Combinatorics, Probability and Computing (1998) 7, 89–110. Printed in the United Kingdom

A Poisson * Geometric Convolution Law for the Number of Components in Unlabelled Combinatorial Structures

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Received 25 October 1995; revised 4 September 1996

Given a class of combinatorial structures $\mathscr{C}$, we consider the quantity $N(n,m)$, the number of multiset constructions $\mathscr{P}$ (of $\mathscr{C}$) of size n having exactly m $\mathscr{C}$-components. Under general analytic conditions on the generating function of $\mathscr{C}$, we derive precise asymptotic estimates for $N(n,m)$, as $n \to \infty$ and m varies through all possible values (in general $1 \leqslant m \leqslant n$). In particular, we show that the number of $\mathscr{C}$-components in a random (assuming a uniform probability measure) $\mathscr{P}$-structure of size n obeys asymptotically a convolution law of the Poisson and the geometric distributions. Applications of the results include random mapping patterns, polynomials in finite fields, parameters in additive arithmetical semigroups, etc. This work develops the 'additive' counterpart of our previous work on the distribution of the number of prime factors of an integer [20].

1. Introduction

The study of the statistical properties of parameters in random combinatorial structures has recently received much attention in the literature. While the methods used may be roughly classified as either *elementary*, *analytic* or *probabilistic*, results obtainable by each of these methods are, in general, of a rather different nature. *Grosso modo*, probabilistic methods are especially useful for understanding the component structures and analytic methods for parameters explicitly definable by generating functions.

The object of this paper is to develop general analytic methods (some of them being new) for characterizing, in a complete manner, the asymptotic behaviour of the number of components in a class of *multiset* combinatorial constructions (see below for a definition). Here the words 'asymptotic' and 'complete' are used in the sense that the first parameter (the size of the structures) tends to infinity, and the second parameter (the number of components) varies through all its possible values. These methods thus constitute, in a certain sense, a concise and effective set of analytic tools. This paper may be regarded as the 'additive' counterpart of Hwang [20]. It should be noted that although the generating

† Research supported by the National Science Council under Grant NSC-85-2121-M-001-007.

https://doi.org/10.1017/S0963548397003295 Published online by Cambridge University Press

functions may present rather complicated forms, the asymptotic results obtained are, somewhat unexpectedly, very neat and explicit.

In general, given a class of combinatorial structures $\mathscr{C}$, we can form the multiset construction $\mathscr{P}$ (of $\mathscr{C}$) whose elements are obtained by taking arbitrary sets of elements of $\mathscr{C}$ (with repetition allowed). For example, an integer partition of n is a multiset of positive integers whose sum is equal to n. Such a construction translates into the relations for generating functions [15, §2.3]:

$$P(w,z) = \sum_{n,m} N(n,m) w^m z^n = \prod_{n\geqslant 1} (1 - wz^n)^{-c_n} = \exp\left(\sum_{j\geqslant 1} \frac{w^j}{j} C(z^j) \right), \tag{1.1}$$

where w marks the number of $\mathscr{C}$-components in a $\mathscr{P}$-structure, $C(z) = \sum_{j\geqslant 1} c_j z^j$ is the generating function of the structures $\mathscr{C}$ and $N(n,m)$ denotes the number of $\mathscr{P}$-structures of size n having exactly m $\mathscr{C}$-components. With the definition of N, we can associate a sequence of random variables $\{\xi_n\}_n$ with probability distributions

$$\Pr\{\xi_n = m\} := \frac{N(n,m)}{\sum_j N(n,j)}, \tag{1.2}$$

provided that the denominator is positive. Note that $\sum_j N(n,j) = P_n$ is the cardinality of the set (denoted by $\mathscr{P}_n$) of elements of $\mathscr{P}$ of size n. Thus ξ_n counts the number of components in a random $\mathscr{P}$-structure of size n, where each element of $\mathscr{P}_n$ is assigned the same probability.

In this paper, we shall consider exclusively the class of combinatorial structures $\mathscr{C}$ whose generating functions are logarithmic [11, 12]; see Section 2 for definition. Roughly, $C(z)$ behaves like a constant times the logarithmic function as z tends to the dominant singularity of C in some connected region. In this case, the asymptotic behaviour of the distribution (1.2) can be completely characterized as $n \to \infty$ and $1 \leqslant m \leqslant n$. Briefly, our results state that the distribution of ξ_n is Poisson when $1 \leqslant m = O(\log n)$ and is geometric for the remaining ranges of m, the transitional behaviour being essentially Gaussian. Moreover, an asymptotic formula incorporating these diverse behaviours (using more primitive approximants) is also derived.

In the next section, we state the main results of this paper. Then we sketch a probabilistic interpretation of the results in Section 3. The proof of the theorems is given in Section 4. Some concrete examples are discussed in Section 5. We conclude this paper by some remarks.

Notation. For notational convenience, we shall represent the class of structures ($\mathscr{C}$), the generating function ($C(z)$) and the counting sequence (c_n) by the same group of letters. The symbol $[z^n]f(z)$ will represent the coefficient of z^n in the Taylor expansion of $f(z)$, the symbol $[w^m z^n]f(w,z)$ being defined similarly. All limits in this paper, including the symbols O, o, $\sim$, unless otherwise specified, will be taken as $n \to \infty$. The letters ε, δ (M) always denote small (large) positive quantities whose values vary from one occurrence to another.

2. Main results

Let us first state the definition of a logarithmic function.

Definition (Logarithmic function). Let $G(z)$ be a generating function† which is analytic at the origin and has a unique dominant singularity ρ, $0 < \rho < \infty$, on its circle of convergence. We say that G is a *logarithmic function* (with *parameters* $(\rho, \alpha, \kappa, \beta)$) if it is analytic inside a domain $\Delta_0 = \Delta_0(\rho, \varepsilon, \phi)$, $\varepsilon > 0$, $0 < \phi < \pi/2$:

$$\Delta_0(\rho, \varepsilon, \phi) := \{z \,:\, |z| \leqslant \rho + \varepsilon \quad \text{and} \quad |\arg(z - \rho)| \geqslant \phi\} \;\setminus\; \{\rho\},$$

being some indented disk in the z-plane, and satisfies there

$$G(z) = \alpha \log \frac{1}{1 - z/\rho} + \kappa + H\left((1 - z/\rho)^{1/\beta}\right), \qquad (z \to \rho, z \in \Delta_0),$$

where $\alpha > 0$, $\kappa \in \mathbb{C}$, $\beta \in \mathbb{Z}^+$, and $H(u)$ is analytic at $u = 0$ with $H(0) = 0$.

Note that although the conditions that we imposed on G are slightly stronger than those used in Flajolet and Soria [11, 12], they are satisfied in almost all applications.

Throughout this paper, we assume that P, N, and C are related by (1.1)*, the function C being logarithmic with parameters* $(\rho, \alpha, \kappa, \beta)$. When $0 < \rho < 1$, Flajolet and Soria [11] established the asymptotic normality of ξ_n whose mean and variance are both asymptotic to $\alpha \log n + O(1)$, as $n \to \infty$. They also showed that the tails of the distribution of ξ_n decrease exponentially. Further limit theorems, starting from the convergence rate in the central limit theorem, are systematically discussed in Hwang [21, Ch. 5]; *cf.* also Gao and Richmond [13]. In particular, the following two theorems are derived in Hwang [21, Ch. 5] as special cases of more general results.

Theorem 2.1 ([21], p. 107). *Let $\varepsilon > 0$ be fixed and set $R = (m-1)/(\alpha \log n)$. Then N satisfies asymptotically the expression*

$$N(n, m) = \rho^{-n} \frac{(\alpha \log n)^{m-1}}{n(m-1)!} \left(g(R) + O_\varepsilon \left(\frac{m}{(\log n)^2} \right) \right),$$

the O-term holding uniformly for $1 \leqslant m \leqslant \alpha(\rho^{-1} - \varepsilon) \log n$, where g is a meromorphic function defined by

$$g(w) = \frac{\alpha e^{\kappa w}}{\Gamma(1 + \alpha w)} \prod_{j \geqslant 1} \left(\frac{e^{-w\rho^j}}{1 - w\rho^j} \right)^{c_j} = \frac{\alpha e^{\kappa w}}{\Gamma(1 + \alpha w)} \exp\left(\sum_{j \geqslant 2} \frac{w^j}{j} C(\rho^j) \right), \tag{2.1}$$

for $|w| < \rho^{-1}$.

