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PERIODS OF DRINFELD MODULES AND LOCAL SHTUKAS WITH COMPLEX MULTIPLICATION

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Abstract Colmez [Périodes des variétés abéliennes a multiplication complexe, *Ann. of Math. (2)* **138**(3) (1993), 625–683; available at http://www.math.jussieu.fr/~colmez] conjectured a product formula for periods of abelian varieties over number fields with complex multiplication and proved it in some cases. His conjecture is equivalent to a formula for the Faltings height of CM abelian varieties in terms of the logarithmic derivatives at $s = 0$ of certain Artin L-functions. In a series of articles we investigate the analog of Colmez's theory in the arithmetic of function fields. There abelian varieties are replaced by Drinfeld modules and their higher-dimensional generalizations, so-called A-motives. In the present article we prove the product formula for the Carlitz module and we compute the valuations of the periods of a CM A-motive at all finite places in terms of Artin L-series. The latter is achieved by investigating the local shtukas associated with the A-motive.

Keywords: Drinfeld modules; local shtukas; complex multiplication; Artin L-series

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1. Introduction

In [10] Colmez considers product formulas for periods of abelian varieties. Let X be an abelian variety defined over a number field K with complex multiplication by the ring of integers in a CM-field E and of CM-type Φ. Let $\mathbb{Q}^{\text{alg}}$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$, let $H_E := \text{Hom}_{\mathbb{Q}}(E, \mathbb{Q}^{\text{alg}})$ be the set of all ring homomorphisms $E \hookrightarrow \mathbb{Q}^{\text{alg}}$ and assume that K contains $\psi(E)$ for every $\psi \in H_E$. For a $\psi \in H_E$ let $\omega_\psi \in \text{H}^1_{\text{dR}}(X, K)$ be a non-zero cohomology class such that $a^*\omega_\psi = \psi(a) \cdot \omega_\psi$ for all $a \in E$. For every embedding $\eta\colon K \hookrightarrow \mathbb{Q}^{\text{alg}}$, let X^η and ω_ψ^η be deduced from X and ω_ψ by base extension. Let $(u_\eta)_\eta \in \prod_{\eta \in H_K} \text{H}_1(X^\eta(\mathbb{C}), \mathbb{Z})$ be a family of cycles compatible with complex conjugation.

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Let v be a place of $\mathbb{Q}$. If $v = \infty$ the de Rham isomorphism between Betti and de Rham cohomology yields a complex number $\int_{u_\eta} \omega_\psi^\eta$ and its absolute value $\left|\int_{u_\eta} \omega_\psi^\eta\right|_\infty \in \mathbb{R}$. If v corresponds to a prime number $p \in \mathbb{Z}$, we fix an inclusion $\mathbb{Q}^{\mathrm{alg}} \hookrightarrow \mathbb{Q}_p^{\mathrm{alg}}$. With this data Colmez [10] associates a period $\int_{u_\eta} \omega_\psi^\eta$ in Fontaine's p-adic period field $\mathbf{B}_{\mathrm{dR}}$ and an absolute value $\left|\int_{u_\eta} \omega_\psi^\eta\right|_v \in \mathbb{R}$. He considers the product $\prod_v \prod_{\eta \in H_K} \left|\int_{u_\eta} \omega_\psi^\eta\right|_v$ and (after some modifications) conjectures that this product evaluates to 1; see [10, Conjecture 0.1] for the precise formulation. This conjecture is equivalent to a conjectural formula for the Faltings height of a CM abelian variety in terms of the logarithmic derivatives at $s = 0$ of certain Artin L-functions. Colmez proves the conjectures when E is an abelian extension of $\mathbb{Q}$. On the way, he computes $\prod_{\eta \in H_K} \left|\int_{u_\eta} \omega_\psi^\eta\right|_v$ at a finite place v in terms of the local factor at v of the Artin L-series associated with an Artin character $a^0_{E,\psi,\Phi} \colon \mathrm{Gal}(\mathbb{Q}^{\mathrm{alg}}/\mathbb{Q}) \to \mathbb{C}$ that only depends on E, ψ and Φ but not on X and v; see [10, Théorème I.3.15]. There has been further progress on Colmez's conjecture by Obus [21], Yang [31], Andreatta, Goren, Howard and Madapusi Pera [2], Yuan and Zhang [32], Barquero-Sanchez and Masri [4] and others.

Our goal in this article is to develop the analog of Colmez's theory in the 'Arithmetic of function fields'. Here abelian varieties are replaced by Drinfeld modules [11, 14] and their higher- dimensional generalizations, so-called A-motives, which also generalize Anderson's *t-motives* [1]. To define them let $\mathbb{F}_q$ be a finite field with q elements, let C be a smooth projective, geometrically irreducible curve over $\mathbb{F}_q$, let $\infty \in C$ be a fixed closed point and let $A := \Gamma(C \setminus \{\infty\}, \mathcal{O}_C)$ be the ring of regular functions on C outside ∞. Let Q be the fraction field of A and let K be a finite field extension of Q contained in a fixed algebraic closure Q^{alg} of Q. We write $A_K := A \otimes_{\mathbb{F}_q} K$ and consider the endomorphism $\sigma := \mathrm{id}_A \otimes \mathrm{Frob}_{q,K}$ of A_K, where $\mathrm{Frob}_{q,K}(b) = b^q$ for $b \in K$. For an A_K-module M we set $\sigma^* M := M \otimes_{A_K,\sigma} A_K$ and for a homomorphism $f \colon M \to N$ of A_K-modules we set $\sigma^* f := f \otimes \mathrm{id}_{A_K} \colon \sigma^* M \to \sigma^* N$. Let $\gamma \colon A \to K$ be the inclusion $A \subset Q \subset K$, and set $\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a) \colon a \in A) \subset A_K$. Then γ can be recovered as the homomorphism $A \to A_K/\mathcal{J} = K$.

Definition 1.1. An *A-motive of rank r over K* is a pair $\underline{M} = (M, \tau_M)$ consisting of a locally free A_K-module M of rank r and an isomorphism of the associated sheaves $\tau_M \colon \sigma^* M|_{\mathrm{Spec}\, A_K \setminus \mathrm{V}(\mathcal{J})} \xrightarrow{\sim} M|_{\mathrm{Spec}\, A_K \setminus \mathrm{V}(\mathcal{J})}$ outside $\mathrm{V}(\mathcal{J}) \subset \mathrm{Spec}\, A_K$. We write $\mathrm{rk}\, \underline{M} := r$. A *morphism* between A-motives $f \colon (M, \tau_M) \to (N, \tau_N)$ is an A_K-homomorphism between the underlying A_K-modules $f \colon M \to N$ with $f \circ \tau_M = \tau_N \circ \sigma^* f$.

Let us first give a rough sketch of the function field analog of Colmez's conjecture, before we explain more details and our main results later in this introduction. An A-motive has various (co-)homology realizations, for example a *de Rham realization* $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$, and if it is uniformizable also a *Betti realization* $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, A)$. For every *place v of Q*, that is a closed point $v \in C$ there is a comparison isomorphism between the Betti and de Rham cohomology of $\underline{M}$, which for $\omega \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$ and $u \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, A)$ is given by a pairing $\langle \omega, u \rangle_v$ and allows to define the absolute value $\left|\int_u \omega\right|_v := \left|\langle \omega, u \rangle_v\right|_v \in \mathbb{R}$. Now we say that $\underline{M}$ has *complex multiplication* if $\mathrm{QEnd}_K(\underline{M}) := \mathrm{End}_K(\underline{M}) \otimes_A Q$ contains a commutative,

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semi-simple Q-algebra E of dimension $\dim_Q E = \operatorname{rk} \underline{M}$. Here semi-simple means that E is a product of fields and we do not assume that E is itself a field. Let $\underline{M}$ be a uniformizable A-motive over a finite Galois extension $K \subset Q^{\mathrm{alg}}$ of Q, which has complex multiplication by a *separable* Q-algebra E. Let $0 \neq \omega_\psi \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$ satisfy $a^*\omega_\psi = \psi(a) \cdot \omega_\psi$ for all $a \in E$, where $\psi : E \to K$ is a Q-homomorphism. Then in Theorem 5.24 and Corollary 5.25 we, for all finite places v, compute $\left|\int_u \omega_\psi\right|_v$ and its average over all Q-automorphisms of K in terms of the local factor at v of an Artin L-series. The question analogous to [10] is then, whether one can make sense of the product $\prod_v \left|\int_u \omega_\psi\right|_v$ over all places v including ∞, and whether this product evaluates to 1.

After this vague sketch let us give more details and precise definitions in order to formulate our main results. We start by introducing the cohomology realizations of an A-motive $\underline{M}$ over K. First, there is the *de Rham realization* $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K) := \sigma^*M/\mathcal{J} \cdot \sigma^*M$ and for each maximal ideal $v \subset A$ a *v-adic étale realization* $\mathrm{H}^1_v(\underline{M}, A_v)$ where A_v is the v-adic completion of A; see Definition 3.3 below. We let Q_v be the fraction field of A_v, and we let Q_∞ be the ∞-adic completion of Q and $\mathbb{C}_\infty$ be the completion of a fixed algebraic closure of Q_∞. We fix a Q-embedding $Q^{\mathrm{alg}} \hookrightarrow \mathbb{C}_\infty$ and consider the base extension of $\underline{M}$ to $\mathbb{C}_\infty$. There is a notion of $\underline{M}$ being *uniformizable* and a uniformizable $\underline{M}$ has a *Betti realization* $\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A)$; see [17, § 3.5]. These realizations are related by period isomorphisms

$$h_{\mathrm{Betti},\, v} \colon \mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A) \otimes_A A_v \xrightarrow{\ \sim\ } \mathrm{H}^1_v(\underline{M}, A_v) \quad \text{and}$$

$$h_{\mathrm{Betti},\, \mathrm{dR}} \colon \mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A) \otimes_A \mathbb{C}_\infty \xrightarrow{\ \sim\ } \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K) \otimes_K \mathbb{C}_\infty;$$

see [17, Theorem 3.23]. Also for every place v of Q, let $\mathbb{F}_v$ be its residue field and set $q_v := \#\mathbb{F}_v = q^{[\mathbb{F}_v:\mathbb{F}_q]}$. Let $z := z_v \in Q$ be a uniformizing parameter at v. Then there is a canonical isomorphism $A_v = \mathbb{F}_v[\![z_v]\!]$. Let $\zeta := \zeta_v := \gamma(z_v)$ denote the image of z_v in K. We simply write z, respectively ζ for the elements $z \otimes 1$, respectively $1 \otimes \zeta$ of $Q \otimes_{\mathbb{F}_q} K$. Then the power series ring $K[\![z - \zeta]\!]$ in the 'variable' $z - \zeta$ is canonically isomorphic to the completion of the local ring of $C_K := C \times_{\mathbb{F}_q} K$ at $\mathrm{V}(\mathcal{J})$; see [17, Lemmas 1.2 and 1.3], and thus independent of v. We always consider the embedding $Q \hookrightarrow K[\![z - \zeta]\!]$ given by $z \mapsto z = \zeta + (z - \zeta)$. The de Rham realization lifts to

$$\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z - \zeta]\!]) := \sigma^*M \otimes_{A_K} K[\![z - \zeta]\!],$$

which is the analog of the (conjectural) q-de Rham cohomology of Bhatt, Morrow and Scholze [5, 6, 24], and the vector space $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z - \zeta]\!])[\frac{1}{z-\zeta}]$ over the field $K(\!(z - \zeta)\!) := K[\![z - \zeta]\!][\frac{1}{z-\zeta}]$ contains the $K[\![z - \zeta]\!]$-lattice $\mathfrak{q}^{\underline{M}} := \tau_M^{-1}(M \otimes_{A_K} K[\![z - \zeta]\!])$, which is called the *Hodge–Pink lattice of $\underline{M}$* and is the analog of the Hodge-filtration of an abelian variety; see [18, Remark 5.13].

If $v \neq \infty$ we also fix a Q-embedding of Q^{alg} into a fixed algebraic closure Q_v^{alg} of Q_v and we let $\mathbb{C}_v$ be the v-adic completion of Q_v^{alg}. Again we denote the image of z_v in Q_v^{alg} and $\mathbb{C}_v$ by ζ_v. We let $K_v \subset Q_v^{\mathrm{alg}}$ be the induced completion of K and we let R be its valuation ring. There is a period isomorphism by [18, Remark 4.16]

$$h_{v,\mathrm{dR}} \colon \mathrm{H}^1_v(\underline{M}, A_v) \otimes_{A_v} \mathbb{C}_v(\!(z_v - \zeta_v)\!) \xrightarrow{\ \sim\ } \mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, K[\![z_v - \zeta_v]\!]\big) \otimes_{K[\![z_v - \zeta_v]\!]} \mathbb{C}_v(\!(z_v - \zeta_v)\!).$$

The field $\mathbb{C}_v(\!(z_v - \zeta_v)\!)$ is the analog of Fontaine's p-adic period field $\mathbf{B}_{\mathrm{dR}}$; see [18, Remark 4.17].

Let $\underline{M}$ have complex multiplication by a commutative, semi-simple Q-algebra E of dimension $\dim_Q E = \operatorname{rk}\underline{M}$. Let $\mathcal{O}_E$ be the integral closure of A in E. It is a locally free A-module of $\operatorname{rk}_A \mathcal{O}_E = \dim_Q E$. We let $H_E := \operatorname{Hom}_Q(E, Q^{\mathrm{alg}})$ be the set of Q-homomorphisms $\psi\colon E \to Q^{\mathrm{alg}}$ and we assume that K contains $\psi(E)$ for every $\psi \in H_E$. Then by Lemma A.3 in the appendix there is a decomposition

$$E \otimes_Q K[\![z-\zeta]\!] = \prod_{\psi\in H_E} K[\![y_\psi - \psi(y_\psi)]\!],$$

where y_ψ is a uniformizer at a place of E such that $\psi(y_\psi) \neq 0$. Again by [17, Lemmas 1.2 and 1.3] the factors are obtained as the completion of $\mathcal{O}_E \otimes_A A_K = \mathcal{O}_E \otimes_{\mathbb{F}_q} K$ along the kernels $(a\otimes 1 - 1\otimes\psi(a)\colon a \in \mathcal{O}_E)$ of the homomorphisms $\psi\otimes\mathrm{id}_K\colon \mathcal{O}_E\otimes_{\mathbb{F}_q} K \to K$ for $\psi \in H_E$. Correspondingly $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z-\zeta]\!])$ decomposes into eigenspaces

$$\mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!]) := \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z-\zeta]\!]) \otimes_{E\otimes_Q K[\![z-\zeta]\!]} K[\![y_\psi - \psi(y_\psi)]\!]$$

each of which is free of rank 1 over $K[\![y_\psi - \psi(y_\psi)]\!]$. There are integers d_ψ such that the Hodge–Pink lattice is $\mathfrak{q}^{\underline{M}} = \prod_\psi (y_\psi - \psi(y_\psi))^{-d_\psi}\, \mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])$. The tuple $\Phi := (d_\psi)_{\psi\in H_E}$ is the *CM-type* of $\underline{M}$.

We fix elements $u \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q) := \operatorname{Hom}_A\big(\mathrm{H}^1_{\mathrm{Betti}}(\underline{M}, A), Q\big)$ and $\omega \in \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K[\![z-\zeta]\!])$. Then we can define

$$\langle\omega, u\rangle_\infty := u\otimes\mathrm{id}_{\mathbb{C}_\infty}\big(h^{-1}_{\mathrm{Betti,dR}}(\omega \bmod z-\zeta)\big) \in \mathbb{C}_\infty \quad\text{and} \tag{1.1}$$

$$\left|\int_u \omega\right|_\infty := |\langle\omega, u\rangle_\infty|_\infty \in \mathbb{R}, \tag{1.2}$$

where $|\,.\,|_v$ is the normalized absolute value on $\mathbb{C}_v$ with $|\zeta_v|_v = (\#\mathbb{F}_v)^{-1} = q_v^{-1}$ for every place v. We also consider the valuation $v\colon \mathbb{C}_v^\times \to \mathbb{Q}$ on $\mathbb{C}_v$ with $v(\zeta_v) = 1$. The expressions in (1.1) and (1.2) only depend on the image of ω in $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$. Also at a finite place $v \neq \infty$ of Q we consider on elements $x \neq 0$ of the discretely valued field $\mathbb{C}_v(\!(z_v-\zeta_v)\!)$ the valuation $\hat v(x) := \operatorname{ord}_{z_v-\zeta_v}(x)$, and in addition we define

$$\begin{aligned}
&|x|_v := \big|\big((z_v-\zeta_v)^{-\hat v(x)}\cdot x\big) \bmod z_v-\zeta_v\big|_v \quad\text{and}\\
&v(x) := -\log|x|_v/\log q_v \quad\text{induced from}\\
&\big((z_v-\zeta_v)^{-\hat v(x)}\cdot x\big) \bmod z_v-\zeta_v \in \mathbb{C}_v.
\end{aligned}$$

Note that $|x|_v$ and $v(x)$ are not a norm, respectively a valuation, because they do not satisfy the triangle inequality. The value $|x|_v$ does not depend on the choice of the uniformizer z_v of A_v, because if $\tilde z_v = \sum_{n=0}^\infty b_n z_v^n =: f(z_v)$ with $b_n \in \mathbb{F}_v$ is another uniformizer and $\tilde\zeta_v = f(\zeta_v)$, then $\frac{\tilde z_v - \tilde\zeta_v}{z_v-\zeta_v} \equiv f'(\zeta_v) \bmod z_v-\zeta_v$ in $\mathcal{O}_{\mathbb{C}_v}[\![z_v]\!] = \mathbb{F}_v[\![z_v]\!]\widehat{\otimes}_{\mathbb{F}_v,\gamma}\mathcal{O}_{\mathbb{C}_v}$ by Lemma A.1 in the appendix and $f'(\zeta_v) \in \mathbb{F}_v[\![\zeta_v]\!]^\times$ with inverse $\frac{dz_v}{d\tilde z_v}\big|_{\tilde z_v=\tilde\zeta_v}$. We define

$$\langle\omega, u\rangle_v := u\otimes_{\mathbb{C}_v(\!(z_v-\zeta_v)\!)}\big(h^{-1}_{\mathrm{Betti},v}\circ h^{-1}_{v,\mathrm{dR}}(\omega)\big) \in \mathbb{C}_v(\!(z_v-\zeta_v)\!) \quad\text{and} \tag{1.3}$$

$$\left|\int_u\omega\right|_v := |\langle\omega,u\rangle_v|_v := \big|\big((z_v-\zeta_v)^{-\hat v(\langle\omega,u\rangle_v)}\cdot\langle\omega,u\rangle_v\big) \bmod z_v-\zeta_v\big|_v \in \mathbb{R}. \tag{1.4}$$

If E is separable over Q and if $\omega \in \mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])$ has non-zero image in $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$, then we will show in Theorem 5.24 below that the absolute value (1.4) only depends on that image.

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With these definitions we can now consider the product $\prod_v \big|\int_u \omega\big|_v$, or equivalently its logarithm $\log \prod_v \big|\int_u \omega\big|_v = -\sum_v v\big(\int_u \omega\big) \log q_v$. Like in Colmez's theory, these products or sums do not converge and one has to give a convergent interpretation to their finite parts $\prod_{v\neq\infty} \big|\int_u \omega\big|_v$, respectively $-\sum_{v\neq\infty} v\big(\int_u \omega\big) \log q_v$; see Convention 1.4 below. To formulate the convention we make the following

Definition 1.2. For $F = Q$ or $F = Q_v$ let F^{sep} be the separable closure of F in F^{alg} and let $\mathscr{G}_F := \operatorname{Gal}(F^{\text{sep}}/F)$. For a finite field extension F' of F let $H_{F'} := \operatorname{Hom}_F(F', F^{\text{alg}})$ be the set of F-homomorphisms $\psi\colon F' \to F^{\text{alg}}$. Let $\mathcal{C}(\mathscr{G}_F, \mathbb{Q})$ be the $\mathbb{Q}$-vector space of locally constant functions $a\colon \mathscr{G}_F \to \mathbb{Q}$ and let $\mathcal{C}^0(\mathscr{G}_F, \mathbb{Q})$ be the subspace of those functions which are constant on conjugacy classes, that is, which satisfy $a(h^{-1}gh) = a(g)$ for all $g, h \in \mathscr{G}_F$. Then the $\mathbb{C}$-vector space $\mathcal{C}^0(\mathscr{G}_F, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is spanned by the traces of representations $\rho\colon \mathscr{G}_F \to \operatorname{GL}_n(\mathbb{C})$ with open kernel for varying n by [26, § 2.5, Theorem 6]. Via the fixed embedding $Q^{\text{sep}} \hookrightarrow Q_v^{\text{sep}}$ we consider the induced inclusion $\mathscr{G}_{Q_v} \subset \mathscr{G}_Q$ and morphism $\mathcal{C}(\mathscr{G}_Q, \mathbb{Q}) \to \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$. If χ is the trace of a representation $\rho\colon \mathscr{G}_Q \to \operatorname{GL}_n(\mathbb{C})$ with open kernel we let $L(\chi, s) := \prod_{\text{all } v} L_v(\chi, s)$, respectively $L^\infty(\chi, s) := \prod_{v\neq\infty} L_v(\chi, s)$ be the Artin L-function of ρ, respectively without the factor at ∞. It only depends on χ and converges for all $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 1$; see [22, pp. 126ff]. We also set

$$Z(\chi, s) := \frac{\frac{d}{ds} L(\chi, s)}{L(\chi, s)} = -\sum_{\text{all } v} Z_v(\chi, s) \log q_v \quad \text{and} \tag{1.5}$$

$$Z^\infty(\chi, s) := \frac{\frac{d}{ds} L^\infty(\chi, s)}{L^\infty(\chi, s)} = -\sum_{v\neq\infty} Z_v(\chi, s) \log q_v \quad \text{with} \tag{1.6}$$

$$Z_v(\chi, s) := \frac{\frac{d}{ds} L_v(\chi, s)}{-L_v(\chi, s) \cdot \log q_v} = \frac{\frac{d}{dq_v^{-s}} L_v(\chi, s)}{q_v^s \cdot L_v(\chi, s)}. \tag{1.7}$$

Moreover, we let $\mathfrak{f}_\chi$ be the Artin conductor of χ. It is an effective divisor $\mathfrak{f}_\chi = \sum_v \mu_{\text{Art},v}(\chi) \cdot (v)$ on C; see [27, Chapter VI, §§ 2, 3], where $\mu_{\text{Art},v}(\chi)$ is denoted $f(\chi, v)$. We set

$$\mu_{\text{Art}}(\chi) := \deg(\mathfrak{f}_\chi) \log q := \sum_{\text{all } v} \mu_{\text{Art},v}(\chi)[\mathbb{F}_v : \mathbb{F}_q] \log q = \sum_{\text{all } v} \mu_{\text{Art},v}(\chi) \log q_v \quad \text{and} \tag{1.8}$$

$$\mu^\infty_{\text{Art}}(\chi) := \sum_{v\neq\infty} \mu_{\text{Art},v}(\chi) \log q_v. \tag{1.9}$$

In particular, only finitely many values $\mu_{\text{Art},v}(\chi)$ are non-zero. By linearity we extend $Z^\infty(\,.\,, s)$ and μ^∞_{Art} to all $a \in \mathcal{C}^0(\mathscr{G}_Q, \mathbb{Q})$ and $Z_v(\,.\,, s)$ and $\mu_{\text{Art},v}$ to all $a \in \mathcal{C}^0(\mathscr{G}_{Q_v}, \mathbb{Q})$. The map $Z_v(\,.\,, s)$ takes values in $\mathbb{Q}(q_v^{-s})$.

