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Affine functions and series with co-inductive real numbers

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We extend the work of A. Ciaffaglione and P. di Gianantonio on the mechanical verification of algorithms for exact computation on real numbers, using infinite streams of digits implemented as a co-inductive type. Four aspects are studied. The first concerns the proof that digit streams correspond to axiomatised real numbers when they are already present in the proof system. The second re-visits the definition of an addition function, looking at techniques to let the proof search engine perform the effective construction of an algorithm that is correct by construction. The third concerns the definition of a function to compute affine formulas with positive rational coefficients. This is an example where we need to combine co-recursion and recursion. Finally, the fourth aspect concerns the definition of a function to compute series, with an application on the series that is used to compute Euler's number *e*. All these experiments should be reproducible in any proof system that supports co-inductive types, co-recursion and general forms of terminating recursion; we used the Coq system (Dowek *et al.* 1993; Bertot and Castéran 2004; Giménez 1994).

1. Introduction

Several proof systems provide data-types to describe real numbers, together with basic operations and theorems giving an ordered, complete and archimedian field (Harrison 1996; Harrison 1998; Mayero 2001). In the Coq system, several approaches have been taken; depending on whether developers wanted to adhere to pure constructive mathematics or more classical approaches. In the classical approach, the type of real numbers is merely 'axiomatised', the existence of the type and the elementary operations is assumed and the properties of these operations are asserted as axioms. This approach has been used extensively to provide a large collection of results, going all the way to the description of trigonometric functions, calculus and the like. However, because the type of real numbers is axiomatised, there is no 'physical representation of numbers' and basic operations do not correspond to any algorithms.

In an alternative approach, a type of *constructive* numbers may be defined as a data-type and the basic operations may be described as algorithms manipulating elements of this data-type. This approach is used, for instance, in C-CoRN (Cruz-Filipe *et al.* 2004). A. Ciaffaglione and P. di Gianantonio (Ciaffaglione and di Gianantonio 2000) showed that a well-known representation of real numbers as infinite sequences of redundant digits could easily be implemented inside theorem provers with co-inductive types. We say the digits are redundant because several representation are possible for every number. In the case of Ciaffaglione and di Gianantonio (2000) the representation is simply inspired

by the usual binary representation of fractional numbers and made more redundant by adding the possibility of using a negative digit. Ciaffaglione and di Gianantonio then provide addition and multiplication, and show that these operations enjoy the properties that are expected from a constructive field of real numbers.

In our approach there is also an extra digit, but its meaning is intermediary between the existing 0 and 1. Edalat and Potts (Edalat and Potts 1998) show that both approaches are special cases of a more general family of representations based on *linear fractional transformations*.

Once we have chosen the way to represent real numbers as a data-type, we proceed by establishing a relation between this data-type and the axiomatised type of real numbers. This is a departure from the prescription of pure constructive mathematics, because we rely on the non-constructive axioms of that theory to state the correctness of our algorithms. Still, we also describe the relation between our representation and constructive approaches to real numbers, by showing how infinite sequences of digits can be produced from constructive Cauchy sequences. We also show the relation between infinite sequences and a constructive subset of Dedekind cuts.

Once we have provided the basic data-type and its relation to the axiomatised theory of real numbers, we proceed by defining an addition function. We rely on proof search tools to construct the addition algorithm: we only provide some guidelines for the construction of the algorithm, without actually describing all 25 cases in the function. The correctness proof then consists of showing that there is a morphism between the data-type and the axiomatised type. Our contribution in Section 3 is to show how to use the proof search engine to construct a well-formed addition function.

We then focus on affine formulas combining two real values with rational coefficients. For these more general operations, we need to combine co-recursion and well-founded recursion. We show that the function responsible for producing the infinite stream of digits representing the result can be decomposed into two recursive functions. One of the functions is a guarded co-recursive function as proposed in Giménez (1994), the other function is a well-founded recursive function as in Nordström (1988). Each function satisfies a different form of constraint: the co-recursive function does not need to be terminating, but it must produce at least one digit at each recursive call, while the well-founded function does not need to produce a digit at each recursive call, but it must terminate. Our main contribution in this part is to show that some functions that appear at first sight to be beyond the expressive power of guarded co-recursion can actually be modelled and proved correct.

In Section 5, we focus on constructively converging infinite sums. We show how to avoid having to consider the infinity of terms that are parts of the sum. We exhibit a framework that can be re-used from one series to the other. We present two examples, we show how to compute the infinite stream representing Euler's number e and how to multiply two real numbers represented as infinite streams of digits. In particular, the algorithm we obtain can be executed directly using the reduction mechanism provided in Coq to compute e to a great precision in a reasonable time.

Our account stops here, although the experiments described in this paper seem to open the door for a more complete study of real functions, especially analytic functions.

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2. Related work

Real numbers are usually represented for the purposes of numerical computation as approximates using floating point numbers. These floating point numbers are composed of a mantissa and an exponent, so the value of the least significant bit in the mantissa varies with the exponent. However, both the exponent and the mantissa have a fixed size, so there is only a finite number of floating point numbers, and real values must be rounded to find the closest floating point numbers. Floating point based computations are thus only approximative and errors stemming from successive rounding operations may accumulate to the point that some computations can become grossly wrong (Muller 1997).

In spite of their limitations, floating point numbers are used extensively: most processors provide a direct implementation of the elementary operations (addition, subtraction, multiplication, division) according to a standard that gives a precise mathematical meaning to the rounding operations (IEEE 1987). This standard provides the basis for implementing computations with a guaranteed precision (Muller 1997; Hanrot *et al.* 2000), sometimes with correctness proofs that can be verified with the help of computer-aided proof tools (Daumas *et al.* 2001; Harrison 2000; Russinoff 1999; Boldo 2004). In particular, some approaches, named *expansions*, make it possible to increase the number of representable numbers drastically by extending the length of the mantissa (Boldo *et al.* 2002).

An alternative to floating point and rounding concentrates on data-structures that support exact computation. Among the possible approaches, the best known are based on continued fractions (Gosper 1972; Vuillemin 1990) or representations with mantissas that grow as needed (Ménissier-Morain 1995). In the latter case, the representation is very close to the floating point representations with rounding modes. One way to understand this 'growing mantissa' is to view it as an infinite sequence of digits, where only a finite prefix is known at any one time. An introduction to exact real arithmetics can be found in Edalat and Potts (1998). Implementations are provided in the setting of conventional programming languages (Lambov 2005; Müller 2005), or in the setting of functional programming (Boehm *et al.* 1986; Ménissier-Morain 1994; Bauer *et al.* 2002).

Formal proofs about computations on infinite data-structures are a privileged ground for the use of co-inductive types (Coquand 1993; Giménez 1994). The first experiments on formalising exact real number computation algorithms using co-inductive types were performed by Ciaffaglione and di Gianantonio (Ciaffaglione and di Gianantonio 2000) who showed that one could represent infinite sequences of digits with co-inductive types and the basic operations of arithmetic (addition, multiplication and comparison) with simple co-recursive functions, as long as the set of digits was extended to allow for enough redundancy. Niqui (Niqui 2004) has also studied the problems of modelling real number arithmetic for use in formal proofs, providing a single point of view to account for both continued fractions and infinite sequences of digits. Our approach is very similar to Ciaffaglione and di Gianantonio's, differing only in the collection of digits that we consider.

The example of the combination of co-recursive functions and well-founded recursive functions that we exhibit in our treatment of affine formulas is related to work by di Gianantonio and Miculan (di Gianantonio and Miculan 2003) and our own work on partial co-recursive functions (Bertot 2005).

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With regard to the computation of series, we are aware through personal communication that computations of e had already been done using the real numbers as they are formalised in the C-CoRN library (Cruz-Filipe *et al.* 2004); it seems that co-inductive presentations make it possible to achieve a higher efficiency. While the C-CoRN library aims to provide a comprehensive study of constructive mathematics, our work abandons some of the foundations of constructive mathematics: we let our logical reasoning depend on non-constructive arguments, although we do provide algorithms working on concrete data structures to represent real numbers and the basic operations. We believe the algorithms we develop will be useful even in the context of constructive mathematics, but the proofs of correctness will probably have to be re-done.