In view of the asymptotic formula (*cf.* Flajolet and Soria [11, 12])

$$P_n = \rho^{-n} n^{\alpha - 1} \left(g(1) + O\left(n^{-1/\beta} \right) \right), \tag{2.2}$$

† By a generating function, we implicitly assume that all the coefficients of its Taylor expansion are non-negative.

we observe that this theorem states roughly that the distribution of ξ_n is asymptotically Poisson with parameter $\alpha \log n$.

The restriction that $m < (\alpha/\rho)\log n$ is a natural one since the function g has a pole of order c_1 at $w = \rho^{-1}$ (when $c_1 > 0$). However, this property of g offers an asymptotic benefit, since we can apply Cauchy's residue theorem to include the contribution of this pole, which yields an asymptotic formula for $N(n,m)$ for larger m. For simplicity we consider only the case of a simple pole.

Theorem 2.2 [21, p. 108]. *Let $M > \rho^{-1} + \varepsilon > \rho^{-1}$ be any finite number. Assume $c_1 = 1$. Then for $\alpha(\rho^{-1}+\varepsilon)\log n \leqslant m \leqslant \alpha M \log n$, we have*

$$N(n,m) = K\rho^{m-n}n^{(\alpha/\rho)-1}\left(1 + O_M\left(n^{-(\alpha/\rho)Q(1+\rho\varepsilon)}\right)\right),$$

where $Q(t) := t\log t - t + 1$, $t > 0$, and K is the residue of $-g(w)$ at $w = \rho^{-1}$:

$$K = \frac{e^{(\kappa/\rho)-1}}{\Gamma(\alpha/\rho)}\prod_{j\geqslant 2}\left(\frac{e^{\rho^{j-1}}}{1-\rho^{j-1}}\right)^{c_j}. \tag{2.3}$$

Roughly, this theorem says that the distribution of ξ_n is geometric (with parameter ρ) when m lies in the range specified in the theorem.

From these two results, we see that there is a drastic change as to the asymptotic behaviour of N as $m/(\alpha\log n)$ crosses the 'critical interval' $[\rho^{-1}-\varepsilon, \rho^{-1}+\varepsilon]$. The following theorem states that the transitional behaviour of N in the critical interval is asymptotically Gaussian.

Theorem 2.3. *If $m \to \infty$ with n in such a way that $m \leqslant \alpha(\rho^{-2} - \varepsilon)\log n$, $\varepsilon > 0$, then the quantity N admits the asymptotic expansion*

$$N(n,m) \sim \frac{\rho^{-n}}{n}\left(K\rho^m n^{\alpha/\rho}\Phi\left(\sqrt{2m((\rho R_1)^{-1} + \log(\rho R_1) - 1)}\right) + \frac{R_1^{-m}e^m}{2\pi\sqrt{m}}\sum_{j\geqslant 0} b_j m^{-j}\right), \tag{2.4}$$

uniformly in m, with the convention that the square root has the sign of $R_1 - \rho$. Here $R_1 = m/(\alpha\log n)$, Φ represents the standard normal distribution:

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{1}{2}t^2}\,\mathrm{d}t, \qquad (x \in \mathbb{R}),$$

and the b_j's are certain bounded coefficients depending upon R_1.

An expression of the coefficients b_j is given in (4.4).

When $m = (\alpha/\rho)\log n + t\sqrt{(\alpha/\rho)\log n}$, $t = o((\log n)^{1/6})$, the main contribution to N comes from the first term in (2.4) and we have the following result.

Corollary 2.1. *If $m = [(\alpha/\rho)\log n + t\sqrt{(\alpha/\rho)\log n}]$, $[y]$ being the integral part of y, then*

$$N(n,m) = K\rho^{m-n}n^{(\alpha/\rho)-1}\Phi(t)\left(1 + O\left(\frac{1+|t|^3}{\sqrt{\log n}}\right)\right),$$

uniformly for $t = o((\log n)^{1/6})$.

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Note that an asymptotic expansion can be obtained by expanding more terms in (2.4). The range of m in Theorem 2.2 can be further extended in the following way.

Theorem 2.4. *If $m \geqslant \alpha(\rho^{-1}+\varepsilon)\log n$, $\varepsilon > 0$, and $n-m \to \infty$, then N satisfies*

$$N(n,m) = K\rho^{m-n}(n-m)^{(\alpha/\rho)-1}\left(1+O\left(n^{-(\alpha/\rho)Q(1+\rho\varepsilon)}+\frac{(\log(n-m))^{\delta_{\beta,1}}}{(n-m)^{1/\beta}}\right)\right),$$

the O-term holding uniformly in m, where $\delta_{p,q}$ denotes Kronecker's symbol. If, furthermore, $m/\log n \to \infty$, then N satisfies the asymptotic expansion

$$N(n,m) \sim K\rho^{m-n}(n-m)^{(\alpha/\rho)-1}\left(1+\sum_{j\geqslant 1}\frac{\varpi_j(\log(n-m))}{(n-m)^{j/\beta}}\right),$$

where the ϖ_js are polynomials of degree $\beta[j/\beta]$.

Thus the geometric behaviour of N subsists in the 'right domain'. There remains the case when $n-m = O(1)$. This is completed by the following combinatorial theorem.

Theorem 2.5. *Let $d > 1$ be the smallest integer such that $c_1c_d > 0$. Then N satisfies the identity*

$$N(n,m) = \sum_{0\leqslant j\leqslant (n-m)/(d-1)} N'(n-m+j,j), \quad \text{for} \quad (1-d^{-1})n < m \leqslant n, \tag{2.5}$$

where

$$\sum_{n,m} N'(n,m)w^m z^n = \prod_{j\geqslant d}\left(1-wz^j\right)^{-c_j}. \tag{2.6}$$

For practical purposes, (2.5) is useful when $n-m = O(1)$. Note that

$$\sum_{0\leqslant j\leqslant (n-m)/(d-1)} N'(n-m+j,j) = [z^{n-m}]\prod_{j\geqslant d}\left(1-z^{j-1}\right)^{-c_j}.$$

Thus when $0 < \rho < 1$ and $c_1 = 1$, the asymptotic behaviour of N is completely characterized. The case when $c_1 > 1$, although technically more complicated, can be treated by the same set of analytic methods used in this paper. A convolution law of the Poisson and negative binomial distributions is then naturally introduced.

The similarity of these results to those of the quantity $\Omega(x,m)$:

$$\Omega(x,m) := \sum_{\substack{1\leqslant n\leqslant x\\ \Omega(n)=m}} 1, \qquad (x\geqslant 1, m\in\mathbb{N}),$$

$\Omega(n)$ being the number (multiplicities counted) of prime factors of n, suggests the possibility of an asymptotic formula unifying the trichotomous behaviours exhibited by N, as was discovered by Balazard, Delange and Nicolas [5] for $\Omega(x,m)$ using more 'primitive' approximants. Such a formula will make the Gaussian transition of N between the Poisson and geometric behaviours clearer and more explicit.

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Theorem 2.6. *Let* $0 < B < \rho^{-2}$ *and set* $X = (\alpha/\rho)\log(n-m)$, $\Pi_k(X) = \sum_{0\leqslant j\leqslant k} X^j/j!$,

$$h(w) = \frac{g(w/\rho)}{\rho}(1-w) = \frac{\alpha e^{(-1+\kappa/\rho)w}}{\rho\Gamma(1+\alpha w/\rho)}\prod_{j\geqslant 2}\left(\frac{e^{-w\rho^{j-1}}}{1-w\rho^{j-1}}\right)^{c_j}, \qquad (|w| < \rho^{-1}),$$

$$\lambda = \max\{1, \min\{B, (m-1)/X\}\},$$

$$r = r(m;X) = 1 - \frac{X^{m-1}}{\Pi_{m-1}(X)(m-1)!}, \qquad (m = 1, 2, 3, \ldots).$$

Then N *satisfies*

$$N(n,m) = h(r)\frac{\rho^{m-n}}{n-m}\Pi_{m-1}(X)\left(1 + O\left(\min\left\{X^{-1}, X^{-1/2}e^{-XQ(\lambda)}\right\}\right)\right), \tag{2.7}$$

for $m \geqslant 1$ *and* $n - m \to \infty$.