In terms of this definition we prove in this article a formula for $\big|\int_u \omega\big|_v$ with $v \neq \infty$ for a uniformizable A-motive $\underline{M}$ over K with complex multiplication by a semi-simple *separable* CM-algebra E of CM-type $\Phi = (d_\varphi)_{\varphi\in H_E}$ as follows. Let us assume that $\mathcal{O}_E \subset \operatorname{End}_K(\underline{M})$, that K is a finite Galois extension of Q which contains $\psi(E)$ for all $\psi \in H_E$, and that $\underline{M}$ has good reduction at all primes of K. (By unpublished results of Schindler [23] this is no restriction of generality, because for every A-motive $\underline{M}'$ with complex multiplication

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by a semi-simple separable CM-algebra E there is an A-motive $\underline{M}$ isogenous to $\underline{M}'$ such that the integral closure $\mathcal{O}_E$ of A in E is contained in $\mathrm{End}_K(\underline{M})$ and $\underline{M}'$ and $\underline{M}$ have good reduction everywhere after replacing K by a finite separable extension. Moreover, every A-motive over a field extension of Q with $\mathcal{O}_E \subset \mathrm{End}_K(\underline{M})$ is already defined over a finite separable extension K of Q). For $\psi \in H_E$ we define the functions

$$a_{E,\psi,\Phi}\colon \mathscr{G}_Q \to \mathbb{Z}, \quad g \mapsto d_{g\psi} \quad \text{and} \tag{1.10}$$

$$a^0_{E,\psi,\Phi}\colon \mathscr{G}_Q \to \mathbb{Q}, \quad g \mapsto \frac{1}{\#H_K} \sum_{\eta \in H_K} d_{\eta^{-1} g \eta \psi} \tag{1.11}$$

which factor through $\mathrm{Gal}(K/Q)$. In particular, $a_{E,\psi,\Phi} \in \mathcal{C}(\mathscr{G}_Q, \mathbb{Q})$ and $a^0_{E,\psi,\Phi} \in \mathcal{C}^0(\mathscr{G}_Q, \mathbb{Q})$ is independent of K.

Note that $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)$ is a free E-module of rank 1 by [7, Lemma 7.2] and that the eigenspace $\mathrm{H}^\psi(\underline{M}, K) := \mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])/(y_\psi - \psi(y_\psi))\,\mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])$ in $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$ of the character $\psi\colon E \to K$ is a K-vector space of dimension 1 by Proposition 4.9 below. For an E-generator $u \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)$ and a generator $\omega_\psi \in \mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])$ as $K[\![y_\psi - \psi(y_\psi)]\!]$-module we next define integers $v(\omega_\psi)$ and $v_\psi(u)$ for all $v \neq \infty$ which are all zero except for finitely many. Let $\mathcal{O}_{E_v} := \mathcal{O}_E \otimes_A A_v$ and let $c \in E_v := E \otimes_Q Q_v$ be such that $c^{-1}u$ is an $\mathcal{O}_{E_v}$-generator of $\mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, A) \otimes_A A_v$, which exists because $\mathcal{O}_{E_v}$ is a product of discrete valuation rings. Then c is unique up to multiplication by an element of $\mathcal{O}_{E_v}^\times$ and we set

$$v_\psi(u) := v(\psi(c)) \in \mathbb{Z}, \tag{1.12}$$

where we extend $\psi \in H_E$ by continuity to $\psi\colon E_v \to Q_v^{\mathrm{alg}}$. Also let $\underline{\mathcal{M}} = (\mathcal{M}, \tau_{\mathcal{M}})$ be an A-motive over $R := \mathcal{O}_{K_v}$ with good reduction and $\underline{\mathcal{M}} \otimes_{\mathcal{O}_{K_v}} K_v \cong \underline{M} \otimes_K K_v$; see Example 3.2. Then there is an element $x \in K_v^\times$, unique up to multiplication by $R^\times$, such that $x^{-1}\omega_\psi \bmod y_\psi - \psi(y_\psi)$ is an R-generator of the free R-module of rank one

$$\mathrm{H}^\psi(\underline{\mathcal{M}}, R) := \left\{\omega \in \mathrm{H}^1_{\mathrm{dR}}(\underline{\mathcal{M}}, R) := \sigma^*\mathcal{M} \otimes_{A_R, \gamma \otimes \mathrm{id}_R} R \colon [b]^*\omega = \psi(b) \cdot \omega \;\forall\, b \in \mathcal{O}_E\right\},$$

and we set

$$v(\omega_\psi) := v(x) \in \mathbb{Z}. \tag{1.13}$$

This value only depends on the image of ω_ψ in $\mathrm{H}^1_{\mathrm{dR}}(\underline{M}, K)$. It also does not depend on the choice of the model $\underline{\mathcal{M}}$ with good reduction, because all such models are isomorphic over R by [13, Proposition 2.13(ii)]. In this situation our first main result is the following

Theorem 1.3. *Let $\omega_\psi \in \mathrm{H}^\psi(\underline{M}, K[\![y_\psi - \psi(y_\psi)]\!])$ be a $K[\![y_\psi - \psi(y_\psi)]\!]$-generator. For every $\eta \in H_K$ let $\underline{M}^\eta$ and $\omega^\eta_\psi \in \mathrm{H}^{\eta\psi}(\underline{M}^\eta, K[\![y_{\eta\psi} - \eta\psi(y_{\eta\psi})]\!])$ be obtained by extension of scalars via η, and choose an E-generator $u_\eta \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}^\eta, Q)$. Then for every place $v \neq \infty$ of C we have*

$$\frac{1}{\#H_K} \sum_{\eta \in H_K} v\left(\int_{u_\eta} \omega^\eta_\psi\right) = Z_v(a^0_{E,\psi,\Phi}, 1) - \mu_{\mathrm{Art},v}(a^0_{E,\psi,\Phi}) - \frac{v(\mathfrak{d}_{\psi(E)/Q})}{[\psi(E) : Q]} + \frac{1}{\#H_K} \sum_{\eta \in H_K} \big(v(\omega^\eta_\psi) + v_{\eta\psi}(u_\eta)\big),$$

where $\mathfrak{d}_{\psi(E)/Q}$ is the discriminant of the field extension $\psi(E)/Q$.

We prove this theorem at the end of § 5 by using the *local shtuka* at v attached to $\underline{M}$. The latter is an analog of the Dieudonné-module of the p-divisible group attached to an abelian variety; see [16, § 3.2]. The theorem allows us to make the following convention which is the analog of [10, Convention 0].

Convention 1.4. Let $(x_v)_{v\neq\infty}$ be a tuple of complex numbers indexed by the finite places v of Q. We give a sense to the (divergent) series $\Sigma \overset{?}{=} \sum_{v\neq\infty} x_v$ in the following situation. We suppose that there exists an element $a \in \mathcal{C}^0(\mathscr{G}_Q, \mathbb{Q})$ such that $x_v = -Z_v(a, 1)\log q_v$ for all v except for finitely many. Then we let $a^* \in \mathcal{C}^0(\mathscr{G}_Q, \mathbb{Q})$ be defined by $a^*(g) := a(g^{-1})$. We further assume that $Z^\infty(a^*, s)$ does not have a pole at $s = 0$, and we define the limit of the series $\sum_{v\neq\infty} x_v$ as

$$\Sigma := -Z^\infty(a^*, 0) - \mu^\infty_{\mathrm{Art}}(a) + \sum_{v\neq\infty} (x_v + Z_v(a, 1)\log q_v) \tag{1.14}$$

inspired by Weil's [30, p. 82] functional equation

$$Z(\chi, 1-s) = -Z(\chi^*, s) - (2\cdot genus(C) - 2)\chi(1)\log q - \mu_{\mathrm{Art}}(\chi)$$

deprived of the summands at ∞, where the genus term is considered as belonging to ∞.

The Convention 1.4 and the Theorem 1.3 allow us to give a convergent interpretation to the sum $-\sum_v \sum_{\eta\in H_K} v\big(\int_{u_\eta} \omega^\eta_\psi\big)\log q_v$ and the product $\prod_v \prod_{\eta\in H_K} \big|\int_{u_\eta} \omega^\eta_\psi\big|_v$, and we can ask whether this product is 1. In § 2 we prove our second main result, namely that the answer to the question is 'yes' in the easiest case of the *Carlitz motive* which is related to the zeta function of $\mathbb{F}_q[t]$ and is the analog of the multiplicative group $\mathbb{G}_{m,\mathbb{Q}}$ considered by Colmez. For general CM A-motives we plan to address the question in a sequel to this article and also discuss its consequences for the Faltings height of CM A-motives similar to [10, Théorème 0.3 and Conjecture 0.4] and conditions under which $Z^\infty(a^*, s)$ does not have a pole at $s = 0$.

Let us describe the structure of this article. In § 3 we recall from [18] the definition of local shtukas, how to attach a local shtuka at $v \subset A$ to an A-motive $\underline{M}$ over L with good reduction, and we discuss its cohomology realizations. In § 4 we define the notions of complex multiplication and CM-type of a local shtuka, and in § 5 we compute the periods and their valuations of a local shtuka with complex multiplication, and we prove Theorem 1.3. Finally in Appendix A we prove the facts used above.

2. The Carlitz motive

Let $A = \mathbb{F}_q[t]$ and $C = \mathbb{P}^1_{\mathbb{F}_q}$. Let $K = \mathbb{F}_q(\vartheta)$ be the rational function field in the variable ϑ and let $\gamma : A \to K$ be given by $\gamma(t) = \vartheta$. We also set $z := z_\infty := \frac{1}{t}$ and $\zeta := \zeta_\infty := \frac{1}{\vartheta}$. It satisfies $|\zeta|_\infty = q^{-1} < 1$. The *Carlitz motive* over K is the A-motive

$$\underline{\mathcal{C}} = (K[t], \tau_{\mathcal{C}} = t - \vartheta)$$

which is associated with the Carlitz module; see [9] or [14, Chapter 3]. It has rank 1 and dimension 1, and complex multiplication by the ring of integers A in $E := Q$ with

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CM-type $\Phi = (d_{\text{id}})$, where $H_E = \{\text{id}\}$ and $d_{\text{id}} = 1$. As is well known, its cohomology satisfies $\mathrm{H}^1_{\text{dR}}(\underline{\mathcal{C}}, K[\![z-\zeta]\!]) = K[\![z-\zeta]\!]$ and $\mathrm{H}^1_{\text{Betti}}(\underline{\mathcal{C}}, A) = A \cdot \beta\ell_\zeta^-$, where $\beta \in \mathbb{C}_\infty$ satisfies $\beta^{q-1} = -\zeta$ and $\ell_\zeta^- := \prod_{i=0}^\infty (1-\zeta^{q^i}t)$; see for example [17, Example 3.34]. We denote the generator 1 of $\mathrm{H}^1_{\text{dR}}(\underline{\mathcal{C}}, K[\![z-\zeta]\!])$ by ω and we take $u \in \mathrm{H}_{1,\text{Betti}}(\underline{\mathcal{C}}, \mathbb{F}_q[t])$ as the generator which is dual to $\beta\ell_\zeta^- \in \mathrm{H}^1_{\text{Betti}}(\underline{\mathcal{C}}, \mathbb{F}_q[t])$. The de Rham isomorphism $h_{\text{Betti,dR}}$ sends $\beta\ell_\zeta^-$ to

$$\sigma^*(\beta\ell_\zeta^-) \cdot \omega = \beta^q \sigma^*(\ell_\zeta^-) \cdot \omega \in \mathrm{H}^1_{\text{dR}}(\underline{M}, \mathbb{C}_\infty[\![z-\zeta]\!]) = \mathbb{C}_\infty[\![z-\zeta]\!] \cdot \omega,$$

respectively to $\beta^q \sigma^*(\ell_\zeta^-)|_{t=\vartheta} \cdot \omega = \beta^q \prod_{i=1}^\infty (1-\zeta^{q^i-1}) \cdot \omega \in \mathrm{H}^1_{\text{dR}}(\underline{M}, \mathbb{C}_\infty) = \mathbb{C}_\infty \cdot \omega$. Here the coefficient $\beta^q \prod_{i=1}^\infty (1-\zeta^{q^i-1})$ is the function field analog of the complex number $(2i\pi)^{-1}$, the inverse of the period of the multiplicative group $\mathbb{G}_{m,\mathbb{Q}}$. We obtain

$$\left|\int_u \omega\right|_\infty = \left|\left(\beta^q \prod_{i=1}^\infty (1-\zeta^{q^i-1})\right)^{-1}\right|_\infty = |\beta|_\infty^{-q} = q^{q/(q-1)}.$$

At a finite place $v \subset \mathbb{F}_q[t]$ let $v = (z_v)$ and $\zeta_v = \gamma(z_v)$. Then $\mathrm{H}^1_v(\underline{M}, A_v) = A_v \cdot (\ell_{\zeta_v}^+)^{-1}$, where $\ell_{\zeta_v}^+ := \sum_{n=0}^\infty \ell_n z_v^n \in \mathbb{C}_v[\![z_v]\!]$ with $\ell_0^{q_v-1} = -\zeta_v$ and $\ell_n^{q_v} + \zeta_v \ell_n = \ell_{n-1}$; see [18, Example 4.19]. This implies $|\ell_n| = |\zeta_v|^{q_v^{-n}/(q_v-1)} < 1$. The v-adic comparison isomorphism $h_{v,\text{dR}}$ sends the generator $\ell_{\zeta_v}^+$ of $\mathrm{H}^1_v(\underline{M}, A_v)$ to

$$(z_v-\zeta_v)^{-1}(\ell_{\zeta_v}^+)^{-1} \cdot \omega \in \mathrm{H}^1_{\text{dR}}\big(\underline{M}, \mathbb{C}_v(\!(z_v-\zeta_v)\!)\big) = \mathbb{C}_v(\!(z_v-\zeta_v)\!) \cdot \omega,$$

where the coefficient of ω is the *v-adic Carlitz period* which has a pole of order one at $z_v = \zeta_v$. So $\langle \omega, u\rangle_v = (z_v-\zeta_v)\ell_{\zeta_v}^+$ has $\hat{v}(\langle \omega, u\rangle_v) = 1$ and

$$\left|\int_u \omega\right|_v = |\ell_{\zeta_v}^+ \bmod z_v-\zeta_v|_v = \left|\sum_{n=0}^\infty \ell_n \zeta_v^n\right|_v = |\ell_0|_v = q_v^{-1/(q_v-1)}.$$

So the product $\prod_v \left|\int_u \omega\right|_v$ of the norms at all places has logarithm

$$\log \prod_{\text{all } v} \left|\int_u \omega\right|_v = \log\left|\int_u \omega\right|_\infty + \log \prod_{v\neq\infty} \left|\int_u \omega\right|_v = \frac{q}{q-1}\log q + \sum_{v\neq\infty} \frac{-1}{q_v-1}\log q_v. \tag{2.1}$$

Note that this series is not convergent, but that the sum over $v \neq \infty$ is equal to $\frac{\zeta_A'(1)}{\zeta_A(1)}$ and the summand at ∞ is equal to $\frac{\zeta_A'(0)}{\zeta_A(0)}$, where ζ_A is the zeta function associated with A, which does not converge at $s = 1$. Namely, the zeta functions are defined as the following products which converge for $s \in \mathbb{C}$ with $\mathcal{R}e(s) > 1$

$$\zeta_C(s) := \prod_{\text{all } v} (1-(\#\mathbb{F}_v)^{-s})^{-1} = \prod_{\text{all } v} (1-q_v^{-s})^{-1} = \frac{1}{(1-q^{-s})(1-q^{1-s})} \quad \text{and}$$

$$\zeta_A(s) := \prod_{v\neq\infty} (1-(\#\mathbb{F}_v)^{-s})^{-1} = \prod_{v\neq\infty} (1-q_v^{-s})^{-1} = \frac{1}{1-q^{1-s}};$$

see for example [28, Chapter V, Example 2.1]. In particular, $(1-q_v^{-s})^{-1} = L_v(\mathbb{1}, 1)$ and $\zeta_A(s) = L^\infty(\mathbb{1}, s)$ where $\mathbb{1}\colon g \mapsto 1$ is the trivial character in $\mathcal{C}^0(\mathscr{G}_Q, \mathbb{Q})$, which equals $a_{Q,\text{id},\Phi} = a^0_{Q,\text{id},\Phi}$. Then $\frac{1}{q_v^s-1} = Z_v(\mathbb{1}, s)$ and $\frac{\zeta_A'(s)}{\zeta_A(s)} = Z^\infty(\mathbb{1}, s)$ in the notation of

Definition 1.2. Applying Convention 1.4 with $a = \mathbb{1}$ and $\mu_{\mathrm{Art}}(\mathbb{1}) = 0$ and $genus(\mathbb{P}^1_{\mathbb{F}_q}) = 0$ we define the limit of the series as

$$\begin{aligned}\frac{q}{q-1}\log q + \sum_{v\neq\infty}\frac{-1}{q_v-1}\log q_v &:= \frac{q}{q-1}\log q - \sum_{v\neq\infty} Z_v(\mathbb{1},1)\log q_v\\ &= \frac{q}{q-1}\log q - Z^{\infty}(\mathbb{1},0)\\ &= \frac{q}{q-1}\log q - \frac{\zeta_A'(0)}{\zeta_A(0)}\\ &= 0.\end{aligned}$$

So the value of the product $\prod_v \left|\int_u \omega\right|_v$ is 1 for the Carlitz motive.

3. Local shtukas

In the rest of the article we fix a place $v \neq \infty$ of Q. We keep the notation from the introduction, except that we write $L = \kappa(\!(\pi_L)\!)$ for the field K_v and let $R = \kappa[\![\pi_L]\!]$ be its valuation ring. We write $z := z_v$. Then $A_v = \mathbb{F}_v[\![z]\!]$ and $Q_v = \mathbb{F}_v(\!(z)\!)$. The homomorphism $\gamma\colon A \to K$ extends by continuity to $\gamma\colon A_v \to L$ and factors through $\gamma\colon A_v \to R$ with $\zeta := \zeta_v = \gamma(z) \in \pi_L R \setminus \{0\}$. Let $R[\![z]\!]$ be the power series ring in the variable z over R and $\hat\sigma$ the endomorphism of $R[\![z]\!]$ with $\hat\sigma(z) = z$ and $\hat\sigma(b) = b^{q_v}$ for $b \in R$, where $q_v = \#\mathbb{F}_v$. For an $R[\![z]\!]$-module $\hat M$ we let $\hat\sigma^*\hat M := \hat M \otimes_{R[\![z]\!],\hat\sigma^*} R[\![z]\!]$ as well as $\hat M[\frac{1}{z-\zeta}] := \hat M \otimes_{R[\![z]\!]} R[\![z]\!][\frac{1}{z-\zeta}]$ and $\hat M[\frac{1}{z}] := \hat M \otimes_{R[\![z]\!]} R[\![z]\!][\frac{1}{z}]$.

Definition 3.1. A *local $\hat\sigma$-shtuka of rank r* over R is a pair $\underline{\hat M} = (\hat M, \tau_{\hat M})$ consisting of a free $R[\![z]\!]$-module $\hat M$ of rank r, and an isomorphism $\tau_{\hat M} : \hat\sigma^*\hat M[\frac{1}{z-\zeta}] \xrightarrow{\sim} \hat M[\frac{1}{z-\zeta}]$. It is *effective* if $\tau_{\hat M}(\hat\sigma^*\hat M) \subset \hat M$ and *étale* if $\tau_{\hat M}(\hat\sigma^*\hat M) = \hat M$. We write $\operatorname{rk}\underline{\hat M}$ for the rank of $\underline{\hat M}$.

A *morphism* of local shtukas $f : \underline{\hat M} = (\hat M, \tau_{\hat M}) \to \underline{\hat N} = (\hat N, \tau_{\hat N})$ over R is a morphism of the underlying modules $f : \hat M \to \hat N$ which satisfies $\tau_{\hat N} \circ \hat\sigma^* f = f \circ \tau_{\hat M}$. We denote the A_v-module of homomorphisms $f\colon \underline{\hat M} \to \underline{\hat N}$ by $\mathrm{Hom}_R(\underline{\hat M}, \underline{\hat N})$ and write $\mathrm{End}_R(\underline{\hat M}) = \mathrm{Hom}_R(\underline{\hat M}, \underline{\hat M})$.

A *quasi-morphism* between local shtukas $f\colon (\hat M, \tau_{\hat M}) \to (\hat N, \tau_{\hat N})$ over R is a morphism of $R[\![z]\!][\frac{1}{z}]$-modules $f\colon M[\frac{1}{z}] \xrightarrow{\sim} N[\frac{1}{z}]$ with $\tau_{\hat N} \circ \hat\sigma^* f = f \circ \tau_{\hat M}$. It is called a *quasi-isogeny* if it is an isomorphism of $R[\![z]\!][\frac{1}{z}]$-modules. We denote the Q_v-vector space of quasi-morphisms from $\underline{\hat M}$ to $\underline{\hat N}$ by $\mathrm{QHom}_R(\underline{\hat M}, \underline{\hat N})$ and write $\mathrm{QEnd}_R(\underline{\hat M}) = \mathrm{QHom}_R(\underline{\hat M}, \underline{\hat M})$.

Note that $\mathrm{Hom}_R(\underline{\hat M}, \underline{\hat N})$ is a finite free A_v-module of rank at most $\operatorname{rk}\underline{\hat M} \cdot \operatorname{rk}\underline{\hat N}$ by [18, Corollary 4.5] and $\mathrm{QHom}_R(\underline{\hat M}, \underline{\hat N}) = \mathrm{Hom}_R(\underline{\hat M}, \underline{\hat N}) \otimes_{A_v} Q_v$. Also every quasi-isogeny $f\colon \underline{\hat M} \to \underline{\hat N}$ induces an isomorphism of Q_v-algebras $\mathrm{QEnd}_R(\underline{\hat M}) \xrightarrow{\sim} \mathrm{QEnd}_R(\underline{\hat N})$, by sending $g \mapsto fgf^{-1}$.