3. Redundant digit representation for real numbers

We are all used to the notation using a decimal point to represent real numbers. For instance, we usually write a number between 0 and 1 as a string of the form $0.1354647\cdots$, and we know that the sequence of digits must be infinite for some numbers, actually all those that are not of the form $\frac{a}{10^b}$, where a and b are positive integers. It is a bit less natural, but still easy to understand, that all numbers can be represented by an infinite sequence: for those that have a finite representation, it suffices to add an infinite sequence of zeros. Moreover, the number 1 can also be represented by the sequence $0.999\cdots$. In general, decimal numbers have two possible infinite representations.

When we know a prefix of an infinite sequence with length p, we actually know precisely the bounds of an interval of width $\frac{1}{10^p}$ that contains the number. We are accustomed to reasoning with these prefixes of infinite sequences, and we expect tools to return correct prefixes of an operation's output when this operation has been fed with correct prefixes for the inputs.

In the conventional representation, the number *ten* plays a special role: it is the *base*. We can change the base and use digits that are between 0 and the base. For instance, we can use *two* as the base, so that the digits are only 0 and 1. The number $\frac{1}{2}$ can then be represented by the sequence $0.1000\cdots$ and the number 1 can be represented by the sequence $0.1111\cdots$. For a sequence $0.d_1d_2\cdots$, the number being represented is:

$$\sum_{i=1}^{\infty} \frac{d_i}{2^i}.$$

The following equalities hold:

$$0.0s = \frac{0.s}{2}$$
$$0.1s = \frac{0.s + 1}{2}$$

A prefix with p digits gives an interval of width $\frac{1}{2^p}$ that contains the number represented by the infinite sequence. In the rest of this paper, we will carry on using base 2 (but the set of digits will change).

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For the computation of basic operations, the base-2 conventional representation is not really well adapted. Here is an example that exhibits the main flaw of this representation. The numbers $\frac{1}{3}$ and $\frac{1}{6}$ add up to give $\frac{1}{2}$. However, the infinite sequences for these numbers are given in the following equations:

$$\begin{aligned}\frac{1}{3} &= 0.01010101\cdots\\ \frac{1}{6} &= 0.00101010\cdots\\ \frac{1}{2} &= 0.10000000\cdots = 0.01111111\cdots\end{aligned}$$

The following reasoning steps justify the first equation:

$$0.01010101\cdots = \sum_{i=1}^{\infty}\frac{1}{2^{2i}} = \frac{1}{1-\frac{1}{4}} - 1 = \frac{1}{3}$$

Similar justifications can be used for the other equations. If w is a prefix of $0.0101\cdots$, then w is also the prefix of all numbers between $w00\cdots$ and $w111\cdots$. These two numbers are rational numbers of the form $\frac{a}{2^b}$ and neither can be equal to $\frac{1}{3}$. Thus, we actually have an interval of possible values that contains both values that are smaller and values that are larger than $\frac{1}{3}$. The same property occurs for the representation of $\frac{1}{6}$. When considering the sum of values in the interval around $\frac{1}{3}$ and values in the interval around $\frac{1}{6}$, the results are in an interval that contains both values that are smaller and values that are larger than $\frac{1}{2}$. However, numbers of the form $0.1\cdots$ can only be larger than or equal to $\frac{1}{2}$ and numbers of the form $0.0\cdots$ can only be smaller than or equal to $\frac{1}{2}$. Thus, even if we know the inputs with a great number of digits, we must indefinitely delay the decision and require more information on the input before choosing the first digit of the result: we need to know the whole infinite sequences of digits for both inputs before deciding the first digit of the output.

We solve this problem by adding a third digit in the notation. This digit provides a way to express the fact that the interval given by a prefix has $\frac{1}{2}$ in its interior. This new digit adds more redundancy to the representation. We now have three digits, even though we still work in base 2. We choose to name the three digits `L` (for *left*), `R` (for *right*), and `C` (for *center*).

— The digit `L` is used like the digit 0. If s is an infinite sequence of digits representing the number x, the sequence `L`s represents $x/2$.

— The digit `R` is used like the digit 1. If s is an infinite sequence of digits representing the number x, the sequence `R`s represents $x/2 + 1/2$.

— The digit `C` is used with the following meaning: if s is an infinite sequence of digits representing the number x, the sequence `C`s represents $x/2 + 1/4$.

The fact that `L` is like 0, `R` is like 1, etcetera, can also be expressed using a function α such that $\alpha(\texttt{L}) = 0$, $\alpha(\texttt{R}) = 1$ and $\alpha(\texttt{C}) = \frac{1}{2}$. With this function we can still interpret a digit

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sequence $0.d_1d_2\cdots$ as an infinite sum:

$$\sum_{i=1}^{\infty} \frac{\alpha(d_i)}{2^i}.$$

From now on, we will only consider numbers in the interval $[0, 1]$ and we drop the first characters '0.' when writing a sequence of digits. We use the same notation for a sequence of digits and the real number it represents. In the same spirit, we use the same notation for a digit d and the function it represents, the function that maps x to $\frac{x+\alpha(d)}{2}$. Finally, we associate the digits `L`, `R`, `C` with the intervals $[0, \frac{1}{2}]$, $[\frac{1}{2}, 1]$, and $[\frac{1}{4}, \frac{3}{4}]$, respectively. These intervals are called the *basic intervals*.

The redundancy of the new digit gives us a very simple property: a number that can be written `CL`x can also be written `LR`x and a number that can be written `CR`x can also be written `RL`x. This property is used several times in this paper.

3.1. *Formal details of infinite sequences and real numbers*

In our formalisation, we benefit from theories that state the main properties of natural numbers (type `nat`), integers (type `Z`) and real numbers (the type is usually written `R`, but in this article we shall write it as `Rdefinitions.R` in Coq excerpts to avoid any ambiguity with the 'digit' `R`). The two integer types come with addition, subtraction and multiplication, while the type of real numbers is also equipped with division. The integer types are actually described as inductive types and the basic operations are implemented as recursive functions. For the real numbers, the existence of the type, two constants 0 and 1, the operations and comparison predicates, and the properties of these operations (associativity, distributivity, inverse, and so on) are assumed. Among the assumed features, there is an axiom that expresses completeness. This axiom states that every bound and non-empty subset of $\mathbb{R}$ has a least upper bound in $\mathbb{R}$. This means that whenever we exhibit a property E and prove that it is bounded, we can construct a function that returns its least upper bound. This completeness axiom is inherently non-constructive. To be more precise, our work describes a collection of algorithms on a representation of real numbers, which is in some sense a construction of real numbers, but many justifications of correctness, which rely on the axiomatised real numbers, are non-constructive.

The axiomatisation of real numbers also provides a few decision procedures. The decision procedure `field` (Delahaye and Mayero 2001) solves equalities between rational expressions, occasionally leaving proof obligations to make sure denominators are non-zero. The decision procedure `fourier` determines when a collection of inequalities concerning affine formulas with rational coefficients is satisfiable.

The type of digits is described as an enumerated type:

```
Inductive idigit: Set := L | R | C.
```

We provide both a *numeric* interpretation (named `alpha`) and a *functional* interpretation (named `lift`) for these digits. These can be defined in Coq with the following text:

```
Definition alpha (d:idigit): Rdefinitions.R :=
  match d with L => 0 | C => 1/2 | R => 1 end.
```

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```
Definition lift (d:idigit)(x:Rdefinitions.R) := (x+ alpha d)/2.
```

The type of infinite sequences of digits is based on a polymorphic type of streams, which is defined as a co-inductive type using a declaration with the following form:

```
CoInductive stream (A:Set): Set := Cons: A -> stream A -> stream A.
```

This definition defines `stream A` to be a data-type for any type `A`. It also defines the constructor `Cons`, with the type given in the definition. This definition is similar to a recursive data-type definition in a conventional functional programming language. In our mathematical notation, we will simply write *ds* instead of `Cons` *d s*. In Coq excerpts, we will also use the notation *d*::*s*.

In proof systems, recursion is seldom unrestricted. In the Coq system, it is mostly provided as a companion to inductive and co-inductive data structures. For inductive structures, the form of recursion provided is called *structural* recursion, and it basically imposes the requirement that a recursive function takes an element of an inductive type as argument and a recursive call can only be performed if the argument is a sub-term of the initial argument. For co-inductive structures, the form of recursion provided is called *guarded co-recursion*, and it basically imposes the requirement that a co-recursive value must be an element of a co-inductive type and that co-recursive calls can only be used to produce sub-terms of the output. More general forms of recursion are also provided, such as *well-founded recursion*, where recursive calls are allowed only if the argument of the recursive call is a predecessor of the initial argument with respect to a relation that is known to be well-founded (which intuitively means that this function contains no infinite chain of predecessors). In fact, well-founded recursion can be shown to be a special case of structural recursion (Nordström 1988; Paulin-Mohring 1993; Bertot and Castéran 2004).

Co-recursive values need not be functions. For instance, 0 and 1 are represented by the infinite sequences `LLL`... and `RRR`..., these are defined as co-recursive values with the following definition:

```
CoFixpoint zero: stream idigit := L::zero.