Note that $h(1) = K$ and $r = \Pi_{m-2}(X)/\Pi_{m-1}(X)$ when $m \geqslant 2$. As in [4], we can derive some interesting consequences of this theorem. For example, since (*cf.* [4, pp. 18–19] or (3.4) below)

$$r \sim r' := \min\left\{1, \frac{m-1}{X}\right\},$$

we can write

$$N(n,m) \sim h(r')\frac{\rho^{m-n}}{n-m}\Pi_{m-1}(X), \qquad (m \geqslant 1,\ n-m \to \infty).$$

Another consequence of this theorem is the *unimodality* (*cf.* Balazard [4, pp. 10–17]) of the sequence $\{N(n,m)\}_m$ for sufficiently large n. Recall that a sequence of positive numbers $\{a_m\}_m$ is called unimodal if there exists an index k such that $a_j \leqslant a_k$ and $a_k \geqslant a_\ell$ for all $j < k$ and $\ell > k$.

Intuitively, the geometric behaviour of ξ_n when $m > (\alpha/\rho)\log n$ is dictated by the number of $\mathscr{C}$-components of size 1, this being so since we count the multiplicity of the occurrences of each component. We may further divide the ranges of m in such a way to make explicit (or isolate) the contributions of the $\mathscr{C}$-components of sizes 2, 3, *etc*., these latter being, however, asymptotically negligible. A more precise probabilistic interpretation is provided in the next section.

It should be noted that, when the multiplicity of each component is not taken into account and counted only once, the generating function being

$$\prod_{n\geqslant 1}\left(1 + \frac{wz^n}{1-z^n}\right)^{c_n},$$

the situation becomes more involved when $m \gg \log n$ as no a single component plays a predominating rôle in the corresponding counting function. In such a case the use of a two-dimensional saddle-point method seems necessary (*cf.* Hildebrand and Tenenbaum [16]). A similar remark applies to structures whose generating functions are of the forms

$$\prod_{n\geqslant 1}(1+wz^n)^{c_n} \qquad \text{and} \qquad \prod_{n\geqslant 1}(1+c_n wz^n),$$

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corresponding, respectively, to the set construction of $\mathscr{C}$ and the construction of $\mathscr{C}$ whose elements have no two components of the same size.

Our complete characterization of N is in spirit similar to the classical work by Moser and Wyman [28, 29] concerning the Stirling numbers of both kinds for which the second parameters are divided into several overlapping ranges to each of which different analytic methods are then applied. We add that uniform asymptotic estimates of these numbers are recently derived by Temme [35]. While the principal tools used in these problems are the saddle-point method and its extensions, our basic tool of attack is the singularity analysis of Flajolet and Odlyzko [10].

Let us briefly describe the methods of proof of these theorems. First, a uniform estimate for $P_n(w) = [z^n]P(w, z)$ is derived by singularity analysis [10]:

$$P_n(w) = wg(w)\rho^{-n}n^{\alpha w-1}\left(1 + O_\varepsilon\left(n^{-1/\beta}\right)\right),$$

for $|w| \leqslant \rho^{-1} - \varepsilon$, $\varepsilon > 0$. From this formula, Theorems 2.1 and 2.2 follow from Cauchy's formula and the saddle-point method and Theorem 2.3 from van der Waerden's method that we used in [20]. The proofs of Theorems 2.4, 2.5, and 2.6 are based on explicitly isolating the contribution of $\mathscr{C}$-components of size 1.

It should be mentioned that although Theorems 2.1–2.4 can be derived as corollaries of Theorem 2.6 by applying the asymptotic properties of $\Pi_{m-1}(X)$ (*cf.* (3.4) below) as in Balazard [3, pp. 109–112], and it suffices that we prove only Theorem 2.6; however, the individual method leading to the result of each theorem has general applicability and is of some interest *per se*, thus from a methodological point of view, it does not seem devoid of interest to present these methods separately.

3. Poisson * geometric law and probabilistic interpretation

Let X be a Poisson random variable with parameter $\lambda > 0$:

$$\pi_j = \Pr\{X = j\} = e^{-\lambda}\frac{\lambda^{j-1}}{(j-1)!}, \qquad (j = 1, 2, 3, \ldots),$$

and Y a geometric random variable independent of X:

$$\gamma_j = \Pr\{Y = j\} = (1-p)p^j, \qquad (0 < p < 1,\ j = 0, 1, 2, \ldots).$$

The convolution of the distributions of X and Y is then defined by $(Z = X + Y)$:

$$\Pr\{Z = k\} = \sum_{1\leqslant j\leqslant k}\pi_j\gamma_{k-j} = (1-p)p^{k-1}e^{-\lambda}\sum_{0\leqslant j\leqslant k-1}\frac{(\lambda/p)^j}{j!}, \tag{3.1}$$

for $k = 1, 2, 3, \ldots$. Such a Poisson * geometric law is a special case of the Poisson * negative binomial convolution law, the latter being known in the actuarial literature as Delaporte distribution (*cf.* Johnson, Kotz and Kemp [22, p. 232] and the references therein). A comparison of (3.1) with (2.7) suggests the following probabilistic interpretation (*cf.* Balazard [4]).

Write

$$\xi_n = \xi_n' + \eta_n,$$

where ξ'_n denotes the number of $\mathscr{C}$-components of size > 1 in a random $\mathscr{P}_n$-structure and η_n the number of $\mathscr{C}$-components of size 1. Thus, in view of (3.1) and (2.7), it is to be expected that

$$\xi'_n \sim \text{Poisson}(\alpha \log n), \quad \eta_n \sim \text{Geometric}(\rho).$$

In fact, we can derive more precise results concerning these two variates (assuming $c_1 = 1$):

$$\Pr\{\xi'_n = m\} = n^{-\alpha} \frac{(\alpha \log n)^{m-1}}{(m-1)!} \left(\frac{g_0(R)}{g(1)} + O_\varepsilon \left(\frac{m}{(\log n)^2} \right) \right), \tag{3.2}$$

for $1 \leqslant m \leqslant (\rho^{-2} - \varepsilon)\alpha \log n$, where $R = (m-1)/(\alpha \log n)$ and

$$g_0(w) = \frac{\alpha e^{(\kappa-\rho)w}}{\Gamma(1+\alpha w)} \prod_{j \geqslant 2} \left(\frac{e^{-w\rho^j}}{1 - w\rho^j} \right)^{c_j} = (1 - \rho w) g(w);$$

and

$$\Pr\{\eta_n = m\} = (1-\rho)\rho^m \left(1 + O\left(\frac{m}{n} + (n-m)^{-1/\beta} \right) \right), \tag{3.3}$$

for $m \geqslant 0$ and $m = o(n)$. Since the proof of (3.2) proceeds along the same line as the proof (see next section) of Theorem 2.1 starting from the relation

$$\Pr\{\xi'_n = m\} = P_n^{-1} [w^m z^n] \prod_{j \geqslant 2} (1 - wz^j)^{-c_j},$$

(only those $\mathscr{C}$-components with size $\geqslant 2$ are 'marked' by w) it is omitted here. As to (3.3), it follows easily from the defining equation

$$\begin{aligned} \Pr\{\eta_n = m\} &= P_n^{-1} [w^m z^n] (1 - wz)^{-1} \prod_{j \geqslant 2} (1 - z^j)^{-c_j} \\ &= \begin{cases} P_n^{-1} (P_{n-m} - P_{n-m-1}), & \text{if } 0 \leqslant m < n; \\ P_n^{-1}, & \text{if } m = n, \end{cases} \end{aligned}$$

and the asymptotic estimate (2.2) for P_n.

The two random variables ξ'_n and η_n are asymptotically independent in view of (3.2) (3.3) and (2.7). We can, of course, derive more precise quantitative results for their joint distribution by a similar method.