Example 3.2. We assume that the A-motive $\underline{M} = (M, \tau_M)$ has *good reduction*, that is, there exist a pair $\underline{\mathcal{M}} = (\mathcal{M}, \tau_{\mathcal{M}})$ consisting of a locally free module $\mathcal{M}$ of finite rank

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over $A_R := A \otimes_{\mathbb{F}_q} R$ and an isomorphism $\tau_{\mathcal{M}} : \sigma^*\mathcal{M}|_{\operatorname{Spec} A_R \setminus \mathrm{V}(\mathcal{J})} \xrightarrow{\sim} \mathcal{M}|_{\operatorname{Spec} A_R \setminus \mathrm{V}(\mathcal{J})}$ of the associated sheaves outside the vanishing locus $\mathrm{V}(\mathcal{J}) \subset \operatorname{Spec} A_R$ of the ideal

$$\mathcal{J} := (a \otimes 1 - 1 \otimes \gamma(a) : a \in A) \subset A_R,$$

such that $\underline{\mathcal{M}} \otimes_R L \cong \underline{M}$. The reduction $\underline{\mathcal{M}} \otimes_R \kappa$ is an A-motive over κ of A-characteristic $v = \ker(\gamma : A \to \kappa)$. The pair $\underline{\mathcal{M}}$ is called an *A-motive over R* and a *good model* of $\underline{M}$.

We consider the v-adic completions $A_{v,R}$ of A_R and

$$\underline{\mathcal{M}} \otimes_{A_R} A_{v,R} := (\mathcal{M} \otimes_{A_R} A_{v,R}, \tau_{\mathcal{M}} \otimes \mathrm{id})$$

of $\underline{\mathcal{M}}$. We let $d_v := [\mathbb{F}_v : \mathbb{F}_q]$ and discuss the two cases $d_v = 1$ and $d_v > 1$ separately. If $d_v = 1$, and hence $q_v = q$ and $\hat{\sigma} = \sigma$, we have $A_{v,R} = R[\![z]\!]$, and $\underline{\mathcal{M}} \otimes_{A_R} A_{v,R}$ is a local $\hat{\sigma}$-shtuka over $\operatorname{Spec} R$ which we denote by $\underline{\hat{M}}_v(\underline{\mathcal{M}})$ and call the *local shtuka at v associated with $\underline{\mathcal{M}}$*.

If $d_v > 1$, the situation is more complicated, because $\mathbb{F}_v \otimes_{\mathbb{F}_q} R$ and $A_{v,R}$ fail to be integral domains. Namely,

$$\mathbb{F}_v \otimes_{\mathbb{F}_q} R = \prod_{\operatorname{Gal}(\mathbb{F}_v/\mathbb{F}_q)} \mathbb{F}_v \otimes_{\mathbb{F}_v} R = \prod_{i \in \mathbb{Z}/d_v\mathbb{Z}} \mathbb{F}_v \otimes_{\mathbb{F}_q} R/(a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v)$$

and σ transports the ith factor to the $(i+1)$th factor. In particular, $\hat{\sigma}$ stabilizes each factor. Denote by $\mathfrak{a}_i$ the ideal of $A_{v,R}$ generated by $\{a \otimes 1 - 1 \otimes \gamma(a)^{q^i} : a \in \mathbb{F}_v\}$. Then

$$A_{v,R} = \prod_{\operatorname{Gal}(\mathbb{F}_v/\mathbb{F}_q)} A_v \widehat{\otimes}_{\mathbb{F}_v} R = \prod_{i \in \mathbb{Z}/d_v\mathbb{Z}} A_{v,R}/\mathfrak{a}_i.$$

Note that each factor is isomorphic to $R[\![z]\!]$ and the ideals $\mathfrak{a}_i$ correspond precisely to the places v_i of $C_{\mathbb{F}_v}$ lying above v. The ideal $\mathcal{J}$ decomposes as follows $\mathcal{J} \cdot A_{v,R}/\mathfrak{a}_0 = (z - \zeta)$ and $\mathcal{J} \cdot A_{v,R}/\mathfrak{a}_i = (1)$ for $i \neq 0$. We define the *local shtuka at v associated with $\underline{\mathcal{M}}$* as

$$\underline{\hat{M}}_v(\underline{\mathcal{M}}) := (\hat{M}, \tau_{\hat{M}}) := \big(\mathcal{M} \otimes_{A_R} A_{v,R}/\mathfrak{a}_0, (\tau_{\mathcal{M}} \otimes 1)^{d_v}\big),$$

where $\tau_{\mathcal{M}}^{d_v} := \tau_{\mathcal{M}} \circ \sigma^*\tau_{\mathcal{M}} \circ \cdots \circ \sigma^{(d_v-1)*}\tau_{\mathcal{M}}$. Of course if $d_v = 1$ we get back the definition of $\underline{\hat{M}}_v(\underline{\mathcal{M}})$ given above. Also if $\underline{\mathcal{M}}$ is effective, then $\mathcal{M}/\tau_{\mathcal{M}}(\sigma^*\mathcal{M}) = \hat{M}/\tau_{\hat{M}}(\hat{\sigma}^*\hat{M})$.

The local shtuka $\underline{\hat{M}}_v(\underline{\mathcal{M}})$ allows to recover $\underline{\mathcal{M}} \otimes_{A_R} A_{v,R}$ via the isomorphism

$$\bigoplus_{i=0}^{d_v-1} (\tau_{\mathcal{M}} \otimes 1)^i \bmod \mathfrak{a}_i : \Big(\bigoplus_{i=0}^{d_v-1} \sigma^{i*}(\mathcal{M} \otimes_{A_R} A_{v,R}/\mathfrak{a}_0), (\tau_{\mathcal{M}} \otimes 1)^{d_v} \oplus \bigoplus_{i \neq 0} \mathrm{id}\Big) \xrightarrow{\sim} \underline{\mathcal{M}} \otimes_{A_R} A_{v,R},$$

because for $i \neq 0$ the equality $\mathcal{J} \cdot A_{v,R}/\mathfrak{a}_i = (1)$ implies that $\tau_{\mathcal{M}} \otimes 1$ is an isomorphism modulo $\mathfrak{a}_i$; see [8, Propositions 8.8 and 8.5] for more details.

Next we define the v-adic realization and the de Rham realization of a local shtuka $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ over R. Since $\tau_{\hat{M}}$ induces an isomorphism $\tau_{\hat{M}} : \hat{\sigma}^*\hat{M} \otimes_{R[\![z]\!]} L[\![z]\!] \xrightarrow{\sim} \hat{M} \otimes_{R[\![z]\!]} L[\![z]\!]$, because $z - \zeta \in L[\![z]\!]^\times$, we can think of $\underline{\hat{M}} \otimes_{R[\![z]\!]} L[\![z]\!]$ as an étale local shtuka over L.

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Definition 3.3. The *v-adic realization* $\mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ of a local $\hat{\sigma}$-shtuka $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ is the $\mathscr{G}_L := \mathrm{Gal}(L^{\mathrm{sep}}/L)$-module of τ- invariants

$$\mathrm{H}^1_v(\underline{\hat{M}}, A_v) := (\hat{M} \otimes_{R[\![z]\!]} L^{\mathrm{sep}}[\![z]\!])^{\tau} := \{m \in \hat{M} \otimes_{R[\![z]\!]} L^{\mathrm{sep}}[\![z]\!] : \tau_{\hat{M}}(\hat{\sigma}^*_{\hat{M}} m) = m\},$$

where we set $\hat{\sigma}^*_{\hat{M}} m := m \otimes 1 \in \hat{M} \otimes_{R[\![z]\!],\hat{\sigma}} R[\![z]\!] =: \sigma^* M$ for $m \in M$. One also writes sometimes $\check{T}_v \underline{\hat{M}} = \mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ and calls this the *dual Tate module of* $\underline{\hat{M}}$. By [18, Proposition 4.2] it is a free A_v-module of the same rank as $\hat{M}$. We also write $\mathrm{H}^1_v(\underline{\hat{M}}, B) := \mathrm{H}^1_v(\underline{\hat{M}}, A_v) \otimes_{A_v} B$ for an A_v-algebra B.

If $\underline{M} = (M, \tau_M)$ is an A-motive over L with good model $\underline{\mathcal{M}}$ and $\underline{\hat{M}} = \underline{\hat{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$, then $\mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ is by [18, Proposition 4.6] canonically isomorphic as a representation of $\mathscr{G}_L$ to the *v-adic realization of* $\underline{M}$, which is defined as

$$\mathrm{H}^1_v(\underline{M}, A_v) := \{m \in M \otimes_{A_L} A_{v,L^{\mathrm{sep}}} : \tau_M(\sigma^*_M m) = m\},$$

where we set $\sigma^*_M m := m \otimes 1 \in M \otimes_{A_R,\sigma} A_R =: \sigma^* M$ for $m \in M$ and where $A_{v,L^{\mathrm{sep}}}$ is the v-adic completion of $A_{L^{\mathrm{sep}}}$.

Definition 3.4. Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ be a local $\hat{\sigma}$-shtuka over R. We define the *de Rham realizations* of $\underline{\hat{M}}$ as

$$\begin{aligned}
\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, R) &:= \hat{\sigma}^*\hat{M}/(z-\zeta)\hat{\sigma}^*\hat{M} = \hat{\sigma}^*\hat{M} \otimes_{R[\![z]\!],z\mapsto\zeta} R, \quad \text{as well as} \\
\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!]) &:= \hat{\sigma}^*\hat{M} \otimes_{R[\![z]\!]} L[\![z-\zeta]\!] \quad \text{and} \\
\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L) &:= \hat{\sigma}^*\hat{M} \otimes_{R[\![z]\!],z\mapsto\zeta} L = \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!]) \otimes_{L[\![z-\zeta]\!]} L[\![z-\zeta]\!]/(z-\zeta) \\
&= \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, R) \otimes_R L.
\end{aligned}$$

It carries the *Hodge–Pink lattice* $\mathfrak{q}^{\underline{\hat{M}}} := \tau^{-1}_{\hat{M}}(\hat{M} \otimes_{R[\![z]\!]} L[\![z-\zeta]\!]) \subset \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!])[\frac{1}{z-\zeta}]$. We also write $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, B) := \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!]) \otimes_{L[\![z-\zeta]\!]} B$ for an $L[\![z-\zeta]\!]$-algebra B.

If $\underline{M} = (M, \tau_M)$ is an A-motive over L with good model $\underline{\mathcal{M}}$ and $\underline{\hat{M}} = \underline{\hat{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$ and $d_v = [\mathbb{F}_v : \mathbb{F}_q]$ is as in Example 3.2, the map

$$\sigma^*\tau_M^{d_v-1} = \sigma^*\tau_M \circ \sigma^{2*}\tau_M \circ \cdots \circ \sigma^{(d_v-1)*}\tau_M : \sigma^{d_v*}M \otimes_{A_R} A_{v,R}/\mathfrak{a}_0 \xrightarrow{\sim} \sigma^* M \otimes_{A_R} A_{v,R}/\mathfrak{a}_0$$

is an isomorphism, because τ_M is an isomorphism over $A_{v,R}/\mathfrak{a}_i$ for all $i \neq 0$. Therefore, it defines canonical isomorphisms of the de Rham realizations

$$\begin{aligned}
\sigma^*\tau_M^{d_v-1} &: \mathrm{H}^1_{\mathrm{dR}}\big(\underline{\hat{M}}, L[\![z-\zeta]\!]\big) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}\big(\underline{M}, L[\![z-\zeta]\!]\big) \quad \text{and} \\
\sigma^*\tau_M^{d_v-1} &: \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(\underline{M}, L),
\end{aligned}$$

which are compatible with the Hodge–Pink lattices.

Remark 3.5. By [18, Theorem 4.15] there is a canonical comparison isomorphism

$$h_{v,\mathrm{dR}} : \ \mathrm{H}^1_v(\underline{\hat{M}}, \mathbb{C}_v(\!(z-\zeta)\!)) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, \mathbb{C}_v(\!(z-\zeta)\!)) \tag{3.1}$$

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which is equivariant for the action of $\mathscr{G}_L$. For our computations below we need an explicit description of $h_{v,\mathrm{dR}}$. It is constructed as follows. The natural inclusion $\mathrm{H}^1_v(\underline{\hat{M}}, A_v) \subset \hat{M} \otimes_{R[\![z]\!]} L^{\mathrm{sep}}[\![z]\!]$ defines a canonical isomorphism of $L^{\mathrm{sep}}[\![z]\!]$-modules

$$\mathrm{H}^1_v(\underline{\hat{M}}, A_v) \otimes_{A_v} L^{\mathrm{sep}}[\![z]\!] \xrightarrow{\sim} \hat{M} \otimes_{R[\![z]\!]} L^{\mathrm{sep}}[\![z]\!], \tag{3.2}$$

which is $\mathscr{G}_L$ and τ-equivariant, where on the left module $\mathscr{G}_L$ acts on both factors and τ is $\mathrm{id} \otimes \hat{\sigma}$ and on the right module $\mathscr{G}_L$ acts only on $L^{\mathrm{sep}}[\![z]\!]$ and τ is $(\tau_{\hat{M}} \circ \hat{\sigma}^*_{\hat{M}}) \otimes \hat{\sigma}$. Since $(L^{\mathrm{sep}})^{\mathscr{G}_L} = L$ we obtain

$$\hat{M} \otimes_{R[\![z]\!]} L[\![z]\!] = (\mathrm{H}^1_v(\underline{\hat{M}}, A_v) \otimes_{A_v} L^{\mathrm{sep}}[\![z]\!])^{\mathscr{G}_L}.$$

It turns out, see [18, Remark 4.3], that the isomorphism (3.2) extends to an equivariant isomorphism

$$h\colon\ \mathrm{H}^1_v(\underline{\hat{M}}, A_v) \otimes_{A_v} L^{\mathrm{sep}}\langle \tfrac{z}{\zeta} \rangle \xrightarrow{\sim} \hat{M} \otimes_{R[\![z]\!]} L^{\mathrm{sep}}\langle \tfrac{z}{\zeta} \rangle, \tag{3.3}$$

where for an $r \in \mathbb{R}_{>0}$ we use the notation

$$L^{\mathrm{sep}}\langle \tfrac{z}{\zeta^r} \rangle := \left\{ \sum_{i=0}^{\infty} b_i z^i :\ b_i \in L^{\mathrm{sep}}, |b_i|\,|\zeta|^{ri} \to 0\,(i \to +\infty) \right\}.$$

These are subrings of $L^{\mathrm{sep}}[\![z]\!]$ and the endomorphism $\hat{\sigma}\colon \sum_i b_i z^i \mapsto \sum_i b_i^{q_v} z^i$ of $L^{\mathrm{sep}}[\![z]\!]$ restricts to a homomorphism $\hat{\sigma}\colon L^{\mathrm{sep}}\langle \frac{z}{\zeta^r} \rangle \to L^{\mathrm{sep}}\langle \frac{z}{\zeta^{rq_v}} \rangle$. Note that the τ-equivariance of h means $h \otimes \mathrm{id}_{L^{\mathrm{sep}}\langle \frac{z}{\zeta^{q_v}} \rangle} = \tau_{\hat{M}} \circ \hat{\sigma}^* h$. Now the period isomorphism is defined as

$$\begin{aligned} h_{v,\mathrm{dR}} := (\tau^{-1}_{\hat{M}} \circ h) \otimes \mathrm{id}_{\mathbb{C}_v((z-\zeta))}\colon\ \mathrm{H}^1_v(\underline{\hat{M}}, \mathbb{C}_v((z-\zeta))) &\xrightarrow{\sim} \hat{\sigma}^* \hat{M} \otimes_{R[\![z]\!]} \mathbb{C}_v((z-\zeta)) \\ &= \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, \mathbb{C}_v((z-\zeta))). \end{aligned} \tag{3.4}$$

4. Local shtukas with complex multiplication

Definition 4.1. Let $\underline{\hat{M}}$ be a local $\hat{\sigma}$-shtuka over R and assume that there is a commutative, semi-simple Q_v-algebra $E_v \subset \mathrm{QEnd}_R(\underline{\hat{M}}) := \mathrm{End}_R(\underline{\hat{M}}) \otimes_{A_v} Q_v$ with $\dim_{Q_v} E_v = \mathrm{rk}\, \underline{\hat{M}}$. Then we say that $\underline{\hat{M}}$ has *complex multiplication (by E_v)*.

Here again semi-simple means that E_v is a direct product $E_v = E_{v,1} \times \cdots \times E_{v,s}$ of finite field extensions of Q_v. We do *not* assume that E_v is itself a field and in § 4 we do *not* assume that the $E_{v,i}$ are separable over Q_v. We let $\mathcal{O}_{E_v}$ be the integral closure of A_v in E_v. It is a product $\mathcal{O}_{E_v} = \mathcal{O}_{E_{v,1}} \times \cdots \times \mathcal{O}_{E_{v,s}}$ of complete discrete valuation rings where $\mathcal{O}_{E_{v,i}}$ is the integral closure of A_v in the field $E_{v,i}$. For every i we write $\mathcal{O}_{E_{v,i}} = \mathbb{F}_{\tilde{v}_i}[\![y_i]\!]$ and set $f_i := [\mathbb{F}_{\tilde{v}_i} : \mathbb{F}_v]$ and $e_i := \mathrm{ord}_{y_i}(z)$. Then $f_i\, e_i = [E_{v,i} : Q_v]$ and e_i is divisible by the inseparability degree of $E_{v,i}$ over Q_v. Also we write $\tilde{q}_i := \#\mathbb{F}_{\tilde{v}_i} = q_v^{f_i}$.

Proposition 4.2. *If $\underline{\hat{M}}$ has complex multiplication by E_v, then there is a local shtuka $\underline{\hat{M}}'$ over R quasi-isogenous to $\underline{\hat{M}}$ with $\mathcal{O}_{E_v} \subset \mathrm{End}_R(\underline{\hat{M}}')$.*

Proof. The A_v-submodule $T' := \mathcal{O}_{E_v} \cdot \mathrm{H}^1_v(\underline{\hat{M}}, A_v) \subset \mathrm{H}^1_v(\underline{\hat{M}}, Q_v)$ is $\mathscr{G}_L$-invariant and contains $\mathrm{H}^1_v(\underline{\hat{M}}, A_v)$. Since $\mathcal{O}_{E_v} \subset \mathrm{QEnd}_R(\underline{\hat{M}}) = \mathrm{End}_R(\underline{\hat{M}}) \otimes_{A_v} Q_v$ there is an element

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$a \in A_v$ with $a \cdot \mathcal{O}_{E_v} \subset \mathrm{End}_R(\underline{\hat{M}})$, and therefore $a \cdot T' \subset \mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ is a finitely generated A_v-module, that is an A_v-lattice. By [18, Proposition 4.22] there is a local shtuka $\underline{\hat{M}}'$ and a quasi-isogeny $f\colon \underline{\hat{M}} \to \underline{\hat{M}}'$ which maps T' isomorphically onto $\mathrm{H}^1_v(\underline{\hat{M}}', A_v)$. In particular, $\mathcal{O}_{E_v}$ acts as $\mathscr{G}_L$-equivariant endomorphisms of $\mathrm{H}^1_v(\underline{\hat{M}}', A_v)$. Since the functor $\underline{\hat{M}}' \mapsto \mathrm{H}^1_v(\underline{\hat{M}}', A_v)$ from local shtukas to $A_v[\mathscr{G}_L]$-modules is fully faithful by [18, Theorem 4.20], we see that $\mathcal{O}_{E_v} \subset \mathrm{End}_R(\underline{\hat{M}}')$. □

Definition 4.3. If $\mathcal{O}_{E_v} \subset \mathrm{End}_R(\underline{\hat{M}})$ we say that $\underline{\hat{M}}$ has *complex multiplication by* $\mathcal{O}_{E_v}$. This makes the underlying module $\hat{M}$ into a module over the ring $\mathcal{O}_{E_v,R} := \mathcal{O}_{E_v} \otimes_{A_v} R[\![z]\!] = \mathcal{O}_{E_v} \widehat{\otimes}_{\mathbb{F}_v} R$. For $a \in \mathcal{O}_{E_v}$ note that $a \otimes 1 \in \mathcal{O}_{E_v,R}$ acts on $\hat{M}$ as the endomorphism a and on $\hat{\sigma}^* \hat{M}$ as the endomorphism $\hat{\sigma}^* a$ and $\tau_{\hat{M}}$ is $\mathcal{O}_{E_v}$-linear because $a \circ \tau_{\hat{M}} = \tau_{\hat{M}} \circ \hat{\sigma}^* a$.

4.4. Let us assume that L contains $\psi(E_v)$ for every $\psi \in H_{E_v}$. This implies $\psi(\mathcal{O}_{E_v}) \subset R$ for every $\psi \in H_{E_v}$. Under this assumption let us describe the ring $\mathcal{O}_{E_v,R} = \prod_{i=1}^s \mathcal{O}_{E_{v,i},R}$. Fix an i and choose and fix an $\mathbb{F}_v$-homomorphism $\mathbb{F}_{\tilde{v}_i} \hookrightarrow \kappa$. Then $H_{\tilde{v}_i} := \mathrm{Hom}_{\mathbb{F}_v}(\mathbb{F}_{\tilde{v}_i}, \kappa) \cong \mathbb{Z}/f_i\mathbb{Z}$ under the map that sends $j \in \mathbb{Z}/f_i\mathbb{Z}$ to the homomorphism $(\lambda \mapsto \lambda^{q_v^j}) \in H_{\tilde{v}_i}$. Also

$$\mathbb{F}_{\tilde{v}_i} \otimes_{\mathbb{F}_v} R = \prod_{H_{\tilde{v}_i}} R = \prod_{j \in \mathbb{Z}/f_i\mathbb{Z}} \mathbb{F}_{\tilde{v}_i} \otimes_{\mathbb{F}_v} R/(\lambda \otimes 1 - 1 \otimes \lambda^{q_v^j} : \lambda \in \mathbb{F}_{\tilde{v}_i})$$

and $\hat{\sigma}^*$ transports the jth factor to the $(j+1)$th factor. Denote by $\mathfrak{b}_{i,j} \subset \mathcal{O}_{E_{v,i}}$ the ideal generated by $(\lambda \otimes 1 - 1 \otimes \lambda^{q_v^j} : \lambda \in \mathbb{F}_{\tilde{v}_i})$. Then

$$\mathcal{O}_{E_{v,i},R} := \mathbb{F}_{\tilde{v}_i}[\![y_i]\!] \otimes_{A_v} R[\![z]\!] = \prod_{j \in \mathbb{Z}/f_i\mathbb{Z}} \mathbb{F}_{\tilde{v}_i}[\![y_i]\!] \otimes_{\mathbb{F}_v[\![z]\!]} R[\![z]\!]/\mathfrak{b}_{i,j} = \prod_{H_{\tilde{v}_i}} R[\![y_i]\!]. \tag{4.1}$$

Definition 4.5. For every $\psi \in H_{E_v}$ we let $i(\psi)$ be such that ψ factors through the quotient $E_v \twoheadrightarrow E_{v,i(\psi)}$ and we let $j(\psi) \in \mathbb{Z}/f_{i(\psi)}\mathbb{Z}$ be the element such that $\psi(\lambda) = \lambda^{q_v^{j(\psi)}}$ for all $\lambda \in \mathbb{F}_{\tilde{v}_{i(\psi)}}$. Then the morphism $\psi\colon \mathcal{O}_{E_v} \to R$ equals the composition $\mathcal{O}_{E_v} \hookrightarrow \mathcal{O}_{E_v,R} \twoheadrightarrow \mathcal{O}_{E_{v,i(\psi)},R}/(\mathfrak{b}_{i(\psi),j(\psi)}, y_{i(\psi)} - \psi(y_{i(\psi)}))$ and $H_{E_{v,i}} = \{\psi \in H_{E_v} : i(\psi) = i\}$.