CoFixpoint one: stream idigit := R::one.
```

To relate infinite streams of digits to real numbers, we define a co-inductive relation:

```
CoInductive represents: stream idigit -> Rdefinitions.R -> Prop:=
  repr: forall d s r,
    represents s r -> 0 <= r <= 1 -> represents (d::s) (lift d r).
```

This definition introduces both the two-place predicate `represents`, and a constructor, named `repr`, which can be used as a theorem to prove instances of this predicate. The statement of this theorem can be read as 'for every `s` and `r`, if the proposition `represents s r` holds and the proposition `0 <= r <= 1` holds, then the proposition `represents (d::s) (lift d r)` holds'. This relation really states that infinite streams are only used to represent numbers between 0 and 1, and it confirms the correspondence between the digits and their function interpretations.

An alternative approach to relating sequences of digits and real numbers is to build a function that maps an infinite sequence to a real value. Every prefix of an infinite sequence

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corresponds to an interval that contains all the values that could have the same prefix. As the prefix grows, the new intervals are included in each other, and the size is divided by 2 at each step, while the bounds remain rational. We can define a function `bounds` to compute the interval corresponding to the prefix of a given length for a given sequence. This function takes as arguments a digit sequence and a number n and it computes the bounds of an interval that contains all the real numbers whose representation shares the same prefix of length n. This function is primitive recursive in n (in other words, it is structural recursive with respect to the conventional representation of natural numbers as an inductive type).

$$\begin{aligned} \texttt{bounds}(\ldots, 0) &= [0, 1] \\ \texttt{bounds}(ds, n+1) &= [\texttt{lift}\ d\ a, \texttt{lift}\ d\ b] \quad \text{where} \quad \texttt{bounds}(s, n) = [a, b]. \end{aligned}$$

In practice, we do not manipulate real numbers in this function, but only integers. The result of the function is a structure $((a, b), k)$ such that the interval is $[\frac{a}{2^k}, \frac{b}{2^k}]$.

We then define a function that maps a stream of digits to a sequence of real numbers, which are the lower bounds of the intervals. This function is called `si_un` and is defined by the following text, where `IZR` is the function that injects integers in the type of real numbers:

```
Definition si_un (s:stream idigit) (n:nat): Rdefinitions.R :=
   let (p,k) := bounds n s in let (a,b) := p in IZR a/IZR(2^k).
```

We can prove that for every d, `lift` d is monotonic, so `si_un` s is a growing sequence bounded by 1. All this leads us to a proof that `si_un` s has a limit and that this limit is in [0,1]. This makes it possible to define the function `real_value` that associates an infinite sequence of digits to the limit.

We then prove that adding a digit in front of a sequence is the same as using this digit as a function:

```
Theorem real_value_lift:
   forall d s, real_value (d::s) = lift d (real_value s).
```

This allows us to show that `real_value` and `represents` follow the same structure and to obtain the following theorem:

```
Theorem represents_real_value: forall s, represents s (real_value s).
```

To complete the correspondence between the two notions, we need to express the fact that the relation `represents` is actually a function. We do this by showing that the distance between two possible values represented by a sequence is smaller than $\frac{1}{2^n}$, which is proved by induction over n.

```
Theorem represent_diff_2pow_n :
  forall n x r1 r2, represents x r1 -> represents x r2 ->
        -1/(2^n) <= r1 - r2 <= 1/(2^n).
```

It is then easy to conclude with the following theorem:

```
Theorem represents_equal: forall s r, represents s r -> real_value s = r.
```

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We thus have two ways to express the fact that a given sequence represents a real number. The function `real_value` is more suited to use in theorem statements, but the co-inductive property makes proofs more elegant. Actually, all proofs of function correctness presented in this paper are performed using a proof by co-induction based on `represents`, even though theorem statements are sometimes expressed using `real_value`.

This raises a third question: *given an arbitrary real number, is there a digit stream that represents it?* The answer to this question is related to question of constructivism in mathematics. Consider a constructive description of your real number, such as a Cauchy sequence of rational numbers and a constructive proof that it satisfies the Cauchy criterion. One way to describe Cauchy sequences is to fix a function h from `nat` to $\mathbb{Q}$, with its limit in 0 when the argument goes to infinity. A Cauchy sequence may then be given by a function f from `Z` to $\mathbb{Q}$, and the Cauchy criterion may be given by a monotonic function g from `Z` to `Z` such that

$$\forall n\ m\ p, g(n) \leqslant m \wedge g(n) \leqslant p \Rightarrow |f(m) - f(p)| < h(n).$$

To construct the infinite list of digits for a given Cauchy sequence, we simply need to repeat the following process:

1 Compute the first n elements of the stream, which actually gives an interval of width $\frac{1}{2^n}$, and compute the lower bound b of this interval,
2 Find the least n such that $h(n) \leqslant \frac{1}{2^{n+3}}$.
3 Compute $f(g(n))$. We know that the distance between this value and the sequence's limit is less than $\frac{1}{2^{n+3}}$.
4 Compute the value $a = 2^n(f(g(n)) - b)$, which lies in [0,1] by an invariant of the recursive process.
5 If $a \leqslant \frac{3}{8}$, we know that for every $m \geqslant g(n)$, $f(m) \leqslant \frac{1}{2}$ and we choose the $n+1$th digit to be `L`.
6 If $\frac{3}{8} \leqslant a \leqslant \frac{5}{8}$, we know that for every $m \geqslant g(n)$, $\frac{1}{4} \leqslant f(m) \leqslant \frac{3}{4}$, and we choose the $n+1$th digit to be `C`.
7 If $\frac{5}{8} \leqslant a$, we know that for every $m \geqslant g(n)$, $\frac{1}{2} \leqslant f(m)$, and we choose the n+1th digit to be `R`.

We contend that this technique gives a constructive process for associating a sequence of digits with a sequence of rational numbers associated with a constructive proof that this sequence satisfies the Cauchy criterion.

When implementing this recursive process as a co-inductive function, we propose to represent rational numbers with pairs of integers and to replace comparisons of rational numbers with comparisons of integers. It is also more convenient to restrict ourselves to the case where $h(n) = \frac{1}{2^n}$. In the recursive process, we do not need to keep the list of the first n digits, we only need to know n and the lower bound of the represented interval, these are given as arguments to the co-recursive function:

```
CoFixpoint stream_of_cauchy (f: Z -> Z*Z)(g: Z -> Z)
  (n:Z)(b:Z*Z) : stream idigit :=
  let (vn,vd) := f(g n) in
  let (bn,bd) := b in
```

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```
let (d, r) :=
  if is_smaller (8*2^n*(vn*bd-vd*bn)) (3*vd*bd) then
     (L,b)
  else if is_smaller (8*2^n*(vn*bd-vd*bn)) (5*vd*bd) then
     (C,(4*2^n*bn+bd,4*2^n*bd))
  else (R, (2*2^n*bn+bd,2*2^n*bd)) in
let (new_bn, new_bd) := r in
   d::stream_of_cauchy f g (n+1) (new_bn, new_bd).
```

Alternatively, some constructive real numbers correspond to a boolean predicate on rational numbers, which corresponds to viewing this number as a Dedekind cut (a boolean predicate that is `false` for every rational number smaller than the represented real number and `true` for any rational number that is larger). For instance, the fractional part of the square root of 2 can be described as a Dedekind cut and represented by the predicate that is true on x if $(x+1)^2 > 2$ and false if $(x+1)^2 < 2$. We can produce the co-recursive value corresponding to any boolean predicate using a co-recursive function. The following is an example where the rational numbers are viewed as pairs of integers (we use the convention that `p (a, b) = true` means that the real number of interest is smaller than or equal to $\frac{a}{b}$):

```
CoFixpoint stream_of_cut (p:Z*Z->bool) : stream idigit :=
  match p (1, 2) with
    true => R::stream_of_cut (fun r => let (a,b) := r in p (a+b, 2*b))
  | false => L::stream_of_cut (fun r => let (a,b) := r in p (a, 2*b))
  end.
```

The functions `stream_of_cut` and `stream_of_cauchy` are only given here to show the feasibility of connecting streams of digits with the usual notions of real number constructions, but more efficient ways to handle rational numbers should be used if these functions were to be used effectively, for instance least common denominators should be computed between fraction numerators and denominators.