For our purposes, we need the following estimates for partial sums of the exponential series $\Pi_k(X) = \sum_{0 \leqslant j \leqslant k} X^j / j!$, as the parameter $X \to \infty$:

$$\Pi_k(X) = \begin{cases} \dfrac{X^k}{k!\,(1 - k/X)} \left(1 + O\left(\dfrac{k}{XM^2} \right) \right), & \text{if } 0 \leqslant k \leqslant X - M\sqrt{X}; \\ e^X \Phi(y) \left(1 + O\left(\dfrac{1 + |y|^3}{\sqrt{X}} \right) \right), & \text{if } y = \dfrac{k - X}{\sqrt{X}} = o(X^{1/6}); \\ e^X - \dfrac{X^k}{k!\,(k/X - 1)} \left(1 + O\left(\dfrac{k}{XM^2} \right) \right), & \text{if } k \geqslant X + M\sqrt{X}. \end{cases} \tag{3.4}$$

Note that there are overlaps in the first and the last two ranges.

Of these, the second one is a consequence of Cramér-type large deviations for sums of Poisson distributions; *cf.* Norton [30] [27, p. 100], or Hwang [18, Ch. 3]. We only prove

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the remaining ones which seem less known in the probability literature. The key idea of the proof, due to Selberg [33], is summarized in the following lemma.

Lemma 3.1. *Let*

$$I := \frac{1}{2i\pi}\oint_{|z|=\zeta} F(z) z^{-m-1} e^{Xz}\,\mathrm{d}z, \tag{3.5}$$

where F *is analytic for* $|z| \leqslant a$*,* $a > \zeta > 0$*,* $m \in \mathbb{N}$ *and* X *is a large parameter. Then* I *satisfies*

$$I = \frac{X^m}{m!}\left(F(\tau) + O\left(L_2 \frac{m}{X^2}\right)\right), \tag{3.6}$$

uniformly for $0 \leqslant m \leqslant aX$*, where* $\tau := m/X$ *and* $L_2 := \sup_{|z|\leqslant a}|F''(z)|$.

Sketch of proof. Expand F at $z = \tau \leqslant a$:

$$F(z) = F(\tau) + F'(\tau)(z-\tau) + (z-\tau)^2 \int_0^1 (1-t)F''(zt + (1-t)\tau)\,\mathrm{d}t,$$

substitute this formula into I and estimate the integral

$$\frac{1}{2i\pi}\oint_{|z|=\zeta} z^{-m-1}e^{Xz}(z-\tau)^2 \int_0^1 (1-t)F''(zt+(1-t)\tau)\,\mathrm{d}t\,\mathrm{d}z$$

by Laplace's method. For details, we refer readers to Balazard [4], Hwang [17] or Tenenbaum [36, pp. 230–231]. □

An extension of I to an asymptotic expansion involving Laguerre polynomials as coefficients was established in Hwang [17].

4. Proof of the Theorems

Adopting a number-theoretic convention, we shall use the symbols $\ll$ and $O(.)$ interchangeably as is convenient.

Theorems 2.1, 2.2 and 2.3
First, by (1.1), we can write

$$P(w,z) = e^{wC(z)}\Psi(w,z), \quad \text{where} \quad \Psi(w,z) = \exp\left(\sum_{j\geqslant 2}\frac{w^j}{j}C(z^j)\right).$$

Since the radius of convergence of C is equal to $\rho < 1$, it follows that the function $z \mapsto \Psi(w,z)$ is analytic for $|z| < \sqrt{\rho}$ when $|zw| < 1$. On the other hand, the assumption that C is logarithmic implies that

$$P(w,z) = \left(1 - \frac{z}{\rho}\right)^{-\alpha w} e^{\kappa w}\Psi(w,\rho)\left(1 + O\left(\left(1-\frac{z}{\rho}\right)^{1/\beta}\right)\right),$$

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as $z \to \rho$ in some Δ_0-region. Thus, applying singularity analysis [10] to $P(w, z)$ yields, in view of (2.1),

$$P_n(w) = [z^n]P(w,z) = wg(w)\rho^{-n}n^{\alpha w-1}\left(1 + O_\varepsilon\left(n^{-1/\beta}\right)\right), \tag{4.1}$$

uniformly in w, $|w| \leqslant \rho^{-1} - \varepsilon$, $\varepsilon > 0$.

Proof of Theorem 2.1. Starting from (4.1) and observing that g is analytic at the origin, we can apply Cauchy's formula and write

$$N(n,m) = [w^m]P_n(w) = \frac{\rho^{-n}}{n}(I_1 + I_2),$$

where

$$I_1 = \frac{1}{2i\pi}\oint_{|w|=R} g(w)w^{-m}n^{\alpha w}\,\mathrm{d}w, \qquad \left(R = (m-1)/(\alpha \log n)\right),$$

is of type (3.5) and

$$\begin{aligned} I_2 &\ll R^{-m+1}n^{-1/\beta}\int_{-\pi}^{\pi} n^{\alpha R\cos t}\,\mathrm{d}t = R^{-m+1}n^{-1/\beta}e^{m-1}\int_{-\pi}^{\pi} e^{-(1-\cos t)(m-1)}\,\mathrm{d}t \\ &\ll R^{-m+1}n^{-1/\beta}e^{m-1}(m-1)^{-1/2} \ll \frac{X^{m-1}}{(m-1)!}n^{-1/\beta}, \quad (m = 1,2,3,\ldots). \end{aligned}$$

Thus Theorem 2.1 follows from applying (3.6) to I_1. □

Proof of Theorem 2.2. First, when $\alpha(\rho^{-1} + \varepsilon)\log n \leqslant m \leqslant \alpha M \log n$, we have easily

$$I_2 \ll R_2^{-m}n^{-(1/\beta)+\alpha R_2},$$

where $R_2 = \rho^{-1} - \delta$, $\delta > 0$. Thus

$$I_2 \ll \rho^m n^{\alpha/\rho} n^{(\alpha/\rho)(1+\rho\delta)\log(1-\rho\delta)-(1/\beta)-\alpha\delta} \ll \rho^m n^{\alpha/\rho} n^{-1/(2\beta)},$$

by choosing δ sufficiently small.

On the other hand, since g has a simple pole at $w = \rho^{-1}$ with residue $-K$, we have, by Cauchy's residue theorem,

$$I_1 = K\rho^m n^{\alpha/\rho} + \frac{1}{2i\pi}\oint_{|w|=\rho^{-1}+\varepsilon} g(w)w^{-m}n^{\alpha w}\,\mathrm{d}w,$$

the last integral being bounded in modulus by

$$\ll \rho^m n^{\alpha/\rho} n^{\alpha\varepsilon-(\alpha/\rho)(1+\rho\varepsilon)\log(1-\rho\varepsilon)} = \rho^m n^{\alpha/\rho} n^{-(\alpha/\rho)Q(1+\rho\varepsilon)},$$

by the definition of Q (in Theorem 2.2). This completes the proof. □

Proof of Theorem 2.3. We again use (4.1), but now with van der Waerden's method [37]; see Hwang [20], where the following lemma is proved.

Lemma 4.1. *Let $a > 0$ and $F(z)$ be an analytic function in $|z| \leqslant A$ with $A > a$ and $F(a) \neq 0$. Then the integral J defined by*

$$J := \frac{1}{2i\pi}\oint_{|z|=\zeta}\frac{F(z)}{a-z}z^{-m-1}e^{Xz}\,\mathrm{d}z \qquad (X \to \infty,\ 0 < \zeta < a)$$

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satisfies, as $m, X \to \infty$ and $m \leqslant AX$, the asymptotic expansion

$$J \sim F(a)a^{-m-1}e^{aX}\Phi\left(\sqrt{2m(a/\tau + \log(\tau/a) - 1)}\right) + \frac{\tau^{-m}e^m}{2\pi\sqrt{m}}\sum_{j\geqslant 0}\varphi_j m^{-j},$$

with the convention that the square root has the sign of $a-\tau$. Here $\tau = m/X$ and the φ_js are certain bounded coefficients.

To prove Theorem (2.3), we rewrite (4.1) as

$$P_n(w) = \rho^{-n}n^{\alpha w-1}\left(\frac{G(w)}{1-\rho w} + V_n(w)\right), \tag{4.2}$$

where

$$G(w) = w(1-\rho w)g(w) = \frac{e^{(\kappa-\rho)w}}{\Gamma(\alpha w)}\prod_{j\geqslant 2}\left(\frac{e^{-w\rho^j}}{1-w\rho^j}\right)^{c_j}, \qquad (|w| < \rho^{-2}),$$

and $V_n(w) \ll_\varepsilon n^{-1/\beta}$ for $|w| \leqslant \rho^{-1}-\varepsilon$, $\varepsilon > 0$. We can then apply Lemma 4.1 to the principal term in (4.2). It remains to show that, for $m \to \infty$ and $m \leqslant \alpha(\rho^{-2}-\varepsilon)\log n$,

$$I_2 = \frac{1}{2i\pi}\oint_{|w|=R'} V_n(w)w^{-m-1}n^{\alpha w}\,\mathrm{d}w \ll I_1, \qquad (0 < R' < \rho^{-1}), \tag{4.3}$$

or, equivalently, for any $L \geqslant 0$,

$$I_2 \ll \rho^m n^{\alpha/\rho}\Phi\left(\sqrt{2m((\rho R_1)^{-1} + \log(\rho R_1) - 1)}\right) + R_1^{-m}e^m m^{-L-1/2} =: T_1 + T_2,$$

say.

