Ergod. Th. & Dynam. Sys. (1999), **19**, 339–362
Printed in the United Kingdom

Gradient-like flows on high-dimensional manifolds

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(*Received* 16 *October* 1996 *and accepted in revised form* 4 *January* 1998)

Abstract. The main purpose of this paper is to study the implications that the homology index of critical sets of smooth flows on closed manifolds M have on both the homology of level sets of M and the homology of M itself. The bookkeeping of the data containing the critical set information of the flow and topological information of M is done through the use of Lyapunov graphs. Our main result characterizes the necessary conditions that a Lyapunov graph must possess in order to be associated to a Morse–Smale flow. With additional restrictions on an abstract Lyapunov graph L we determine sufficient conditions for L to be associated to a flow on M.

1. *Introduction*

In this paper we propose to study gradient-like flows on n-dimensional manifolds M where $n \geq 4$. However, our results hold for $n = 2, 3$ but would yield weaker results than those in [**dRF93**] and [**dR93**]. Throughout this paper M will denote a smooth connected closed orientable n-manifold. We take a Morse theoretical approach by working with a Lyapunov function associated to a flow on M and by considering the level sets associated to regular values of f. We propose to study the changes in topology that are forced on the level sets as we pass through critical sets. By *critical set* we mean an invariant chain transitive piece (see [**Fra82**]) of the chain recurrent set $\mathcal{R}$. Note that $\mathcal{R}$ need not be hyperbolic. From this point on, we refer to chain transitive piece as *chain recurrent component*. The changes in topology depend not only on the critical set but also on the connections of stable and unstable manifolds. These connections, as we shall see later, are in part detected by the homology boundary map.

† Supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico under Grant 300647/91.3.
‡ Supported in part by the Conselho Nacional de Desenvolvimento Científico e Tecnológico under Grant 300072/90.2.

https://doi.org/10.1017/S0143385799120893 Published online by Cambridge University Press

We will isolate a critical set S of a flow ϕ_t defined on M in a *basic block*. A basic block N for S can be defined using a Lyapunov function $f : M \to \mathbb{R}$ associated to ϕ_t. Let $f(S) = c$ and choose $\epsilon > 0$ small enough so that c is the only critical value in $(c-\epsilon, c+\epsilon)$ and S is the only critical set at level c. Define the basic block as the component N of the pre-image $f^{-1}(c - \epsilon, c + \epsilon)$ which contains S. For a hyperbolic flow we can assume that there is only one chain recurrent component per critical level. We now analyze the changes in topology of the entering $(\partial_+ N = f^{-1}(c + \epsilon) \cap N)$ and exiting components $(\partial_- N = f^{-1}(c-\epsilon) \cap N)$ of the flow restricted to N. In order to do this we use the Conley homology index of the critical set and we set up a natural long exact homology sequence which involves this index. Our results in §3 will therefore, detect homology changes, more specifically changes in the Betti numbers of $\partial_+ N$ and $\partial_- N$. Hence, given a flow on M we study the changes in $\partial_+ N$ and $\partial_- N$ for each critical set and record all this information in a Lyapunov graph. In this process we determine necessary conditions imposed by the flow on this graph.

In §4 our aim is to determine if the necessary conditions on a Lyapunov graph L found in the previous section are sufficient to determine the manifold and a flow with a Lyapunov graph equivalent to L. That is, if we impose these conditions on an abstract Lyapunov graph do they determine the ambient manifold and a flow with a Lyapunov graph equivalent to L? It turns out that this is not the case in general and we must impose additional restrictions on abstract Lyapunov graphs. We then realize a class of abstract Lyapunov graphs as flows on manifolds which are connected sums of generalized tori $S^p \times S^q$ with critical sets which are hyperbolic singularities. We later consider flows with hyperbolic periodic orbits as well.

2. *Background material*

2.1. *Gradient-like flows.* Given a smooth flow $\phi_t : M \to M$, there is a smooth function $f : M \to \mathbb{R}$ associated to this flow with the properties that it strictly decreases along the orbits outside of the chain recurrent set $\mathcal{R}$; that is, if $x \notin \mathcal{R}$ then $f(\phi_t(x)) < f(\phi_s(x))$ whenever $t > s$ and is constant on the chain recurrent components (chain transitive pieces) of $\mathcal{R}$ (see [**Con78, Fra82**]). This function f is called a *Lyapunov function*. Since f decreases along orbits of ϕ_t not in $\mathcal{R}$, we say that ϕ_t is a *gradient-like flow* with respect to f.

A Morse function is an example of a Lyapunov function. Recall that a C^∞ function $f : M \to \mathbb{R}$ is a *Morse function* provided that each of its critical points is non-degenerate. Associated to each f, the flow ϕ_t is a gradient flow, a particular case of a gradient-like flow, defined by $(\partial/\partial t)\phi_t = -\nabla f \circ \phi_t$. We refer to this flow, for short, as a *Morse flow*. Note that f decreases along orbits of ϕ_t for all x not in the critical set of f. The unstable set of p is defined as $W^u(p) = \{y \in M : \lim_{t\to-\infty} \phi_t(y) = p\}$ and is a submanifold of M. The *Morse index* of a critical point p is the number of negative eigenvalues of $\nabla^2 f(p)$ or, equivalently, the dimension of the unstable set of the flow at p, $\lambda = \mathrm{ind}(p) = \dim W^u(p)$.

However, we will be working with an even more general notion of index due to Conley which generalizes the classical Morse index and which is not concerned with the non-degeneracy or hyperbolicity of the critical set.

2.2. *Lyapunov graphs.* The notion of Lyapunov graphs was introduced by Franks [**Fra85**]. A *Lyapunov graph* is a finite, connected, oriented graph with no oriented cycles and with the vertices labelled by chain recurrent flows.

A Lyapunov function $f : M \to \mathbb{R}$ associated to a flow determines a Lyapunov graph by defining the following equivalence relation on M: $x \sim_f y$ if and only if x and y belong to the same connected component of a level set of f. Hence $M/\sim_f$ is a Lyapunov graph. If the flow is hyperbolic, it is possible to choose f so that it contains one chain recurrent component per critical level. A point on $M/\sim_f$ is a *vertex point* if under the equivalence relation it is a component of a level set containing a chain recurrent component. All other points are *edge points*. Each edge represents a codimension one submanifold of M, $Q \times I$. Hence, in order for an edge to retain some of the topological information of $Q \times I$, we label it with the Betti numbers of Q. $M/\sim_f$ is oriented by the flow.

In this paper, homology is always taken using $\mathbb{Z}_2$ coefficients. In order to have a finite graph, we must place the restriction that the chain recurrent set has finitely many components, which we refer to as a *finite component chain recurrent set*.

An *abstract Lyapunov graph* is a finite connected oriented graph with no oriented cycles, with each vertex labelled with index information on a chain recurrent flow on a compact set and each edge labelled with the Betti numbers of a closed $(n-1)$-dimensional manifold. Of course, there is the question of which abstract Lyapunov graphs can be realized as a flow on a manifold. We will answer this in a later section. We note that abstract Lyapunov graphs are neither more nor less general than Lyapunov graphs. As defined, the vertices of a Lyapunov graph are labelled with chain recurrent flows whereas the vertices of an abstract Lyapunov graph are labelled with index information of the chain recurrent flows; a weaker notion in regard to this feature. On the other hand, Lyapunov graphs do not have labelled edges whereas abstract Lyapunov graphs do; a stronger notion in regard to this feature.

2.3. *Conley index theory.* Let $\phi : M \times \mathbb{R} \to M$ be a smooth flow on M which we denote by ϕ_t. A set $S \subset M$ is *invariant* if $\phi_t(S) = S$ for all $t \in \mathbb{R}$. The closure of a bounded open set $N \subset M$ is an *isolating neighborhood* for ϕ_t if for every $x \in \partial N$ there is a $t \in \mathbb{R}$ such that $\phi_t(x) \notin N$. An invariant set is called an *isolated invariant set* if it is the maximal invariant set in some isolating neighborhood.

A chain recurrent component R for the flow ϕ_t is an example of an isolated invariant set. If f is a Lyapunov function associated to a flow and $c = f(R)$, then for a small enough $\varepsilon > 0$, the component of $f^{-1}[c-\varepsilon, c+\varepsilon]$ which contains R is an isolating neighborhood for R which we refer to as a *basic block* for R.

An *index pair* for an isolated invariant set S is a pair of compact spaces (N_1, N_0) such that:

(1) $\mathrm{cl}(N_1 - N_0)$ is an isolating neighborhood for S;
(2) N_0 is positively invariant in N_1, i.e. if $x \in N_0$ and $\phi_{[0,T]}(x) \subset N_1$ then $\phi_{[0,T]}(x) \subset N_0$;
(3) N_0 is the exit set for the flow, i.e. if $x \in N_1$ and $\phi_{[0,\infty)}(x) \not\subset N_1$ then there exists a $T > 0$ such that $\phi_{[0,T]}(x) \subset N_1$ and $\phi_T(x) \in N_0$.

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For a chain recurrent component with $f(R) = c$ and $\varepsilon > 0$ such that there are no other critical values in $[c-\varepsilon, c+\varepsilon]$, define N_1 as the component of $f^{-1}[c-\varepsilon, c+\varepsilon]$ which contains R and $N_0 = f^{-1}(c-\varepsilon) \cap N_1$. (N_1, N_0) is an index pair for R which we will refer to as a *block defined index pair*.

Given any index pair (N_1, N_0) consider the quotient space N_1/N_0 as a pointed space with the equivalence class of N_0 as the distinguished point. In [**Con78**], it is shown that given two index pairs (N_1, N_0) and (N_1', N_0') for the same isolated invariant set S, there is a flow defined homotopy equivalence between N_1/N_0 and N_1'/N_0'. Hence, we define the *Conley index* of S, $h(S)$, as the homotopy type of N_1/N_0 for any index pair for S. The *homology index* of S, $CH(S)$, is the homology of the Conley index.

Block defined index pairs will be used throughout this paper and the main advantage of their usage is that since $f^{-1}(c-\varepsilon)$ is a deformation retract of a neighborhood of itself in $f^{-1}[c-\varepsilon, c+\varepsilon]$, it follows that N_0 is a deformation retract of a neighborhood of itself in N_1 and by standard results in algebraic topology, $H_*(N_1/N_0) \cong H_*(N_1, N_0)$ which in turn is isomorphic to $CH_*(R)$.