For an arbitrary real number between 0 and 1 given by its binary representation (an infinite sequence of 0 and 1 digits), this real number is simply represented by the corresponding infinite stream where 0 is replaced by `L` and 1 is replaced by `R`.

3.2. *Addition*

It is well known[†] that adding two infinite sequences of redundant digits can be described as a simple automaton that reads digits from both inputs and produces digits at every recursive call. Two approaches can be taken: either this automaton is understood as a program that keeps a carry as it processes the inputs, or it can be viewed as a program that performs a little look-ahead before outputting the result and processing the rest, maybe

[†] P. Martin-Löf suggested to the author that Cauchy had devised an algorithm for addition in a similar representation. Di Gianantonio refers to Cauchy and Leslie, but the reference to Cauchy's work is wrong, and the reference to Leslie could not be found at the time of writing.

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with a slight modification of the first digit in both inputs. The algorithm we describe follows the second approach.

Our algorithm has two parts. The first part is a function that computes the arithmetic mean of two values (in other words, the half-sum). The second part is a function that computes the double of a value. The first algorithm maps two real numbers in $[0, 1]$ to a value in $[0, 1]$. The doubling function only returns a meaningful value when the input is smaller than or equal to $\frac{1}{2}$.

For the half-sum, the structure of the algorithm is as follows: if the inputs have the form $d_1 d_2 x$ and $d_3 d_4 y$, then the algorithm outputs a digit d and calls itself recursively with the new arguments $d_5 x$ and $d_6 y$. Written as an equation, this yields the following formula:

$$\texttt{half\_sum}(d_1 d_2 x, d_3 d_4 y) = d(\texttt{half\_sum}(d_5 x, d_6 y)).$$

As an example, suppose that $d_1 = \texttt{L}$ and $d_3 = \texttt{R}$. In this case we can choose $d = \texttt{C}$ and $d_5 = d_2$ and $d_6 = d_4$, because the following equalities hold, using the interpretations of digits as functions:

$$\begin{aligned}
\texttt{half_sum}(\texttt{L}d_2 x, \texttt{R}d_4 y) &= \frac{\texttt{L}d_2 x + \texttt{R}d_4 y}{2} \\
&= \frac{\frac{d_2 x}{2} + \frac{d_4 y}{2} + \frac{1}{2}}{2} \\
&= \frac{d_2 x}{4} + \frac{d_4 y}{4} + \frac{1}{4} \\
&= \frac{\frac{d_2 x + d_4 y}{2}}{2} + \frac{1}{4} \\
&= \texttt{C}(\texttt{half_sum}(d_2 x, d_4 y)).
\end{aligned}$$

In this case, it is not necessary to scrutinise d_2 and d_4 to decide the value of d and the arguments for the recursive call. The equation can be re-written as

$$\texttt{half\_sum}(\texttt{L}x, \texttt{R}y) = \texttt{C}(\texttt{half\_sum}(x, y)).$$

As a second example, consider the case where d_5 and d_6 are modified with respect to d_2 and d_4. We suppose $d_1 = \texttt{C}$, $d_2 = \texttt{L}$, and $d_3 = \texttt{L}$. In this case, the following equalities hold:

$$\begin{aligned}
\texttt{half_sum}(\texttt{CL}x, \texttt{L}d_4 y) &= \frac{\frac{x}{4} + \frac{1}{4} + \frac{d_4 y}{2}}{2} \\
&= \frac{\frac{x}{2} + \frac{1}{2} + d_4 y}{4} \\
&= \texttt{L}(\texttt{half_sum}(\texttt{R}x, d_4 y))
\end{aligned}$$

In this case, it is not necessary to scrutinise d_4, but the value d_5 is modified with respect to d_6. The equation can be re-written as

$$\texttt{half\_sum}(\texttt{CL}x, \texttt{L}y) = \texttt{L}(\texttt{half\_sum}(\texttt{R}x, y)).$$

If we had designed the algorithm to scrutinise two digits in each input, there would be 81 cases, but since some cases can be handled with a scrutiny of only the first digit in

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each input, or only one digit in one of the inputs, the number of cases is brought down to 25 cases. Moreover, the exact behaviour of the algorithm in each case can be found automatically with the help of proof search procedures.

3.3. *Automatic generation of function code*

We can use the proof engine to construct the half-sum function by making this program use its proof search facility to construct a term with the right type, which should be

```
stream idigit -> stream idigit -> stream idigit.
```

We have enough control on the proof search mechanism to express the fact that the value of this type that we want to construct should be a co-recursive function and that it should analyse the first digit of the two input streams. This is simply done by stating that a case analysis on these arguments should be performed. Doing a case analysis on the first digit of the first input yields three arguments, doing a case analysis on the first digit of the second argument also gives three cases, so there are at least nine cases to consider. Some cases are easily solved directly by simply finding an output digit that makes the addition correct. For instance, if the two inputs are dx and dy (in other words, they have the same initial digit), the result should be d(`half_sum` x y). This is because the following formula holds:

$$\frac{(\frac{x}{2} + \frac{\alpha(d)}{2}) + (\frac{y}{2} + \frac{\alpha(d)}{2})}{2} = \frac{\frac{x+y}{2}}{2} + \frac{\alpha(d)}{2}.$$

When no output digit can be computed to make the formula work directly, more information is gathered from the inputs by performing more case analysis. When this case analysis is performed, we look at possible values of the second digit of one of the inputs and decide if we have enough information to decide what the first output digit should be. This decision is taken by performing some interval reasoning. If two digits of one of the inputs are fixed, this input belongs in an interval of length $\frac{1}{4}$. Then, adding this interval with an input for which only one digit is known gives an interval of length $\frac{3}{4}$, and taking half of this gives an interval of length $\frac{3}{8}$. If the lower bound of this interval is 0, $\frac{1}{16}$, $\frac{1}{8}$, $\frac{1}{4}$,..., we know what the output first digit can be, but if the lower bound of this interval is $\frac{3}{16}$ (this happens when one of the inputs is `LC`x and the other is `C`y), the upper bound is $\frac{9}{16}$ and we cannot conclude because this interval may contain values lower than $\frac{1}{4}$ (which should not start with a `C` or a `R`) and values larger than $\frac{1}{2}$ (which should not start with a `L`): for these cases an extra case analysis on the second input is required.

When the first digit of the output is fixed, we still need to decide what the first digit of the arguments to recursive calls will be. This may include a change with the initial second digit of the input. This difference is often related to the equivalence between `LR` and `CL` prefixes on the one hand and between `RL` and `CR` on the other.

For instance, if the first input has the form `CL`x and the second input has the form `L`y, the function can return `L(half_sum(R`x y`))`, because the half sum of the two inputs is equivalent to the half sum of `LR`x and `L`y.

To determine the first digits of inputs in recursive calls, we must first respect an important rule: variables that represent sub-streams should appear behind the same

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number of digits in the input pattern and in the output pattern. This rule comes from the fact that the real number represented by these variables is divided by 2 every time a digit is added in front of it. If we want the half-sum equation to be satisfied, we must make sure that the number of divisions by 2 is preserved between the inputs and the output. Thus, we can only prescribe the first digit of one of the arguments to the co-recursive call of `half_sum` if the corresponding input had two digits in the input pattern.

