omega)$;

(iii) $w \in \omega(w_0)$ implies that $w(t, w_0) \in \omega(w_0)$ for all $t > 0$.

Finally, we have the following results:

Theorem 4.5 *The function $\omega(w_0)$ coincides with one of the equilibrium solutions.*

Proof Define

$$V(w) = \int_\Omega [\tfrac{1}{2}\{(\mathscr{D}(w))'\}^2 - \mathscr{H}(w)].$$

Since the function $t \to V(w(t))$ is decreasing and bounded from below, it has a limit V_∞ as $t \to \infty$. Let $\overline{w} \in \omega(u_0)$. Then there exists $\{t_n\}$ such that

$$\lim_{t_n \to \infty} \|w(t_n, u_0) - \overline{w}\|_{L^2(\Omega)} = 0. \tag{4.23}$$

Since $\mathscr{D}$ is Lipschitz continuous, (4.23) implies that

$$\lim_{t_n \to \infty} \|\mathscr{D}(w(t_n, u_0)) - \mathscr{D}(\overline{w})\|_{L^2(\Omega)} = 0. \tag{4.24}$$

Furthermore, we deduce from (4.21) that, as $t_n \to \infty$,

$$w(t_n, u_0) \rightharpoonup \overline{w} \quad \text{weakly in} \quad H^1(\Omega), \tag{4.25}$$

and that

$$\mathscr{D}(w(t_n, u_0)) \rightharpoonup \mathscr{D}(\overline{w}) \quad \text{weakly in} \quad H^1(\Omega). \tag{4.26}$$

We write

$$V(w) = V_1(w) - V_2(w), \tag{4.27 a}$$

where

$$V_1(w) = \frac{1}{2}\int_\Omega [\tfrac{1}{2}\{(\mathscr{D}(w))'\}^2] \quad \text{and} \quad V_2(w) = \int_\Omega \mathscr{H}(w), \tag{4.27 b}$$

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and $w_n = w(t_n, u_0)$. By (4.23) the sequence $\{w_n\}$ converges strongly in $L^2(\Omega)$ to $\overline{w}$ as $n \to \infty$, and by Lemma 2.1, it is uniformly bounded in $L^\infty(\Omega)$. Thus, since $\mathscr{H}$ is locally Lipschitz continous, it is clear that

$$V_2(w_n) \to V_2(\overline{w}) \quad \text{as} \quad n \to \infty. \tag{4.28}$$

We now use the observation that the functional

$$u \to \int_\Omega u_x^2$$

is continuous and convex in $H^1(\Omega)$, and hence that it is weakly lower semicontinuous in $H^1(\Omega)$. Thus, in view of (4.26) we conclude that V_2 is weakly lower semicontinuous, and hence

$$V_1(\overline{w}) \leqslant \liminf_{n\to\infty} V_1(w_n). \tag{4.29}$$

Putting (4.28) and (4.29) together, we conclude that

$$V(\overline{w}) \leqslant \liminf_{n\to\infty} V(w(t_n)) = V_\infty.$$

It is standard (see, for instance, Dafermos [D, proposition 3.1]) that V is constant on $\omega(u_0)$. Therefore,

$$V(w(t, \overline{w})) = V_\infty \quad \text{for all} \quad t > 0.$$

Combining this with Lemma 4.4, we deduce that

$$(\mathscr{D}(w, (t, \overline{w}))_t \equiv 0$$

so that

$$\mathscr{D}(w(t, \overline{w})) \equiv \mathscr{D}(\overline{w}),$$

and thus

$$w(t, \overline{w}) \equiv \overline{w}.$$

Definition 3.3 then implies that

$$(\mathscr{D}(\overline{w}))'' + h(\overline{w}) = 0 \quad \text{in} \quad \mathscr{D}'(\Omega).$$

Since $\overline{w} \in L^\infty$ and h is Lipschitz continuous, this equation holds classically, i.e. $\mathscr{D}(\overline{w}) \in C^2(\overline{\Omega})$ and $\overline{w}_x = 0$ at the points 0 and L. Hence, $\overline{w}$ is an equilibrium solution.

Finally, it follows from the connectedness of $\omega(u_0)$, and the fact that the set of equilibria is finite, that $\omega(u_0) = \{\overline{w}\}$. □

We note that most of the results of this section also hold in higher space dimension. However, since we do not know in that case that the equilibria are isolated, we can only conclude that $\omega(u_0)$ is included in the set of equilibrium solutions.

Finally, we comment on the relation between the stability of equilibrium solutions of the reaction-diffusion Problem (P_k) and of the limiting Problem (P) in higher space dimension. It follows easily from a slight variant of a result of Dancer & Zongming Guo [DZ] that a stationary state of (2.1) with $u - (1/\alpha)v$ close to a stationary state w of (P), with w not identically zero, is unstable if the formal linearization of Problem (P) at w has a negative eigenvalue, and is stable if all the eigenvalues of the formal linearization are strictly positive.

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5 Concluding remarks

We have considered competition–diffusion systems for two species in the case that the interspecific competition rates are sufficiently large. This situation involves the occurrence of strong segregation between two competing species. To study the dynamics of spatially segregating patterns, we have taken the spatial segregation limit in the system and derived the corresponding free boundary problem. This problem is similar to the classical two-phase Stefan problem. Essential differences are that (i) systems describing the two species include reaction terms; and (ii) the latent heat is zero. In this paper, we have restricted our discussion to homogeneous Neumann boundary conditions but the result is also valid for other boundary conditions as well.

Systems involving three competing species are also interesting. Taking different scaling limits for the interspecific competition rates among the three species yields different types of Stefan-like problems. These will be described in forthcoming papers.

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