The following proposition is a particular case of a result of Conley's which generalizes the classical Morse inequalities. We refer the reader to [**dR93**] for a proof of Proposition 2.1.

PROPOSITION 2.1. *Let M be a closed orientable n-manifold, $\phi_t : M \to M$ a smooth flow with a finite component chain recurrent set $\mathcal{R} = \cup_{i\in I} R_i m$, where I is a finite indexing set for the chain recurrent components. Let h^i_j be the dimensions of the homology indices of R_i, for $j = 0, \ldots, n$. Then*

$$\sum_{i,j} (-1)^j h^i_j = \chi(M),$$

where $\chi(M)$ is the Euler characteristic of M. Moreover, the result also holds if $\partial M \neq \emptyset$. In this case, let ∂M^- be the part of the boundary of M through which the flow exits, then the above equality holds provided we consider $\chi(M, \partial M^-)$.

For more background material on the Conley index theory see [**Con78**] and for material on gradient-like flows in low dimensions see [**Fra85, Fra82, dR87, dRF93, dR93**].

2.4. *Handle theory.* Here we will give definitions pertinent to handle theory since in §4 we will be working with the attachment of handles in order to construct Morse flows on manifolds. We will also mention the notion of round handles which will be used in constructing Morse–Smale flows on manifolds. However, we refer the reader to [**RS82, Mil65**] and [**Asi75**] for a detailed exposition of this material.

We denote by D^j, the closed unit j-disk, i.e. the j-ball. Also, the boundary of D^j, ∂D^j is the $(j-1)$-sphere S^{j-1}.

Let N be an n-manifold and $H = D^\ell \times D^{n-\ell}$. Let $\theta : \partial D^\ell \times D^{n-\ell} \to \partial N$ be an embedding which defines the new manifold $N' = N \cup_\theta H$, which is the result of attaching an ℓ-handle to N. The following loose notations are also used: $N' = N \cup H^{(\ell)}$ or $N' = N \cup H$. The map θ is an embedding which we will simply refer to as an attaching map. The ℓ-handle is the pair (H, θ) which we denote more loosely as $H^{(\ell)}$. The *core* of

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the handle is $D^\ell \times 0$ and the *cocore* is $0 \times D^{n-\ell}$. Also, $\partial D^\ell \times 0$ is the *attaching sphere* (*a-sphere*) and $0 \times \partial D^{n-\ell}$ is the *belt sphere* (*b-sphere*). The *attaching region* (*a-region*) is $\partial_- H^{(\ell)} = (\partial D^\ell) \times D^{n-\ell}$ and the *belt* is $\partial_+ H^{(\ell)} = D^\ell \times \partial D^{n-\ell}$. The index of a singularity is the index of the handle which contains it and hence the dimension of the unstable manifold of the singularity must equal the dimension of the core of the handle.

If N is a smooth compact n-manifold and ∂N is the disjoint union of two open and closed codimension one submanifolds N_0 and N_1, (N, N_1, N_0) is a *smooth manifold triad*. In this case N is said to be a *cobordism* from N_0 to N_1 and N_0 and N_1 are said to be *cobordant*. Note that the manifolds in a cobordism are not assumed connected. We can create a further cobordism by attaching a handle H to N_1.

According to the above definition N_1 can be thought of as the boundary components of N which the flow enters through, i.e. $N_1 = \partial_+ N$, and N_0 as the boundary components which the flow exits through, i.e. $N_0 = \partial_- N$. Consider a Morse function $f : (N, \partial_+ N, \partial_- N) \to ([0, 1], 1, 0)$, where $\partial_+ N = f^{-1}(1)$ and $\partial_- N = f^{-1}(0)$ since the function decreases along orbits of the flow. Given the above function and $0 < c < 1$ which is not a critical value then both $W_1 = f^{-1}[c, 1]$ and $W_2 = f^{-1}[0, c]$ are smooth manifolds with boundary. Hence the cobordism from $(N, \partial_+ N, \partial_- N)$ is the *composition* of the cobordism $(W_1, \partial_+ N, f^{-1}(c))$ and $(W_2, f^{-1}(c), \partial_- N)$. More generally, if $(N, \partial_+ N, \partial_- N)$ and $(N', \partial_+ N', \partial_- N')$ are two smooth manifold triads and $h : \partial_- N \to \partial_+ N'$ is a diffeomorphism then a third triad is formed: $(N \cup_h N', \partial_+ N, \partial_- N')$, where $N \cup_h N'$ is the space formed from N and N' by identifying points of $\partial_- N$ and $\partial_+ N'$ under h. See [**Mil65**] for more details.

An *elementary cobordism* is a triad (N, N_1, N_0) possessing a Morse function with exactly one critical point. A triad (N, N_1, N_0) is a *product cobordism* if it is diffeomorphic to the triad $(N_0 \times [0, 1], N_0 \times 1, N_0 \times 0)$. Any cobordism can be expressed as a composition of elementary cobordisms. In other words, an elementary cobordism (N, N_1, N_0) containing an index ℓ critical point is the result of attaching an index ℓ handle $H^{(\ell)}$ to the product cobordism $(N_0 \times [0, 1], N_0 \times 1, N_0 \times 0)$ along $N_0 \times 1$. Alternatively, since a product cobordism is a closed collar $C(N_0)$, if (N, N_1, N_0) is an elementary cobordism containing an index ℓ critical point, then $N = C(N_0) \cup H^{(\ell)}$, where $H^{(\ell)}$ is attached to $C(N_0)$ along $\partial C(N_0) - N_0$.

We can easily make a parallel between cobordisms and Lyapunov graphs. A Lyapunov graph can be viewed as a composition of cobordisms (possibly elementary cobordisms if the graph is associated to a Morse flow) each of which contains exactly one chain recurrent component. Hence a vertex of a Lyapunov graph together with its labelled incident edges represents such a cobordism N. The incoming edges represent $\partial_+ N \times J$ and the outgoing edges represent $\partial_- N \times J$, where J is an open interval.

We will also work with flows which possess hyperbolic periodic orbits and hence introduce the notion of round handles [**Asi75**]. Let N be an n-manifold and $R = S^1 \times D^j \times D^{n-j-1}$. Let $\theta : \partial(S^1 \times D^j) \times D^{n-j-1} \to \partial N$ be a diffeomorphism which defines the new manifold $N' = N \cup_\theta R$, which is the result of attaching a j-round handle to N. We refer to θ as the attaching map. The j-round handle is the pair (R, θ). The *core* is $1 \times D^j \times 0$, the *cocore* is $1 \times 0 \times D^{n-j-1}$. The *attaching* and *belt spheres* are the boundaries of the core and cocore respectively. The *attaching region* is $\partial_- R = S^1 \times S^{j-1} \times D^{n-j-1}$

and $\partial_+ R = S^1 \times D^j \times S^{n-j-2}$. The index of an untwisted periodic orbit is the index of the round handle which contains it and hence the dimension of the unstable manifold of the periodic orbit must equal the dimension of the core of the round handle.

The attaching of a j-round handle R to a manifold W to form $N = W \cup_\theta R$ where $\theta : \partial_- R \to \partial_+ W$ also creates a cobordism $(N, \partial_+ N, \partial_- N)$ where $\partial_+ N = \partial_+(W \cup_\theta R)$ and $\partial_- N = \partial_- W$. An *elementary round cobordism* is a triad (N, N_1, N_0) such that a flow ϕ_t is defined on N entering through N_1, exiting through N_0 and containing an untwisted periodic orbit as the only critical set in N. It can be shown that N is a closed collar of N_0, $C(N_0)$ together with a round handle R attached along $\partial C(N_0) - N_0$.

3. *Lyapunov graphs for gradient-like flows*

In this section we will determine properties that a Lyapunov graph associated to a gradient-like flow possesses.

The next result, which we refer to as the Poincaré–Hopf equality is a direct consequence of Proposition 2.1 and characterizes the relation between the Euler characteristic of the incoming and the outgoing boundary components of a basic block containing a chain recurrent component R and the homology indices h_j, $j = 0, \dots, n$.

COROLLARY 3.1. (Poincaré-Hopf equality) *Let* (N_1, N_0) *be an index pair for a chain recurrent component* R *of a smooth flow on* M^n*,* n *odd, and let* h_j*,* $j = 0, \dots, n$*, be the dimensions of the homology indices of* R*. Then*

$$\sum_j (-1)^j h_j(R) = \tfrac{1}{2}(\chi(\partial N_1^+) - \chi(\partial N_1^-)).$$

Proof.

$$\begin{aligned}
\sum_{j=0}^{n} (-1)^j h_j &= \chi(N_1, N_0) \\
&= \chi(N_1) - \chi(N_0) \\
&= \tfrac{1}{2}\chi(\partial N_1) - \chi(N_0) \\
&= \tfrac{1}{2}(\chi(\partial N_1^+) + \chi(\partial N_1^-)) - \chi(N_0) \\
&= \tfrac{1}{2}(\chi(\partial N_1^+) - \chi(\partial N_1^-)).
\end{aligned}$$

The second equality follows from the exact sequence of the pair, whereas the third follows from considering the Euler characteristic of the double of N_1, DN_1. That is, $0 = \chi(DN_1) = 2\chi(N_1) - \chi(\partial N_1)$. The fourth equality follows since $\partial N_1^- = N_0$. □

We can interpret Corollary 3.1 as an equality which relates the Betti numbers with which the edges of a Lyapunov graph are labelled with the degree (indegree and outdegree) and the homology index labelling of v. This is the content of the following corollary.