Consider first I_2 and T_2. We divide the comparison into two cases. Recall that $R_1 = m/(\alpha\log n)$.

1. $0 < R_1 \leqslant \rho^{-1}-\varepsilon$. Taking $R' = R_1$ in (4.3), we find easily

$$I_2 \ll n^{-1/\beta}R_1^{-m}e^m \ll R_1^{-m}e^m m^{-L-1/2} = T_2,$$

   for all $L \geqslant 0$.

2. $(\rho^{-1}-\varepsilon) \leqslant R_1 < \rho^{-2}$. We take $R' = \rho^{-1}-\varepsilon$ and we obtain, by choosing ε sufficiently small,

$$I_2 \ll n^{-1/\beta}(\rho^{-1}-\varepsilon)^{-m}n^{(\alpha/\rho)-\alpha\varepsilon}(\log n)^{-1/2} \ll R_1^{-m}e^m m^{-L-1/2} = T_2,$$

   for all $L \geqslant 0$.

As to T_1, we also distinguish two cases.

1. $|m - (\alpha/\rho)\log n| \leqslant M\sqrt{\log n}$. In this case, we have $|R_1 - \rho^{-1}| \ll (\log n)^{-1/2}$ and $T_1 \sim \Phi(t)\rho^m n^{\alpha/\rho}$, since

$$2m\left(\frac{1}{\rho R_1} + \log(\rho R_1) - 1\right) = t^2 - \frac{t^3}{3\sqrt{(\alpha/\rho)\log n}} + \cdots,$$

   where $m = (\alpha/\rho)\log n + t\sqrt{(\alpha/\rho)\log n}$. Taking $R' = \rho^{-1}-\varepsilon$, we obtain easily

$$I_2 \ll n^{-1/\beta}(\rho^{-1}-\varepsilon)^{-m}n^{(\alpha/\rho)-\alpha\varepsilon} \ll \rho^m n^{\alpha/\rho}(\log n)^{-L},$$

   for all $L \geqslant 0$, by taking ε sufficiently small.

2. $|m-(\alpha/\rho)\log n| \geqslant M\sqrt{\log n}$. We have $2m(1/(\rho R_1)+\log(\rho R_1)-1) = m(\rho^2(R_1-\rho^{-1})^2+\cdots)$. Observe that Φ satisfies the asymptotic expansion (by integration by parts)

$$1-\Phi\left(\sqrt{2\lambda}\right) \sim \frac{e^{-\lambda}}{2\sqrt{\pi\lambda}}\left(1+\sum_{j\geqslant 1}\frac{e_j}{(2\lambda)^j}\right), \qquad (\lambda\to\infty),$$

the e_js being real coefficients. Thus, it follows that

$$\begin{aligned}\rho^m n^{\alpha/\rho}\Phi\left(\sqrt{2m(1/(\rho R_1)+\log(\rho R_1)-1)}\right) &\ll \frac{\rho^m n^{\alpha/\rho}e^{-m(1/(\rho R_1)+\log(\rho R_1)-1)}}{2\sqrt{\pi m|1/(\rho R_1)+\log(\rho R_1)-1|}}\\ &= \frac{R_1^{-m}e^m}{2\sqrt{\pi m|1/(\rho R_1)+\log(\rho R_1)-1|}},\end{aligned}$$

From the comparison of T_2 with I_2 above, we conclude that

$$I_2 \ll \rho^m n^{\alpha/\rho}\Phi\left(\sqrt{2m(1/(\rho R_1)+\log(\rho R_1)-1)}\right).$$

This completes the proof of Theorem 2.3. □

From [20], we can compute the coefficients b_j as follows. Set $u := \sqrt{-(e^{it}-1-it)} = t/\sqrt{2}+it^2/(6\sqrt{2})+\cdots$ and

$$\phi(u) := \frac{G(R_1e^{it})}{\rho^{-1}-R_1e^{it}}\frac{2u}{i(1-e^{it})} - \frac{i\rho G(\rho^{-1})}{u-\sqrt{(\rho R_1)^{-1}+\log(\rho R_1)-1}}, \qquad (R_1 = m/(\alpha\log n)),$$

the function ϕ being analytic at $u=0$ and $u=\sqrt{(\rho R_1)^{-1}+\log(\rho R_1)-1}$. Then we have

$$b_j = b_j(R_1) := \Gamma(j+1/2)[u^{2j}]\phi(u), \qquad (j=0,1,2,\ldots), \tag{4.4}$$

the coefficient of u^{2j} in the Taylor expansion of ϕ. In particular, $\rho_1 = 1/\rho$,

$$\begin{aligned}\frac{b_0}{\sqrt{\pi}} &= \frac{\sqrt{2}\,G(R_1)}{\rho^{-1}-R_1} + \frac{G(\rho^{-1})}{\sqrt{(\rho R_1)^{-1}+\log(\rho R_1)-1}}\\ \frac{b_1}{\sqrt{\pi}} &= \frac{G(R_1)(13R_1^2-2R_1\rho^{-1}+\rho^{-2})-12R_1^2G'(r)(R_1-\rho^{-1})+6R_1^2G''(R_1)(R_1-\rho^{-1})^2}{6\sqrt{2}\,(R_1-\rho^{-1})^3}\\ &\quad - \frac{\rho\,G(\rho^{-1})}{2((\rho R_1)^{-1}+\log(\rho R_1)-1)^{3/2}}.\end{aligned}$$

Theorems 2.4 and 2.5

We first prove Theorem 2.5. *Assume, throughout this and the next section, that* $c_1 = 1$ *and* d *is the least integer* >1 *such that* $c_1c_d > 0$.

Lemma 4.2. *Let* N' *be defined as in (2.6). Then, for any* $n, m \geqslant 1$,

$$N(n,m) = \sum_{0\leqslant j\leqslant m} N'(n-j,m-j) = \sum_{j\geqslant 0} N'(n-m+j,j) - \sum_{j\geqslant 1} N'(n+j,m+j). \tag{4.5}$$

Proof. The first equality follows immediately from the definitions of N and N':

$$\sum_{n,m} N(n,m)w^m z^n = \frac{1}{1-wz}\sum_{n,m} N'(n,m)w^m z^n,$$

and the second from rearranging the indices of the summations. □

Proof of Theorem 2.5. Note that $N'(n,m) = 0$ if $m > n/d$. Thus the series on the right-hand side of (4.5) have in fact only a finite number of terms, namely,

$$N(n,m) = \sum_{0\leqslant j\leqslant (n-m)/(d-1)} N'(n-m+j,j) - \sum_{1\leqslant j\leqslant (n-dm)/(d-1)} N'(n+j,m+j). \tag{4.6}$$

Thus, when $m > (n-m)/(d-1)$, the second sum is identically zero and we find exactly Theorem 2.5. □

An important corollary of this decomposition is that when m is sufficiently large, the first sum in (4.6) gives the principal term in the asymptotic expansion of N. What we need is the following analytic version of (4.5).

Corollary 4.1. *For $n,m \in \mathbb{Z}^+$, the quantity N satisfies*

$$N(n,m) = I_3 + I_4, \tag{4.7}$$

where

$$I_3 = \frac{1}{2i\pi}\oint_{|z|=\zeta} z^{-n+m-1}\prod_{k\geqslant d}\left(1-z^{k-1}\right)^{-c_k}\,\mathrm{d}z,$$

and

$$I_4 = -\sum_{j\geqslant 1}\frac{1}{2i\pi}\oint_{|w|=v} w^{-m-j-1}\frac{1}{2i\pi}\oint_{|z|=\zeta'} z^{-n-j-1}\prod_{k\geqslant d}\left(1-wz^k\right)^{-c_k}\,\mathrm{d}z\,\mathrm{d}w,$$

with $0<\zeta,\zeta'<\rho$ and $0<v<\rho^{-1}$.