Lemma 4.6. *Let p^m be the inseparability degree of $E_{v,i}$ over Q_v. Then in the jth component $R[\![y_i]\!]$ of (4.1) we have*

$$z - \zeta = \epsilon \cdot \prod_{\psi \in H_{E_v} : (i,j)(\psi) = (i,j)} (y_i - \psi(y_i))^{p^m} \tag{4.2}$$

for a unit $\epsilon \in R[\![y_i]\!]$.

Proof. Set $y_i' := y_i^{p^m}$ and let $P = P(z, Y) = \sum_{\mu,\nu} b_{\mu\nu} z^\mu Y^\nu \in \mathbb{F}_{\tilde{v}_i}[\![z]\!][Y]$ with $b_{\mu\nu} \in \mathbb{F}_{\tilde{v}_i}$ be the minimal polynomial of y_i' over $\mathbb{F}_{\tilde{v}_i}(\!(z)\!)$. It is an Eisenstein polynomial of degree e_i/p^m, because $\mathbb{F}_{\tilde{v}_i}(\!(y_i')\!)$ is purely ramified and separable over $\mathbb{F}_{\tilde{v}_i}(\!(z)\!)$ by Lemma A.2 in the appendix. In particular, $b_{0,\nu} = 0$ for $0 \leqslant \nu < e_i/p^m$, and $b_{1,0} \neq 0$. Consider the polynomials $P^{(j)}(z, Y) := \sum_{\mu,\nu} b_{\mu\nu}^{q_v^j} z^\mu Y^\nu \in \mathbb{F}_{\tilde{v}_i}[\![z]\!][Y] \subset R[\![z]\!][Y]$ and $P^{(j)}(\zeta, Y) \in R[Y]$.

If $\psi \in H_{E_v}$ satisfies $(i,j)(\psi)=(i,j)$ then $P^{(j)}\big(\zeta,\psi(y_i')\big)=\psi\big(P(z,y_i')\big)=\psi(0)=0$. These zeros $\psi(y_i')$ of $P^{(j)}(\zeta,Y)$ in L are pairwise different, because if $\psi(y_i')=\widetilde{\psi}(y_i')$ then $(i,j)(\psi)=(i,j)(\widetilde{\psi})$ implies that ψ and $\widetilde{\psi}$ coincide on $E_{v,i}$ and hence on E_v. It follows that $P^{(j)}(\zeta,Y)=\prod_{\psi\colon (i,j)(\psi)=(i,j)}(Y-\psi(y_i'))$ in $L[Y]$, whence already in $R[Y]$. In the jth component $R[\![y_i]\!]$ of (4.1) we have $0=\sum_{\mu,\nu} b_{\mu\nu}z^\mu (y_i')^\nu\otimes 1=\sum_{\mu,\nu}(y_i')^\nu\otimes b_{\mu\nu}^{q_v^j}z^\mu = P^{(j)}(z,y_i')$, and we compute

$$\begin{aligned}\prod_{\psi\colon (i,j)(\psi)=(i,j)} (y_i-\psi(y_i))^{p^m} &= P^{(j)}(\zeta,y_i')\\ &= P^{(j)}(\zeta,y_i')-P^{(j)}(z,y_i')\\ &= \sum_{\mu,\nu} b_{\mu\nu}^{q_v^j}(\zeta^\mu - z^\mu)(y_i')^\nu\\ &= (\zeta-z)\cdot\sum_{\mu,\nu} b_{\mu\nu}^{q_v^j}(\zeta^{\mu-1}+\zeta^{\mu-2}z+\cdots+z^{\mu-1})(y_i')^\nu.\end{aligned}$$

The factor $\sum_{\mu,\nu} b_{\mu\nu}^{q_v^j}(\zeta^{\mu-1}+\zeta^{\mu-2}z+\cdots+z^{\mu-1})(y_i')^\nu$ is congruent to $b_{1,0}^{q_v^j}\neq 0$ modulo the maximal ideal $(\pi_L,y_i)\subset R[\![y_i]\!]$ and therefore a unit in $R[\![y_i]\!]$. This finishes the proof. Note that since

$$P^{(j)}(\zeta,y_i')-P^{(j)}(z,y_i')=\sum_{n\geqslant 1}\frac{1}{n!}\frac{\partial^n P^{(j)}}{\partial z^n}(z,y_i')\cdot(\zeta-z)^n,$$

the proof could also be phrased by saying that $\frac{\partial P^{(j)}}{\partial z}(z,y_i')=\sum_{\mu,\nu}\mu b_{\mu\nu}^{q_v^j}z^{\mu-1}(y_i')^\nu$ lies in $\mathcal{O}^\times_{E'_{v,i}}$. In fact this partial derivative is congruent to $b_{1,0}^{q_v^j}\neq 0$ modulo $y_i'\cdot\mathcal{O}_{E'_{v,i}}$. □

Let us draw a direct corollary from the proof of this lemma. To formulate it, recall that if $E_{v,i}$ is separable over Q_v, the different $\mathfrak{D}_{E_{v,i}/Q_v}$ of $E_{v,i}$ over Q_v is defined as the ideal in $\mathcal{O}_{E_{v,i}}$ which annihilates the module $\Omega^1_{\mathcal{O}_{E_{v,i}}/A_v}$ of relative differentials.

Corollary 4.7. *If $E_{v,i}$ is separable over Q_v then $\mathfrak{D}_{\varphi(E_{v,i})/Q_v}=\Big(\frac{z-\zeta}{y_i-\varphi(y_i)}\Big|_{y_i=\varphi(y_i)}\Big)$ in $\mathcal{O}_{\varphi(E_{v,i})}$ for every $\varphi\in H_{E_{v,i}}$.*

Proof. By [27, Chapter III, §4, Proposition 8] the different is multiplicative, that is $\mathfrak{D}_{E_{v,i}/Q_v}=\mathfrak{D}_{E_{v,i}/\mathbb{F}_{\tilde{v}_i}((z))}\cdot\mathfrak{D}_{\mathbb{F}_{\tilde{v}_i}((z))/Q_v}$. Moreover, $\mathfrak{D}_{\mathbb{F}_{\tilde{v}_i}((z))/Q_v}=1$ because $\mathbb{F}_{\tilde{v}_i}[\![z]\!]$ is unramified over A_v. As in the proof of the preceding lemma let $P(z,Y)$ be the minimal polynomial of y_i' over $\mathbb{F}_{\tilde{v}_i}((z))$ and note that $y_i'=y_i$ under our separability assumption. Then $\frac{\partial P}{\partial z}(z,y_i)\in\mathcal{O}^\times_{E_{v,i}}$ and

$$\begin{aligned}\Omega^1_{\mathcal{O}_{E_{v,i}}/\mathbb{F}_{\tilde{v}_i}[\![z]\!]} &= \mathcal{O}_{E_{v,i}}\langle dz,dy_i\rangle\Big/\Big(dz,\frac{\partial P}{\partial z}(z,y_i)\,dz+\frac{\partial P}{\partial Y}(z,y_i)\,dy_i\Big)\\ &= \mathcal{O}_{E_{v,i}}\cdot dy_i\Big/\Big(\frac{\partial P}{\partial Y}(z,y_i)\,dy_i\Big).\end{aligned}$$

We write $z = f(y_i) \in \mathbb{F}_{\tilde{v}_i}[\![y_i]\!]$. Then

$$0 = \frac{d}{dy_i} P(f(y_i), y_i) = \frac{\partial P}{\partial z}(f(y_i), y_i)\frac{df(y_i)}{dy_i} + \frac{\partial P}{\partial Y}(f(y_i), y_i)$$

and $\mathfrak{D}_{E_{v,i}/Q_v} = \big(\frac{\partial P}{\partial Y}(z, y_i)\big) = \big(\frac{df(y_i)}{dy_i}\big)$. Now Lemma A.1 in the appendix implies that $\mathfrak{D}_{\varphi(E_{v,i})/Q_v} = \varphi\big(\frac{df(y_i)}{dy_i}\big) = \big(\frac{z-\zeta}{y_i-\varphi(y_i)}\big)\big|_{y_i=\varphi(y_i)}$. □

Now we explore the consequences of these decompositions for local shtukas with complex multiplication.

Proposition 4.8. *Let $\underline{\hat{M}} = (\hat{M}, \tau_{\hat{M}})$ have complex multiplication by $\mathcal{O}_{E_v}$. Then the $\mathcal{O}_{E_v,R}$-module $\hat{M}$ is free of rank 1. In particular, $\underline{\hat{M}}_i := \underline{\hat{M}} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_{v,i}}$ is a local $\hat{\sigma}$-shtuka over R with $\operatorname{rk} \underline{\hat{M}}_i = [E_{v,i} : Q_v]$ and $\underline{\hat{M}} = \bigoplus_{i=1}^{s} \underline{\hat{M}}_i$.*

Proof. By faithfully flat descent [15, IV$_2$, Proposition 2.5.2], we may replace R by a finite extension of discrete valuation rings. Therefore, it suffices to prove the proposition in the case where R contains $\psi(\mathcal{O}_{E_v})$ for all $\psi \in H_{E_v}$. In this case $\mathcal{O}_{E_v,R}$ is a product of two-dimensional regular local rings $R[\![y_i]\!]$ by (4.1). By [25, § 6, Lemme 6] a finitely generated module M over such a ring is free if and only if it is reflexive, that is M is isomorphic to its bidual $M^{\vee\vee}$, where $M^{\vee} = \operatorname{Hom}_{R[\![y_i]\!]}(M, R[\![y_i]\!])$. In particular, $M^{\vee\vee}$, which is isomorphic to $(M^{\vee\vee})^{\vee\vee}$ is free. We apply this to $M := \hat{M} \otimes_{\mathcal{O}_{E_v,R}} R[\![y_i]\!]$ and consider the base changes $M \otimes_{R[\![y_i]\!]} L[\![y_i]\!] = M \otimes_{R[\![z]\!]} L[\![z]\!]$ and $M \otimes_{R[\![y_i]\!]} R[\![y_i]\!][\frac{1}{y_i}] = M \otimes_{R[\![z]\!]} R[\![z]\!][\frac{1}{z}]$. Like $L[\![y_i]\!]$ also $R[\![y_i]\!][\frac{1}{y_i}]$ is a principal ideal domain, because it is a factorial ring of dimension 1. Using [12, Proposition 2.10] and that both base changes of M are torsion free, whence free, we see that the canonical morphism $M \to M^{\vee\vee}$ is an isomorphism after both base changes. Since $R[\![z]\!] = L[\![z]\!] \cap R[\![z]\!][\frac{1}{z}] \subset L(\!(z)\!)$ and M and $M^{\vee\vee}$ are free $R[\![z]\!]$-modules, M equals the intersection $(M \otimes_{R[\![z]\!]} L[\![z]\!]) \cap (M \otimes_{R[\![z]\!]} R[\![z]\!][\frac{1}{z}])$ inside $M \otimes_{R[\![z]\!]} L(\!(z)\!)$, and likewise for $M^{\vee\vee}$. This shows that $M \to M^{\vee\vee}$ is an isomorphism and M is free over $R[\![y_i]\!]$.

It remains to compute the rank. Let $r_{i,j} := \operatorname{rk}_{R[\![y_i]\!]}(\hat{M} \otimes_{\mathcal{O}_{E_v,R}} \mathcal{O}_{E_{v,i},R}/\mathfrak{b}_{i,j})$ for all $i = 1, \ldots, s$ and all $j \in \mathbb{Z}/f_i\mathbb{Z}$. Then $\sum_{i,j} r_{i,j} \cdot e_i = \operatorname{rk} \underline{\hat{M}}$. We first prove that for a fixed i all $r_{i,j}$ are equal. Since $(\hat{\sigma}^*\hat{M}) \otimes_{\mathcal{O}_{E_v,R}} \mathcal{O}_{E_{v,i},R}/\mathfrak{b}_{i,j} = \hat{\sigma}^*(\hat{M} \otimes_{\mathcal{O}_{E_v,R}} \mathcal{O}_{E_{v,i},R}/\mathfrak{b}_{i,j-1}) \cong R[\![y_i]\!]^{r_{i,j-1}}$, we can write the isomorphism $\tau_{\hat{M}} : \hat{\sigma}^*\hat{M}[\frac{1}{z-\zeta}] \xrightarrow{\sim} \hat{M}[\frac{1}{z-\zeta}]$ in the form

$$\prod_j R[\![y_i]\!][\tfrac{1}{z-\zeta}]^{r_{i,j-1}} \xrightarrow{\sim} \prod_j R[\![y_i]\!][\tfrac{1}{z-\zeta}]^{r_{i,j}},$$

which gives us $r_{i,j-1} = r_{i,j} =: r_i$ for all j, and hence $\sum_i r_i\, f_i\, e_i = \operatorname{rk} \underline{\hat{M}} = \dim_{Q_v} E_v = \sum_i \dim_{Q_v} E_{v,i} = \sum_i f_i\, e_i$. Thus if we prove that $r_i \neq 0$ then all r_i must be 1 and so $\hat{M}$ is a free $\mathcal{O}_{E_v,R}$-module of rank 1 and $\operatorname{rk} \underline{\hat{M}}_i = f_i\, e_i = [E_{v,i} : Q_v]$. Now $r_i = 0$ means that $\hat{M} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_{v,i}} = (0)$, and hence $E_{v,i}$ acts as zero on $\underline{\hat{M}}$ in contradiction to $E_v \subset \operatorname{QEnd}_R(\underline{\hat{M}})$. This finishes the proof. □

Proposition 4.9. *If $\underline{\hat{M}}$ has complex multiplication by a commutative semi-simple Q_v-algebra E_v then $\mathrm{H}^1_v(\underline{\hat{M}}, Q_v)$ is a free E_v-module of rank 1 and $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!])$ is a free $E_v \otimes_{Q_v} L[\![z-\zeta]\!]$-module of rank one, where the homomorphism $Q_v = \mathbb{F}_v((z)) \to L[\![z-\zeta]\!]$ is given by $z \mapsto z = \zeta + (z-\zeta)$. If we assume that $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$ then the decomposition (A 1) induces a decomposition*

$$\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!]) = \bigoplus_{\psi \in H_{E_v}} \mathrm{H}^{\psi}(\underline{\hat{M}}, L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!]), \tag{4.3}$$

where $\mathrm{H}^{\psi}(\underline{\hat{M}}, L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!])$ is free of rank 1 over $L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!]$. In particular,

$$\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L) = \bigoplus_{\psi \in H_{E_v}} \mathrm{H}^{\psi}(\underline{\hat{M}}, R) \otimes_R L \tag{4.4}$$

is the decomposition into generalized eigenspaces of the E_v-action. Here

$$\mathrm{H}^{\psi}(\underline{\hat{M}}, R) := \left\{\omega \in \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, R) \colon \left([a]^* - \psi(a)\right)^{[E_{v,i(\psi)}:Q_v]_{\mathrm{insep}}} \cdot \omega = 0 \ \forall\, a \in E_v \cap \mathrm{End}_R(\underline{\hat{M}})\right\}$$

is a free R-module of rank equal to the inseparability degree $[E_{v,i(\psi)} : Q_v]_{\mathrm{insep}}$ of $E_{v,i(\psi)}$ over Q_v.

Proof. By the faithfulness of the functor $\underline{\hat{M}} \to \mathrm{H}^1_v(\underline{\hat{M}}, Q_v)$ we have an inclusion of rings $E_v \subset \mathrm{End}_{Q_v} \mathrm{H}^1_v(\underline{\hat{M}}, Q_v)$. So the first statement follows from [7, Lemma 7.2].

Since $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!])$ is an isogeny invariant, we may by Proposition 4.2 assume that $\mathcal{O}_{E_v} \subset \mathrm{End}_R(\underline{\hat{M}})$ and then $\hat{M}$ is free of rank 1 over $\mathcal{O}_{E_v,R}$ by Proposition 4.8. It follows that $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!]) := \hat{\sigma}^*\hat{M} \otimes_{R[\![z]\!]} L[\![z-\zeta]\!] \cong E_v \otimes_{Q_v} L[\![z-\zeta]\!]$. Now we use Lemma A.3. In particular, (4.4) and the statement about $\mathrm{H}^{\psi}(\underline{\hat{M}}, R)$ follow from (A 2) and the equation $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L) = \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, R) \otimes_R L$. □

The proposition allows us to make two definitions.

Definition 4.10. Let $\underline{\hat{M}}$ have complex multiplication by $\mathcal{O}_{E_v}$ and assume that $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$. Fix a $\psi \in H_{E_v}$ and let $i := i(\psi)$. Let $\omega^{\circ}_{\psi} \in \mathrm{H}^{\psi}(\underline{\hat{M}}, L[\![y_i - \psi(y_i)]\!])$ be an $L[\![y_i - \psi(y_i)]\!]$-generator whose reduction

$$\omega^{\circ}_{\psi} \bmod (y_i - \psi(y_i)) \in \mathrm{H}^{\psi}(\underline{\hat{M}}, L) \big/ (y_i - \psi(y_i))\, \mathrm{H}^{\psi}(\underline{\hat{M}}, L)$$

is a generator of the free R-module of rank one $\mathrm{H}^{\psi}(\underline{\hat{M}}, R) \big/ (y_i - \psi(y_i))\, \mathrm{H}^{\psi}(\underline{\hat{M}}, R)$. Such a ω°_{ψ} is uniquely determined up to multiplication by $R^{\times} + (y_i - \psi(y_i))L[\![y_i - \psi(y_i)]\!]$. Note that if $E_{v,i}$ is separable over Q_v then $y_i - \psi(y_i)$ acts trivially on $\mathrm{H}^{\psi}(\underline{\hat{M}}, L)$ and $\mathrm{H}^{\psi}(\underline{\hat{M}}, R)$ is a free R-module of rank 1. Also $L[\![y_i - \varphi(y_i)]\!] = L[\![z-\zeta]\!]$.

If $\omega_{\psi} \in \mathrm{H}^{\psi}(\underline{\hat{M}}, L[\![y_i - \psi(y_i)]\!])$ is any generator, there is an element $x \in L[\![y_i - \psi(y_i)]\!]^{\times}$ with $\omega_{\psi} = x\, \omega^{\circ}_{\psi}$. We define the *valuation of* ω_{ψ} as $v(\omega_{\psi}) := v\big(x \bmod y_i - \psi(y_i)\big)$. It only depends on the image of ω_{ψ} in $\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L)$ and is also independent of the choice of ω°_{ψ}.

Note that if $\underline{M} = (M, \tau_M)$ is an A-motive over L with good model $\underline{\mathcal{M}}$ over R, and $\underline{\hat{M}} = \underline{\hat{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$ as in Example 3.2, then for an $L[\![y_i - \psi(y_i)]\!]$-generator $\omega_{\psi} \in \mathrm{H}^{\psi}(\underline{M}, L[\![y_i - \psi(y_i)]\!]) = \mathrm{H}^{\psi}(\underline{\hat{M}}, L[\![y_i - \psi(y_i)]\!])$ the present definition of $v(\omega_{\psi})$ coincides with the definition of $v(\omega_{\psi})$ from (1.13).

Definition 4.11. A *local CM-type* at v is a pair (E_v, Φ) with E_v a semi-simple commutative Q_v-algebra and $\Phi = (d_\psi)_{\psi\in H_{E_v}}$ a tuple of integers $d_\psi \in \mathbb{Z}$.

If $\underline{\hat{M}}$ is a local shtuka with complex multiplication by a commutative semi-simple Q_v-algebra E_v and if $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$ then the Hodge–Pink lattice $\mathfrak{q}^{\underline{\hat{M}}} = \tau_{\hat{M}}^{-1}(\hat{M} \otimes_{R[\![z]\!]} L[\![z-\zeta]\!])$ of $\underline{\hat{M}}$ satisfies

$$\mathfrak{q}^{\underline{\hat{M}}} = \prod_{\psi\in H_{E_v}} \big(y_{i(\psi)} - \psi(y_{i(\psi)})\big)^{-d_\psi}\, \mathrm{H}^\psi(\underline{\hat{M}}, L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!])$$

for integers d_ψ under the decomposition (4.3). We call $\Phi = (d_\psi)_{\psi\in H_{E_v}}$ the *local CM-type* of $\underline{\hat{M}}$.

Note that if $\underline{M} = (M, \tau_M)$ is an A-motive over L with good model $\underline{\mathcal{M}}$ over R, and $\underline{\hat{M}} = \underline{\hat{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$ as in Example 3.2, and $E_v := E \otimes_Q Q_v$, then we see from the isomorphism $\mathrm{H}^\psi(\underline{M}, L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!]) = \mathrm{H}^\psi(\underline{\hat{M}}, L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!])$ that the local CM-type of $\underline{\hat{M}}$ is equal to the CM-type of $\underline{M}$ under the identification $H_E \xrightarrow{\sim} H_{E_v}$, which extends $\psi\colon E \to Q^{\mathrm{alg}} \subset Q_v^{\mathrm{alg}}$ to the completion $\psi\colon E_v \to Q_v^{\mathrm{alg}}$.