To determine which first digit can be used for the recursive call on an argument, the proof search procedure compares the lower bound of the output as prescribed by its first digit to the lower bound of possible results that we can predict from the shape of the input patterns. In the example, the lower bound of the output as prescribed by the first digit `L` is 0, and the lower bound predicted from the half-sum of `CL`x and `L`y is $\frac{1}{8}$. The discrepancy must be resolved by making sure that the sum of all the digits appearing at the head of recursive call arguments adds up to $\frac{1}{2}$ (which does fit with a target $\frac{1}{8}$ since we are computing a half-sum and place the output's first digit `L` in front). Here there is only one digit available, and we can only choose its value to be `R`.

Although the half-sum function is obtained by mechanical means to be correct by construction, its type is only

```
stream idigit -> stream idigit -> stream idigit.
```

This type does not express what the function does. We need to add a theorem to state that it has the right behaviour with respect to the real numbers represented by the inputs and outputs. The statement has the following shape:

```
Theorem half_sum_correct :
  forall x y u v, represents x u -> represents y v ->
    represents (half_sum x y) ((u + v)/2).
```

The proof of this statement relies on a proof by co-induction: we assume that the statement is already satisfied for any output stream that is a strict sub-stream of the current output and we show that this is enough to prove the result for the current output. The proof analyses the behaviour of the `half_sum` function and explores all the 25 cases that were found at the time the function was constructed. In all cases, it is a simple matter to verify the equality between the algebraic formulas corresponding to the half-sum of the inputs and the output pattern present in the `half_sum` function. The tactic named `field` (Delahaye and Mayero 2001) solves this kind of equality between rational expressions in a field. A second statement that needs to be verified is that the half-sum of the inputs does belong to [0,1] if the two inputs do. The tactic named `fourier`, which solves affine comparisons between real values with rational coefficients, is suitable for this task.

We believe that our definition technique can be easily reproduced for different sets of digits or for other simple binary operations, like subtraction.

3.4. *Multiplication by 2*

We also need to provide a function to multiply the output of `half_sum` by 2. This function is based on the following remarks.

— The double of a number of the form `L`x is simply x.

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— The double of a number of the form `R`x is either 1 or outside the interval [0,1].
— The double of a number of the form `C`x is a number of the form `R`x' where x' is the double of x (so the algorithm exhibits a co-recursive call).

The formal definition is:

```
Cofixpoint mult2 (v:stream idigit): stream idigit :=
  match v with L::x => x | C::x => R::mult2 x | R::x => one end.
```

The correctness of this function is expressed by the following statement:

```
Theorem mult2_correct :  forall x u,
    0 <= u <= 1/2 -> represents x u -> represents (mult2 x)(2*u).
```

Please note that this theorem explicitly states that the result value is specified correctly only when the input is at most $\frac{1}{2}$.

3.5. *Subtraction*

In this section we discuss several approaches to subtraction. A first approach uses a few intermediary functions. The first intermediary function mimics the opposite function. Of course, the opposite function cannot be defined from [0,1] to [0,1], but we can define the function that maps x to $1 - x$. The general definition, where we name the function `minus_aux`, is:

$$\begin{aligned}
\texttt{minus_aux}(\texttt{L}(x)) &= \texttt{R}(\texttt{minus_aux}(x)) \\
\texttt{minus_aux}(\texttt{C}(x)) &= \texttt{C}(\texttt{minus_aux}(x)) \\
\texttt{minus_aux}(\texttt{R}(x)) &= \texttt{L}(\texttt{minus_aux}(x))
\end{aligned}$$

These equations are justified through simple computations. For instance, the last equation is justified with the following reasoning steps:

$$\begin{aligned}
\texttt{minus_aux}(\texttt{R}(x)) &= 1 - \left(\frac{x}{2} + \frac{1}{2}\right) \\
&= \frac{1}{2} - \frac{x}{2} \\
&= \frac{1}{2}(1 - x) \\
&= \texttt{L}(\texttt{minus_aux}(x)).
\end{aligned}$$

Combining `minus_aux` with addition, we can easily compute the binary function that maps x and y to $x + (1 - y) = 1 + x - y$. Of course, this function returns a meaningful result only when x is smaller than y.

Alternatively, we can combine `minus_aux` with `half_sum` to have a function that maps x and y to $\frac{1+x-y}{2}$. Now, if we really want to have a subtraction, we can remove the $\frac{1}{2}$ offset. We use another auxiliary function, which we name `minus_half`, defined by the

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following equations:

$$\begin{aligned}\texttt{minus\_half}(\texttt{R}x) &= \texttt{L}x\\ \texttt{minus\_half}(\texttt{L}x) &= \texttt{zero}\\ \texttt{minus\_half}(\texttt{C}x) &= \texttt{L}(\texttt{minus\_half}(x)).\end{aligned}$$

The first of these equations is trivial to justify. The second is justified by the fact that the only value inside $[0, \frac{1}{2}]$ for which $x - \frac{1}{2}$ belongs to $[0, 1]$ is $\frac{1}{2}$, and the result is 0 in this case. The third equation is justified by the following reasoning steps:

$$\begin{aligned}\texttt{minus\_half}(\texttt{C}x) &= \frac{x}{2} + \frac{1}{4} - \frac{1}{2}\\ &= \frac{x}{2} - \frac{1}{4}\\ &= \frac{x - \frac{1}{2}}{2}\\ &= \texttt{L}(\texttt{minus\_half}(x)).\end{aligned}$$

4. Parameterised affine operations

In this section we study another approach to addition, with the encoding of a more general function that computes affine formulas in two real values with rational coefficients. More precisely, we want to compute the value

$$\frac{a}{a'}x + \frac{b}{b'}y + \frac{c}{c'}.$$

When a, b, c are non-negative integers and a', b', c' are positive integers (a, b, c may be null, but the others must not be), and x and y are real numbers, given as infinite sequences of digits.

The interpretation of digits as affine functions (using our function `lift`) makes them a special case of what Edalat and Potts call *Linear Fractional Transformations* (Edalat and Potts 1998). They actually show that a more general form of two argument operations can be programed on this form of real number representation, namely the computation of the following operation, called a *Möbius transform*, where a, b, and so on, are integers:

$$\frac{axy + bx + cy + d}{exy + fx + gy + h}.$$

Restricting consideration purely to affine formulas corresponds to restricting the general study proposed by Edalat and Potts to the case where e, f and g are 0. A good reason for studying this restricted case separately is that the formal proofs stay within the realm of proofs about equalities and comparisons of affine formulas with rational coefficients, a realm for which automatic proof tools exist at the time of writing this article, while verifying the correctness of the general Möbius transform requires incursions into the realm of proofs about equalities and comparisons of polynomial formulas, a domain for which proof procedures are still only under development (Mahboubi and Pottier 2002; McLaughlin and Harrison 2005; Mahboubi 2007).

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4.1. *Main structure of the algorithm*

The choice of digits for the result is based on the following observations:

1 Even without observing x and y, we already know that they lie between 0 and 1. The result lies between the extrema

$$\frac{c}{c'} \quad \text{and} \quad \frac{ab'c' + a'bc' + abc'}{a'b'c'}.$$

2 If the extrema belong to the same basic interval, it is possible to produce a digit and perform a co-recursive call with a new affine formula. In this sense, the output does not depend on reading more digits from the input, and the algorithm can be described as a streaming algorithm in the sense of Gibbons (2004).

3 If the extrema are badly placed, we cannot choose an interval associated to a digit that is sure to contain the result. In this case, we scrutinise x and y and observe their first digit. As a result, we obtain a new estimate of the interval that may contain the result, and its size is half the previous size. We can then perform a recursive call with a new affine formula. In the long run, we are forced to arrive at a situation where the extrema are within a basic interval and a co-recursive call can be performed. In fact, this condition is guaranteed as soon as the distance between extrema is less than $1/4$.

Let us study two examples. In the first example, suppose that the property $\frac{c}{c'} \geqslant 1/2$ holds. We know that the result is larger than $1/2$, so we can produce a $\mathtt{R}$ digit. The following computation takes place:

$$\begin{aligned}\frac{a}{a'}x + \frac{b}{b'}y + \frac{c}{c'} &= \mathtt{R}\left(2 \times \left(\frac{a}{a'}x + \frac{b}{b'}y + \frac{c}{c'}\right) - 1\right)\\ &= R\left(\frac{2a}{a'}x + \frac{2b}{b'}y + \frac{2c - c'}{c'}\right).\end{aligned}$$

There is a recursive call with a new affine formula, where all the coefficients are positive integers or non-negative integers, as required.