Proof. It is easily seen that

$$\sum_{\ell,j\geqslant 0} N'(\ell+j,j)z^\ell = \prod_{k\geqslant d}\left(1-z^{k-1}\right)^{-c_k},$$

and the required result follows from (4.5) and Cauchy's integral formula. □

Remarks.

1. As in Hwang [20], a purely formal proof of (4.7) is to use the integral representation

$$N(n,m) = \frac{1}{2i\pi}\oint_{|w|=v} w^{-m-1}\frac{1}{2i\pi}\oint_{|z|=\zeta} z^{-n-1}\frac{1}{1-wz}\prod_{k\geqslant d}(1-wz^k)^{-c_k}\,\mathrm{d}z\,\mathrm{d}w, \tag{4.8}$$

   by first interchanging the order of integrations, computing the residue of the integrand at $w=1/z$ and then expanding the factor $(1-wz)^{-1}$ in descending powers of wz.

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2. There is yet another way of looking into the above formal argument. It is based on the observation

$$[w^m z^n] \prod_{j\geqslant 1} (1-wz^j)^{-c_j} = [w^m z^n] \prod_{1\leqslant j\leqslant n} (1-wz^j)^{-c_j},$$

and the latter expression can then be expanded into sums of partial fractions (in the variable w).
3. The validity of the formulae (4.5) and (4.7) is independent of the assumption that $\rho < 1$.

Lemma 4.3. *If $n-m\to\infty$, then I_3 admits the asymptotic expansion*

$$I_3 = K\rho^{m-n}(n-m)^{(\alpha/\rho)-1}\left(1+\sum_{1\leqslant k<v}\frac{\varpi_k(\log(n-m))}{(n-m)^{k/\beta}} + O\left(\frac{(\log(n-m))^{[v/\beta]}}{(n-m)^{v/\beta}}\right)\right), \quad (4.9)$$

for any positive integer v, where $\varpi_j(u)$ is a polynomial in u of degree $[j/\beta]$.

Proof. We have

$$\prod_{k\geqslant d}\left(1-z^{k-1}\right)^{-c_k} = e^{-1+C(z)/z}\Lambda(z),$$

where

$$\Lambda(z) = \exp\left(\sum_{\ell\geqslant 2}\frac{1}{\ell}\left(z^{-\ell}C(z^\ell)-1\right)\right),$$

is analytic for $|z|<\sqrt{\rho}$. Now using the formal identity ($a_0 = e_0 = 1$)

$$\left(1+\sum_{j\geqslant 1}a_j y^j\right)\left(1+\sum_{j\geqslant 1}e_j y^{j/\beta}\right) = 1+\sum_{k\geqslant 1}y^{k/\beta}\sum_{0\leqslant j\leqslant [k/\beta]}a_j e_{k-\beta j},$$

for any positive integer β, we deduce the local expansion

$$\begin{aligned} e^{-1+C(z)/z}\Lambda(z) &= e^{-1+\kappa/\rho}\left(1-\frac{z}{\rho}\right)^{-\alpha/\rho}\Lambda(\rho)\Bigg\{1 \\ &\quad + \sum_{1\leqslant k<v}\left(1-\frac{z}{\rho}\right)^{j/\beta}\sum_{0\leqslant j\leqslant[k/\beta]}\pi_j\left(\log\frac{1}{1-z/\rho}\right)h'_{k-\beta j} \\ &\quad + O\left(\left(1-\frac{z}{\rho}\right)^{v/\beta}\left(\log\frac{1}{1-z/\rho}\right)^{[v/\beta]}\right)\Bigg\}, \end{aligned} \quad (4.10)$$

as $z\to\rho$ in some Δ_0-region, where $v\in\mathbb{Z}^+$,

$$\exp\left(\frac{\alpha}{z}\log\frac{1}{1-z/\rho}\right) \sim \left(1-\frac{z}{\rho}\right)^{-\alpha/\rho}\left(1+\sum_{j\geqslant 1}\left(1-\frac{z}{\rho}\right)^j\pi_j\left(\log\frac{1}{1-z/\rho}\right)\right),$$

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$\pi_j(u)$ being a polynomial in u of degree j, and

$$\Lambda(z)\exp\left(\frac{1}{z}H\left(\left(1-z/\rho\right)^{1/\beta}\right)\right)\sim\Lambda(\rho)\left(1+\sum_{j\geqslant 1}h'_j\left(1-z/\rho\right)^{j/\beta}\right),\qquad (z\sim\rho).$$

Thus (4.9) follows from applying singularity analysis [10] to (4.10). Note that, in view of (2.3), we have $K=e^{-1+\alpha/\rho}\Lambda(\rho)/\Gamma(\alpha/\rho)$. □

Lemma 4.4. *Let $m\in\mathbb{Z}^+$ and $\rho^{-1}<v<\rho^{-2}$. Then I_4 satisfies the estimate*

$$I_4\ll\rho^{-n}v^{-m}n^{\alpha v-1}. \tag{4.11}$$

Proof. First, since C is logarithmic, we have available the estimate

$$\prod_{k\geqslant d}\left(1-wz^k\right)^{-c_k}\ll\left(1-\frac{z}{\rho}\right)^{-\alpha\Re(w)},\qquad \left(|w|\leqslant\rho^{-2}-\varepsilon\right),$$

uniformly for z in any compact set in some Δ_0-region. Thus, again, by singularity analysis,

$$[z^{n+j}]\prod_{k\geqslant d}\left(1-wz^k\right)^{-c_k}\ll\rho^{-n-j}(n+j)^{\alpha\Re(w)-1},\qquad (j\in\mathbb{Z}^+),$$

for $|w|\leqslant\rho^{-2}-\varepsilon$, $\varepsilon>0$. Hence, for I_4, we obtain, for any $\rho^{-1}<v<\rho^{-2}$ and $j\in\mathbb{Z}^+$,

$$I_4\ll\sum_{j\geqslant 1}v^{-m-j}\rho^{-n-j}(n+j)^{\alpha v-1}\ll\rho^{-n}v^{-m}n^{\alpha v-1},$$

as required. □

Corollary 4.2. *For $m\geqslant\alpha(\rho^{-1}+\varepsilon)\log n$, I_4 satisfies*

$$I_4\ll\rho^{m-n}(n-m)^{(\alpha/\rho)-1}n^{-(\alpha/\rho)Q(1+\rho\varepsilon)}.$$

Proof. Distinguish two cases: (i) $m=o(n)$ and (ii) $m\asymp n$. The result follows from (4.11) by straightforward computations. □

From Lemmas 4.2, 4.3, 4.4, and Corollary 4.2, Theorem 2.4 follows.

Theorem 2.6

To prove Theorem 2.6, we again use Lemma 4.2 but with the following analytic version which *formally* corresponds to the change of variables $u=wz$ in (4.8).

Corollary 4.3. *For any $n,m\in\mathbb{Z}^+$, N satisfies*

$$N(n,m)=\frac{1}{2i\pi}\oint_{|u|=v}\frac{u^{-m-1}}{1-u}\frac{1}{2i\pi}\oint_{|z|=\zeta}z^{-n+m-1}\prod_{k\geqslant d}\left(1-uz^{k-1}\right)^{-c_k}\,\mathrm{d}z\,\mathrm{d}u, \tag{4.12}$$

where $0<v<1$ and $0<\zeta<\rho^{-1}$.

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Proof. Observe that

$$N'(n-j,m-j) = [w^{m-j}z^{n-j}]\prod_{k\geqslant d}\left(1-wz^k\right)^{-c_k} = [w^{m-j}z^{n-m}]\prod_{k\geqslant d}\left(1-wz^{k-1}\right)^{-c_k},$$

the required formula follows from expanding the factor $(1-u)^{-1}$ in ascending powers of u and (4.5). □

The following lemma is derived by refining the same analyses in the proof of Lemma 4.3.