COROLLARY 3.2. *For a smooth flow on* M^n*,* n *odd, with Lyapunov function* $f : M \to \mathbb{R}$ *let* h_j*,* $j = 0, \dots, n$*, be the dimensions of the homology indices of a chain recurrent component* R *of the flow. Let* v *be the vertex on the associated Lyapunov graph* L *which corresponds to* R *and which is labelled with* $\{h_j\}$*. Denote the indegree of* v *by*

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$e^+(v)$ and the outdegree by $e^-(v)$. Then the Betti numbers $\{(\beta^+_{(n-1)/2}, \dots, \beta^+_1, \beta^+_0)\}_k$ and $\{(\beta^-_{(n-1)/2}, \dots, \beta^-_1, \beta^-_0)\}_k$ on the incident edges to v must satisfy:

$$\mathcal{B}^- - \mathcal{B}^+ = e^-(v) - e^+(v) + \sum_{j=0}^{n} (-1)^j h_j,$$

where

$$\mathcal{B}^- = \sum_{k=1}^{e^-(v_i)} ((\beta^-_{(n-1)/2})_k \pm \cdots - (\beta^-_1)_k)$$

and

$$\mathcal{B}^+ = \sum_{k=1}^{e^+(v_i)} ((\beta^+_{(n-1)/2})_k \pm \cdots - (\beta^+_1)_k).$$

Proof. Note that the Poincaré–Hopf equality in Corollary 3.1 relates the Euler characteristic of the incoming and outgoing boundary components of a block defined index pair (N_1, N_0) of a chain recurrent component R to the dimensions of the homology indices of R. Note that R corresponds to a vertex v on the Lyapunov graph and that each connected component of ∂N_1^+ (∂N_1^-) corresponds to an incoming (outgoing) edge incident to v. Hence the zeroth Betti number of $\partial_+ N$ $(\partial_- N)$ is being represented on the graph by the number of incoming (outgoing) edges of v. Also, $\chi(\partial N_1^+)$ $(\chi(\partial N_1^-))$ which is the alternating sum of the Betti numbers corresponds to $\mathcal{B}^+$ $(\mathcal{B}^-)$ excluding the zeroth Betti numbers which have already been taken into account. □

The next result will determine upper bounds on the indegree of v, $e^+(v)$ and the outdegree of v, $e^-(v)$.

THEOREM 3.1. *Consider a smooth flow $\phi_t : M \to M$, with Lyapunov function $f : M \to \mathbb{R}$ and Lyapunov graph L. Let h_j, $j = 0, \dots, n$, be the dimensions of the homology indices of a chain recurrent component R of ϕ_t and let v be the vertex corresponding to R. Then v must satisfy:*

(1) $e^+(v) \le (h_1)^* + 1$ *($*$ indicates the index of the time-reversed flow);*

(2) $e^-(v) \le h_1 + 1$.

Proof. Let (N_1, N_0) be an index pair for R. Consider the long exact sequence where $\mathbb{Z}_2$ coefficients are used for the homology groups:

$$\cdots \to H_1(N_0) \to H_1(N_1) \to CH_1(R) \xrightarrow{\partial_1} \widetilde{H}_0(N_0) \xrightarrow{i_0} 0.$$

Since $\dim \widetilde{H}_0(N_0) = e^-(v) - 1$ and by exactness $\dim \ker i_0 = \dim \operatorname{Im} \partial_1$ we have that ∂_1 is surjective hence $\dim CH_1(R) = h_1^i \ge e^-(v) - 1$. By reversing the flow we obtain that $e^+(v) \le (h_1)^* + 1$. □

By taking less general flows we will obtain sharper results and for the most part this is the content of the following sections.

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3.1. *Lyapunov graphs for Morse–Smale flows.* In this section we will characterize a Lyapunov graph associated to a Morse-Smale flow, i.e. we will determine necessary properties the graph must possess in order to be associated to a Morse–Smale flow. We will work with the Conley homology index of hyperbolic singularities and periodic orbits. The following proposition computes this index and is due to Conley. We provide a simple proof based on the fact that for these cases the Conley index is a Thom space.

PROPOSITION 3.1. *Let (N, N_0) be an index pair with N containing a unique critical set. If N contains a hyperbolic singularity of index ℓ then $H_i(N, N_0) \cong \mathbb{Z}_2$ for $i = \ell$ and 0 otherwise. If N contains a hyperbolic periodic orbit of index ℓ then $H_i(N, N_0) \cong \mathbb{Z}_2$ for $i = \ell, \ell + 1$ and 0 otherwise.*

Proof. Let S be the singularity or the periodic orbit. Denote by ξ and ζ the unstable and stable bundles of S respectively. Denote by $D(\cdot)$ and $S(\cdot)$ the total spaces of the associated disk and sphere bundles respectively. Without loss of generality, we may assume $N = D(\xi \oplus \zeta)$. By sliding along the fibers of ζ it follows that

$$(N/N_0, N_0/N_0) \sim (D(\xi)/S(\xi), S(\xi)/S(\xi)) = (\Gamma(\xi), t_0),$$

a pointed Thom space. By Thom's isomorphism theorem,

$$H_i(N, N_0) \cong H_i(\Gamma(\xi), t_0) \cong H_{i-\ell}(D(\xi)) \cong H_{i-\ell}(S),$$

from which the proposition follows. We are following the convention that for a space X, $H_i(X) \cong 0$ for $i < 0$. □

One can think of a hyperbolic periodic orbit of index ℓ as the joining of two hyperbolic singularities p and q of adjacent indices ℓ and $\ell+1$ respectively. The following proposition is due to Franks [**Fra82**].

PROPOSITION 3.2. *Suppose ϕ_t is a Morse–Smale flow on an orientable manifold with a periodic orbit γ of index ℓ. Then given a neighborhood U of γ there exists a new Morse–Smale flow ψ_t whose vector field agrees with that of ϕ_t outside U and which has two singularities of index ℓ and $\ell + 1$ in U but no other chain recurrent points in U.*

The singularity q having one more unstable direction than p enables us to use an appropriate 1-submanifold of $W^u(q)$ to join with an appropriate 1-submanifold of $W^s(p)$, obtaining an index ℓ periodic orbit. Topologically this means that it should be possible to break a round handle of index ℓ into two handles of indices ℓ and $\ell + 1$.

A handle and its corresponding singularity is called *ℓ-disconnecting*, *ℓ-connecting* or *β-invariant*, in short, ℓ-d, ℓ-c, β-i, if and only if this handle has the algebraic effect of increasing or decreasing the ℓth Betti number of $\partial_+ N$ in relation to $\partial_- N$ in the first two cases respectively and keeping constant all Betti numbers in the latter case. See Figure 1.

The attachment of a single round handle to form $(N, \partial_+ N, \partial_- N)$ has the algebraic effect of altering the Betti numbers of $\partial_+ N$ in relation to $\partial_- N$. This effect is specified by the two singularities given by Proposition 3.2. Thus, if the first singularity is ℓ-d and the second $(\ell+1)$-d we refer to the round handle and the corresponding periodic orbit as ℓ-d, $(\ell+1)$-d and so on. However, if the first singularity is ℓ-d and the second ℓ-c then the round handle

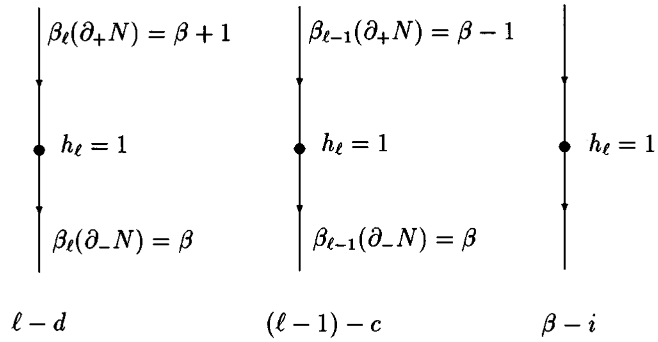

FIGURE 1. The three possible algebraic effects.

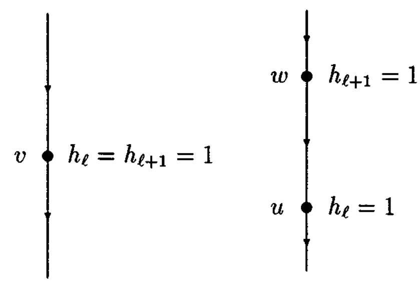

FIGURE 2. w and u are derived vertices from v.

and the corresponding periodic orbit are said to be β-i. For an example of a β-i handle see §4.1. Also, we refer to these singularities as the singularities *derived* from the periodic orbit.

Given a Morse–Smale flow ϕ_t on a boundaryless oriented smooth manifold M, let $f : M \to \mathbb{R}$ be a Lyapunov function and form the associated Lyapunov graph L. By repeated use of Proposition 3.2 it is possible to change ϕ_t to a Morse flow ϕ'_t with Lyapunov function $f' : M \to \mathbb{R}$. The associated Lyapunov graph is called the *derived graph* L'. If a Lyapunov graph L is the graph of a Morse flow its derived graph $L' = L$.

In order to obtain L' from L, a vertex v labelled with an index ℓ periodic orbit is removed from L and replaced by an oriented subgraph I which respects orientations. The subgraph I contains two vertices: w labelled with an index $\ell + 1$ singularity and u labelled with an index ℓ singularity. These vertices are connected by a directed edge e from w to u. Note that in L' all incoming edges of w are the incoming edges of v in L. Similarly, all the outgoing edges of u in L' are the outgoing edges of v in L. See Figure 2.

This description permits us to define the derived graph L' for an abstract Lyapunov graph L.

3.2. *Main result.* Theorem 3.2 is the main result in this section and for expository reasons it is a compendium of several propositions which will appear subsequently in §3.3.

We hope with this to give the reader a clearer picture of the global result.

Theorem 3.1 asserts that the number of boundary components of an isolating neighborhood in a smooth gradient-like flow is controlled by the one-dimensional homology index. Our main result, Theorem 3.2, shows how the higher-dimensional homology of the boundary components of an isolating neighborhood of a critical set in a Morse–Smale flow is controlled by the higher-dimensional homology indices.