In a second example, suppose that the properties $x = \mathtt{L}x'$ and $y = \mathtt{R}y'$ hold. The following computation can take place:

$$\begin{aligned}\frac{a}{a'}x + \frac{b}{b'}y + \frac{c}{c'} &= \left(\frac{a}{a'}\frac{x'}{2}\right) + \left(\frac{b}{b'}\frac{y' + 1}{2}\right) + \frac{c}{c'}\\ &= \frac{a}{2a'}x' + \frac{b}{2b'}y' + \frac{bc' + 2b'c}{2b'c'}.\end{aligned}$$

Here, again, we can have a recursive call with a new affine formula, no digit has been produced (therefore the recursive call cannot be a co-recursive call) but the distance between the extrema in the new formula is $a/2a' + b/2b'$, which is half of $a/a' + b/b'$, which was the distance between extrema for the initial affine formula.

4.2. *Formal details for affine formulas*

When providing the formal description of the recursive algorithm for the computation of affine formulas, we need to pay attention to the following two important aspects:

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1 The function must be partial, because we must ensure the sign conditions on the coefficients of the affine formula.
2 Not all recursive calls are acceptable co-recursive calls, because recursive calls after the consumption of digits from the two input streams are not associated with the production of an output digit.

We define a record type named `affine_data` that collects the eight elements of an affine formula and a predicate named `positive_coefficients` that states that the coefficients satisfy the sign conditions.

The computation is then represented by a main function called `axbyc` to compute the affine formula. This function has a dependent type: it takes as first argument an affine formula and as second argument a proof that its coefficients satisfy the sign conditions:

```
axbyc: forall x: affine_data, positive_coefficients x -> stream idigit.
```

This function is defined as a co-recursive function. The constraints on recursive programming require that this function can only perform the recursive calls that are associated with the production of a digit in the output (phase 2 in the previous section). We need to define an auxiliary function, not a co-recursive one, that takes charge of the recursive calls that are only associated with the consumption of digits from the input streams (phase 3 in the previous section).

The auxiliary function is named `axbyc_rec`. It takes as arguments an affine formula and a proof that it satisfies the predicate `positive_coefficients`. It returns an equivalent affine formula, for which the decision to produce the next output digit can be taken. This is represented by the fact that output of this function is in a type with three constructors, called `caseR`, `caseL`, or `caseC`. Each constructor contains as its first field the new affine formula, and as its second field a proof that this new formula satisfies the sign conditions. The third field (for the constructors `caseR` and `caseL`) or the third and fourth fields (for the constructor `caseC`) express the fact that the right interval conditions are satisfied to output a digit. The final field is a proof that the new affine formula is equivalent to the initial one.

The recursive structure of the function `axbyc_rec` is based on well-founded recursion. More precisely, it relies on the fact that the distance between extrema decreases at each recursive step. This can be translated into an integer formula that decreases while remaining positive. When this integer formula is 0, we can prove that one of the interval conditions to output a digit is necessarily satisfied.

Two other collections of auxiliary functions perform the elementary operations. Functions named `prod_R`, `prod_L` and `prod_C` perform the coefficients' transformations that should be performed after producing an output digit. For instance `prod_R` maps the affine formula

$$\frac{a}{a'}x + \frac{b}{b'}y + \frac{c}{c'}$$

to the formula

$$\frac{2a}{a'}x + \frac{2b}{b'}y + \frac{2c - c'}{c'},$$

as we have already justified in the first example of the previous section.

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The lemmas named `A.prod_R_pos`, `A.prod_L_pos`, and `A.prod_C_pos` ensure that the functions `prod_R`, `prod_L`, and `prod_C`, respectively, preserve the sign conditions on the coefficients. These lemmas rely on the interval conditions produced by `axbyc_rec`. For instance, for `A.prod_R_pos` the extra interval condition is $c' \leqslant 2c$. In our description of the co-recursive function, this information is transferred from `axbyc_rec` to `A.prod_R_pos` with the help of a variable named `Hc`.

With these auxiliary functions, the main function can be given a simple structure:

```
CoFixpoint axbyc (x:affine_data)
   (h:positive_coefficients x):stream idigit :=
  match axbyc_rec x h with
    caseR y Hpos Hc _ =>
     R::(axbyc (prod_R y) (A.prod_R_pos y Hpos Hc))
  | caseL y Hpos _ _ =>
     L::(axbyc (prod_L y) (A.prod_L_pos y Hpos))
  | caseC y Hpos H1 H2 _ =>
     C::(axbyc (prod_C y) (A.prod_C_pos y Hpos H2))
  end.
```

With the help of the function `real_value` we can also define a function `af_real_value` that maps any affine formula represented by an element of `affine_data` to the real number it represents. This function is used to express the correctness of our algorithm in computing the affine formula:

```
axbyc_correct:
  forall x, forall H :positive_coefficients x,
   0 <= af_real_value x <= 1 ->
   real_value (axbyc x H) = af_real_value x.
```

This proof is based on a lemma that is proved by co-induction:

```
axbyc_correct_aux :
  forall x:affine_data, forall H :positive_coefficients x,
  0 <= (af_real_value x) <= 1 ->
   represents (axbyc x H) (af_real_value x).
```

This algorithm is interesting because it provides us with ways to add two real numbers, to multiply them by rational numbers, and to encode rational numbers as real numbers. Having formalised this algorithm, it may seem that the direct implementation of addition described earlier is now redundant. However, this is not the case, since the direct implementation of addition makes no use of dependent types, proof arguments or well-founded recursion. As a result, the direct addition can be executed directly within the theorem prover using its internal reduction mechanism, while the affine formula computation can only be executed outside the theorem prover as extracted code. Algorithms that can be reduced within the proof system may play a role in reflection-based proof procedures (Boutin 1997).

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5. Computing series

A series is an infinite sum of values. Knowing how to compute series can help in the computation of famous constants, like e (Euler's number) or π, and in implementing the multiplication of two real numbers.

5.1. *Main structure of the algorithm*

We consider the problem of computing values of the form $\sum_{i=0}^{\infty} a_i$ where the a_i terms are real numbers.

Studying series is very close to studying converging sequences, since it is enough to consider the sequence $u_n = \sum_{i=0}^{n} a_i$. Each element of the sequence can then be computed as a finite combination of additions.

Computing the first p digits of the limit implies computing the bounds of an interval of width 2^{-p} containing this limit. If we know that a given element u_n is closer than ε to the limit, and if we can compute an interval $[a, b]$ of width 2^{-p} such that $\varepsilon \leqslant u_n - a$ and $\varepsilon \leqslant b - u_n$, then the sequence of digits that corresponds to $[a, b]$ is a correct prefix for a representation of the limit. In particular, this means that we need to compute an interval containing u_n of width strictly smaller than 2^{-p}: in practice we need to compute an interval containing u_n with width $2^{-(p+1)}$ or $2^{-(p+2)}$. This approach shows that we can avoid considering the whole infinite sum before producing the first digit of the output.

We restrict our study to series whose convergence is described constructively by a function μ that satisfies the following properties:

$$\forall m.\quad n \leqslant m \Rightarrow \left|\sum_{i=m}^{\infty} a_i\right| < \mu(n), \qquad \lim_{n\to\infty} \mu(n) = 0.$$

We actually formalise the computation of a function f that has the following informal specification:

$$f(x, y, n) = x + y \times \sum_{i=n}^{\infty} a_i,$$

where x is a real number, y is a rational number and n is a natural number. Intuitively, y represents the inverse of the interval size that is reached in the computation ($y = 2^p$). If we know that $y \times \mu(n) \leqslant \frac{1}{16}$, and we know three digits of x, then we are able to choose the first digit d of $x + y \sum_{i=n}^{\infty} a_i$. We can then perform the following computation:

$$f(x, y, n) = d\, f(2x - \alpha(d), 2y, n).$$

In most cases, we also have $x = dx'$ for some x', and the value $2x - \alpha(d)$ is simply represented by x'. If $y \times \mu(n) > \frac{1}{16}$, we compute a new value $\phi(y, n)$ such that $y \times \mu(\phi(y, n)) \leqslant \frac{1}{16}$. This value is bound to exist because μ converges to 0 at infinity. We can then proceed with the following step:

$$f(x, y, n) = f\left(x + y \times \sum_{i=n}^{\phi(y,n)-1} a_i, y, \phi(y, n)\right).$$