Lemma 4.5. *If $n-m\to\infty$, then the asymptotic relation*

$$[z^{n-m}]\prod_{k\geqslant d}\left(1-uz^{k-1}\right)^{-c_k} = uh(u)\rho^{m-n}(n-m)^{(\alpha u/\rho)-1}\left(1+T(u)\right), \tag{4.13}$$

holds, where

$$T(u) \ll_\varepsilon \frac{(\log(n-m))^{[1/\beta]}}{(n-m)^{1/\beta}}, \qquad (|u|\leqslant \rho^{-1}-\varepsilon), \tag{4.14}$$

and h is defined as in Theorem 2.6.

Thus, from (4.12) and (4.13), the evaluation of N is decomposed into two terms:

$$N(n,m) = \frac{\rho^{m-n}}{n-m}\left(I_5+I_6\right),$$

where (*throughout this section* $X=(\alpha/\rho)\log(n-m)$)

$$I_5 = \frac{1}{2i\pi}\oint_{|u|=v}\frac{u^{-m}}{1-u}h(u)e^{Xu}\,\mathrm{d}u, \qquad (0<v<1),$$

and

$$I_6 = \frac{1}{2i\pi}\oint_{|u|=v'}\frac{u^{-m}}{1-u}T(u)e^{Xu}\,\mathrm{d}u, \qquad \left(0<v'<1\right). \tag{4.15}$$

Before the evaluation of these integrals, let us first prove a simple lemma. Recall that $r=1-X^{m-1}/(\Pi_{m-1}(X)(m-1)!)$. Set $r_0:=(m-1)/X$.

Lemma 4.6. *Let $Y(t)=t^{1-m}e^{tX}$. Then Y satisfies*

$$Y(r)\leqslant\begin{cases} Y(r_0)\left(1+O\left(\dfrac{m}{XM^2}\right)\right), & \text{if } 1\leqslant m\leqslant X-M\sqrt{X};\\[2mm] Y(r_0)\left(1+O\left(\dfrac{1+|y|^3}{\sqrt{X}}\right)\right), & \text{if } y=\dfrac{m-1-X}{\sqrt{X}}=o(X^{1/6}).\end{cases}$$

Proof. First of all, observe that $Y'(t)=Y(t)(X-(m-1)/t)$, $t>0$. By the first mean value theorem, we have

$$\begin{aligned} Y(t) &= Y(r_0)\left(1-\frac{r_0-t}{Y(r_0)}Y'(r_0-\theta(r_0-t))\right) \qquad (0<\theta<1, 0<t\leqslant R)\\ &= Y(r_0)\left(1-\frac{Y(r_0-\theta(r_0-t))}{Y(r_0)}(r_0-t)\left(X-\frac{m-1}{t}\right)\right)\\ &\leqslant Y(r_0)\left(1-(r_0-t)\left(X-\frac{m-1}{t}\right)\right), \end{aligned}$$

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since Y attains the minimum at $r_0 = (m-1)/X$. Thus

$$Y(r) \leqslant Y(r_0)\left(1 + \frac{X}{r}(r_0 - r)^2\right).$$

The lemma follows from the first two estimates in (3.4). □

We now show that the principal contribution to N comes from I_5.

Lemma 4.7. *Let r, F, λ be defined as in Theorem 2.6. Then I_5 satisfies*

$$I_5 = F(r)\frac{\rho^{m-n}}{n-m}\Pi_{m-1}(X)\left(1 + O\left(\min\left\{X^{-1}, X^{-1/2}e^{-XQ(\lambda)}\right\}\right)\right),$$

uniformly for $m \geqslant 1$ and $n - m \to \infty$.

Proof. The proof uses again Selberg's idea that we mentioned in Lemma 3.1. We shall follow Balazard's proof with some simplifications [3, pp. 102–109].

First, as in the proof of Lemma 3.1, expand $h(u)$ at $u = r < 1$:

$$h(u) = h(r) + h'(r)(u-r) + (u-r)^2 f(u,r), \quad f(u,r) = \int_0^1 (1-t)h''(r + t(u-r))\,\mathrm{d}t.$$

Substituting this formula into I_5 and taking $\upsilon = r$, we find

$$I_5 = h(r)\Pi_{m-1}(X) + 0 + I_7,$$

where

$$I_7 = \frac{1}{2i\pi}\oint_{|u|=r}\frac{(u-r)^2}{1-u}f(u,r)u^{-m}e^{Xu}\,\mathrm{d}u.$$

We shall show that

$$I_7 \ll \Pi_{m-1}(X)\min\left\{X^{-1} + X^{-1/2}e^{-XQ(\lambda)}\right\}, \qquad (m \geqslant 1, n - m \to \infty).$$

To this aim, we divide the estimation into three cases.

1. $1 \leqslant m \leqslant X - M\sqrt{X}$. We have

$$\begin{aligned} I_7 &\ll \frac{r^{3-m}e^{rX}}{1-r}\int_{-\pi}^{\pi} t^2 e^{-rX(1-\cos t)}\,\mathrm{d}t < \frac{r^{-m+3/2}e^{rX}}{1-r}X^{-3/2} \\ &= \Pi_{m-1}(X)\frac{(m-1)!}{X^{m-1}}r^{-m+3/2}e^{rX}X^{-3/2}. \end{aligned}$$

Applying Lemma 4.6, we have

$$\begin{aligned} I_7 &\ll \Pi_{m-1}(X)\frac{(m-1)!}{X^{m-1}}Y(r_0)r^{1/2}X^{-3/2} \\ &\ll \Pi_{m-1}(X)\sqrt{m-1}\,r^{1/2}X^{-3/2} \ll \Pi_{m-1}(X)X^{-1}, \end{aligned}$$

since $r \sim r_0 = (m-1)/X$ in this case; *cf.* (3.4).

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2. $m \geqslant X + M\sqrt{X}$. By Cauchy's residue theorem,

$$I_7 = (1-r)^2 f(1,r)e^X + \frac{1}{2i\pi}\oint_{|u|=\lambda} \frac{(u-r)^2}{1-u} f(u,r)u^{-m}e^{Xu}\,\mathrm{d}u.$$

First of all, by Stirling's formula, we have

$$\begin{aligned} 1-r &= \frac{X^{m-1}}{\Pi_{m-1}(X)(m-1)!} \ll e^{-X}\frac{(eX)^{m-1}}{(m-1)^{m-1/2}} \\ &= e^{-XQ(r_0)}(r_0X)^{-1/2} \ll e^{-XQ(\lambda)}(r_0X)^{-1/2}. \end{aligned}$$

Thus

$$(1-r)^2 f(1,r)e^X \ll \Pi_{m-1}(X)\frac{e^{-XQ(\lambda)}}{r_0X}.$$

Now, for $|u| = \lambda$,

$$\left|\frac{(u-r)^2}{1-u}\right| \leqslant |u-r|\left(1+\frac{1-r}{\lambda-1}\right) \ll 1,$$

it follows that

$$\begin{aligned} \frac{1}{2i\pi}\oint_{|u|=\lambda} \frac{(u-r)^2}{1-u} f(u,r)u^{-m}e^{Xu}\,\mathrm{d}u &\ll e^{\lambda X}\lambda^{m-1}X^{-1/2} \\ &\ll e^{X-XQ(\lambda)}X^{-1/2} \sim \Pi_{m-1}(X)e^{-XQ(\lambda)}X^{-1/2}, \end{aligned}$$

in view of (3.4).

3. $|m-X| \leqslant M\sqrt{X}$. Using the elementary inequality

$$|u-r|^2 = |(u-r)(u-1)+(u-r)(1-r)| \leqslant 2|u-r||1-u|,$$

and Lemma 4.6, we have

$$\begin{aligned} I_7 &\ll r^{2-m}e^{rX}\int_{-\pi}^{\pi} |t|e^{-rX(1-\cos t)}\,\mathrm{d}t \ll r^{2-m}e^{rX}X^{-1} \\ &\ll \Pi_{m-1}(X)r^{2-m}e^{(r-1)X}X^{-1} \ll \Pi_{m-1}(X)r_0^{2-m}e^{r_0X}X^{-1} \\ &\ll \Pi_{m-1}(X)X^{-1}. \end{aligned}$$

This completes the proof of Lemma 4.7. □

Following similar but simpler analyses as above with the aid of (4.14), we can show that I_6 (defined in (4.15)) is negligible comparing with I_7.