THEOREM 3.2. *Let $\phi_t : M \to M$ be a Morse–Smale flow with Lyapunov function $f : M \to \mathbb{R}$. Let v be a vertex of the associated Lyapunov graph L. Let N be a basic block containing only one singularity or one periodic orbit that corresponds to v. Let $\partial_- N$, $\partial_+ N$ be the exiting and entering components of N contained in ∂N. If v is labelled as an:*

(1) *index ℓ singularity, then:*
  - (a) *v is ℓ-d, $(\ell - 1)$-c or β-i;*
  - (b) *the sum of the labels on the incoming edges incident to v (i.e. the total Betti number of $\partial_+ N$) changes with respect to the sum of the labels on the outgoing edges incident to v (i.e. the total Betti number of $\partial_- N$) by ± 2 or 0;*
  - (c) *v cannot be β-i if $n \neq 2\ell$;*

(2) *index ℓ periodic orbit, then:*
  - (a) *if v has u and w as its derived vertices then v is one of the combinations in the table below:*

w/u	ℓ-d	$(\ell - 1)$-c	β-i
$(\ell + 1)$-d	$(\ell + 1)$-d; ℓ-d	$(\ell + 1)$-d; $(\ell - 1)$-c	$(\ell + 1)$-d; β-i
ℓ-c	ℓ-c; ℓ-d	ℓ-c; $(\ell - 1)$-c	ℓ-c; β-i
β-i	β-i; ℓ-d	β-i; $(\ell - 1)$-c	

  - (b) *the sum of the labels on the incoming edges incident to v (i.e. the total Betti number of $\partial_+ N$) changes with respect to the sum of the labels on the outgoing edges incident to v (i.e. the total Betti number of $\partial_- N$) by adding the changes of the derived vertices;*
  - (c) *v cannot be in the last column of the table if $n \neq 2\ell$ and cannot be in the last row of the table if $n \neq 2(\ell + 1)$.*

This theorem is a consequence of the results obtained in §3.3 and will be proved within that section.

Theorem 3.2 describes the variation of the Betti numbers of the level sets as we pass through the critical set. In particular, it gives us the variation of the zeroth Betti number in terms of e^+ and e^-.

COROLLARY 3.3. *Consider a Morse–Smale flow ϕ_t, an associated Lyapunov graph L and a vertex v on L.*

(1) *If v is labelled as a sink ($h_0 = 1$) or an attracting periodic orbit ($h_1 = h_0 = 1$) then $e^+ = 1$ and $e^- = 0$.*

(2) *If v is labelled as a source ($h_n = 1$) or a repelling periodic orbit ($h_n = h_{n-1} = 1$) then $e^- = 1$ and $e^+ = 0$.*

(3) *If v is labelled as a saddle of Morse index* 1 $(h_1 = 1)$ *or a saddle type periodic orbit of index* 1 $(h_2 = h_1 = 1)$ *then* $e^+ = 1$. *Moreover, if v is labelled as* 0-*c singularity or as a* 0-*c*, 2-*d periodic orbit then* $e^- = 2$. *Otherwise,* $e^- = 1$.

(4) *If v is labelled as a saddle of Morse index* $n-1$ $(h_{n-1} = 1)$ *or a saddle type periodic orbit of index* $n-2$ $(h_{n-1} = h_{n-2} = 1)$ *then* $e^- = 1$. *Moreover, if v is labelled as* $(n-1)$-*d singularity or as a* $(n-3)$-*c*, $(n-1)$-*d periodic orbit or as a* $(n-2)$-*d*, $(n-1)$-*d periodic orbit then* $e^+ = 2$. *Otherwise,* $e^+ = 1$.

(5) *If v is labelled as any other saddle or periodic orbit then* $e^+ = e^- = 1$.

Proof. Theorem 3.2 specifies the effect singularities and periodic orbits have on the Betti numbers of the level hypersurfaces of a basic block N. Recall that e^- and e^+ correspond to the number of exiting and entering boundary components of N, i.e. to the number of components of $\partial_- N$ and $\partial_+ N$. Hence, this number is determined by the zeroth-dimensional Betti numbers of $\partial_- N$ and $\partial_+ N$. The proof follows directly from an analysis of the zeroth-dimensional Betti numbers in Theorem 3.2. □

So far, our theorems have determined locally necessary conditions on a Lyapunov graph so that it is associated to a flow. The following theorem of Franks determines a necessary global condition on a Lyapunov graph so that it is associated to a smooth flow. We refer the reader to [**Fra85**] for a proof of this theorem.

THEOREM 3.3. *Let M be a compact oriented manifold. Suppose that $\phi_t : M \to M$ is a smooth flow and $f : M \to \mathbb{R}$ is a Lyapunov function with a finite associated Lyapunov graph L. If $H_1(M; Q) = 0$ then the graph L is a tree.*

3.3. *Basic blocks for singularities.* This and the next section involve the analysis of exact sequences and hence give a more precise result than the table in Theorem 3.2. The results below will specify the role of the homology boundary maps in the change of Betti numbers of the level surfaces.

Throughout this section all propositions will have as underlying hypotheses:

(1) M is an n-dimensional manifold, $n \geq 2$;

(2) p is a singularity of index ℓ (i.e. $\dim W^u(p) = \ell$ and $\dim W^s(p) = n - \ell$);

(3) $(N, \partial_- N)$ is an index pair for p where $\partial N = \partial_- N \cup \partial_+ N$ and $\partial_- N \cap \partial_+ N = \emptyset$;

(4) $\partial_+ N$ and $\partial_- N$ are denoted by entering and exiting boundary components for the flow respectively;

(5) $\partial_i : H_i(N, \partial_- N) \to H_{i-1}(\partial_- N)$ denotes the homology boundary map.

The following propositions will determine the relationship between the Betti numbers of $\partial_+ N$ and $\partial_- N$. The way in which they are related will depend on the index of the singularity as well as the homology boundary map. It is also interesting to observe that whenever the index of the singularity coincides with half the dimension of an even-dimensional manifold or with either of the two middle dimensions of an odd-dimensional manifold the analysis is slightly more elaborate. We refer to these as the middle-dimensional cases and they are dealt with in §3.3.1.

In the following proofs we will make systematic use of the long exact sequences for the pairs $(N, \partial_- N)$ and $(N, \partial_+ N)$.

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Consider the long exact sequence for the pair $(N, \partial_- N)$, which we will label by LES−:

$$0 \to H_\ell(\partial_- N) \xrightarrow{i_\ell} H_\ell(N) \xrightarrow{p_\ell} H_\ell(N, \partial_- N) \xrightarrow{\partial_\ell^-} H_{\ell-1}(\partial_- N) \xrightarrow{i_{\ell-1}} H_{\ell-1}(N) \to 0.$$

Also consider the long exact sequence for the pair $(N, \partial_+ N)$ which we will label by LES+:

$$0 \to H_{n-\ell}(\partial_+ N) \xrightarrow{i_{n-\ell}} H_{n-\ell}(N) \xrightarrow{p_{n-\ell}} H_{n-\ell}(N, \partial_+ N)$$
$$\xrightarrow{\partial_{n-\ell}^+} H_{n-\ell-1}(\partial_+ N) \xrightarrow{i_{n-\ell-1}} H_{n-\ell-1}(N) \to 0.$$

PROPOSITION 3.3. *Under the hypotheses stated at the beginning of this section, let $n = 2i$ and $\ell \neq i$ or $n = 2i + 1$ and $\ell \neq i, i + 1$. Then either:*

(1) $\partial_\ell^- = 0$ *and* $\partial_{n-\ell}^+ \neq 0$ *in which case* (ℓ-*d*)

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \text{for all } k \neq \ell, n-\ell-1 \\ \beta_k(\partial_+ N) - 1, & \text{for } k = \ell, n-\ell-1 \end{cases}$$

or else,

(2) $\partial_\ell^- \neq 0$ *and* $\partial_{n-\ell}^+ = 0$ *in which case* (($\ell - 1$)-*c*)

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \text{for all } k \neq \ell-1, n-\ell \\ \beta_k(\partial_+ N) + 1, & \text{for } k = \ell-1, n-\ell. \end{cases}$$

Proof. Since p is a singularity of index ℓ with basic block N with entering and exiting boundary components $\partial_+ N$ and $\partial_- N$ respectively, $H_k(N, \partial_- N) = 0$ for all $k \neq \ell$. If we reverse the flow on N, p will now have index $n - \ell$ and the roles of $\partial_+ N$ and $\partial_- N$ will be interchanged, i.e. the exiting boundary component for the time reversed flow will now be $\partial_+ N$ and the entering boundary component will be $\partial_- N$. Once again, $H_k(N, \partial_+ N) = 0$ for all $k \neq n - \ell$.

Combining both long exact sequences LES− and LES+ we obtain that $H_k(\partial_- N) \cong H_k(N)$ for $k \neq \ell$ and $H_k(\partial_+ N) \cong H_k(N)$ for $k \neq n - \ell$. This implies that $H_k(\partial_- N) \cong H_k(\partial_+ N)$ for $k \neq \ell, n - \ell$.

For the cases when $k = \ell, n - \ell$, we must consider whether the homology boundary maps ∂_ℓ^- and $\partial_{n-\ell}^+$ are zero or non-zero.

If both boundary maps are zero or both are non-zero, it is an easy computation to see that we contradict the Poincaré duality. Take, for instance, the case where both boundary maps are zero. Since $\partial_\ell^- = 0$, LES− implies that $H_{\ell-1}(\partial_- N) \cong H_{\ell-1}(N)$. We have shown above that $H_k(\partial_+ N) \cong H_k(N)$ for $k \neq n - \ell$, hence we obtain that $H_{\ell-1}(\partial_+ N) \cong H_{\ell-1}(N) \cong H_{\ell-1}(\partial_- N)$. In other words, this implies that $\beta_{\ell-1}(\partial_- N) = \beta_{\ell-1}(\partial_+ N)$. On the other hand, since $\partial_{n-\ell}^+ = 0$, LES+ implies that $H_{n-\ell}(N) \cong H_{n-\ell}(N, \partial_+ N) \oplus H_{n-\ell}(\partial_+ N)$ and using the fact observed above that $H_k(\partial_- N) \cong H_k(N)$ for $k \neq \ell$, we obtain $H_{n-\ell}(\partial_- N) \cong H_{n-\ell}(N)$, hence $H_{n-\ell}(\partial_- N) \cong H_{n-\ell}(N, \partial_+ N) \oplus H_{n-\ell}(\partial_+ N)$. Thus, $\beta_{n-\ell}(\partial_- N) = \beta_{n-\ell}(\partial_+ N) + 1$. Since $\partial_+ N$ and $\partial_- N$ are closed $(n - 1)$-dimensional manifolds, the indices $n - \ell$ and $\ell - 1$ are complementary in this dimension. Hence the homology groups with these indices of $\partial_+ N$ and also of $\partial_- N$ are Poincaré duals. However, this contradicts the Poincaré duality since $\beta_{\ell-1}(\partial_- N) =$

$\beta_{\ell-1}(\partial_+ N)$ and $\beta_{n-\ell}(\partial_- N) = \beta_{n-\ell}(\partial_+ N) + 1$. The case where both boundary maps are non-zero is treated similarly.