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In practice we first compute $\phi(y,n)$, then we compute $v = x + y \times \sum_{i=n}^{\phi(y,n)-1} a_i$ by repeated binary additions. Let us assume that ρ is defined as $y \times \sum_{i=\phi(y,n)}^{\infty} a_i$. The number we want to compute is $v + \rho$ and we know that $|\rho| \leqslant \frac{1}{16}$. We then perform the following case analysis:

1 If v has the form `R`v' and v' has the form `R`v'', `C`v'', `LC`v'' or `LR`v'', we can deduce that $v \geqslant 9/16$, and, therefore, $v + \rho \geqslant \frac{1}{2}$, and the first digit of the result can be `R`. The result is `R`$(f(v', 2y, \phi(y,n)))$.
2 If v has the form `C`v', where v' has the form `C`v'', `LC`v'', `LR`v'', `RL`v'' or `RC`v'', we are certain that $\frac{5}{16} \leqslant v \leqslant \frac{11}{16}$, and therefore $\frac{1}{4} \leqslant v + \rho \leqslant \frac{3}{4}$, so the result is `C`$(f(v', 2y, \phi(y,n)))$.
3 If v has the form `L`v', where v' has the form `L`v'', `C`v'', `RL`v'' or `RC`v'', we are certain that $v \leqslant \frac{7}{16}$ and $v + \rho \leqslant \frac{1}{2}$, so the result is `L`$(f(v', 2y, \phi(y,n)))$.
4 If v has the form `RLL`v'', then v could also be represented using `CRL`v'' and this case has already been considered above. The same goes for the cases `LRR`, `CLL` and `CRR`, using `CLR`, `LRL` and `RLR`, respectively, as alternatives.

The number $\phi(y,n)$ is chosen so that $y \times \mu(\phi(y,n)) \leqslant \frac{1}{16}$, because $\frac{1}{16}$ is the shortest distance between the bounds of the intervals for `CLC` and `C`, `RLC` and `R`, or `LRC` and `L`. The computation of $\phi(y,n)$ and $\sum_{i=n}^{\phi(y,n)-1} a_i$ depends on the series being studied. Because recursive calls to f always have $2y$ and $\phi(y,n)$ as arguments, we can also assume that the invariant $y \times \mu(n) < \frac{1}{8}$ is maintained through recursive calls.

5.2. *Series with positive terms*

When we know that the a_i terms are all positive, we do not need to use $\frac{1}{16}$ to bound the infinite remainder of the series. The computation technique can be simplified.

We first compute $\phi(y,n)$ so that $y \times \mu(\phi(y,n)) \leqslant \frac{1}{8}$ and $v = x + \sum_{i=n}^{\phi(y,n)-1} a_i$. In what follows, let ρ be defined as $y \times \sum_{i=\phi(y,n)}^{\infty} a_i$; we know $|\rho| \leqslant \frac{1}{8}$. We perform the following case analysis:

1 If v has the form `R`v', we are sure that the result is larger than or equal to $1/2$, so the result is `R`$(f(v', 2y, \phi(y,n)))$.
2 If v has the form `C`v' but not `CR`v', we can deduce $v \in [\frac{1}{4}, \frac{5}{8}]$ and $v + \rho \in [\frac{1}{4}, \frac{3}{4}]$, so the result is `C`$(f(v', 2y, \phi(y,n)))$.
3 If v has the form `L`v', but not `LR`v', we can deduce $v \in [0, \frac{3}{8}]$ and $v + \rho \in [0, \frac{1}{2}]$, so the result is `L`$(f(v', 2y, \phi(y,n)))$.
4 If v has the form `CR`v'' or `LR`v'', we can use the equivalences with `RL`v'' and `CL`v'', respectively, to switch to one of the previous cases.

For positive series, there is also the invariant

$$y \times \mu(n) \text{ is always smaller than } \tfrac{1}{4},$$

which plays a role in the correctness proofs.

5.3. *Formal details for positive series*

The whole case analysis on v described in the previous section is common between all the series with positive terms that we may want to consider. To avoid code duplication, we

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define a function `series_body` that can be re-used from one series with positive terms to another. This function is higher-order, as it takes the function `series` as its argument. The argument function `series` is supposed to perform the computations that are required specifically for each series (computing $\phi(y,n)$ for instance), the values y and n are not passed directly as arguments, but they should be computable from the value `a` of type `A`. This type `A` can be chosen arbitrarily for each series.

```
Definition series_body (A:Set)
  (series: stream idigit -> A -> stream idigit)
  (v: stream idigit) (a:A): stream idigit :=
match v with
  R::v' => R::series v' a
| L::R::v'' => C::series (L::v'') a
| L::v' => L::series v' a
| C::R::v'' => R::series (L::v'') a
| C::v' => C::series v' a
end.
```

5.4. *Application to computing e*

The number e is defined by the formula

$$e = \sum_{k=0}^{\infty} \frac{1}{k!}.$$

Of course, this number is larger than 2, but we only want to compute its fractional part, so we actually compute $\sum_{k=2}^{\infty} \frac{1}{k!}$. The following properties are easy to obtain, by remembering that $n!n^{k-n} < k!$ for every $k \geqslant n$, so

$$0 < \sum_{k=n}^{\infty} \frac{1}{k!} < \frac{1}{(n-1)!(n-1)}.$$

For this series, we choose $\mu(n)$ to be the value $\frac{1}{(n-1)!(n-1)}$. When $2 < n$, we have $\mu(n+1) < \frac{\mu(n)}{2}$, which implies the following property:

$$\forall n, y, 0 < y \wedge 2 < n \wedge y \times \mu(n) < \frac{1}{4} \Rightarrow \phi(y,n) \leqslant n+1.$$

Thus, we never need to absorb more than one term from the infinite sum into x at each co-recursive call.

The type `A` that appears in our use of `series_body` is a triple type. The triple given as argument has the form (y,n,θ), where θ is the precomputed value $\theta = (n-1)!$. This invariant is maintained through recursive calls so that factorials are not recomputed from scratch each time. Here the μ function is given by the formula

$$\mu(n) = \frac{1}{(n-1)!(n-1)} = \frac{1}{\theta \times (n-1)},$$

and computing $\phi(y,n)$ is easy, because we know that this value is always n or $n+1$. To decide whether $\phi(y,n) = n$, we simply need to compare $y \times \frac{1}{\theta(n-1)}$ with $\frac{1}{8}$, in other words

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to compare $8y$ with $\mu' = \frac{1}{\mu} = \theta(n-1)$. In the following code, `rat_to_stream` builds the infinite sequence of digits for a rational number given by its numerator and denominator.

```
CoFixpoint e_series (x:stream idigit)(s :Z*Z*Z) :stream idigit :=
  let (aux, theta) := s in let (y, n) := aux in let mu' := theta*(n-1) in
  let (v, phi, theta') :=
   if Z_le_gt_dec (8*y) mu' then
      mk_triple x n theta
   else
      let theta' := mu'+theta in
      mk_triple (x+(rat_to_stream y theta'))(n+1) theta' in
   series_body  _ e_series v (2*y, phi, theta').
```

To express the fact that this function computes the series correctly, we rely on a predicate `infinite_sum`, which takes a function f from `Z` to $\mathbb{R}$ and a value v in $\mathbb{R}$ as arguments and means $\sum_{i=0}^{\infty} f(i)$ *converges and the limit is* v. The correctness statement has the following shape:

```
Theorem e_correct1 :
 forall v vr r y n,
   4 * y <= fact(n-1) * (n-1) -> 2 <= n -> 1 <= y ->
   represents v vr ->
   infinite_sum (fun i => 1/IZR(fact (i+n))) r ->
   vr + (IZR y)*r <= 1 ->
   represents (e_series v (y, n, fact(n-1))) (vr+(IZR y)*r).
```

This statement is made less clear by the simultaneous use of two types of numbers and the function `IZR` is the natural injection of integers into the type of real numbers. In this statement the formula `fact(n-1)*(n-1)` represents the inverse of $\mu(n)$, and the formula `4 * y <= fact(n-1) * (n-1)` corresponds to the invariant of the series. Note that this theorem expresses the fact that the series is computed correctly only if the series really converges to a value that is smaller than or equal to 1. The proof that the series converges has to be done independently.