Lemma 4.8. *For $m \geqslant 1$ and $n - m \to \infty$, we have*

$$I_6 \ll \Pi_{m-1}(X)\min\left\{X^{-1}, X^{-1/2}e^{-XQ(\lambda)}\right\}.$$

Proof. Omitted. □

By collecting the results of the lemmas in this section, we complete the proof of Theorem 2.6.

5. Applications

In this section, we indicate some applications of our theorems. For further examples, see Flajolet and Soria [11] and Knopfmacher [24, 25]. As in previous sections, we still use the same group of letters to represent the structures ($\mathscr{C}$), the generating function ($C(z)$), and the counting function (c_n).

Example 1: Random mapping patterns. By random mapping (Kolchin [26]), we mean a random singled-valued mapping of the set $\{1, 2, 3, \ldots, n\}$ into itself. These structures have been extensively studied in the literature due both to their intrinsic interest and to their wide applications to many different fields (*cf.* Kolchin [26], Flajolet and Odlyzko [9], Donnelly, Ewens and Padmadisastra [7] and *Advances in Applied Probability* [1]).

Two mappings μ_1 and μ_2 are said to be equivalent if there exists a permutation π of $\{1, 2, 3, \ldots, n\}$ such that $\mu_1(i) = j$ iff $\mu_2(\pi(i)) = \pi(j)$ for all pairs (i, j). *Random mapping patterns* are equivalence classes of mapping functions. Structurally, they are multisets of cycles of rooted unlabelled trees with bivariate generating function [11]

$$P(w, z) = \prod_{j\geqslant 1} \left(1 - wz^j\right)^{-c_j},$$

where w marks the number of connected components and

$$\begin{aligned} C(z) &= \sum_{j\geqslant 1} c_j z^j = \sum_{j\geqslant 1} \frac{\varphi(j)}{j} \log \frac{1}{1 - S(z^j)} \\ &= z + 2z^2 + 4z^3 + 9z^4 + 20z^5 + 51z^6 + 125z^7 + 329z^8 + 862z^9 + \cdots, \end{aligned}$$

$\varphi(j)$ being Euler's totient function and S satisfying

$$\begin{aligned} S(z) &= z \exp\left(\sum_{j\geqslant 1} \frac{S(z^j)}{j}\right) \\ &= z + z^2 + 2z^3 + 4z^4 + 9z^5 + 20z^6 + 48z^7 + 115z^8 + 286z^9 + \cdots. \end{aligned}$$

It is known (*cf.* [6, 11]) that C is logarithmic with $\alpha = 1/2$, $\beta = 2$, $0 < \rho < 1$, and

$$\kappa = -\log \sigma + \sum_{j\geqslant 2} \frac{\varphi(j)}{j} \log \frac{1}{1 - S(\rho^j)},$$

where σ occurs in the local expansion of S at $z = \rho$:

$$S(z) = 1 - \sigma\sqrt{1 - z/\rho} + \sigma_1(1 - z/\rho) + \cdots.$$

Since $c_1 = 1$ and $0 < \rho < 1$, all our theorems apply and the results so obtained are new.

Example 2: Antisymmetric necklaces. A necklace containing n black and n white beads is said to be antisymmetric (Salvy [32, pp. 94–95]) if the diametrically opposed bead of a black bead is white. The generating function of the number of such necklaces satisfies

$$\begin{aligned} C(z) &= \sum_{j\geqslant 1} c_j z^j = \frac{1}{2} \sum_{j\geqslant 1} \frac{\varphi(2j-1)}{2j-1} \log \frac{1}{1 - 2z^{2j-1}} \\ &= z + z^2 + 2z^3 + 2z^4 + 4z^5 + 6z^6 + 10z^7 + 16z^8 + 30z^9 + \cdots, \end{aligned}$$

https://doi.org/10.1017/S0963548397003295 Published online by Cambridge University Press

where c_n denotes the number of antisymmetric necklaces with n black and n white beads. Forming the multiset construction of such objects with w marking the number of antisymmetric necklaces, we find the bivariate generating function

$$P(w,z) = \prod_{j\geqslant 1} \left(1 - wz^j\right)^{-c_j}.$$

Obviously, C is logarithmic with $\alpha = 1/2$, $\beta = 1$, $\rho = 1/2$ and

$$\kappa = \frac{1}{2} \sum_{j\geqslant 2} \frac{\varphi(2j-1)}{2j-1} \log \frac{1}{1-4^{1-j}},$$

and all our results again apply.

Example 3: Factorizations of characteristic polynomials. Fix a finite field F_q and consider the total number of irreducible factors $\Omega_n(T)$ (multiplicities counted) in the (polynomial) factorization of the characteristic polynomial of a random matrix $T \in GL_n(F_q)$, where a uniform probability distribution on $GL_n(F_q)$ is assumed. The bivariate generating function of Ω_n satisfies [34, 14]

$$\begin{aligned} P(w,z) &= \sum_{n,m} \Pr\{\Omega_n = m\} w^m z^n = \prod_{k\geqslant 1} \left(1 + \sum_{j\geqslant 1} \frac{q^{kj(j-1)/2} w^j z^{kj}}{(q^k-1)(q^{2k}-1)\cdots(q^{kj}-1)} \right)^{e(k)} \\ &= \exp\left(\sum_{k,j\geqslant 1} \frac{e(k) w^j z^{kj}}{j(q^{kj}-1)} \right), \end{aligned}$$

where

$$\begin{aligned} E(z) &= \sum_{k\geqslant 1} e(k) z^k = \sum_{j\geqslant 1} \frac{\mu(j)}{j} \log \frac{1}{1-qz^j} - z \\ &= (q-1)z + \frac{q(q-1)}{2} z^2 + \frac{q(q^2-1)}{3} z^3 + \frac{q^2(q^2-1)}{4} z^4 + \cdots, \end{aligned}$$

$\mu(j)$ being Möbius function. We observe that although these generating functions do not bear the same form as that we studied in this paper, there are essentially few differences. For we can write (*cf.* Stong [34] and Goh and Schmutz [14])

$$P(w,z) = e^{wE(z)} Q(w,z),$$

where E is logarithmic and the function $z \mapsto Q(w,z)$ has a larger radius of convergence. In particular, when $q = 2$, the asymptotic behaviour of $\Pr\{\Omega_n = m\}$ can be fully characterized by the same types of results that we derived in previous sections. Note that Theorem 2.1 does not require that $c_1 = 1$.

6. Concluding remarks

First, the same underlying principle of the proof techniques used in this paper can be applied, for example, to the case when C satisfies

$$C(z) = \frac{\alpha}{1 - z/\rho} + H(z), \qquad (\alpha > 0, 0 < \rho < 1),$$

where H is analytic in $|z| \leqslant \rho + \varepsilon$, $\varepsilon > 0$. Such a case, besides its combinatorial source (Hwang [21, Ch. 6]), also occurs in the problem (Knopfmacher, Knopfmacher and Warlimont [23]) of 'factorisatio numerorum' in (additive) arithmetical semigroups (when we consider the distribution of the number of components). There appears a convolution law of the Bessel and the negative binomial distributions for the number of components (*cf.* Hwang [21, Ch. 6]).

As we have mentioned, at the expense of more computations, we can treat the case $c_1 > 1$ by the same set of analytic tools. This case occurs, for example, in counting the number of irreducible factors in the factorization of a random monic polynomial over a finite field (Flajolet and Soria [11]).

The case when $\rho = 1$, which occurs ubiquitously in integer partition problems (Andrews [2]), is more involved and requires more delicate analysis (*cf.* Richmond [31] and the references therein). Our tools (with singularity analysis replaced by the saddle-point method) can still be applied but with less satisfactory results than those in this paper (*cf.* Hwang [19]).

We can impose further arithmetical constraints on either the number of components or the sizes of the components, these are systematically studied in Hwang [21, Ch. 5].

Another frequently encountered exponential scheme [11, 12] is $P(w,z) = e^{wC(z)}$, where C is logarithmic. The asymptotic behaviour of the coefficient

$$[w^m z^n]P(w,z) = \frac{1}{m!}[z^n]C^m(z),$$

when $n \to \infty$ and $1 \leqslant m \leqslant M \log n$, can be obtained by the same method used in the proof of Theorem 2.1, namely, singularity analysis and Selberg's method (*cf.* Hwang [17]). The case when $m \geqslant M \log n$ requires the use of the saddle-point method and is discussed in detail in Drmota and Soria [8].

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