Hence we are left with the cases where one of the boundary maps is non-zero and the other is zero. Let us consider one of these cases where $\partial_\ell^- = 0$ and $\partial_{n-\ell}^+ \neq 0$. If $\partial_\ell^- = 0$, then exactness of LES− implies that $H_\ell(N) \cong H_\ell(\partial_- N) \oplus H_\ell(N, \partial_- N)$ and $H_{\ell-1}(\partial_- N) \cong H_{\ell-1}(N)$. Now if $\partial_{n-\ell}^+ \neq 0$, then exactness of LES+ implies that $H_{n-\ell}(N) \cong H_{n-\ell}(\partial_+ N)$ and $H_{n-\ell-1}(\partial_+ N) \cong H_{n-\ell}(N, \partial_+ N) \oplus H_{n-\ell-1}(N)$. Since the Betti numbers of $H_\ell(N, \partial_- N)$ and $H_{n-\ell}(N, \partial_+ N)$ are equal to 1 we have that $\beta_\ell(N) = \beta_\ell(\partial_- N) + 1$ and $\beta_k(N) = \beta_k(\partial_- N)$ for all $k \neq \ell$. Also, $\beta_{n-\ell-1}(N) = \beta_{n-\ell-1}(\partial_+ N) - 1$ and $\beta_k(N) = \beta_k(\partial_+ N)$ for $k \neq n - \ell - 1$. Combining these results we prove (1). Also, (2) is proved by a similar analysis. □

This analysis of the long exact sequences LES− and LES+ can be summarized in the following tables:

$(N, \partial_- N)$	$\partial_\ell^- = 0$	$\partial_\ell^- \neq 0$
$\beta_\ell(N) =$	$\beta_\ell(\partial_- N) + 1$	$\beta_\ell(\partial_- N)$
$\beta_{\ell-1}(N) =$	$\beta_{\ell-1}(\partial_- N)$	$\beta_{\ell-1}(\partial_- N) - 1$

TABLE 1.

In addition to this

$$\beta_k(N) = \beta_k(\partial_- N), \quad \forall k \neq \ell, \ell - 1.$$

$(N, \partial_+ N)$	$\partial_{n-\ell}^+ = 0$	$\partial_{n-\ell}^+ \neq 0$
$\beta_{n-\ell}(N) =$	$\beta_{n-\ell}(\partial_+ N) + 1$	$\beta_{n-\ell}(\partial_+ N)$
$\beta_{n-\ell-1}(N) =$	$\beta_{n-\ell-1}(\partial_+ N)$	$\beta_{n-\ell-1}(\partial_+ N) - 1$

TABLE 2.

In addition to this

$$\beta_k(N) = \beta_k(\partial_+ N), \quad \forall k \neq n - \ell, n - \ell - 1.$$

3.3.1. *Middle-dimensional cases.*

PROPOSITION 3.4. *Under the hypotheses stated at the beginning of this section, let $n = 2i$ and $\ell = i$. Then either:*

(1) $\partial_i^- = 0, \partial_i^+ = 0$ *or* $\partial_i^- \neq 0, \partial_i^+ \neq 0$ *in which case* (β-i)

$$\beta_k(\partial_- N) = \beta_k(\partial_+ N) \quad \textit{for all } k$$

or else,

(2) $\partial_i^- = 0, \partial_i^+ \neq 0$ *in which case* (ℓ-d)

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \textit{for all } k \neq i, i-1 \\ \beta_k(\partial_+ N) - 1, & \textit{for } k = i, i-1. \end{cases}$$

or else,

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(3) $\partial_i^- \neq 0$, $\partial_i^+ = 0$ *in which case* $((\ell - 1)\text{-}c)$

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \text{for all } k \neq i, i-1 \\ \beta_k(\partial_+ N) + 1, & \text{for } k = i, i-1. \end{cases}$$

Proof. Since $n = 2i$ and $\ell = i$, we substitute these values for the indices in LES$-$ and LES$+$ The analysis is similar to Proposition 3.3, i.e. we analyse the four possible cases arising from the choices of the homology boundary maps, ∂_i^- and ∂_i^+ being zero or non-zero. It is interesting to note that in this case if both boundary maps are zero or both are non-zero, we do not contradict the Poincaré duality. In what follows we detail this analysis.

We first observe that since p is a singularity of index i with basic block N, with entering and exiting boundary components, $\partial_+ N$ and $\partial_- N$ respectively, $H_k(N, \partial_- N) = 0$ for all $k \neq i$. If we reverse the flow on N, p will now continue to be an index i singularity with the roles of $\partial_+ N$ and $\partial_- N$ interchanged, i.e. the exiting boundary component will now be $\partial_+ N$ and the entering boundary component will be $\partial_- N$. Once again, $H_k(N, \partial_+ N) = 0$ for all $k \neq i$.

There are four cases to consider, however, we take two for illustrative purposes, the others being entirely similar in nature. The proofs always reduce to an analysis of LES$-$ and LES$+$ with ℓ substituted for i in those sequences. So it suffices to consider the four combinations of choices for ∂_ℓ^- and $\partial_{n-\ell}^+$ in the tables above with $\ell = i$. □

In the next proposition we will treat the case where M is odd-dimensional and the index of p is in the middle-dimensional range, i.e. $\ell = i$ and $\ell = i + 1$.

PROPOSITION 3.5. *Under the hypotheses stated at the beginning of this section, let* $n = 2i + 1$. *Then either:*

(1) $\ell = i$ *and*

(a) $\partial_i^- = 0$, $\partial_{i+1}^+ \neq 0$ *in which case,* $(\ell\text{-}d)$

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \text{for all } k \neq i \\ \beta_k(\partial_+ N) - 2, & \text{for } k = i; \end{cases}$$

or else,

(b) $\partial_i^- \neq 0$, $\partial_{i+1}^+ = 0$ *in which case,* $((\ell - 1)\text{-}c)$

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \text{for all } k \neq i+1, i-1 \\ \beta_k(\partial_+ N) + 1, & \text{for } k = i+1, i-1; \end{cases}$$

(2) *or* $\ell = i + 1$ *and*

(a) $\partial_{i+1}^- \neq 0$, $\partial_i^+ = 0$ *in which case,* $((\ell - 1)\text{-}c)$

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \text{for all } k \neq i \\ \beta_k(\partial_+ N) + 2, & \text{for } k = i; \end{cases}$$

or else,

(b) $\partial^-_{i+1} = 0$, $\partial^+_i \neq 0$ *in which case,* (ℓ-*d*)

$$\beta_k(\partial_- N) = \begin{cases} \beta_k(\partial_+ N), & \textit{for all } k \neq i+1, i-1 \\ \beta_k(\partial_+ N) - 1, & \textit{for } k = i+1, i-1. \end{cases}$$

Proof. Since $n = 2i + 1$ and $\ell = i$, we substitute these values for the indices in LES$-$ and LES$+$. The analysis is similar to Proposition 3.3, i.e. we analyse the four possible cases arising from the choices of the homology boundary maps, ∂^-_i and ∂^+_{i+1} being zero or non-zero. Here again if both boundary maps are zero or both are non-zero, we contradict the Poincaré duality and we illustrate one of these cases below. If this is not the case we are led to the results in this proposition.

We first observe that since p is a singularity of index i with basic block N with entering and exiting boundary components, $\partial_+ N$ and $\partial_- N$ respectively, $H_k(N, \partial_- N) = 0$ for all $k \neq i$. If we reverse the flow on N, p will now be an index $i + 1$ singularity with the roles of $\partial_+ N$ and $\partial_- N$ interchanged, i.e. the exiting boundary component will now be $\partial_+ N$ and the entering boundary component will be $\partial_- N$. Once again, $H_k(N, \partial_+ N) = 0$ for all $k \neq i + 1$.

The analysis of LES$-$ and LES$+$, with $\ell = i$ and consequently $n - \ell = 2i + 1 - i = i + 1$, can be done by substituting these values into the tables above.

In the case when $\partial^-_i = 0$ and $\partial^+_{i+1} = 0$ we obtain from column one of Tables 1 and 2 that $\beta_{i-1}(\partial_- N) = \beta_{i-1}(\partial_+ N)$ and that $\beta_{i+1}(\partial_- N) = 1 + \beta_{i+1}(\partial_+ N)$. However, we have contradicted Poincaré duality which asserts that $\beta_{i+1}(\partial_- N) = \beta_{i-1}(\partial_- N)$ and $\beta_{i+1}(\partial_+ N) = \beta_{i-1}(\partial_+ N)$. The case when $\partial^-_i \neq 0$ and $\partial^+_{i+1} \neq 0$ is treated similarly.

In the case when $\partial^-_i = 0$ and $\partial^+_{i+1} \neq 0$ we combine the results in column one of Table 1 with those in column two of Table 2. We obtain that $\beta_k(\partial_- N) = \beta_k(N)$ for all $k \neq i$ and $\beta_i(\partial_+ N) - 1 = \beta_i(\partial_- N) + 1$. Hence $\beta_i(\partial_- N) = \beta_i(\partial_+ N) - 2$. The case where $\partial^-_i \neq 0$ and $\partial^+_{i+1} = 0$ is an entirely similar analysis.

The results in (2) can be obtained from (1) by considering the time reversed flow where the roles of $\partial_+ N$ and $\partial_- N$ are interchanged. □

The next propositions narrow down the possibilities of manifolds where cases (1) and (2) of Proposition 3.4 may occur.