5. Periods of local shtukas with complex multiplication

5.1. In this section we let $\underline{\hat{M}}$ be a local $\hat{\sigma}$-shtuka over R with complex multiplication by $\mathcal{O}_{E_v}$ where E_v is a commutative semi-simple Q_v-algebra as in the preceding section. From Theorem 5.13 on we assume that the factors $E_{v,i}$ of E_v are *separable* field extensions of Q_v. Throughout we assume that $L \supset \psi(E_v)$ for all $\psi \in H_{E_v}$. Using Proposition 4.8 we may choose a basis of $\underline{\hat{M}}$ and write it under the decomposition (4.1) as

$$\underline{\hat{M}} \cong \prod_i \prod_{j\in\mathbb{Z}/f_i\mathbb{Z}} (R[\![y_i]\!], \tau_{i,j}) \quad \text{with } \tau_{i,j} \in R[\![y_i]\!][\tfrac{1}{z-\zeta}]^\times.$$

Let $c \in \mathbb{N}_0$ be such that $(z-\zeta)^c\tau_{ij}, (z-\zeta)^c\tau_{ij}^{-1} \in R[\![y_i]\!]$. Since the $y_i - \varphi(y_i)$ for $\varphi \in H_{E_v}$ with $(i,j)(\varphi) = (i,j)$ are prime elements in the factorial ring $R[\![y_i]\!]$, Lemma 4.6 applied to $(z-\zeta)^c\tau_{ij}\cdot(z-\zeta)^c\tau_{ij}^{-1} = (z-\zeta)^{2c}$ shows that

$$\tau_{i,j} = \epsilon_{i,j}\cdot \prod_{\varphi\in H_{E_v}\colon (i,j)(\varphi)=(i,j)} (y_i - \varphi(y_i))^{d_\varphi} \tag{5.1}$$

for a unit $\epsilon_{i,j} \in R[\![y_i]\!]^\times$ and integers $d_\varphi \in \mathbb{Z}$. By Definition 4.11 the tuple $\Phi = (d_\varphi)_\varphi$ is the local CM-type of $\underline{\hat{M}}$.

Note that we can view $\underline{\hat{M}}$ as the tensor product $\underline{\hat{M}}_{E_v,0} \otimes \bigotimes_\varphi \underline{\hat{M}}_{E_v,\varphi}{}^{\otimes d_\varphi}$ over $\mathcal{O}_{E_v,R}$ of $\underline{\hat{M}}_{E_v,0} := (\mathcal{O}_{E_v,R}, \tau_0 = \prod_{i,j}\epsilon_{i,j})$ and all the d_φth powers of $\underline{\hat{M}}_{E_v,\varphi} := (\mathcal{O}_{E_v,R}, \prod_{i,j}\tau_{\varphi,i,j})$ where

$$\tau_{\varphi,i,j} = \begin{cases} 1 & \text{if } (i,j) \neq (i,j)(\varphi), \\ y_i - \varphi(y_i) & \text{if } (i,j) = (i,j)(\varphi). \end{cases}$$

Likewise the cohomology realizations decompose as tensor products

$$\mathrm{H}^1_v(\underline{\hat{M}}, A_v) \cong \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,0}, A_v) \otimes \bigotimes_{\varphi\in H_{E_v}} \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)^{\otimes d_\varphi},$$

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where the tensor product is over $\mathcal{O}_{E_v}$, and

$$\mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, L[\![z-\zeta]\!]) \cong \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}_{E_v,0}, L[\![z-\zeta]\!]) \otimes \bigotimes_{\varphi\in H_{E_v}} \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}_{E_v,\varphi}, L[\![z-\zeta]\!])^{\otimes d_\varphi},$$

where the tensor product is over $E_v \otimes_{Q_v} L[\![z-\zeta]\!]$. For the purpose of computing the period isomorphism $h_{v,\mathrm{dR}}$, we may therefore treat all factors of $\tau_{i,j}$ separately; see 5.4 below.

5.2. We first treat the case of $\underline{\hat{M}}_{E_v,0} = (\mathcal{O}_{E_v,R}, \tau_0 = (\epsilon_{i,j})_{i,j})$, where $\epsilon_{i,j} \in R[\![y_i]\!]^\times$. We compute the τ-invariants $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,0}, A_v)$ as the set of tuples $(c_{i,j})_{i,j}$ with $c_{i,j} := \sum_{n=0}^\infty c_{i,j,n}\, y_i^n \in L^{\mathrm{sep}}[\![y_i]\!]$ subject to the condition

$$(c_{i,j})_{i,j} = \tau_0 \circ \hat{\sigma}((c_{i,j})_{i,j}), \quad \text{that is } c_{i,j} = \epsilon_{i,j}\cdot\hat{\sigma}(c_{i,j-1}) \quad \text{for all } i, j.$$

The latter implies $c_{i,j} = \epsilon_{i,j}\cdot\hat{\sigma}(\epsilon_{i,j-1})\cdot\ldots\cdot\hat{\sigma}^{j-1}(\epsilon_{i,1})\cdot\hat{\sigma}^j(c_{i,0})$ and $c_{i,0} = \epsilon_i\cdot\hat{\sigma}^{f_i}(c_{i,0})$, where we set $\epsilon_i := \epsilon_{i,0}\cdot\hat{\sigma}(\epsilon_{i,f_i-1})\cdot\ldots\cdot\hat{\sigma}^{f_i-1}(\epsilon_{i,1}) = \sum_{n=0}^\infty b_{i,n}\, y_i^n \in R[\![y_i]\!]^\times$. In particular, $b_{i,0} \in R^\times$. The resulting formulas for the coefficients

$$c_{i,0,0} = b_{i,0}\cdot c_{i,0,0}^{\tilde{q}_i} \quad \text{and} \quad c_{i,0,n} - b_{i,0}\cdot c_{i,0,n}^{\tilde{q}_i} = \sum_{\ell=1}^n b_{i,\ell}\cdot c_{i,0,n-\ell}^{\tilde{q}_i},$$

where $\tilde{q}_i = q_v^{f_i}$, lead to the formulas

$$c_{i,0,0}^{\tilde{q}_i-1} = b_{i,0}^{-1} \quad \text{and} \quad \frac{c_{i,0,n}}{c_{i,0,0}} - \left(\frac{c_{i,0,n}}{c_{i,0,0}}\right)^{\tilde{q}_i} = \sum_{\ell=1}^n \frac{b_{i,\ell}}{b_{i,0}}\cdot\left(\frac{c_{i,0,n-\ell}}{c_{i,0,0}}\right)^{\tilde{q}_i},$$

which have solutions $c_{i,0,n} \in \mathcal{O}_{L^{\mathrm{sep}}}$ with $c_{i,0,0} \in \mathcal{O}_{L^{\mathrm{sep}}}^\times$. In particular, the field extension of L generated by the $c_{i,j,n}$ is unramified. Then $(c_{i,j})_{i,j}$ is an $\mathcal{O}_{E_v}$-basis of $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,0}, A_v)$. Under the period isomorphism $h_{v,\mathrm{dR}}$ it is mapped to

$$(\epsilon_{i,j}^{-1} c_{i,j})_{i,j} \in (\mathcal{O}_{E_v} \otimes_{A_v} \mathcal{O}_{\mathbb{C}_v}[\![z]\!])^\times \subset E_v \otimes_{Q_v} \mathbb{C}_v[\![z-\zeta]\!] = \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}_{E_v,0}, \mathbb{C}_v[\![z-\zeta]\!]).$$

5.3. Next we compute the period isomorphism for the local shtuka $\underline{\hat{M}}_{E_v,\varphi}$ from above. For an element $0 \neq \xi \in (\pi_L) \subset R$ we consider the equation

$$\hat{\sigma}^{f_i}(\ell^+_{y_i,\xi}) = (y_i - \xi)\cdot\ell^+_{y_i,\xi} \quad \text{for } \ell^+_{y_i,\xi} := \sum_{n=0}^\infty \ell_n y_i^n \in L^{\mathrm{sep}}[\![y_i]\!]. \tag{5.2}$$

The equation can be solved by taking $\ell_n \in L^{\mathrm{sep}}$ with $\ell_0^{\tilde{q}_i-1} = -\xi$ and $\ell_n^{\tilde{q}_i} + \xi\ell_n = \ell_{n-1}$. This implies that $|\ell_n| = |\xi|^{\tilde{q}_i^{-n}/(\tilde{q}_i-1)} < 1$ and $\ell_n \in \mathcal{O}_{L^{\mathrm{sep}}}$. Note that this solution is not unique, but that every other solution $\tilde{\ell}^+_{y_i,\xi}$ of (5.2) is obtained by multiplying $\ell^+_{y_i,\xi}$ by an element of $\mathbb{F}_{\tilde{v}_i}[\![y_i]\!] = \mathcal{O}_{E_{v,i}}$, because $\hat{\sigma}^{f_i}\left(\frac{\tilde{\ell}^+_{y_i,\xi}}{\ell^+_{y_i,\xi}}\right) = \frac{(y_i-\xi)\cdot\tilde{\ell}^+_{y_i,\xi}}{(y_i-\xi)\cdot\ell^+_{y_i,\xi}} = \frac{\tilde{\ell}^+_{y_i,\xi}}{\ell^+_{y_i,\xi}} \in L^{\mathrm{sep}}[\![y_i]\!]$ is invariant under $\hat{\sigma}^{f_i}$ and hence lies in $\mathbb{F}_{\tilde{v}_i}[\![y_i]\!]$.

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According to the decomposition (4.1) the τ-invariants $\check{u} \in \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)$ of $\underline{\hat{M}}_{E_v,\varphi}$ have the form $\check{u} = (\check{u}_{i,j})_{i,j} \in \prod_{i,j} L^{\mathrm{sep}}[\![y_i]\!]$ with $\check{u} = \tau_\varphi \cdot \hat{\sigma}(\check{u})$, that is

$$\check{u}_{i,j} = \begin{cases} \hat{\sigma}(\check{u}_{i,j-1}) & \text{if } (i,j) \neq (i,j)(\varphi), \\ (y_i - \varphi(y_i)) \cdot \hat{\sigma}(\check{u}_{i,j-1}) & \text{if } (i,j) = (i,j)(\varphi). \end{cases}$$

For $j, j' \in \mathbb{Z}/f_i\mathbb{Z}$ we denote by (j, j') the representative of $j - j'$ in $\{0, \ldots, f_i - 1\}$. This implies that $\check{u}_{(i,j)(\varphi)} = (y_{i(\varphi)} - \varphi(y_{i(\varphi)})) \cdot \hat{\sigma}^{f_{i(\varphi)}}(\check{u}_{(i,j)(\varphi)})$, and $\check{u}_{i(\varphi),j} = \hat{\sigma}^{(j,j(\varphi))}(\check{u}_{(i,j)(\varphi)})$, as well as $\check{u}_{i,j} \in \mathbb{F}_{\tilde{v}_i}[\![y_i]\!]$ for all $i \neq i(\varphi)$ and all j. In particular, an $\mathcal{O}_{E_v}$-basis of $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)$ is given by

$$\check{u} = (\check{u}_{i,j})_{i,j} \quad \text{with } \check{u}_{i,j} = \hat{\sigma}^{(j,j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})^{-\delta_{i,i(\varphi)}} = (\ell^+_{y_i,\varphi(y_i)^{q_v^{(j,j(\varphi))}}})^{-\delta_{i,i(\varphi)}} \tag{5.3}$$

where $\delta_{i,i(\varphi)}$ is the Kronecker δ. The comparison isomorphism $h_{v,\mathrm{dR}}$ sends this $\check{u}$ to the element

$$\tau_\varphi^{-1} \cdot \check{u} = \left(\left((y_i - \varphi(y_i))^{\delta_{j,j(\varphi)}} \cdot \hat{\sigma}^{(j,j(\varphi))}(\ell^+_{y_i,\varphi(y_i)}) \right)^{-\delta_{i,i(\varphi)}} \right)_{i,j} \tag{5.4}$$

of $E_v \otimes_{Q_v} \mathbb{C}_v(\!(z-\zeta)\!) = \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}_{E_v,\varphi}, \mathbb{C}_v(\!(z-\zeta)\!))$.

5.4. Putting everything together we see that our $\underline{\hat{M}} \cong (\mathcal{O}_{E_v,R}, \prod_{i,j} \tau_{i,j})$ with $\tau_{i,j}$ from (5.1) has

$$\check{u} = (\check{u}_{i,j})_{i,j} = \left(c_{i,j} \cdot \prod_{\varphi \in H_{E_{v,i}}} \hat{\sigma}^{(j,j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})^{-d_\varphi} \right)_{i,j}$$

as an $\mathcal{O}_{E_v}$-basis of $\mathrm{H}^1_v(\underline{\hat{M}}, A_v) \cong \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,0}, A_v) \otimes \bigotimes_{\varphi \in H_{E_v}} \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)^{\otimes d_\varphi}$, where the tensor product is over $\mathcal{O}_{E_v}$. Under $h_{v,\mathrm{dR}}$ this $\check{u}$ is mapped to the element

$$\tau_{\hat{M}}^{-1} \cdot \check{u} = \left(\epsilon_{i,j}^{-1} c_{i,j} \cdot \prod_{\varphi \in H_{E_{v,i}}} \left((y_i - \varphi(y_i))^{\delta_{j,j(\varphi)}} \cdot \hat{\sigma}^{(j,j(\varphi))}(\ell^+_{y_i,\varphi(y_i)}) \right)^{-d_\varphi} \right)_{i,j} \tag{5.5}$$

of

$$\begin{aligned} E_v \otimes_{Q_v} \mathbb{C}_v(\!(z-\zeta)\!) &\cong \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}, \mathbb{C}_v(\!(z-\zeta)\!)) \\ &\cong \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}_{E_v,0}, \mathbb{C}_v[\![z-\zeta]\!]) \otimes \bigotimes_{\varphi \in H_{E_v}} \mathrm{H}^1_{\mathrm{dR}}(\underline{\hat{M}}_{E_v,\varphi}, \mathbb{C}_v[\![z-\zeta]\!])^{\otimes d_\varphi}, \end{aligned}$$

where the tensor product is over $E_v \otimes_{Q_v} \mathbb{C}_v[\![z-\zeta]\!]$.

Remark 5.5. Note that $\underline{\hat{M}}_{E_v,\varphi} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_{v,i}}$ with $i = i(\varphi)$ is the local $\hat{\sigma}$-shtuka associated with a Lubin–Tate formal group, and so our treatment is analogous to Colmez's [10, § I.2]. Namely, let $\hat{G} = \hat{\mathbb{G}}_{a,R} = \mathrm{Spf}\, R[\![X]\!]$ be the formal additive group over R with an action of $\mathcal{O}_{E_{v,i}} = \mathbb{F}_{\tilde{v}_i}[\![y_i]\!]$ given by

$$\begin{aligned} [\lambda] : X &\longmapsto \varphi(\lambda) \cdot X = \lambda^{q_v^{j(\varphi)}} \cdot X \quad \text{for } \lambda \in \mathbb{F}_{\tilde{v}_i}, \\ [y_i] : X &\longmapsto X^{\tilde{q}_i} + \varphi(y_i) \cdot X. \end{aligned}$$

Then $\hat{G}$ is the Lubin–Tate formal group over R associated with $\mathcal{O}_{\varphi(E_{v,i})}$; see [20]. It is a *z-divisible local Anderson module* in the sense of [19, Definition 7.1]. For an element $a \in \mathcal{O}_{E_{v,i}}$ let $\hat{G}[a] := \ker[a]$. Under the anti-equivalence between z-divisible local Anderson modules and effective local $\hat{\sigma}$-shtukas over $S = \operatorname{Spec} R$ from [19, Theorem 8.3] the associated local shtuka is

$$\hat{M} := \hat{M}(\hat{G}) := \varprojlim_n \operatorname{Hom}_R(\hat{G}[z^n], \mathbb{G}_{a,R}) = \varprojlim_n \operatorname{Hom}_R(\hat{G}[y_i^{ne_i}], \mathbb{G}_{a,R}) = \bigoplus_{k=0}^{f_i-1} R[\![y_i]\!]\tau^k$$

with $\tau^0 := \mathrm{id}\colon \hat{G} \xrightarrow{\sim} \hat{\mathbb{G}}_{a,R}$ and $\tau^k := \operatorname{Frob}_{q_v,\hat{\mathbb{G}}_{a,R}} \circ \tau^0\colon X \mapsto X^{q_v^k}$. It is an $\mathcal{O}_{E_{v,i},R}$-module via the $\mathcal{O}_{E_{v,i}}$-action on $\hat{G}[z^n]$ and the R-action on $\mathbb{G}_{a,R}$, and is equipped with the Frobenius $\tau_{\hat{M}}\colon \hat{\sigma}^*\hat{M} \to \hat{M}$ given by $\hat{\sigma}^*_{\hat{M}} m \mapsto \operatorname{Frob}_{q_v,\mathbb{G}_{a,R}} \circ m$ for $m \in \hat{M}$. We set $\underline{\hat{M}}(\hat{G}) := (\hat{M}, \tau_{\hat{M}})$. In particular, we see that $\lambda \in \mathbb{F}_{\tilde{v}_i}$ acts on $R[\![y_i]\!]\tau^k$ as $\lambda^{q_v^{k+j(\varphi)}}$ and so $\hat{M}/\mathfrak{b}_{i,j}\hat{M} = R[\![y_i]\!]\tau^{(j,j(\varphi))}$ under the decomposition (4.1). Since $\tau_{\hat{M}}(\hat{\sigma}^*_{\hat{M}}\tau^{f_i-1}) = \tau^{f_i} = [y_i] - \varphi(y_i)\colon X \mapsto X^{\tilde{q}_i} = \big([y_i] - \varphi(y_i)\big)(X)$, we see that

$$\tau_{\hat{M}} = (\tau_{\hat{M},j})_j \quad \text{with } \tau_{\hat{M},j} = \begin{cases} 1 & \text{if } j \neq j(\varphi), \\ y_i - \varphi(y_i) & \text{if } j = j(\varphi), \end{cases}$$

that is, $\underline{\hat{M}}(\hat{G}) = \underline{\hat{M}}_{E_v,\varphi} \otimes_{\mathcal{O}_{E_v}} \mathcal{O}_{E_{v,i}}$.

Moreover, if we want to also consider the other components of $\underline{\hat{M}}_{E_v,\varphi}$ for $i \neq i(\varphi)$ we take the divisible local Anderson module $\hat{G}_{E_v,\varphi} := \hat{G} \times \prod_{i \neq i(\varphi)} (E_{v,i}/\mathcal{O}_{E_{v,i}})_R$. It has local shtuka $\underline{\hat{M}}(\hat{G}_{E_v,\varphi}) = \underline{\hat{M}}(\hat{G}) \oplus \bigoplus_{i \neq i(\varphi)} \underline{\hat{M}}\big((E_{v,i}/\mathcal{O}_{E_{v,i}})_R\big) = \underline{\hat{M}}_{E_v,\varphi}$, because $\underline{\hat{M}}\big((E_{v,i}/\mathcal{O}_{E_{v,i}})_R\big) = (\mathcal{O}_{E_{v,i},R}, \tau = 1)$.

5.6. We want to describe the Galois action of $\mathscr{G}_L$ on $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)$. Recall from [18, Definition 4.8, Proposition 4.9 and Remark 4.10] that the Tate module of $\hat{G}$ is defined as $T_v\hat{G} := \operatorname{Hom}_{A_v}\big(Q_v/A_v, \hat{G}(L^{\mathrm{sep}})\big)$ and that there is a perfect pairing of A_v-modules

$$T_v\hat{G} \times \mathrm{H}^1_v(\underline{\hat{M}}(\hat{G}), A_v) \longrightarrow \operatorname{Hom}_{\mathbb{F}_v}(Q_v/A_v, \mathbb{F}_v), \quad (f, m) \longmapsto m \circ f, \tag{5.6}$$

which is equivariant for the actions of $\mathscr{G}_L$ and $\operatorname{End}_R(\underline{\hat{M}}(\hat{G})) = \operatorname{End}_R(\hat{G})^{\mathrm{op}}$. Here the A_v-module $\operatorname{Hom}_{\mathbb{F}_v}(Q_v/A_v, \mathbb{F}_v) \cong \widehat{\Omega}^1_{A_v/\mathbb{F}_v} \cong \mathbb{F}_v[\![z]\!]dz$ is free of rank one; see [18, Equation (4.5) before Proposition 4.9]. We have already computed $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v) = E_{v,i} \cdot (\check{u}_{i,j})_{i,j}$ in (5.3). We now compute $T_v\hat{G}_{E_v,\varphi}$ and the action of $\mathscr{G}_L$ on both $T_v\hat{G}_{E_v,\varphi}$ and $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)$. Let again $i = i(\varphi)$. Since $\mathcal{O}_{E_{v,i}}$ acts on $\hat{G}(L^{\mathrm{sep}})$ we have

$$\begin{aligned} T_v\hat{G} &= \operatorname{Hom}_{\mathcal{O}_{E_{v,i}}}(\mathcal{O}_{E_{v,i}} \otimes_{A_v} (Q_v/A_v), \hat{G}(L^{\mathrm{sep}})) \\ &= \operatorname{Hom}_{\mathcal{O}_{E_{v,i}}}(E_{v,i}/\mathcal{O}_{E_{v,i}}, \hat{G}(L^{\mathrm{sep}})) && \ni f \\ &= \Big\{(P_n)_n \in \prod_{n \in \mathbb{N}_0} \hat{G}[y_i^n](L^{\mathrm{sep}})\colon [y_i](P_n) = P_{n-1}\Big\} && \ni (P_n)_n := (f(y_i^{-n}))_n, \end{aligned}$$

where f is reconstructed from $(P_n)_n$ as $f(ay_i^{-n}) := [a](P_n)$ for $a \in \mathcal{O}_{E_{v,i}}^\times$. From equation (5.2) we see that $\ell^+_{y_i,\varphi(y_i)} = \sum_{n=0}^\infty \ell_n y_i^n$ satisfies

$$[y_i](\ell_0) = \ell_0^{\tilde{q}_i} + \varphi(y_i)\ell_0 = 0 \quad \text{and} \quad [y_i](\ell_n) = \ell_n^{\tilde{q}_i} + \varphi(y_i)\ell_n = \ell_{n-1}.$$

Thus $\ell_{n-1} \in \hat{G}[y_i^n](L^{\mathrm{sep}})$ and $T_v\hat{G} = \mathcal{O}_{E_{v,i}} \cdot (\ell_{n-1})_n$. To compute the $\mathscr{G}_L$-action on $T_v\hat{G}$ we need the following

Proposition 5.7. *Let* $\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty := \mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))(\ell_n : n \in \mathbb{N}_0)$. *Then there is an isomorphism of topological groups*

$$\chi :\ \mathrm{Gal}(\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty / \mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))) \xrightarrow{\sim} \mathbb{F}_{\tilde{v}_i}[\![y_i]\!]^\times = \mathcal{O}_{E_{v,i}}^\times$$

satisfying $g(\ell^+_{y_i,\varphi(y_i)}) := \sum_{n=0}^\infty g(\ell_n) y_i^n = \chi(g)\cdot \ell^+_{y_i,\varphi(y_i)}$ *in* $\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty[\![y_i]\!]$ *for* g *in the Galois group. The isomorphism* χ *is independent of the choice of* $\ell^+_{y_i,\varphi(y_i)}$ *and is called the* cyclotomic character *of the field* $E_{v,i} = \mathbb{F}_{\tilde{v}_i}((y_i))$.

