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ON M-SYMMETRIC LATTICES

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Introduction. Every $\perp$-symmetric relatively semi-orthocomplemented lattice is M-symmetric. This answers the Problem 1 in [2] in the affirmative and provides a new proof to a result on $\perp$-symmetric lattices proved in [2] (Corollary below).

The notation and terminology are as in [2].

Let $\langle L; \wedge, \vee\rangle$ be a lattice. Two elements a and b of L are said to form a modular pair, in symbols aMb, if

$$(c \vee a) \wedge b = c \vee (a \wedge b) \quad \text{for every} \quad c \leq b$$

The relation aM^*b is defined dually.

With each element k of L we associate two mappings of L into L, thus: $x\varphi_k = x \wedge k$ and $x\psi_k = x \vee k$. The following lemma [1, p. 82], connecting the modular relation with certain properties of the above mappings is basic to our discussion.

LEMMA [1]. *For any two elements a, b of a lattice L, the following statements are equivalent:*

(1) *The mapping $\varphi_b\colon [a, a \vee b] \to [a \wedge b, b]$ is onto*
(2) *$y\psi_a\varphi_b = y$ for all $y \in [a \wedge b, b]$*
(3) *The mapping $\psi_a\colon [a \wedge b, b] \to [a, a \vee b]$ is one-to-one*
(4) *aMb*

A lattice L is called *M-symmetric* if whenever aMb holds we have bMa. A lattice L with 0 is called *$\perp$-symmetric* if $a \wedge b = 0$ and aMb implies bMa.

If, in a lattice L with 0, there exists a binary relation $\perp$ satisfying $a \perp a \Rightarrow a = 0$, $a \perp b \Rightarrow b \perp a$, $a \perp b$, $a_1 \leq a \Rightarrow a_1 \perp b$ and $a \perp b$, $a \vee b \perp c \Rightarrow a \perp b \vee c$ then the system $\langle L; \vee, \wedge, \perp\rangle$ is called a *semi-ortholattice*. A semi-ortholattice is *relatively semi-orthocomplemented* if for every pair of elements (a, b) with $a \leq b$, there exists an element c such that

$$b = a \vee c \quad \text{and} \quad a \perp c$$

Finally, let a and b be elements of a lattice L with 0. An element c is a *left complement* within b of a in $a \vee b$ if

(1) $$a \vee b = a \vee c, \qquad a \wedge c = 0, \qquad c\,M\,a \quad \text{and} \quad c \leq b$$

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THEOREM. *For a relatively semi-orthocomplemented lattice L the following statements are equivalent:*

(i) *L is M-symmetric*
(ii) *L is $\perp$-symmetric*
(iii) *If aMb then a has a left-complement within b.*

Proof. (i)$\Rightarrow$(ii) is trivial.

(ii)$\Rightarrow$(iii): Let aMb. If $a \wedge b=0$ then bMa and b itself is a left-complement of a. Let $a \wedge b>0$. Choose a relative semi-orthocomplement c of $a \wedge b$ in b. So we have $(a \wedge b) \vee c=b$ and $a \wedge b \perp c$. Hence $a \wedge b \wedge c=0$ and $a \wedge bMc$ [2, §2]. It follows that $a \vee c=a \vee (a \wedge b) \vee c=a \vee b$ and $a \wedge c=0$. Since aMb and $a \wedge bMc$, Lemma 1 assures that the maps $\varphi_b: [a, a \vee b]\to[a \wedge b, b]$ and $\varphi_c: [a \wedge b, b]\to[0, c]$ are onto. Hence $x\varphi_b\varphi_c=x\varphi_c$ and thus the map $\varphi_c: [a, a \vee c]\to[a \wedge c, c]$ is onto. Again, from Lemma 1, aMc. Since $a \wedge c=0$ and L is $\perp$-symmetric, cMa follows. Thus c is a left complement within b of a in $a \vee b$.

(iii)$\Rightarrow$(i): Let aMb. By (iii) we can choose a left complement c of a in b. Let $y \in [a \wedge b, a] \subseteq [a \wedge c, a]$. By cMa and Lemma 1, there exists an $x \in [c, a \vee c]$ such that $x\varphi_a=y$. Thus $x\geq y\geq a \wedge b$ and hence $x \vee c\geq(a \wedge b) \vee c=b$ (i.e.) $x\geq b$ which shows that this x indeed belongs to the interval $[b, a \vee b]$ and hence the mapping $\varphi_a: [b, a \vee b]\to[a \wedge b, a]$ is onto. Thus we get bMa.

COROLLARY [2, Theorem 1.14]. *A $\perp$-symmetric lattice L with* 1 *satisfying the condition that every element a of L has a complement a′ such that aMa′ and a′M*a is M-symmetric.*

Proof. Such a lattice is relatively semi-orthocomplemented by Lemma 3.6 of [2].

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