For a $(2k + 1)$-dimensional closed manifold X, the mod 2 *semi-characteristic* with coefficients in the field $\mathbb{F}$ is defined to be:

$$\chi_{1/2}(X; \mathbb{F}) = \sum_{i=0}^{k} \beta_i(X; \mathbb{F}) \mod 2.$$

PROPOSITION 3.6. *Under the hypotheses of Proposition 3.4 with $n = 2\ell$ and N a basic block, if cases (1) or (2) occur (i.e. $\partial^-_\ell = \partial^+_\ell = 0$ or both are not equal to zero) then at least a* mod 2 *homology class in $H_\ell(M)$ has self-intersection number non-zero. In particular, M cannot be S^n or the connected sum of generalized tori $S^i \times S^j$, $i + j = n$.*

Proof. If every mod 2 homology class $\xi \in H_\ell(N)$ has self-intersection $\xi \cdot \xi = 0$ then arguing as in Lemma 5.10 in [**KM63**] it follows that $\chi_{1/2}(\partial_+ N, \mathbb{Z}_2) \neq \chi_{1/2}(\partial_- N, \mathbb{Z}_2)$. However, since we assume that cases (1) and (2) occur, this implies that all Betti numbers of $\partial_+ N$ and $\partial_- N$ are the same thus the semi-characteristics must be equal. Hence, there must be a mod 2 class $\xi \in H_\ell(N)$ such that $\xi \cdot \xi \neq 0$. Since $N \subset M$ is a codimension 0 embedding, if η is the image of ξ under the map induced by inclusion $H_\ell(N) \to H_\ell(M)$ it follows that $\xi \cdot \xi = \eta \cdot \eta$. Thus, $\eta \cdot \eta \neq 0$. □

PROPOSITION 3.7. *Under the hypotheses of Proposition 3.4 with $n = 2$ mod 4, N a basic block and M orientable, then cases (1) and (2) do not occur.*

Proof. It follows from Lemma 5.8 in [**KM63**] that $\chi_{1/2}(\partial_+ N, \mathbb{Q}) \neq \chi_{1/2}(\partial_- N, \mathbb{Q})$. In the case $n = 2\ell \geq 4$, $1 < \ell < n - 1$, by Corollary 3.3, $\partial_+ N$ and $\partial_- N$ are level surfaces at regular values of a Morse function. Thus, $\partial_+ N$ and $\partial_- N$ are null-cobordant. By a theorem in [**LMP69**] since $\dim \partial_+ N = \dim \partial_- N = 1 \bmod 4$, $\chi_{1/2}(N^\pm, \mathbb{Q}) = \chi_{1/2}(N^\pm, \mathbb{Z}_2)$. Thus, $\chi_{1/2}(\partial_+ N, \mathbb{Z}_2) \neq \chi_{1/2}(\partial_- N, \mathbb{Z}_2)$. Now, if cases (1) or (2) of Proposition 3.4 occur, then all Betti numbers of $\partial_+ N$ and $\partial_- N$ are the same. Hence the semi-characteristics must be equal, a contradiction. In the case $n = 2$, $\partial_+ N$ and $\partial_- N$ are disjoint unions of circles and hence null-cobordant. □

The following two-dimensional example illustrates the necessity of the hypothesis of the ambient manifold being orientable in Proposition 3.7. Let N be a Mobius band with a disk removed from its interior. N is a basic block for a saddle with $\partial_+ N$ and $\partial_- N$ being homeomorphic to S^1.

3.3.2. *Proof of Theorem 3.2.* We now prove Theorem 3.2 of §3.2.

Proof. The proof of (1) of Theorem 3.2 follows directly from Propositions 3.3–3.5.

The proof of (2) follows from Proposition 3.2 where we can 'substitute' a hyperbolic periodic orbit of index ℓ for two singularities of index ℓ and $\ell + 1$. Hence, if v represents a vertex on a Lyapunov graph labelled with a hyperbolic periodic orbit of index ℓ, let w and u be the derived vertices labelled as hyperbolic singularities of index $\ell + 1$ and ℓ respectively. Thus, if ℓ is not the mid-dimension of the ambient manifold we combine the two cases of Propositions 3.3 for w (($\ell + 1$)-d and ℓ-c) with the two cases of Propositions 3.3 for u (ℓ-d and ($\ell - 1$)-c) to obtain the four possibilities for the vertex v.

w/u	ℓ-d	$(\ell - 1)$-c
$(\ell + 1)$-d	$(\ell + 1)$-d; ℓ-d	$(\ell + 1)$-d; $(\ell - 1)$-c
ℓ-c	ℓ-c; ℓ-d	ℓ-c; $(\ell - 1)$-c

If ℓ is the mid-dimension of the ambient n-manifold M then we must consider the cases when $n = 2\ell + 1$ and $n = 2\ell$. In the first case, $n = 2\ell + 1$, we combine the two cases in (1) of Proposition 3.5 for u (ℓ-d and ($\ell - 1$)-c) with the two cases in (2) of Proposition 3.5 for w (($\ell + 1$)-d and ℓ-c) generating the four possibilities for v listed in the table above.

If the ambient n-manifold M has dimension $n = 2\ell$ we combine the three cases of Proposition 3.4 for u (ℓ-d and ($\ell - 1$)-c and β-i) with the two cases in Proposition 3.3 for w (($\ell + 1$)-d and ℓ-c) generating the six possibilities for v listed in the table below.

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w/u	ℓ-d	$(\ell-1)$-c	β-i
$(\ell+1)$-d	$(\ell+1)$-d; ℓ-d	$(\ell+1)$-d; $(\ell-1)$-c	$(\ell+1)$-d; β-i
ℓ-c	ℓ-c; ℓ-d	ℓ-c; $(\ell-1)$-c	ℓ-c; β-i

Finally, if $n = 2(\ell+1)$ we generate the last row of the table in Theorem 3.2. □

4. *Realizing Lyapunov graphs*

In the previous section we obtained results which determine necessary conditions on a Lyapunov graph in order for it to be associated to a smooth flow. In this section, we propose to establish sufficient conditions on an abstract Lyapunov graph L so that it can be realized as a smooth flow on some differentiable manifold.

Ideally, it would be desirable that the necessary conditions coincide with the sufficient conditions. However, the necessary conditions obtained within this paper are too weak for this purpose and hence we must impose additional restrictions on L in order to construct a flow with an equivalent Lyapunov graph.

The first natural consideration is to restrict the class of abstract Lyapunov graphs to those which satisfy the necessary conditions determined in §3. An abstract Lyapunov graph L whose vertices are labelled with singularities or periodic orbits, is called *admissible* if and only if L satisfies the conclusions of Theorem 3.2.

Two Lyapaunov graphs L_1 and L_2 are said to be *equivalent* if and only if there is a vertex and edge preserving bijection $\varphi : L_1 \to L_2$ such that:

(1) v and $\varphi(v)$ are labelled with topologically equivalent chain recurrent flows;
(2) e and $\varphi(e)$ are labelled with the same Betti numbers.

Also, at times, we will deal with Lyapunov graphs as topological one complexes.

4.1. *Flows on basic blocks and some handle decompositions.*

We will first describe two different ways to build a basic block for an ℓ-handle. We wish to construct a basic block $N = C(\partial_- N) \cup H^{(\ell)}$, where $C(\partial_- N)$ is a closed collar on $\partial_- N$ and $H^{(\ell)}$ is an index ℓ handle attached to $C(\partial_- N)$ along $\partial C(\partial_- N) - \partial_- N$ by the attaching map θ. We can take $\partial_- N$ to be either S^{n-1} or $S^{\ell-1} \times S^{n-\ell}$. In both cases the attaching map θ is given by the inclusion: in the prior case by

$$\theta : S^{\ell-1} \times D^{n-\ell} \to S^{\ell-1} \times D^{n-\ell} \cup_{\mathrm{id}} D^{\ell} \times S^{n-\ell-1} = S^{n-1}$$

and in the latter case by

$$\theta : S^{\ell-1} \times D^{n-\ell} \to S^{\ell-1} \times D^{n-\ell} \cup_{\mathrm{id}} S^{\ell-1} \times D^{n-\ell} = S^{\ell-1} \times S^{n-\ell}.$$

Also, $\partial_+ N = \partial N - \partial_- N$: in the first case,

$$\partial_+ N = D^{\ell} \times S^{n-\ell-1} \cup_{\phi} D^{\ell} \times S^{n-\ell-1} = S^{\ell} \times S^{n-\ell-1}$$

and in the second by

$$\partial_+ N = D^{\ell} \times S^{n-\ell-1} \cup_{\phi} S^{\ell-1} \times D^{n-\ell} = S^{n-1},$$

https://doi.org/10.1017/S0143385799120893 Published online by Cambridge University Press

where ϕ is induced by the diffeomorphism on $D^\ell \times D^{n-\ell}$ which exchanges factors. In the first case the handle is ℓ-d and in the second case it is $(\ell - 1)$-c.

One can easily construct a Morse function on N with a unique non-degenerate critical point of index ℓ in $H \subset N$. Associated to this Morse function there is a Morse flow which we can assume is transverse not only to ∂H but also to ∂N [**Mil65**].

This construction is completely general, in the sense that if we have a connected sum of generalized tori as $\partial_- N$ we can attach an ℓ-d handle to an S^{n-1} factor which is always present in the connected sum and an $(\ell - 1)$-c handle to an $S^{\ell-1} \times S^{n-\ell}$ factor if it is part of the connected sum.

We now will describe a well known example illustrating case (1a) of Proposition 3.4 which corresponds to the attachment of β-i k-handles within a 4-manifold. We remark that if $N' = N \cup_\theta (D^\ell \times D^{n-\ell})$, where $\theta : S^{\ell-1} \times D^\ell \to \partial_- N$ is an embedding, then by definition $\chi(\partial_- N, \theta) = \partial N' - (\partial N - \partial_- N)$ is the result of performing surgery on $\partial_- N$ by means of the embedding θ. Note that surgeries and the attachment of handles are in one-to-one correspondence since they both use the same embedding (see [**Mil65**]). It is well known [**Rol76**] that if L is a three-dimensional Lens space, then $L = \chi(S^3, \theta)$ where $\theta : S^1 \times D^2 \to S^3$ is a suitable embedding. Take L such that $\pi_1(L)$ has odd order. Then L and S^3 have the same mod 2 homology. This surgery corresponds to the attachment of a 2-handle $H^{(2)}$ on a collar of S^3, $C(S^3)$. Set $N = C(S^3) \cup H^{(2)}$, and hence $\partial_- N = S^3$ and $\partial_+ N = L$. The flow defined on N is obtained as above and because $H_1(L) \cong H_1(S^3) \cong 0$ it follows that $H^{(2)}$ is a β-i 2-handle.