Proof. The existence of χ follows from the equation $\hat{\sigma}^{f_i}(\ell^+_{y_i,\varphi(y_i)}) = (y_i - \varphi(y_i))\cdot \ell^+_{y_i,\varphi(y_i)}$, which implies that $\chi(g) := \frac{g(\ell^+_{y_i,\varphi(y_i)})}{\ell^+_{y_i,\varphi(y_i)}}$ is $\hat{\sigma}^{f_i}$-invariant, that is $\chi(g) \in \mathbb{F}_{\tilde{v}_i}[\![y_i]\!]^\times$. Furthermore, χ is an isomorphism because ℓ_{n-1} is a uniformizing parameter of $\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))(\ell_0, \ldots, \ell_{n-1})$ and so the equations defining the ℓ_n are irreducible by Eisenstein. Every other solution of (5.2) is of the form $a \cdot \ell^+_{y_i,\varphi(y_i)}$ with $a \in \mathbb{F}_{\tilde{v}_i}[\![y_i]\!]$ and so $g(a\cdot \ell^+_{y_i,\varphi(y_i)}) = a \cdot g(\ell^+_{y_i,\varphi(y_i)}) = \chi(g)\cdot a \cdot \ell^+_{y_i,\varphi(y_i)}$. This shows that $\chi(g)$ does not depend on the solution $\ell^+_{y_i,\varphi(y_i)}$. □

Let $\mathcal{I}_L \subset \mathscr{G}_L$ be the inertia subgroup and similarly for other fields. By local class field theory, see Lubin and Tate [20, Corollary on p. 386], the image of $g \in \mathcal{I}_{\varphi(E_{v,i})}$ in $\mathscr{G}^{\mathrm{ab}}_{\varphi(E_{v,i})}$ equals the norm residue symbol $\big(\chi(g)^{-1}|_{y_i=\varphi(y_i)}, \varphi(E_{v,i})^{\mathrm{ab}}/\varphi(E_{v,i})\big)$ where $\varphi(E_{v,i})^{\mathrm{ab}}$ is the maximal abelian extension of $\varphi(E_{v,i})$ in Q_v^{sep}. In general, the homomorphism $\chi_L : \mathcal{I}_L \to \mathcal{O}_L^\times$ with $g|_{L^{\mathrm{ab}}} = (\chi_L(g)^{-1}, L^{\mathrm{ab}}/L)$ is sometimes called the *character of local class field theory* of the field L. So we see that $\chi(g)|_{y_i=\varphi(y_i)} = \chi_{\varphi(E_{v,i})}(g)$. If L is separable over $\varphi(E_{v,i})$ these characters are compatible for $g \in \mathcal{I}_L$ in the sense that $\chi_{\varphi(E_{v,i})}(g) = N_{L/\varphi(E_{v,i})}(\chi_L(g))$.

5.8. From $T_v\hat{G} = \mathcal{O}_{E_{v,i}} \cdot (\ell_{n-1})_n$ it follows that g acts on $T_v\hat{G}$ in the same way as an endomorphism in $\mathcal{O}_{E_{v,i}}^\times$. Let us compute this endomorphism. We write $\chi(g) = \sum_{k=0}^\infty a_k y_i^k$ with $a_k \in \mathbb{F}_{\tilde{v}_i}$. Then the expansion $g(\ell^+_{y_i,\varphi(y_i)}) = \chi(g)\cdot \ell^+_{y_i,\varphi(y_i)} = \sum_{n=0}^\infty \sum_{k=0}^n a_k \ell_{n-k} y_i^n$ implies that $g(\ell_n) = \sum_{k=0}^n a_k \ell_{n-k} = \sum_{k=0}^n \varphi(a_k^{q_v^{-j(\varphi)}})[y_i^k](\ell_n)$. Thus every element $g \in \mathcal{I}_L$ acts on $T_v\hat{G}$ as the endomorphism $\sum_{k=0}^\infty a_k^{q_v^{-j(\varphi)}} y_i^k = \hat{\sigma}^{-j(\varphi)}(\chi(g)) = \varphi^{-1}(\chi(g)|_{y_i=\varphi(y_i)}) = \varphi^{-1} \circ \chi_{\varphi(E_{v,i})}(g) \in \mathcal{O}_{E_{v,i}}^\times$ and on $T_v\hat{G}_{E_v,\varphi}$ as the endomorphism $(\hat{\sigma}^{-j(\varphi)}(\chi(g))^{\delta_{i,i(\varphi)}})_i \in \mathcal{O}_{E_v}^\times$.

Definition 5.9. We define the character $\chi_{E_v,\varphi} := ((\varphi^{-1} \circ \chi_{\varphi(E_{v,i})})^{\delta_{i,i(\varphi)}})_i : \mathcal{I}_L \to \mathcal{O}_{E_v}^\times$ by mapping $g \mapsto (\hat{\sigma}^{-j(\varphi)}(\chi(g))^{\delta_{i,i(\varphi)}})_i = (1, \ldots, 1, \varphi^{-1} \circ \chi_{\varphi(E_{v,i})}(g), 1, \ldots, 1)$.

5.10. Due to the equivariance of the pairing (5.6) under $\mathscr{G}_L$ and $\operatorname{End}_R(\hat{G}_{E_v,\varphi})$ the action of $g \in \mathscr{G}_L$ on $\mathrm{H}^1_v(\underline{\hat{M}}(\hat{G}_{E_v,\varphi}), A_v)$ is given by the endomorphism $\chi_{E_v,\varphi}(g)^{-1}$. We can also compute this action directly as follows. It factors through the restriction of g to $\operatorname{Gal}(\mathbb{F}_{\tilde{v}_i}((\varphi(y_i)))_\infty / \mathbb{F}_{\tilde{v}_i}((\varphi(y_i))))$ which we denote again by g. Then on the basis $(\check{u}_{i,j})_j$ of $\mathrm{H}^1_v(\underline{\hat{M}}(\hat{G}), A_v)$ from (5.3) we compute

$$\begin{aligned} g(\check{u}_{i,j})_j &= g\big(\hat{\sigma}^{(j,j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})^{-\delta_{i,i(\varphi)}}\big)_j \\ &= \big(\hat{\sigma}^{(j,j(\varphi))}(\chi(g)\cdot \ell^+_{y_i,\varphi(y_i)})^{-\delta_{i,i(\varphi)}}\big)_j \\ &= \big(\hat{\sigma}^{j-j(\varphi)}(\chi(g)^{-1})^{\delta_{i,i(\varphi)}}\cdot \hat{\sigma}^{(j,j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})^{-\delta_{i,i(\varphi)}}\big)_j \\ &= \big(\hat{\sigma}^{-j(\varphi)}(\chi(g)^{-1})\otimes 1\big)^{\delta_{i,i(\varphi)}}\cdot(\check{u}_{i,j})_j \end{aligned}$$

for the element $\hat{\sigma}^{-j(\varphi)}(\chi(g)^{-1})\otimes 1 \in \mathcal{O}_{E_{v,i}}\otimes_{A_v} R[\![z]\!]$. That is, the action of $g \in \mathscr{G}_L$ on $\mathrm{H}^1_v(\underline{\hat{M}}(\hat{G}), A_v)$ coincides with the endomorphism $\hat{\sigma}^{-j(\varphi)}(\chi(g)^{-1})$ and the action of $g \in \mathcal{I}_L$ on $\mathrm{H}^1_v(\underline{\hat{M}}(\hat{G}_{E_v,\varphi}), A_v) = \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)$ coincides with the endomorphism $\chi_{E_v,\varphi}(g)^{-1} \in \mathcal{O}^\times_{E_v}$.

Proposition 5.11. *Let $\underline{\hat{M}}$ have complex multiplication by a commutative, semi-simple Q_v-algebra E_v with local CM-type $\Phi = (d_\varphi)_{\varphi\in H_{E_v}}$. Then the action of $g \in \mathcal{I}_L$ on $\mathrm{H}^1_v(\underline{\hat{M}}, A_v)$ coincides with the endomorphism $\prod_{\varphi\in H_{E_v}} \chi_{E_v,\varphi}(g)^{-d_\varphi} \in \mathcal{O}^\times_{E_v}$.*

Proof. This follows from the computations in 5.4, 5.10 and 5.2 by observing that $\mathcal{I}_L$ acts trivially on $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,0}, A_v)$, because its generator $(c_{i,j})_{i,j}$ is defined over the maximal unramified extension of L. □

5.12. To compute the absolute value $\left|\int_u \omega\right|_v$ we again treat each factor $\underline{\hat{M}}_{E_v,0}$ and $\underline{\hat{M}}_{E_v,\varphi}$ of $\underline{\hat{M}}$ separately. We begin with $\underline{\hat{M}}_{E_v,\varphi}$ and set $i := i(\varphi)$. Let

$$\omega^\circ_\psi := 1 \in \mathrm{H}^\psi(\underline{\hat{M}}_{E_v,\varphi}, L[\![y_i - \psi(y_i)]\!]).$$

It is a generator as $L[\![y_i - \psi(y_i)]\!]$-module as in Definition 4.10, and is mapped under the period isomorphism of $\underline{\hat{M}}_{E_v,\varphi}$ from (5.4) to

$$h^{-1}_{v,\mathrm{dR}}(\omega^\circ_\psi) = \Big(0,\ldots,\Big((y_{i(\psi)} - \varphi(y_{i(\psi)}))^{\delta_{j(\psi),j(\varphi)}}\cdot \hat{\sigma}^{(j(\psi),j(\varphi))}(\ell^+_{y_{i(\psi)},\varphi(y_{i(\psi)})})\Big)^{\delta_{i(\psi),i(\varphi)}},\ldots,0\Big)\cdot\check{u}, \tag{5.7}$$

where the non-zero entry is in component ψ. We denote this entry by $\Omega(E_v,\varphi,\psi)$. It is analogous to Colmez's [10, Théorème I.2.1] element of $\mathbf{B}_{\mathrm{dR}}$ with the same name. It satisfies the following

Theorem 5.13. *Let $\varphi,\psi \in H_{E_v}$ satisfy $i(\varphi) = i(\psi) =: i$ and assume that $E_{v,i}$ is separable over Q_v. Then the element*

$$\Omega(E_v,\varphi,\psi) := (y_i - \varphi(y_i))^{\delta_{j(\psi),j(\varphi)}}\cdot\hat{\sigma}^{(j(\psi),j(\varphi))}(\ell^+_{y_i,\varphi(y_i)}) \in \mathbb{C}_v((y_i - \psi(y_i))) = \mathbb{C}_v((z-\zeta))$$

satisfies

(a) $\hat{v}(\Omega(E_v,\varphi,\psi)) = 1$ *if* $\varphi=\psi$ *and* $\hat{v}(\Omega(E_v,\varphi,\psi)) = 0$ *if* $\varphi\neq\psi$.

(b)

$$v(\Omega(E_v,\varphi,\psi)) = \begin{cases} \dfrac{1}{e_i(\tilde{q}_i-1)} - v(\mathfrak{D}_{\psi(E_{v,i})/Q_v}) & \text{if } \varphi=\psi,\\ \dfrac{1}{e_i(\tilde{q}_i-1)} + v(\psi(y_i)-\varphi(y_i)) & \text{if } \varphi\neq\psi \text{ and } j(\varphi)=j(\psi),\\ \dfrac{q_v^{(j(\psi),j(\varphi))}}{e_i(\tilde{q}_i-1)} & \text{if } j(\varphi)\neq j(\psi), \end{cases}$$

where $\mathfrak{D}_{\psi(E_{v,i})/Q_v}$ *is the different of* $\psi(E_{v,i})$ *over* Q_v.

(c) *If* $g\in\mathcal{I}_L$, *then* $g(\Omega(E_v,\varphi,\psi)) = \psi(\chi_{E_v,\varphi}(g))\cdot\Omega(E_v,\varphi,\psi)$. *Note that if* L *is separable over* Q_v *then* $\psi(\chi_{E_v,\varphi}(g)) = \psi(\varphi^{-1}(N_{L/\varphi(E_{v,i})}\chi_L(g)))$.

(d) *Let* $u\in \mathrm{H}_{1,v}(\underline{\hat{M}}_{E_v,\varphi},A_v) := \mathrm{Hom}_{A_v}(\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi},A_v),A_v)$ *be a generator as* $\mathcal{O}_{E_v}$*-module and let* ω°_ψ *be an* $L[\![y_i-\psi(y_i)]\!]$*-generator of* $\mathrm{H}^\psi(\underline{\hat{M}}_{E_v,\varphi},L[\![y_i-\psi(y_i)]\!])$ *subject to the conditions in Definition 4.10, that is subject to the condition that* $\omega^\circ_\psi \bmod y_i-\varphi(y_i)\in \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi},L)$ *is an* R*-generator of the free* R*-module of rank one* $\mathrm{H}^\psi(\underline{\hat{M}},R)/(y_i-\psi(y_i))\,\mathrm{H}^\psi(\underline{\hat{M}},R)$. *Moreover, let* D_ψ *be a generator as* $\psi(\mathcal{O}_{E_{v,i}})$*-module of the different* $\mathfrak{D}_{\psi(E_{v,i})/Q_v}$. *Then*

$$\int_u \omega^\circ_\psi := u\otimes \mathrm{id}_{\mathbb{C}_v(\!(z-\zeta)\!)}(h^{-1}_{v,\mathrm{dR}}(\omega^\circ_\psi)) \in \mathbb{C}_v(\!(z-\zeta)\!)$$

equals $\Omega(E_v,\varphi,\psi)\cdot D_\psi^{-1}$ *up to multiplication by* $R^\times + (z-\zeta)\cdot L[\![z-\zeta]\!]$.

Remark 5.14. Note that in contrast to the number field case [10, Théorème I.2.1] the element $\Omega(E_v,\varphi,\psi)\in\mathbb{C}_v(\!(z-\zeta)\!)$ is by (a)–(c) uniquely determined only up to multiplication by an element of $\mathcal{O}^\times_{\tilde{L}} + (z-\zeta)\cdot\tilde{L}[\![z-\zeta]\!]$, where $\tilde{L}$ is the completion of the compositum of $\mathbb{F}_q^{\mathrm{alg}}$ with the perfect closure of L in Q_v^{alg}, because the fixed field of $\mathcal{I}_L$ in $\mathbb{C}_v(\!(z-\zeta)\!)$ equals $\tilde{L}(\!(z-\zeta)\!)$ by the Ax–Sen–Tate theorem [3].

Proof of Theorem 5.13. In 5.3 we have seen that the coefficients of the series $\ell^+_{y_i,\varphi(y_i)} = \sum_{n=0}^\infty \ell_n y_i^n$ satisfy $v(\ell_n) = v(\varphi(y_i))\cdot\tilde{q}_i^{-n}/(\tilde{q}_i-1)$. From $v(\psi(y_i)) = 1/e_i = v(\varphi(y_i))$ it follows that the evaluation of $\hat{\sigma}^{(j(\psi),j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})|_{y_i=\psi(y_i)} = \sum_{n=0}^\infty \ell_n^{q_v^{(j(\psi),j(\varphi))}}\psi(y_i)^n$ at $y_i=\psi(y_i)$ satisfies

$$v(\ell_n^{q_v^{(j(\psi),j(\varphi))}}\psi(y_i)^n) = \frac{1}{e_i}\cdot\Big(n+\frac{q_v^{(j(\psi),j(\varphi))}}{\tilde{q}_i^n(\tilde{q}_i-1)}\Big). \tag{5.8}$$

Since $0\leqslant (j(\psi),j(\varphi))\leqslant f_i-1$ the second fraction in the parenthesis is strictly smaller than 1, and so the valuations in (5.8) are strictly increasing with n and attain their minimum $\frac{q_v^{(j(\psi),j(\varphi))}}{e_i(\tilde{q}_i-1)}$ for $n=0$. This shows that $\hat{\sigma}^{(j(\psi),j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})|_{y_i=\psi(y_i)}$ is non-zero in L and

$$v\Big(\hat{\sigma}^{(j(\psi),j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})\Big|_{y_i=\psi(y_i)}\Big) = \frac{q_v^{(j(\psi),j(\varphi))}}{e_i(\tilde{q}_i-1)}.$$

In particular, the valuation $\hat{v}\big(\hat{\sigma}^{(j(\psi),j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})\big) = 0$.

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(a) Lemma A.1 implies that in the ψ-component of $E_v \otimes_{Q_v} L[\![z-\zeta]\!]$ we have $\mathrm{ord}_{y_i-\psi(y_i)} = \mathrm{ord}_{z-\zeta}$. If $j(\varphi) = j(\psi)$, that is $\varphi|_{\mathbb{F}_{\tilde{v}_i}} = \psi|_{\mathbb{F}_{\tilde{v}_i}}$ then $\varphi \neq \psi$ implies that $\psi(y_i) - \varphi(y_i) \neq 0$ in L, because $E_{v,i} = \mathbb{F}_{\tilde{v}_i}(\!(y_i)\!)$. Therefore, the valuation $\hat{v}$ of

$$y_i - \varphi(y_i) = (\psi(y_i) - \varphi(y_i)) + (y_i - \psi(y_i))$$

equals zero for $\varphi \neq \psi$ and $j(\varphi) = j(\psi)$. This implies (a).

(b) We calculate $v\big((\Omega(E_v, \varphi, \psi)\big)$ in three different cases separately as follows.

Case 1: $\psi = \varphi$. In this case $\hat{v}\big(\Omega(E_v, \varphi, \psi)\big) = 1$ and so

$$\begin{aligned} v(\Omega(E_v, \varphi, \psi)) &= v\bigg(\Big(\frac{y_i - \varphi(y_i)}{z-\zeta} \cdot \ell^+_{y_i,\varphi(y_i)}\Big)\Big|_{y_i=\varphi(y_i)}\bigg) \\ &= v\bigg(\frac{y_i - \varphi(y_i)}{z-\zeta}\bigg|_{y_i=\varphi(y_i)}\bigg) + v\Big(\ell^+_{y_i,\varphi(y_i)}|_{y_i=\psi(y_i)}\Big) \\ &= -v(\mathfrak{D}_{\psi(E_{v,i})/Q_v}) + \frac{1}{e_i(\tilde{q}_i - 1)} \end{aligned}$$

by Corollary 4.7.

Case 2: $\psi \neq \varphi$ and $j(\psi) = j(\varphi)$. In this case $\hat{v}\big(\Omega(E_v, \varphi, \psi)\big) = 0$ and so

$$\begin{aligned} v(\Omega(E_v, \varphi, \psi)) &= v\Big(\big((y_i - \varphi(y_i)) \cdot \ell^+_{y_i,\varphi(y_i)}\big)\big|_{y_i=\psi(y_i)}\Big) \\ &= v\big(\psi(y_i) - \varphi(y_i)\big) + v\big(\ell^+_{y_i,\varphi(y_i)}|_{y_i=\psi(y_i)}\big) \\ &= v(\psi(y_i) - \varphi(y_i)) + \frac{1}{e_i(\tilde{q}_i - 1)}. \end{aligned}$$

Case 3: $j(\psi) \neq j(\varphi)$. In this case $\hat{v}\big(\Omega(E_v, \varphi, \psi)\big) = 0$ and so

$$v(\Omega(E_v, \varphi, \psi)) = v\bigg(\hat{\sigma}^{(j(\psi),j(\varphi))}(\ell^+_{y_i,\varphi(y_i)})\bigg|_{y_i=\psi(y_i)}\bigg) = \frac{q_v^{(j(\psi),j(\varphi))}}{e_i(\tilde{q}_i - 1)}.$$

(c) For the $\mathcal{O}_{E_v}$-basis $\check{u}$ of $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)$ from (5.3) we have seen in 5.10 that $g(\check{u}) = \chi_{E_v,\varphi}(g)^{-1} \cdot \check{u}$ and $g(0, \dots, \Omega(E_v, \varphi, \psi), \dots, 0) \cdot g(\check{u}) = h^{-1}_{v,\mathrm{dR}}(g(\omega^\circ_\psi)) = h^{-1}_{v,\mathrm{dR}}(\omega^\circ_\psi) = (0, \dots, \Omega(E_v, \varphi, \psi), \dots, 0) \cdot \check{u}$. Thus g acts on the coefficient $(0, \dots, \Omega(E_v, \varphi, \psi), \dots, 0)$ as multiplication with $\chi_{E_v,\varphi}(g)$ and on its ψ-component $\Omega(E_v, \varphi, \psi)$ by multiplication with $\psi(\chi_{E_v,\varphi}(g))$.