We initialise the recursive computation with $x = \frac{1}{2}$, $y = 1$ and $n = 3$, so that the invariant on $y \times \mu(n)$ is satisfied.

```
Definition number_e_minus2: stream idigit :=
  e_series (rat_to_stream 1 2+rat_to_stream 1 6) (1, 4, 6).
```

The correctness theorem then allows us to obtain the following statement:

```
Theorem e_correct :
  infinite_sum (fun i => 1/IZR(fact(i+2)))(real_value number_e_minus2).
```

This statement really means $\sum_{i=2}^{\infty} \frac{1}{i!} =$ `number_e_minus2`.

The value `number_e_minus2` can then be used to construct rational approximations of $e-2$, using the `bounds` function. Actually, given n, we compute (a, b, k) so that

$$\frac{a}{2^k} \leqslant e-2 \leqslant \frac{b}{2^k} \qquad \frac{b}{2^k} - \frac{a}{2^k} = \frac{1}{2^n}.$$

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For $n = 320$, this computation takes approximately a minute with the standard version of Coq[†]. These functions can be used in proof procedures to compute approximations of e. The code can also be extracted to both Ocaml and Haskell. Running the Ocaml extracted code, we can compute 2000 redundant digits of $e - 2$ in about a minute.

5.5. *Multiplication as a special case of series*

When u is the infinite sequence $d_1 d_2 \ldots$, we have uu' is a series:

$$uu' = \sum_{i=1}^{\infty} \frac{\alpha(d_i)u'}{2^i}.$$

This is a series where all terms are positive. Moreover, two simplifications can be made with respect to the general approach. First, while y is multiplied by 2 at every recursive call, a_i contains a divisor that is also multiplied by 2, so the two multiplications by 2 cancel out. Second, it is reasonable to simply consume one element from the infinite sum at each recursive call, without scrutinising the value of u'. If this approach is followed, the argument y is no longer necessary: only the digits d_i and u' are needed. We can re-use the general function `series_body` as follows:

```
Definition sum_mult_d (d:idigit) (u v:stream idigit) :=
  match d with L => u | C => u+L::L::v | R => u+L::v end.

CoFixpoint mult_a (x:stream idigit)(p:stream idigit*stream idigit)
  : stream idigit :=
 let (u,v) := p in
 match u with d::w => series_body _ mult_a (sum_mult_d d x v)(w,v) end.
```

The function `mult_a` x (u, u') computes $x + uu'$ only when $uu' < \frac{1}{4}$ (here again we see the invariant of the general approach). To obtain multiplication for the general case, we divide the second operand by 4 before the multiplication and multiply the result by 4. Here is a naive implementation:

```
Definition mult (x y:stream idigit): stream idigit :=
  mult2(mult2(mult_a zero (x,L::L::y))).
```

The following theorem can then be verified formally:

```
mult_correct
   : forall (x y: stream idigit) (vx vy: Rdefinitions.R),
     represents x vx -> represents y vy -> represents (x*y)(vx*vy).
```

In this statement the notation `x * y` represents our multiplication as an operation on infinite digit streams, while the notation `vx * vy` represents the multiplication of real numbers, as they are axiomatised in the Coq system.

[†] Coq version `8.0pl2`, Intel Pentium(R) M 1700Mhz.

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It turns out that our approach to multiplication, based on series, yields an implementation of multiplication that is very close to the implementation in Ciaffaglione and di Gianantonio (2000).

We can also try this multiplication directly within Coq, for instance, we can compute $(e-2)^2$. With the current standard version of Coq no effort is made to exploit possible sharing in the lazy computation of values, so the same value may be computed several times. For this reason, we cannot compute this number to a high precision as easily as for $e-2$. For example, our few experiments showed that it takes approximately 10 seconds to compute an interval of width $\frac{1}{2^{30}}$ and a minute to compute an interval of width $\frac{1}{2^{50}}$ containing the value $(e-2)^2$.

6. Conclusion

The work described in this paper is available at

`http://hal.inria.fr/inria-00001171`

This is both an extension and a departure from the work of Ciaffaglione and di Gianantonio. Like Ciaffaglione and di Gianantonio, we work in base 2 with an extra digit, but they interpret the three digits as -1, 0, and 1, while we interpret them as 0, $\frac{1}{2}$ and 1. Because the difference is so systematic, there is a direct correspondence between the number represented by the same stream in both settings. One advantage of our experiment is that some algorithms are easier to design and easier to prove correct because we avoid having to deal with negative numbers. One important drawback is that this approach is less well adapted to expanding to the full real line.

We believe that the decision to rely on an existing axiomatisation of real numbers was instrumental in making the proofs in this experiment quicker to obtain. In particular, we relied on a collection of automatic decision procedures for equalities between polynomials and sets of inequalities between affine formulas. This axiomatisation relies on classical mathematics, and we do not know how the decision procedures rely on the classical axioms or how they could be re-implemented in constructive mathematics. Admittedly, we only provide algorithms for computing representations of real values that are constructively definable. We believe that our experiment should be reproducible in the context of constructive mathematics, but we were content with working in a classical setting and taking advantage of a larger body of existing definitions and theorems. The question of constructive or non-constructive mathematics was secondary in this experiment.

It is characteristic that the definition of series appears to be more basic than multiplication, but this is simply due to the fact that the digit-based representation of numbers is already naturally interpretable as a series and that this structure is preserved by multiplication thanks to distributivity.

Now that we have a multiplication for our representation of real numbers, we can consider the task of implementing other functions, such as division and analytic functions. For division, we expect to use a method of range reduction: to compute $\frac{x}{y}$, we should compute $\frac{x}{2^k y}$ or $\frac{x}{y}-k$ where k should be chosen so that the result is within $[0,1]$, but finding the right value for k is only possible when we have a constructive way to prove that y is non-zero.

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An alternative approach to division is provided by generalising our approach to affine formulas to consider Möbius transforms (which have already been studied in Edalat and Potts (1998)):

$$(x, y) \mapsto \frac{axy + bx + cy + d}{exy + fx + gy + h}.$$

However, we suspect that the proofs of correctness for these transformations are less easy to automatise, because they do not rely only on affine formulas (which are easily treated with the Fourier–Motzkin decision procedure).

One of the interesting features of this experiment is that some series can be computed directly within the theorem prover. This is an important feature if a proof requires us to produce an accurate approximation of a series value.

When it comes to the computation of mathematical constants, we already have all the ingredients to compute π using a Machin formula (we used the formula $\frac{\pi}{4} = \arctan(\frac{1}{2}) + \arctan(\frac{1}{3})$, which can be easily proved and computed as the sum of two series).

Several questions will be studied in future work. First, the choice of a digit set is arbitrary. We have already experimented with a digit set that contains only two digits, corresponding to intervals $[0, \frac{2}{3}]$ and $[\frac{1}{3}, 1]$; although some functions seem to be simpler (because there are fewer cases to consider, for instance when considering affine formulas), other problems arise because equalities between patterns do not exist for short patterns (thus we cannot as easily mimic the equality `CL = LR`). di Gianantonio also studied a two-digit representation and he proposes to work with intervals whose length is based on the golden number (di Gianantonio 1996). However, the price to pay is that we now need to solve a polynomial system of degree 2 to determine whether a given value belongs to one of the basic intervals, formal proofs in this setting are again made more complex because we are no longer in the realm of affine formulas.

In the long run, we wish to choose digit sets that are closer to the bounded integers that are usually handled in computers, so that regular integer addition, subtraction, multiplication and comparison, or even bitwise operations can be used to establish the basic relations between various digits.

The second important question is related to the efficiency of co-recursive computation within the theorem prover. While recent evolutions of the Coq system have brought drastic improvements in the computation of recursive functions, it is not certain that the co-recursive question is as well treated. In particular, lazy computation is needed to achieve reasonable speeds, while the current version of Coq can only implement call-by-name. This question is particularly difficult because the reduction mechanism also needs to retain the property of strong normalisation for non-closed terms.

Eventually, we hope to develop a reasonably efficient and formally verified library for mathematical computation. We believe this will be a stepping stone for more ambitious projects like the Flyspeck project (Hales 2000; 2004).

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