In Propositions 4.1 and 4.2, we describe some very specific handle decompositions for compact n-manifolds which are either S^n or generalized tori $S^p \times S^q$ and their connected sums.

PROPOSITION 4.1. *Let $M^n = S^p \times S^q$. This manifold possesses the handle decomposition:*

$$M^n = H^{(0)} \cup H^{(p)} \cup H^{(q)} \cup H^{(n)},$$

where the a-tubes of $H^{(p)}$, $H^{(q)}$ are disjoint and contained in $\partial H^{(0)} \approx S^{n-1}$. Furthermore, the attaching spheres of $H^{(p)}$, $H^{(q)}$ are unknotted in $\partial H^{(0)}$ and bound p- and q-dimensional disks B^p and B^q in $H^{(0)}$ that meet in a single point.

Before we prove this proposition we remark that since $S^p \times S^q \approx S^q \times S^p$, the handles $H^{(p)}$ and $H^{(q)}$ can be attached in reverse order. Also, if $c(H^{(p)})$ and $c(H^{(q)})$ are the cores of $H^{(p)}$ and $H^{(q)}$ respectively, then the p- and q-dimensional subspheres $B^p \cup c(H^{(p)})$ and $B^q \cup c(H^{(q)})$ meet transversally in the single point $B^p \cap B^q$. Thus, the intersection form evaluated at those subspheres (more precisely on the homology class represented by a triangulation of these subspheres) is one. Because of this we refer to the attachment of $H^{(q)}$ as dual to $H^{(p)}$ and we refer to these handles as a *dual pair*.

Proof. Write $S^p = D^p_- \cup D^p_+$, where $D^p_\pm = \{(x_1, \ldots, x_{p+1}) \in S^p \mid \pm x_{p+1} \geq 0\}$. Similarly, write $S^q = D^q_- \cup D^q_+$. Then

$$\begin{aligned} S^p \times S^q &= (D^p_- \cup D^p_+) \times (D^q_- \cup D^q_+) \\ &= D^p_- \times D^q_- \cup D^p_+ \times D^q_- \cup D^p_- \times D^q_+ \cup D^p_+ \times D^q_+. \end{aligned}$$

We would like to see this union as a 0-handle with a p-handle and a q-handle attached so that the resulting boundary bounds an n-handle in $S^p \times S^q$. For this purpose, set $H^{(0)} = D^p_- \times D^q_-$. Now,

$$\begin{aligned} H^{(0)} \cap D^p_+ \times D^q_- &= D^p_- \times D^q_- \cap D^p_+ \times D^q_- \\ &= (\partial D^p_-) \times D^q_- \cap (\partial D^p_+) \times D^q_- \subset (\partial H^{(0)}) \cap (\partial D^p_+) \times D^q_-. \end{aligned}$$

This means that $D^p_+ \times D^q_-$ is a p-handle $H^{(p)}$ attached to $\partial H^{(0)}$. The attaching sphere is $(\partial D^p_+) \times 0 \subset \partial H^{(0)}$. It can be isotoped in $\partial H^{(0)}$ to $(\partial D^p_+) \times 1$ which bounds $D^p_- \times 1$. Thus, the attaching sphere is unknotted in $\partial H^{(0)} = S^{n-1}$.

Now, since

$$H^{(0)} \cup H^{(p)} = D^p_- \times D^q_- \cup D^p_+ \times D^q_- = (D^p_- \cup D^p_+) \times D^q_- = S^p \times D^q_-.$$

It follows that

$$\begin{aligned} (H^{(0)} \cup H^{(p)}) \cap D^p_- \times D^q_+ = S^p \times D^q_- \cap D^p_- \times D^q_+ &= S^p \times \partial D^q_- \cap D^p_- \times \partial D^q_+ \\ &= \partial(H^{(0)} \cup H^{(p)}) \cap D^p_- \times \partial D^q_+. \end{aligned}$$

We conclude that $D^p_- \times D^q_+$ is a q-handle $H^{(q)}$ attached to $\partial(H^{(0)} \cup H^{(p)})$. It is actually attached to $\partial H^{(0)}$ since

$$S^p \times \partial D^q_- \cap D^p_- \times \partial D^q_+ = D^p_- \times \partial D^q_- \cap D^p_- \times \partial D^q_+ \subset \partial H^{(0)} \cap D^p_- \times \partial D^q_+.$$

Note that $\partial(H^{(0)} \cup H^{(p)} \cup H^{(q)}) = \partial(D^p_+ \times D^q_+) = S^{n-1}$. Hence, $D^p_+ \times D^q_+$ is a n-handle $H^{(n)}$ attached to $\partial(H^{(0)} \cup H^{(p)} \cup H^{(q)})$.

Consider $H^{(p)} \cap H^{(q)} = D^p_+ \times D^q_- \cap D^p_- \times D^q_+ = \partial D^p_+ \times \partial D^q_+$. This intersection equals the intersection of the boundaries of the attaching regions $(\partial D^p_+) \times D^q_-$ and $D^p_- \times (\partial D^q_+)$ of $H^{(p)}$ and $H^{(q)}$, respectively. That is,

$$H^{(p)} \cap H^{(q)} = (\partial D^p_+) \times D^q_- \cap (\partial D^p_-) \times D^q_+.$$

As $H^{(p)}$ is a closed tubular neighborhood of its core, the tubular neighborhood theorem provides an ambient isotopy leaving $H^{(0)} \cup c(H^{(p)}) \cup c(H^{(q)})$ fixed and that shrinks $H^{(p)}$ eliminating the intersection. The ambient isotopy will of course change $H^{(n)}$ as well.

Next, set $B^p = D^p_- \times 0 \subset H^{(p)}$ and $B^q = 0 \times D^q_- \subset H^{(q)}$. Then B^p and B^q meet transversally in a single point. The attaching spheres of $H^{(p)}$ and $H^{(q)}$ are ∂B^p and ∂B^q respectively. Also, $B^p \cup c(H^{(p)})$ and $B^q \cup c(H^{(q)})$ are p- and q-dimensional spheres intersecting transversally in $B^p \cap B^q$. Finally, it is necessary to round corners. □

LEMMA 4.1. *Consider the link $L = S^{p-1} \times 0 \cup 0 \times S^{q-1}$ on S^{n-1}, where $n = p+q$. Then $S^{p-1} \times 0$ and $0 \times S^{q-1}$ are unknotted and their linking number is one. Also, L is equivalent to $K^{p-1} \cup K^{q-1}$, where $K^{p-1} \subset S^{n-1}$ is an unknotted sphere, $K^{q-1} = \varphi(\mathrm{pt} \times S^{q-1})$, $\mathrm{pt} \in K^{p-1}$, and $\varphi : K^{p-1} \times D^q \to T$ is a diffeomorphism and T a tubular neighborhood of K^{p-1}.*

Proof. Note that $S^{p-1} \times 0 = \partial D^p \times 0$ and $0 \times S^{q-1} = 0 \times \partial D^q$. It follows that $S^{p-1} \times 0$ and $0 \times S^{q-1}$ are unknotted. L is equivalent to $S^{p-1} \times 1 \cup 0 \times S^{q-1}$. The linking number of $S^{p-1} \times 1$ and $0 \times S^{q-1}$ equals the intersection number of $D^{p-1} \times 1$ and $0 \times S^{q-1}$ which

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intersect transversally in 0×1, hence having intersection number one. Without loss of generality we can assume orientations were chosen to yield 1 and not -1.

The second assertion of the proposition follows from the tubular neighborhood theorem together with the fact that L can be isotoped to $S^{p-1} \times 0 \cup \mathrm{pt} \times S^{q-1}$, where $\mathrm{pt} \in S^{p-1}$ and $S^{p-1} \times D^q$ is a closed tubular neighborhood of $S^{p-1} \times 0$. □

The following proposition asserts that to form M^n, a connected sum of generalized tori, all handles other than $H^{(0)}$ and $H^{(n)}$ can be attached in any order since they are pairwise disjoint with attaching spheres unknotted.

PROPOSITION 4.2. *Let $M^n = \#_i M_i^n$ where for each i, M_i^n is a generalized torus $S^p \times S^q$ where p and q depend on i. Then M^n possesses the following handle decomposition: one* 0*-handle $H^{(0)}$, one n-handle $H^{(n)}$, and for each $M_i^n = S^p \times S^q$ a pair of handles $H^{(p)}$ and $H^{(q)}$ with attaching spheres unknotted in $\partial H^{(0)}$ and attached dually. Furthermore, all handles, with the exception of $H^{(0)}$ and $H^{(n)}$, are pairwise disjoint.*

Proof. For simplicity, we shall consider the case $M^n = (S^p \times S^q)\#(S^r \times S^s)$. The general case is entirely similar. Write

$$S^p \times S^q = H^{(0)} \cup H^{(p)} \cup H^{(q)} \cup H'^{(n)}$$

and

$$S^r \times S^s = H'^{(0)} \cup H^{(r)} \cup H^{(s)} \cup H^{(n)}.$$

To form the connected sum, remove the handles $H'^{(n)}$ and $H'^{(0)}$, obtaining

$$M^n = H^{(0)} \cup H^{(p)} \cup H^{(q)} \cup H^{(r)} \cup H^{(s)} \cup H^{(n)}.$$

We can assume without loss of generality that $p \leq q$ and $r \leq s$ and $r \leq p$. By the reordering lemma [**RS82**], an ambient isotopy will slide $H^{(r)}$ and attach it in $\partial H^{(0)}$ away from the a-regions of both $H^{(p)}$ and $H^{(q)}$. Next, by using Lemma 4.1 together with the tubular neighborhood theorem, we may assume that the a-regions of $H^{(p)}$, $H^{(q)}$, $H^{(r)}$ and $H^{(s)}$ are all disjoint and contained in $\partial H^{(0)}$. □

4.2. *Canonical Lyapunov graphs.* In this section we will define attaching labels for the edges of an admissible Lyapunov graph L. This is analogous to the gluing map labels in [**dR93**].