(d) Again we consider the $\mathcal{O}_{E_v}$-basis $\check{u}$ of $\mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v)$ from (5.3). Let $D = (D_i)_i \in \mathcal{O}_{E_v} = \prod_i \mathcal{O}_{E_{v,i}}$ be a generator of the different $\mathfrak{D}_{E_v/Q_v} = \prod_i \mathfrak{D}_{E_{v,i}/Q_v} = D \cdot \mathcal{O}_{E_v}$ and let $c = (c_i)_i \in \mathcal{O}^\times_{E_v} = \prod_i \mathcal{O}^\times_{E_{v,i}}$ be the element(s) from Lemma 5.15 below for which the pairing $\langle\, .\, ,\, .\,\rangle\colon \mathrm{H}_{1,v}(\underline{\hat{M}}_{E_v,\varphi}, A_v) \times \mathrm{H}^1_v(\underline{\hat{M}}_{E_v,\varphi}, A_v) \to A_v$ takes the value $\langle a\,u\,,\, b\,\check{u}\rangle = \mathrm{Tr}_{E_v/Q_v}(abcD^{-1})$ for $a, b \in \mathcal{O}_{E_v}$. If $\omega^\circ_\psi = 1$ is the generator from 5.12 then

$$\int_u \omega^\circ_\psi = \mathrm{Tr}_{E_v/Q_v}(0, \dots, \Omega(E_v, \varphi, \psi) \cdot \psi(cD^{-1}), \dots, 0) = \Omega(E_v, \varphi, \psi) \cdot \psi(c_i D_i^{-1}).$$

Any other generator ω°_ψ differs from $\omega^\circ_\psi = 1$ by multiplication by an element

$$x \in R^\times + (y_i - \varphi(y_i)) \cdot L[\![y_i - \varphi(y_i)]\!] \subset E_v \otimes_{Q_v} L[\![y_i - \varphi(y_i)]\!].$$

Under the pairing $\langle\, .\,,\, .\,\rangle$ this leads to $\int_u \omega^\circ_\psi = \Omega(E_v, \varphi, \psi) \cdot \psi(c_i x D_i^{-1})$ with $\psi(D_i) = D_\psi$ and

$$\psi(c_i x) \in R^\times + (y_i - \varphi(y_i)) \cdot L[\![y_i - \varphi(y_i)]\!] = R^\times + (z - \zeta) \cdot L[\![z - \zeta]\!]. \qquad \square$$

It remains to record the following well known

Lemma 5.15. *If $E_{v,i}/Q_v$ is separable let $D_i \in \mathcal{O}_{E_{v,i}}$ be a generator of the different $\mathfrak{D}_{E_{v,i}/Q_v} = D_i \cdot \mathcal{O}_{E_{v,i}}$. Then for any perfect pairing $\langle\, .\,,\, .\,\rangle : \mathcal{O}_{E_{v,i}} \times \mathcal{O}_{E_{v,i}} \to A_v$ satisfying $\langle a, b\rangle = \langle ab, 1\rangle = \langle 1, ab\rangle$, there is an element $c_i \in \mathcal{O}^\times_{E_{v,i}}$ with $\langle a, b\rangle = \mathrm{Tr}_{E_{v,i}/Q_v}(abc_i D_i^{-1})$.*

Proof. The set of bilinear forms $\mathcal{O}_{E_{v,i}} \times \mathcal{O}_{E_{v,i}} \to A_v$ equals $\mathrm{Hom}_{A_v}(\mathcal{O}_{E_{v,i}} \otimes_{A_v} \mathcal{O}_{E_{v,i}}, A_v)$ and the condition $\langle a, b\rangle = \langle ab, 1\rangle = \langle 1, ab\rangle$ implies that $\langle\, .\,,\, .\,\rangle$ lies in

$$\mathrm{Hom}_{A_v}(\mathcal{O}_{E_{v,i}} \otimes_{\mathcal{O}_{E_{v,i}}} \mathcal{O}_{E_{v,i}}, A_v) = \mathrm{Hom}_{A_v}(\mathcal{O}_{E_{v,i}}, A_v).$$

The condition that $\langle\, .\,,\, .\,\rangle$ is perfect implies that

$$\mathcal{O}_{E_{v,i}} \xrightarrow{\sim} \mathrm{Hom}_{A_v}(\mathcal{O}_{E_{v,i}}, A_v), \quad a \longmapsto [b \mapsto \langle a, b\rangle] \tag{5.9}$$

is an isomorphism of $\mathcal{O}_{E_{v,i}}$-modules. On the other hand, by definition of the different in [27, § III.3], there are also isomorphisms of $\mathcal{O}_{E_{v,i}}$-modules

$$\mathfrak{D}^{-1}_{E_{v,i}/Q_v} = D_i^{-1} \cdot \mathcal{O}_{E_{v,i}} \xrightarrow{\sim} \mathrm{Hom}_{A_v}(\mathcal{O}_{E_{v,i}}, A_v), \quad \tilde{a} D_i^{-1} \longmapsto [b \mapsto \mathrm{Tr}_{E_{v,i}/Q_v}(\tilde{a} b D_i^{-1})]$$

for $\tilde{a} \in \mathcal{O}_{E_{v,i}}$, and

$$\mathcal{O}_{E_{v,i}} \xrightarrow{\sim} \mathrm{Hom}_{A_v}(\mathcal{O}_{E_{v,i}}, A_v), \quad \tilde{a} \longmapsto [b \mapsto \mathrm{Tr}_{E_{v,i}/Q_v}(\tilde{a} b D_i^{-1})]. \tag{5.10}$$

Comparing (5.9) and (5.10) yields a unit $c_i \in \mathcal{O}^\times_{E_{v,i}}$ with $\tilde{a} = c_i a$ and $\langle a, b\rangle = \mathrm{Tr}_{E_{v,i}/Q_v}(abc_i D_i^{-1})$. $\square$

5.16. Analogously to [10, § I.2], we can give a uniform formula for $v\big(\Omega(E_v, \varphi, \psi)\big)$ by introducing certain measures on $\mathscr{G}_{Q_v}$. Let $\mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$ be the $\mathbb{Q}$-vector space of locally constant functions $a : \mathscr{G}_{Q_v} \to \mathbb{Q}$. If K is a finite separable extension of Q_v, and $\varphi, \psi \in H_K$, let $a_{K,\varphi,\psi} \in \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$ be the function given by

$$a_{K,\varphi,\psi}(g) := \begin{cases} 1 & \text{if } g\varphi = \psi, \\ 0 & \text{otherwise.} \end{cases}$$

Note that the $a_{K,\varphi,\psi}$ span $\mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$.

If $L \subset Q_v^{\mathrm{sep}}$ is a finite Galois extension of Q_v, let $\mu_L \in \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$ be the function given by the formula

$$\mu_L(g) := \begin{cases} 0 & \text{if } g \notin \mathcal{I}_{Q_v}, \\ -v(g(\pi_L) - \pi_L) & \text{if } g \in \mathcal{I}_{Q_v} \text{ and } g(\pi_L) \neq \pi_L, \\ v(\mathfrak{D}_{L/Q_v}) & \text{if } g \in \mathcal{I}_{Q_v} \text{ and } g(\pi_L) = \pi_L, \end{cases}$$

where π_L is a uniformizer of L and $\mathfrak{D}_{L/Q_v}$ is the different of L over Q_v. Moreover, we let e_K be the ramification index of K over Q_v and f_K the degree of the residue field of K over $\mathbb{F}_v$. We let $W_L^n := \{g \in \mathscr{G}_{Q_v} : g(x) \equiv x^{q_v^n} \bmod \mathfrak{m}_{Q_v^{\mathrm{alg}}} \; \forall\, x \in \mathcal{O}_{Q_v^{\mathrm{alg}}}\}/\mathcal{I}_L$. It is in bijection with $\{g \in \mathrm{Gal}(L/Q_v) : g(x) \equiv x^{q_v^n} \bmod (\pi_L)\}$ under the map $\mathscr{G}_{Q_v} \twoheadrightarrow \mathrm{Gal}(L/Q_v)$.

Lemma 5.17. *Let $K, L \subset Q_v^{\mathrm{sep}}$ be finite separable extensions of Q_v with L finite Galois over Q_v containing all the conjugates of K, and let $\varphi, \psi \in H_K$. The function $a_{K,\varphi,\psi}$ is constant modulo $\mathscr{G}_L$ and hence can be considered as a function on $G_L := \mathrm{Gal}(L/Q_v)$. Then*

$$\sum_{g \in G_L} a_{K,\varphi,\psi}(g) \cdot \mu_L(g) = \begin{cases} 0 & \text{if } j(\varphi) \neq j(\psi), \\ v(\mathfrak{D}_{\psi(K)/Q_v}) & \text{if } \varphi = \psi, \\ -v(\psi(\pi_K) - \varphi(\pi_K)) & \text{if } j(\varphi) = j(\psi) \text{ and } \varphi \neq \psi, \end{cases}$$

and

$$\frac{1}{e_L} \sum_{n=1}^{\infty} \sum_{g \in W_L^n} \frac{a_{K,\varphi,\psi}(g)}{q_v^{ns}} = \frac{1}{e_K} \frac{q_v^{(j(\psi), j(\varphi))s}}{q_v^{f_K s} - 1}.$$

In particular, the left-hand side of both equations does not depend on the choice of L.

Proof. The proof follows in the same way as [10, Lemma I.2.4]. □

Since the $a_{K,\varphi,\psi}$ generate the vector space $\mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$, we get the following proposition.

Proposition 5.18. *There exist $\mathbb{Q}$-linear homomorphisms $Z_v(\,.\,, s) : \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q}) \to \mathbb{C}$ if $s \in \mathbb{C}$ and $\mu_{\mathrm{Art},v} : \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q}) \to \mathbb{Q}$ defined by the following formulas: if $a \in \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$ and if $L \subset Q_v^{\mathrm{sep}}$ is a finite Galois extension of Q_v such that a is constant modulo $\mathscr{G}_L$, then*

$$\mu_{\mathrm{Art},v}(a) = \sum_{g \in G_L} a(g) \cdot \mu_L(g)$$

with $G_L := \mathrm{Gal}(L/Q_v)$, and $Z_v(a, s)$ is obtained by meromorphic extension from the following formula, valid for $\mathcal{R}e(s) > 0$:

$$Z_v(a, s) = \frac{1}{e_L} \sum_{n=1}^{\infty} \sum_{g \in W_L^n} \frac{a(g)}{q_v^{ns}}$$ □

Remark 5.19. If V is a finite-dimensional $\mathbb{C}$-vector space, $\rho : \mathscr{G}_{Q_v} \to \mathrm{Aut}_{\mathbb{C}}(V)$ is a continuous complex representation of $\mathscr{G}_{Q_v}$, and if $\chi \in \mathcal{C}^0(\mathscr{G}_{Q_v}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}$ is the character of ρ, then $\mu_{\mathrm{Art},v}(\chi)$ is nothing else than the degree at v of the conductor $\mathfrak{f}_\chi$ of χ; cf. [27, Chapter VI, § 2], where $\mu_{\mathrm{Art},v}(\chi)$ is denoted by $f(\chi)$. And if W is the sub-vector space of V stable by I_{Q_v}, we have

$$Z_v(\chi, a) \log q_v = -\frac{d}{ds} \log(\det(1 - q_v^{-s} \rho(\mathrm{Frob}_{L/Q_v})|_W)^{-1})$$

by [29, Chapter 0, § 4] or [22, Lemma 9.14]. So the linear maps $\mu_{\mathrm{Art},v}$ and $Z_v(\,.\,, s)$ coincide with the maps with the same names in Definition 1.2 in the introduction.

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As a direct consequence of Theorem 5.13, Proposition 5.18 and Lemma 5.17 we get the following

Theorem 5.20. *If $\varphi, \psi \in H_{E_v}$ satisfy $i(\varphi) = i(\psi) =: i$ and $E_{v,i}$ is separable over Q_v then*

$$v(\Omega(E_v, \varphi, \psi)) = Z_v(a_{E_v,\psi,\varphi}, 1) - \mu_{\mathrm{Art},v}(a_{E_v,\psi,\varphi}),$$

where we set $a_{E_v,\psi,\varphi} := a_{E_{v,i},\psi,\varphi} : g \mapsto \delta_{g\psi,\varphi}$. □

Definition 5.21. If $u \in \mathrm{H}_{1,v}(\hat{\underline{M}}, Q_v) := \mathrm{Hom}_{A_v}(\mathrm{H}^1_v(\hat{\underline{M}}, A_v), Q_v)$ is an E_v-generator there is an $a \in E_v^\times$, unique up to multiplication with an element of $\mathcal{O}_{E_v}^\times$, such that $a^{-1}u$ is an $\mathcal{O}_{E_v}$-generator of $\mathrm{H}_{1,v}(\hat{\underline{M}}, A_v)$. Then we define the valuation $v_\psi(u) := v\big(\psi(a)\big) \in \mathbb{Z}$.

Note that if $\underline{M} = (M, \tau_M)$ is a uniformizable A-motive over L with good model $\underline{\mathcal{M}}$ and $\hat{\underline{M}} = \hat{\underline{M}}_v(\underline{\mathcal{M}})$ is the local shtuka at v associated with $\underline{\mathcal{M}}$ as in Example 3.2, then for an E-generator $u \in \mathrm{H}_{1,\mathrm{Betti}}(\underline{M}, Q)$ the present definition of $v_\psi\big(h_{\mathrm{Betti},v}(u)\big)$ coincides with the definition of $v_\psi(u)$ from (1.12).

Corollary 5.22. *Let $\varphi, \psi \in H_{E_v}$ with $i(\varphi) = i(\psi) =: i$ and assume that $E_{v,i}$ is separable over Q_v. Let $u \in \mathrm{H}_{1,v}(\hat{\underline{M}}_{E_v,\varphi}, Q_v)$ be an E_v-generator and let ω_ψ be an $L[\![y_i - \psi(y_i)]\!]$-generator of $\mathrm{H}^\psi(\hat{\underline{M}}_{E_v,\varphi}, L[\![y_i - \psi(y_i)]\!])$. Then $\int_u \omega_\psi := u \otimes \mathrm{id}_{\mathbb{C}_v((z-\zeta))}(h^{-1}_{v,\mathrm{dR}}(\omega_\psi))$ has valuation*

$$v\left(\int_u \omega_\psi\right) = Z_v(a_{E_v,\psi,\varphi}, 1) - \mu_{\mathrm{Art},v}(a_{E_v,\psi,\varphi}) - v(\mathfrak{D}_{\psi(E_v)/Q_v}) + v(\omega_\psi) + v_\psi(u),$$

where $v(\omega_\psi)$ and $v_\psi(u)$ were defined in Definitions 4.10 and 5.21, and $\mathfrak{D}_{\psi(E_v)/Q_v}$ is the different.

Proof. Let $a \in E_v$ be such that $u^\circ := a^{-1}u$ is an $\mathcal{O}_{E_v}$-generator of $\mathrm{H}_{1,v}(\hat{\underline{M}}_{E_v,\varphi}, A_v)$ and let ω_ψ° be an $L[\![y_i - \psi(y_i)]\!]$-generator of $\mathrm{H}^\psi(\hat{\underline{M}}_{E_v,\varphi}, L[\![y_i - \psi(y_i)]\!])$ such that the residue $\omega_\psi^\circ \bmod y_i - \varphi(y_i) \in \mathrm{H}^1_{\mathrm{dR}}(\hat{\underline{M}}_{E_v,\varphi}, L)$ is an R-generator of $\mathrm{H}^1_{\mathrm{dR}}(\hat{\underline{M}}_{E_v,\varphi}, R)$. Let $x \in L[\![y_i - \psi(y_i)]\!]^\times$ such that $\omega_\psi = x \cdot \omega_\psi^\circ$. Moreover, let D_ψ be a generator as $\psi(\mathcal{O}_{E_v})$-module of the different $\mathfrak{D}_{\psi(E_{v,i})/Q_v}$. Then

$$\int_u \omega_\psi = (a \otimes 1)x \cdot \int_{u^\circ} \omega_\psi^\circ = (a \otimes 1)x \cdot \Omega(E_v, \varphi, \psi) \cdot D_\psi^{-1} \in \mathbb{C}_v((z-\zeta))$$

up to multiplication by an element of $R^\times + (z-\zeta) \cdot L[\![z-\zeta]\!]$ by Theorem 5.13(d). The element $(a \otimes 1)x \in E_v \otimes_{Q_v} L[\![z-\zeta]\!]$ lies in the ψ-component $L[\![y_i - \psi(y_i)]\!]$ of the product decomposition (A 1), and in that component $a \otimes 1$ is congruent to $\psi(a)$ modulo $y_i - \psi(y_i)$. Therefore,

$$\begin{aligned} v\left(\int_u \omega_\psi\right) &= v(\psi(a)x \cdot \Omega(E_v, \varphi, \psi) \cdot D_\psi^{-1}) \\ &= Z_v(a_{E_v,\psi,\varphi}, 1) - \mu_{\mathrm{Art},v}(a_{E_v,\psi,\varphi}) - v(\mathfrak{D}_{\psi(E_v)/Q_v}) + v(\omega_\psi) + v_\psi(u). \end{aligned}$$

by Theorem 5.20. □

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To compute $v(\int_u \omega_\psi)$ for general $\underline{\hat{M}}$ we need the following

Definition 5.23. Let E_v be separable over Q_v and let $\Phi = (d_\varphi)_{\varphi\in H_{E_v}}$ be a local CM-type. For $\psi \in H_{E_v}$ let $a_{E_v,\psi,\Phi} \in \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$ and $a^0_{E_v,\psi,\Phi} \in \mathcal{C}^0(\mathscr{G}_{Q_v}, \mathbb{Q})$ be given by the formulas

$$a_{E_v,\psi,\Phi}(g) := \sum_{\varphi\in H_{E_v}} d_\varphi \cdot a_{E_v,\psi,\varphi}(g) = d_{g\psi} \quad \text{and} \tag{5.11}$$

$$a^0_{E_v,\psi,\Phi}(g) := \frac{1}{\#H_L} \sum_{\eta\in H_L} d_{\eta^{-1}g\eta\psi}. \tag{5.12}$$

Note that $a_{E_v,\psi,\Phi}$ and $a^0_{E_v,\psi,\Phi}$ factor through $\mathrm{Gal}(E_v^{\mathrm{nor}}/Q_v)$ where E_v^{nor} is the Galois closure of $\psi(E_v)$ in Q_v^{alg}. In particular, $a^0_{E_v,\psi,\Phi}$ does not depend on the field L provided $\psi(E_v) \subset L$ for all $\psi \in H_{E_v}$.

These functions are the local counterparts to the functions $a_{E,\psi,\Phi} \in \mathcal{C}(\mathscr{G}_Q, \mathbb{Q})$ and $a^0_{E,\psi,\Phi} \in \mathcal{C}^0(\mathscr{G}_Q, \mathbb{Q})$ which were defined in (1.10) and (1.11). The membership in $\mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$, respectively $\mathcal{C}(\mathscr{G}_Q, \mathbb{Q})$ is indicated by the index which gives reference to the Q_v-algebra E_v, respectively the Q-algebra E. In fact, if $E_v = E \otimes_Q Q_v$ and hence $H_{E_v} = H_E$, then $a_{E_v,\psi,\Phi}$ is equal to the image of $a_{E,\psi,\Phi}$ under the map $\mathcal{C}(\mathscr{G}_Q, \mathbb{Q}) \to \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$ from Definition 1.2. However, this is in general not true for $a^0_{E_v,\psi,\Phi}$ and $a^0_{E,\psi,\Phi}$, because if L is the closure of K in Q_v^{alg}, then H_L is in general strictly contained in H_K.

For general $\underline{\hat{M}}$ we can now prove the following

Theorem 5.24. *Let $\underline{\hat{M}}$ be a local shtuka over R with complex multiplication by the ring of integers $\mathcal{O}_{E_v}$ in a commutative, semi-simple, separable Q_v-algebra E_v with local CM-type Φ, and assume that $\psi(E_v) \subset L$ for all $\psi \in H_{E_v}$ and that L is separable over Q_v. Let $u \in \mathrm{H}_{1,v}(\underline{\hat{M}}, Q_v)$ be an E_v-generator and let ω_ψ be an $L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!]$-generator of $\mathrm{H}^\psi(\underline{\hat{M}}, L[\![y_{i(\psi)} - \psi(y_{i(\psi)})]\!])$. Then the period $\int_u \omega_\psi := u \otimes \mathrm{id}_{\mathbb{C}_v((z-\zeta))}(h^{-1}_{v,\mathrm{dR}}(\omega_\psi))$ has valuation*

$$v\left(\int_u \omega_\psi\right) = Z_v(a_{E_v,\psi,\Phi}, 1) - \mu_{\mathrm{Art},v}(a_{E_v,\psi,\Phi}) - v(\mathfrak{D}_{\psi(E_v)/Q_v}) + v(\omega_\psi) + v_\psi(u),$$

where $v(\omega_\psi)$ and $v_\psi(u)$ were defined in Definitions 4.10 and 5.21, and $\mathfrak{D}_{\psi(E_v)/Q_v}$ is the different.

Proof. As in 5.1 the local shtuka $\underline{\hat{M}}$ is isomorphic to the tensor product of local shtukas $\underline{\hat{M}}_{E_v,0} \otimes \bigotimes_\varphi \underline{\hat{M}}_{E_v,\varphi}{}^{\otimes d_\varphi}$ over $\mathcal{O}_{E_v,R}$. Let $i := i(\psi)$ and $j := j(\psi)$. For every $\underline{\hat{M}}_{E_v,\varphi}$ we fix the $L[\![y_i - \psi(y_i)]\!]$-generator $\omega^\circ_{\psi,\varphi} := 1 \in \mathrm{H}^\psi(\underline{\hat{M}}_{E_v,\varphi}, L[\![y_i - \psi(y_i)]\!])$. In addition, we let $\omega^\circ_{\psi,0} := 1 \in \mathrm{H}^\psi(\underline{\hat{M}}_{E_v,0}, L[\![y_i - \psi(y_i)]\!])$. Then we can take the tensor product $\omega^\circ_\psi := \omega^\circ_{\psi,0} \otimes \bigotimes_{\varphi\in H_{E_v}} (\omega^\circ_{\psi,\varphi})^{\otimes d_\varphi}$ in

$$\mathrm{H}^\psi(\underline{\hat{M}}, L[\![y_i - \psi(y_i)]\!]) \cong \mathrm{H}^\psi(\underline{\hat{M}}_{E_v,0}, L[\![y_i - \psi(y_i)]\!]) \otimes \bigotimes_{\varphi\in H_{E_v}} \mathrm{H}^\psi(\underline{\hat{M}}_{E_v,\varphi}, L[\![y_i - \psi(y_i)]\!])^{\otimes d_\varphi}.$$

It is an $L[\![y_i-\psi(y_i)]\!]$-generator as in Definition 4.10. Let $x\in L[\![y_i-\psi(y_i)]\!]^\times$ be such that $\omega_\psi=x\cdot\omega_\psi^\circ$, and let further $a\in E_v$ be such that $u^\circ:=a^{-1}u$ is an $\mathcal{O}_{E_v}$-generator of $\mathrm{H}_{1,v}(\underline{\hat{M}},A_v)$. Moreover, let D_ψ be a generator as $\psi(\mathcal{O}_{E_v})$-module of the different $\mathfrak{D}_{\psi(E_{v,i})/Q_v}$. Then (5.5), (5.7) and Lemma 5.15 imply that

$$\int_{u^\circ}\omega_\psi^\circ=D_\psi^{-1}\cdot\epsilon_{i,j}c_{i,j}^{-1}\prod_{\varphi\in H_{E_{v,i}}}\Omega(E_v,\varphi,\psi)^{d_\varphi}$$

up to multiplication by $R^\times+(z-\zeta)\cdot L[\![z-\zeta]\!]$. Since $\epsilon_{i,j}c_{i,j}^{-1}\in\big(\mathcal{O}_{E_v}\otimes_{A_v}\mathcal{O}_{\mathbb{C}_v}[\![z]\!]\big)^\times$ we conclude as in the proof of Corollary 5.22 that $\int_u\omega_\psi=(a\otimes1)x\cdot\int_{u^\circ}\omega_\psi^\circ$ and

$$\begin{aligned}v\Big(\int_u\omega_\psi\Big)&=v\Big(D_\psi^{-1}\cdot\psi(a)x\cdot\epsilon_{i,j}c_{i,j}^{-1}\prod_{\varphi\in H_{E_{v,i}}}\Omega(E_v,\varphi,\psi)^{d_\varphi}\Big)\\&=v(\omega_\psi)+v_\psi(u)-v(\mathfrak{D}_{\psi(E_v)/Q_v})+\sum_{\varphi\in H_{E_{v,i}}}(Z_v(a_{E_v,\psi,\varphi},1)-\mu_{\mathrm{Art},v}(a_{E_v,\psi,\varphi}))\cdot d_\varphi\\&=Z_v(a_{E_v,\psi,\Phi},1)-\mu_{\mathrm{Art},v}(a_{E_v,\psi,\Phi})-v(\mathfrak{D}_{\psi(E_v)/Q_v})+v(\omega_\psi)+v_\psi(u),\end{aligned}$$

because $i(\varphi)\neq i(\psi)$ implies that $a_{E_v,\psi,\varphi}(g)=\delta_{g\psi,\varphi}=0$ for all $g\in\mathscr{G}_{Q_v}$. □

Corollary 5.25. *Keep the situation of Theorem 5.24. For every $\eta\in H_L$ note that $i(\eta\psi)=i(\psi)$, let $\underline{\hat{M}}^\eta$ and $\omega_\psi^\eta\in\mathrm{H}^{\eta\psi}(\underline{\hat{M}}^\eta,L[\![y_{i(\psi)}-\eta\psi(y_{i(\psi)})]\!])$ be obtained by extension of scalars via η, and choose an E_v-generator $u_\eta\in\mathrm{H}_{1,v}(\underline{\hat{M}}^\eta,Q_v)$. Then*

$$\frac{1}{\#H_L}\sum_{\eta\in H_L}v\Big(\int_{u_\eta}\omega_\psi^\eta\Big)=Z_v(a^0_{E_v,\psi,\Phi},1)-\mu_{\mathrm{Art},v}(a^0_{E_v,\psi,\Phi})-\frac{v(\mathfrak{d}_{\psi(E_v)/Q_v})}{[\psi(E_v):Q_v]}+\frac{1}{\#H_L}\sum_{\eta\in H_L}(v(\omega_\psi^\eta)+v_{\eta\psi}(u_\eta)),$$

where $\mathfrak{d}_{\psi(E_v)/Q_v}\subset A_v$ is the discriminant of the field extension $\psi(E_v)/Q_v$.

Proof. Since $\underline{\hat{M}}^\eta$ has complex multiplication by $\mathcal{O}_{E_v}$ with local CM-type $\eta\Phi:=(d'_\varphi)_{\varphi\in H_{E_v}}$ with $d'_\varphi=d_{\eta^{-1}\varphi}$, Theorem 5.24 implies

$$v\Big(\int_{u_\eta}\omega_\psi^\eta\Big)=Z_v(a_{E_v,\eta\psi,\eta\Phi},1)-\mu_{\mathrm{Art},v}(a_{E_v,\eta\psi,\eta\Phi})-v(\mathfrak{D}_{\eta\psi(E_v)/Q_v})+v(\omega_\psi^\eta)+v_{\eta\psi}(u_\eta).\tag{5.13}$$

We sum over all $\eta\in H_L$, divide by $\#H_L=[L:Q_v]$, and observe that $a_{E_v,\eta\psi,\eta\Phi}(g)=d'_{g\eta\psi}=d_{\eta^{-1}g\eta\psi}$ and $\mathfrak{D}_{\eta\psi(E_v)/Q_v}=\eta(\mathfrak{D}_{\psi(E_v)/Q_v})$, and hence

$$\begin{aligned}\sum_{\eta\in H_L}v(\mathfrak{D}_{\eta\psi(E_v)/Q_v})&=v\Big(\prod_{\eta\in H_L}\eta(\mathfrak{D}_{\psi(E_v)/Q_v})\Big)=v(N_{L/Q_v}(\mathfrak{D}_{\psi(E_v)/Q_v}))\\&=v(N_{\psi(E_v)/Q_v}(N_{L/\psi(E_v)}(\mathfrak{D}_{\psi(E_v)/Q_v})))=[L:\psi(E_v)]\cdot v(\mathfrak{d}_{\psi(E_v)/Q_v}).\end{aligned}$$

This proves the corollary. □

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Finally we are ready to give the

Proof of Theorem 1.3. The proof proceeds like the one of the previous corollary applied to $\underline{\hat{M}} := \underline{\hat{M}}_v(\underline{\mathcal{M}})$ for a model $\underline{\mathcal{M}}$ of $\underline{M}$ with good reduction. Setting $E_v := E \otimes_Q Q_v$, still $\underline{\hat{M}}^\eta := \underline{\hat{M}}_v(\underline{\mathcal{M}}^\eta)$ has complex multiplication by $\mathcal{O}_{E_v}$ with local CM-type $\eta\Phi := (d'_\varphi)_{\varphi\in H_E}$ with $d'_\varphi = d_{\eta^{-1}\varphi}$, where η runs over all elements in H_K. To translate the global situation to the local one, we also use the elements $h_{\mathrm{Betti},v}(u_\eta) \in \mathrm{H}_{1,v}(\underline{\hat{M}}^\eta, Q_v)$, and $\omega^\eta_\psi \otimes_K L \in \mathrm{H}^{\eta\psi}(\underline{M}, L[\![y_{i(\eta\psi)} - \eta\psi(y_{i(\eta\psi)})]\!]) = \mathrm{H}^{\eta\psi}(\underline{\hat{M}}, L[\![y_{i(\eta\psi)} - \eta\psi(y_{i(\eta\psi)})]\!])$. Then the function $a_{E_v,\psi,\Phi} \in \mathcal{C}(\mathscr{G}_{Q_v}, \mathbb{Q})$ is the image of $a_{E,\psi,\Phi} \in \mathcal{C}(\mathscr{G}_Q, \mathbb{Q})$, and (5.13) takes the form

$$v\left(\int_{u_\eta} \omega^\eta_\psi\right) = Z_v(a_{E,\eta\psi,\eta\Phi}, 1) - \mu_{\mathrm{Art},v}(a_{E,\eta\psi,\eta\Phi}) - v(\mathfrak{D}_{\eta\psi(E_v)/Q_v}) + v(\omega^\eta_\psi) + v_{\eta\psi}(u_\eta), \tag{5.14}$$

because the value of $v_{\eta\psi}\big(h_{\mathrm{Betti},v}(u_\eta)\big)$ from Definition 5.21 used in (5.13), coincides with the value of $v_{\eta\psi}(u_\eta)$ from (1.12). This time we sum over all $\eta \in H_K$, divide by $\#H_K = [K : Q]$, and observe that $a_{E,\,\eta\psi,\,\eta\Phi}(g) = d'_{g\eta\psi} = d_{\eta^{-1}g\eta\psi}$, and hence

$$\frac{1}{\#H_K} \sum_{\eta\in H_K} Z_v(a_{E,\eta\psi,\eta\Phi}, 1) - \mu_{\mathrm{Art},v}(a_{E,\eta\psi,\eta\Phi}) = Z_v(a^0_{E,\psi,\Phi}, 1) - \mu_{\mathrm{Art},v}(a^0_{E,\psi,\Phi}). \tag{5.15}$$

For every place w of the field $\psi(E)$ above v let $\psi(E)_w$ be the completion. Then $\psi(E) \otimes_Q Q_v = \prod_{w|v} \psi(E)_w$. Via the fixed inclusion $Q^{\mathrm{alg}} \subset Q_v^{\mathrm{alg}}$ we consider every $\eta \in H_K$ as a morphism $\eta\colon K \to Q^{\mathrm{alg}} \subset Q_v^{\mathrm{alg}}$. The induced morphism $\eta \otimes \mathrm{id}_{Q_v}\colon K \otimes_Q Q_v \to Q_v^{\mathrm{alg}}$, when restricted to a morphism $\psi(E) \otimes_Q Q_v \to Q_v^{\mathrm{alg}}$ factors over $\psi(E)_w$ for a unique w which we denote by $w(\eta)$. We set $H_{K,w} := \{\eta \in H_K : w(\eta) = w\}$ and consider the map

$$H_{K,w} \longrightarrow H_{\psi(E)_w}, \quad \eta \longmapsto (\eta \otimes \mathrm{id}_{Q_v})|_{\psi(E)_w}. \tag{5.16}$$

The map is surjective, because every element of $H_{\psi(E)_w}$ can be restricted to a Q-homomorphism $\psi(E) \to Q^{\mathrm{alg}} \subset Q_v^{\mathrm{alg}}$, which extends to a Q-homomorphism $(\eta\colon K \to Q^{\mathrm{alg}}) \in H_K$ that automatically lies in $H_{K,w}$. We claim that two elements $\eta, \tilde\eta \in H_{K,w}$ have the same image under the map (5.16) if and only if $\tilde\eta = \eta \circ \alpha$ for an element $\alpha \in \mathrm{Gal}\big(K/\psi(E)\big)$. Indeed the latter condition is sufficient, because $\alpha \otimes \mathrm{id}_{Q_v}$ induces the identity on $\psi(E)_w$. To see that it is necessary let $\eta, \tilde\eta \in H_{K,w} \subset H_K = \mathrm{Gal}(K/Q)$ have the same image. Then their restrictions to $\psi(E) \subset \psi(E)_w$ coincide, and hence $\alpha := \eta^{-1} \circ \tilde\eta \in \mathrm{Gal}(K/Q)$ lies in $\mathrm{Gal}\big(K/\psi(E)\big)$ as claimed. For every $\eta \in H_{K,w}$ the different $\mathfrak{D}_{\eta\psi(E_v)/Q_v}$ equals $(\eta \otimes \mathrm{id}_{Q_v})(\mathfrak{D}_{\psi(E)_w/Q_v})$ and only depends on the image of η under the map (5.16). Therefore, we compute

$$\begin{aligned}\sum_{\eta\in H_{K,w}} v(\mathfrak{D}_{\eta\psi(E_v)/Q_v}) &= v\Big(\prod_{\eta\in H_{K,w}} (\eta \otimes \mathrm{id}_{Q_v})(\mathfrak{D}_{\psi(E)_w/Q_v})\Big)\\ &= v\Big(N_{\psi(E)_w/Q_v}(\mathfrak{D}_{\psi(E)_w/Q_v})^{\#\mathrm{Gal}(K/\psi(E))}\Big)\\ &= [K : \psi(E)] \cdot v(\mathfrak{d}_{\psi(E)_w/Q_v}).\end{aligned}$$

Summing over all $w|v$ and using that $\sum_{w|v} v(\mathfrak{d}_{\psi(E)_w/Q_v}) = v(\mathfrak{d}_{\psi(E)/Q})$ by [27, § III.4, Corollary to Proposition 10] we obtain from (5.14) and (5.15)

$$\frac{1}{\#H_K}\sum_{\eta\in H_K} v\left(\int_{u_\eta}\omega_\psi^\eta\right) = Z_v(a^0_{E,\psi,\Phi},1) - \mu_{\mathrm{Art},v}(a^0_{E,\psi,\Phi}) - \frac{v(\mathfrak{d}_{\psi(E)/Q})}{[\psi(E):Q]} + \frac{1}{\#H_K}\sum_{\eta\in H_K}(v(\omega_\psi^\eta) + v_{\eta\psi}(u_\eta)).$$

This proves Theorem 1.3. □

A. Appendix: Product decompositions of certain rings

In this appendix we establish certain product decompositions for the rings used in this article. We begin with the following

Lemma A.1. *Let k be a field and let $z = \sum_{n=0}^\infty b_n y^n \in k[\![y]\!]$. Let $\psi\colon k[\![y]\!] \to R$ be a ring homomorphism into a k-algebra R. Then in $k[\![y]\!]\widehat{\otimes}_{k,\psi} R := \varprojlim_n k[\![y]\!]/(y^n)\otimes_{k,\psi} R \cong R[\![y]\!]$ the fraction $\frac{z\otimes 1 - 1\otimes\psi(z)}{y\otimes 1 - 1\otimes\psi(y)}$ exists and is congruent to $1\otimes\psi\big(\frac{dz}{dy}\big)$ modulo $y\otimes 1 - 1\otimes\psi(y)$.*

Proof. The lemma follows from the computation

$$\begin{aligned} z\otimes 1 - 1\otimes\psi(z) &= \sum_{n=0}^\infty (b_n y^n\otimes 1 - 1\otimes\psi(b_n)\psi(y)^n) \\ &= \sum_{n=1}^\infty (1\otimes\psi(b_n))\cdot\sum_{\nu=0}^{n-1}(y^\nu\otimes\psi(y)^{n-1-\nu})\cdot(y\otimes 1 - 1\otimes\psi(y)) \\ &= (y\otimes 1 - 1\otimes\psi(y))\cdot\sum_{\nu=0}^\infty y^\nu\otimes\psi\left(\sum_{n=\nu+1}^\infty b_n y^{n-1-\nu}\right), \end{aligned}$$

where the second factor converges in $k[\![y]\!]\widehat{\otimes}_{k,\psi} R$. Modulo $y\otimes 1 - 1\otimes\psi(y)$ this factor equals

$$\begin{aligned} \sum_{n=1}^\infty (1\otimes\psi(b_n))\cdot\sum_{\nu=0}^{n-1}(y^\nu\otimes\psi(y)^{n-1-\nu}) &= \sum_{n=1}^\infty (1\otimes\psi(b_n))\cdot n(1\otimes\psi(y)^{n-1}) \\ &= 1\otimes\psi\left(\sum_{n=1}^\infty n b_n y^{n-1}\right) \\ &= 1\otimes\psi\left(\frac{dz}{dy}\right). \end{aligned}$$

□

We need the following well known fact from field theory. For the convenience of the reader we include a proof.

Lemma A.2. *Let E be a finite field extension of Q (or of Q_v) of inseparability degree p^m. Then the separable closure E' of Q (respectively of Q_v) in E equals $E^{p^m} := \{x^{p^m} : x\in E\}$. If y is a uniformizing parameter at a place $\tilde v$ of E then $y' := y^{p^m}$ is a uniformizing parameter at the place $\tilde v'$ of E' below $\tilde v$ and $E = E'(y) = E'[X]/(X^{p^m} - y')$.*

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Proof. This is due to the fact that Q has transcendence degree one over $\mathbb{F}_q$, respectively that Q_v is a discretely valued field. Namely, consider the case for Q_v. Then $E = k((y))$ where k is the finite residue field of E. Clearly $E^{p^m} = k((y'))$ and $E = E^{p^m}(y) = E^{p^m}[X]/(X^{p^m} - y')$, because $X^{p^m} - y'$ is irreducible in $E^{p^m}[X]$ by Eisenstein. In particular, $[E : E^{p^m}] = p^m$. On the other hand, the minimal polynomial $f(X)$ of y over E' is of the form $g(X^{p^{m'}})$ for a separable, irreducible polynomial g over E' and an integer $m' \geqslant 0$. Therefore, the minimal polynomial of $y^{p^{m'}}$ over E' is g and $y^{p^{m'}}$ is separable over E'. This implies $y^{p^{m'}} \in E'$ and $\deg g = 1$, whence $\deg f = p^{m'} \leqslant p^m$. Therefore, $m' \leqslant m$ and $y^{p^m} \in E'$, and hence $E^{p^m} \subset E'$. Since $[E : E'] = p^m = [E : E^{p^m}]$ it follows that $E' = E^{p^m}$. This proves the lemma for Q_v.

For Q a proof for the equality $E' = E^{p^m}$ can be found for example in [28, Chapter II, Corollary 2.12]. If $\mathcal{O}_{E,\tilde{v}}$ is the valuation ring of E at $\tilde{v}$ then $(\mathcal{O}_{E,\tilde{v}})^{p^m}$ equals the valuation ring $\mathcal{O}_{E',\tilde{v}'}$ of E' at $\tilde{v}'$ and so y' is a uniformizing parameter of $\mathcal{O}_{E',\tilde{v}'}$. The last equality follows from the fact that the polynomial $X^{p^m} - y' \in \mathcal{O}_{E',\tilde{v}'}[X]$ is irreducible by Eisenstein. □

In the next lemma we consider the embeddings $Q \hookrightarrow K[[z_v - \zeta_v]]$ and $Q_v \hookrightarrow L[[z_v - \zeta_v]]$ given by $z_v \mapsto z_v = \zeta_v + (z_v - \zeta_v)$.

Lemma A.3. *Let $E = E_1 \times \cdots \times E_s$ be a product of finite field extensions of Q and let $K \subset Q^{\mathrm{alg}}$ be a field extension of Q with $\psi(E) \subset K$ for all $\psi \in H_E := \mathrm{Hom}_Q(E, Q^{\mathrm{alg}})$. Let $i(\psi)$ be such that ψ factors through $E \twoheadrightarrow E_{i(\psi)}$ and let $y_{i(\psi)} \in E_{i(\psi)}$ be a uniformizing parameter at a place of $E_{i(\psi)}$ above v. Then*

$$E \otimes_Q K[[z_v - \zeta_v]] = \prod_{\psi \in H_E} K[[y_{i(\psi)} - \psi(y_{i(\psi)})]] \quad \textit{and}$$

$$E \otimes_Q K = \prod_{\psi \in H_E} K[[y_{i(\psi)} - \psi(y_{i(\psi)})]]/(y_{i(\psi)} - \psi(y_{i(\psi)}))^{[E_{i(\psi)} : Q]_{\mathrm{insep}}},$$

where $[E_{i(\psi)} : Q]_{\mathrm{insep}}$ is the inseparability degree of $E_{i(\psi)}$ over Q.

Likewise, let $E_v = E_{v,1} \times \cdots \times E_{v,s}$ be a product of finite field extensions of Q_v and let $L \subset Q_v^{\mathrm{alg}}$ be a field extension of Q_v with $\psi(E_v) \subset L$ for all $\psi \in H_{E_v} := \mathrm{Hom}_{Q_v}(E_v, Q_v^{\mathrm{alg}})$. Let $i(\psi)$ be such that ψ factors through $E \twoheadrightarrow E_{v,i(\psi)}$ and let $y_{i(\psi)} \in E_{v,i(\psi)}$ be a uniformizing parameter. Then

$$E_v \otimes_{Q_v} L[[z_v - \zeta_v]] = \prod_{\psi \in H_{E_v}} L[[y_{i(\psi)} - \psi(y_{i(\psi)})]] \quad \textit{and} \tag{A 1}$$

$$E_v \otimes_{Q_v} L = \prod_{\psi \in H_{E_v}} L[[y_{i(\psi)} - \psi(y_{i(\psi)})]]/(y_{i(\psi)} - \psi(y_{i(\psi)}))^{[E_{v,i(\psi)} : Q_v]_{\mathrm{insep}}}, \tag{A 2}$$

where $[E_{v,i(\psi)} : Q_v]_{\mathrm{insep}}$ is the inseparability degree of $E_{v,i(\psi)}$ over Q_v.

Proof. Fix a ψ, set $i := i(\psi)$ and let E'_i, respectively $E'_{v,i}$ be the separable closure of Q in E_i, respectively of Q_v in $E_{v,i}$. Then $H_{E_i} = H_{E'_i}$, respectively $H_{E_{v,i}} = H_{E'_{v,i}}$, and

$$E'_i \otimes_Q K \xrightarrow{\sim} \prod_{\psi \in H_{E_i}} K, \quad \text{respectively} \quad E'_{v,i} \otimes_{Q_v} L \xrightarrow{\sim} \prod_{\psi \in H_{E_{v,i}}} L. \tag{A 3}$$

Let $p^m := [E_i : Q]_{\text{insep}} = [E_i : E_i']$, respectively $p^m := [E_{v,i} : Q_v]_{\text{insep}} = [E_{v,i} : E_{v,i}']$, and let $y_i' := y_i^{p^m}$. Then Lemma A.2 implies that $y_i' \in E_i'$ is a uniformizing parameter at a place above v. By Hensel's lemma the decompositions (A 3) extend to decompositions

$$E_i' \otimes_Q K[\![z_v - \zeta_v]\!] \xrightarrow{\sim} \prod_{\psi \in H_{E_i}} K[\![z_v - \zeta_v]\!] = \prod_{\psi \in H_{E_i}} K[\![y_i' - \psi(y_i')]\!], \quad \text{respectively}$$

$$E_{v,i}' \otimes_{Q_v} L[\![z_v - \zeta_v]\!] \xrightarrow{\sim} \prod_{\psi \in H_{E_{v,i}}} L[\![z_v - \zeta_v]\!] = \prod_{\psi \in H_{E_{v,i}}} L[\![y_i' - \psi(y_i')]\!].$$

Here the last identifications in each line follow from [17, Lemmas 1.2 and 1.3] which states that both $K[\![z_v - \zeta_v]\!]$ and $K[\![y_i' - \psi(y_i')]\!]$ are canonically isomorphic to the completion of the ring $\mathcal{O}_{E_i'} \otimes_{\mathbb{F}_q} K$ at the ideal $(a \otimes 1 - 1 \otimes \psi(a) \colon a \in \mathcal{O}_{E_i'})$. The identification in the second line also follows from Lemma A.1 by observing that the derivative $\frac{dz_v}{dy_i'}$ equals $-\frac{\partial}{\partial Y_i'} m(z_v, Y_i') \big/ \frac{\partial}{\partial z_v} m(z_v, Y_i')\big|_{Y_i' = y_i'}$ where $m(z_v, Y_i') \in \mathbb{F}_v[\![z_v]\!][Y_i']$ is the minimal polynomial of y_i' over Q_v, and hence $\psi\big(\frac{dz_v}{dy_i'}\big)$ is non-zero by the separability of y_i' over Q_v, and the injectivity of ψ on $E_{v,i}$. Now $E_i = E_i'(y_i)$, respectively $E_{v,i} = E_{v,i}'(y_i)$, and hence

$$E_i \otimes_{E_i'} K[\![y_i' - \psi(y_i')]\!] = K[\![y_i' - \psi(y_i')]\!][y_i - \psi(y_i)] = K[\![y_i - \psi(y_i)]\!],$$

respectively $E_{v,i} \otimes_{E_{v,i}'} L[\![y_i' - \psi(y_i')]\!] = L[\![y_i' - \psi(y_i')]\!][y_i - \psi(y_i)] = L[\![y_i - \psi(y_i)]\!]$ with $\big(y_i - \psi(y_i)\big)^{p^m} = y_i' - \psi(y_i')$. Since $H_E = \bigcup_i H_{E_i}$, respectively $H_{E_v} = \bigcup_i H_{E_{v,i}}$, the lemma follows. □

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