Let $\mathbf{S}_n$ be the class of closed n-dimensional manifolds obtained from all connected sums of generalized tori $S^p \times S^q$, where $p + q = n$ and their disjoint unions. We will specify certain handle decompositions $\mathbf{H}$ of $M^n \in \mathbf{S}_n$, where M^n is connected. For such a $\mathbf{H}$, a given handle H will be attached by one of the following processes.

(1) Process (t): if the index of H is non-zero then the a-sphere of H is contained in the boundary of an index 0 handle, $H^{(0)}$, and is unknotted. Furthermore, the a-region of H which is a closed tubular neighborhood of the a-sphere in $\partial H^{(0)}$ does not intersect the a-regions of previously attached handles. We will also assume that H does not form a null or dual pair (see §4.1) with a previously attached handle. By null pair H, H' we mean that $H \cup H'$ is a cancelling pair of handles. We refer to these handles

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as *trivial*. By definition, 0-handles are always attached by process (t). We will say that the H ℓ-handle is attached by process (t) to $H^{(0)}$ by labelling the corresponding edge with $t(H^{(0)})$.

(2) Process (n): assuming the index of H is non-zero it forms a *cancelling* or *null* pair with a handle H' previously attached by process (t). Furthermore, if $H^{(0)}$ is such that $\partial H^{(0)}$ contains the a-region of H' then the a-region of H in $\partial(H^{(0)} \cup H')$ does not intersect the a-regions of handles other than H'. Note that the index of H equals the index of H' plus one. We will also say that the ℓ-handle H is attached by process $n(H')$.

(3) Process (d): if the index of H is not n, it forms a *dual* pair with a handle H' previously attached by process (t). Furthermore, if $H^{(0)}$ is such that $\partial H^{(0)}$ contains the a-region of H' and H, then the a-region of H does not intersect the a-regions of other handles. Note that the indices of H' and H add up to n. We will denote an ℓ-handle attached by process (d) by $d(H^{(\ell)})$. If the index of H is n, H forms a dual pair with a 0-handle H' if and only if by the removal of all the null pairs of $\mathbf{H}$ one obtains a handle decomposition whose only index 0 and n handles are H' and H respectively.

Given a vertex v of the derived Lyapunov graph L' labelled with a singularity of index ℓ, we will associate to it an index ℓ handle $H(v)$. Let e be an outgoing edge of v. The attaching label of e specifies how $H(v)$ is attached.

Let w be an index 0 vertex connected to v by an oriented path $\gamma : [0, 1] \to L$, such that $\gamma(0) = w$, $\gamma(1) = v$. If v is ℓ-d, the label $t(w)$ for e means that $H(v)$ is attached by process $t(H(w))$. If v is ℓ-c, $\ell \neq 0$, let $u \in \gamma(0, 1)$ be an ℓ-d vertex such that its only outgoing edge is attached by process $t(w)$. If the index of v is $\ell + 1$ the label $n(u)$ for e means that $H(v)$ is attached by process $n(H(u))$, where the index of u is ℓ. If the index of v is $n - \ell$ the label $d(u)$ for e means that $H(v)$ is attached by process $d(H(u))$ where the index of u is ℓ.

If v is 0-c, as M^n is connected, the index of v must be one. In this case, v has two outgoing edges. Let u be an index 0 vertex which is not w. Let e_1 be the outgoing edge of v contained in $\gamma(0, 1)$. Let e_2 be the other outgoing edge of v. We label e_1 by $t(w)$ and e_2 by $n(u)$. These attachments mean that $H(v)$ is attached so that the a-sphere meets $\partial H(w)$ and also that $H(v)$ and $H(u)$ form a null pair.

We will assume that if u is an ℓ-d vertex, $0 \leq \ell \leq n$, then exactly one ℓ-c vertex v will possess an outgoing edge e, w labelled as $n(u)$ or $d(u)$.

A *canonical Lyapunov graph* L is an admissible Lyapunov graph which is a tree, with the property that L and its derived graph L' contain no β-i vertices and all edges are endowed with attaching labels. See §2.4 for the definition of derived graphs.

The manifolds $M \in \mathbf{S}_n$ have the property that all possess minimal Morse flows, i.e. a flow with $c_i = \beta_i(M)$. For this class it is also easy to see which handles contribute to the homology of M.

Given a Lyapunov graph L, consider its derived graph L' and for L', let t_i be the number of index i vertices with attaching label t, n_i the number of index i vertices with attaching label n, d_i the number of index i vertices with attaching label d. Altogether, the number of index i vertices is $c_i = t_i + n_i + d_i$. The number of trivial handles is equal to

$t_i = n_{i+1} + d_{n-i}$ which reflects the fact that these handles cancel with the $(i+1)$-handles of type n and pair up dually with the $(n-i)$-handles of type d.

Let $M \in \mathbf{S}_n$ be the manifold constructed via the attaching instructions of L', then $\beta_i(M) = t_i - n_{i+1} + d_i$.

We can go from a minimal flow on M to any other Morse flow by adding appropriate null handles. Algebraically this means that we can go from the list $\beta_0(M), \ldots, \beta_n(M)$ to any other list $c_0(M), \ldots, c_n(M)$ by adding to $\beta_i(M)$, $n_{i+1} + n_i$ i-handles where n_i i-handles are of null type and n_{i+1} i-handles are of trivial type to obtain c_i.

What we are doing in the process above is adding to a minimal Morse flow cancelling pairs of singularities of index i and $(i+1)$. That is, n_{i+1} trivial i-handles and n_{i+1} null $(i+1)$-handles. It is easy to see that these additions do not affect the Morse inequalities since c_{i+1} and c_i always appear with opposite signs except for the inequality $c_i - c_{i-1} + \cdots \pm c_0 \geq \beta_i - \beta_{i-1} + \cdots \pm \beta_0$ which is also not affected since we would only increase c_i.

4.3. *Realizing canonical Lyapunov graphs.* It is easy to see that canonical Lyapunov graphs specify handle decompositions $\mathbf{H}$. Such a handle decomposition $\mathbf{H}$ is *admitted* by $M \in \mathbf{S}_n$ if the number of dual i-handles, d_i, and the number of dual $(n-i)$-handles, d_{n-i}, in $\mathbf{H}$ is equal to $\beta_i(M)$.

THEOREM 4.1. *Let L be a canonical Lyapunov graph such that its vertices are labelled with hyperbolic singularities and let $M \in \mathbf{S}_n$ be a simply connected manifold. Then there is a gradient flow and a Morse function $f : M \to \mathbb{R}$ such that L is equivalent to the associated Lyapunov graph of f if the handle decomposition specified by L is a handle decomposition admitted by M.*

Proof. Given that there is a handle decomposition $\mathbf{H}$ of M specified by L which is admitted by M, then L is realizable on M. Also, from the previous section, each handle $H \in \mathbf{H}$ has a Morse function defined on it and the definition of a global Morse function on M and the associated gradient flow is a standard procedure. All that remains for us to show is how these handles are added so as to define a flow on M with equivalent Lyapunov graph. For this purpose, we use the attaching labels on L. Note that the flow is transverse on the attaching regions of each handle. □

We can also view the flow on M by gluing basic blocks as in [**dR87**]. If we attach handles following the order specified by the orientation of L, starting by the index zero singularities we construct a submanifold X_k of M for each handle H_k added. Let $N = C(\partial X) \cup H$ and $\partial_- N = \partial X$, then $(N, \partial_- N)$ is an index pair for the singularity x_k. The basic block is the component of $\overline{N - \partial_- N}$ which contains x_k.

The following construction is a restatement of Asimov's fundamental lemma of round handles. Let $\partial_- N$ be an $(n-1)$-dimensional closed manifold. Attach to a collar of $\partial_- N$, $C(\partial_- N)$ two handles $H^{(\ell)}$ and $H^{(\ell+1)}$. If the attaching regions are disjoint, then by Asimov's lemma [**Asi75**]

$$N = C(\partial_- N) \cup H^{(\ell)} \cup H^{(\ell+1)} = C(\partial_- N) \cup R^{(\ell)},$$

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where $R^{(\ell)}$ is an index ℓ round handle. Lemma 5 of [**Asi75**] implies, in particular, that there is a gradient-like flow on the basic block N whose chain recurrent set is a periodic orbit of index ℓ.

THEOREM 4.2. *Let L be a canonical Lyapunov graph such that its vertices are labelled with hyperbolic singularities or periodic orbits. Let $M^n \in \mathbf{S}_n$ be a simply connected manifold. Then there is a gradient flow and a Morse function $f : M \to \mathbb{R}$ such that L is equivalent to the associated Lyapunov graph of f if the handle decomposition specified by L' is a handle decomposition admitted by M.*

Proof. By Theorem 4.1 it is possible to realize L' by a Morse function $g : M \to \mathbb{R}$ together with a flow $\psi_t : M \to M$. Let $v \in L$ be a vertex labelled with a hyperbolic periodic orbit of index not equal to 0 or $n-1$. Let $u, w \in L'$ be the derived vertices and let $H(u)$ and $H(w)$ be the corresponding handles. These handles cannot be of null type, otherwise v would be β-i. The technique used to prove Theorem 4.1 ensure that $H(u)$ and $H(w)$ are disjoint. By the fundamental lemma of round handles $H(u)$ and $H(w)$ can be replaced by a round handle $R(v)$.

If the index of v is 0, the derived singularities u and w will have indices 0 and 1 respectively, and the corresponding pair of handles $H(u)$ and $H(w)$ will be of null type. The a-sphere of $H(w)$ is a pair of points x, y. One of them, say x, belongs to $\partial H(u)$. The other point, y, does not. However, after an ambient isotopy h we can assume that y does. The effect is to replace $H(u)$ and $H(w)$ by a new pair of handles $H(u)$ and $H'(w)$ such that this pair can be replaced by a round handle $R(v)$.

It is necessary to conjugate the flow ψ_t by this ambient isotopy and compose g with it. L' is equivalent to the graph of $h^{-1}\psi_t h$ with Lyapunov function gh. Indeed, the only change is that the 0-handle that contains y has changed. If the index of v is $n-1$, one can work with the reverse flow to achieve the same result.

After this is done to all the vertices labelled with periodic orbits, we obtain a decomposition of M in handles and round handles. The flow $\phi_t : M \to M$ that corresponds to this decomposition is obtained by using Lemma 5 of [**Asi75**] repeatedly. □

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