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Multidimensional regular C-fraction with independent variables corresponding to formal multiple power series

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In the paper the correspondence between a formal multiple power series and a special type of branched continued fractions, the so-called 'multidimensional regular C-fractions with independent variables' is analysed providing with an algorithm based upon the classical algorithm and that enables us to compute from the coefficients of the given formal multiple power series, the coefficients of the corresponding multidimensional regular C-fraction with independent variables. A few numerical experiments show, on the one hand, the efficiency of the proposed algorithm and, on the other, the power and feasibility of the method in order to numerically approximate certain multivariable functions from their formal multiple power series.

Keywords: multidimensional regular C-fraction with independent variables; multiple power series; correspondence; algorithm

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1. Introduction

Representation of special functions by means of continued fractions has been an interesting matter of study during the two last centuries and giving rise to other important related topics like Padé approximants, orthogonal polynomials, quadrature formulas, differential equations and so on. In comparison with power series, continued fractions have wider convergence domain and endowed with the property of numerical stability. An analytical theory of continued fractions is described in [**11**].

Construction of the rational approximations of function is based on the concept of correspondence between the approximants of continued fraction and the formal power series, which is representing this function. A general theory of correspondence is elaborated and described in [**11**, pp. 148–160].

In the 1960s, for the construction of rational approximations of multivariable functions, V. Skorobogatko proposed a multidimensional generalization of continued fractions – branched continued fractions. These fractions have found their application in various fields, in particular, in numerical theory to express algebraic irrational numbers, in computational mathematics for the solution of systems of linear algebraical equations, in engineering for constructing mathematical models of transistors, in the analysis for approximating multivariable functions. Fundamentals of an analytical theory of general branched continued fractions is described in [2].

The problem of constructing of the corresponding branched continued fractions for the formal multiple power series (FMPS) contributed to the emergence of two-dimensional continued fractions [4, 5, 12–15], since within the general branched continued fractions using the concept of correspondence did not give a clear solution. Constructions of the fractions with increase in numbers of variables were significantly complicated. Therefore, two approaches are used for the construction of the branched continued fractions, which corresponds to the FMPS: to overlay additional conditions on the elements of the branched continued fraction [3, 6] or to variate constructions of the fraction [1, 7–10]. We give here a few facts and definitions that are used.

Let N be a fixed natural number. The following standard notation will be used: $\mathbb{Z}$ denotes the integer numbers, $\mathbb{C}$ denotes the complex numbers; $\mathbf{k} = (k_1, k_2, \ldots, k_N)$ denotes an element of $\mathbb{Z}^N$; $\mathbf{z} = (z_1, z_2, \ldots, z_N)$ is an element of $\mathbb{C}^N$; and for $\mathbf{k} \in \mathbb{Z}^N$ and $\mathbf{z} \in \mathbb{C}^N$

$$|\mathbf{k}| = k_1 + k_2 + \cdots + k_N, \quad \mathbf{z}^{\mathbf{k}} = z_1^{k_1} z_2^{k_2} \ldots z_N^{k_N}.$$

Let

$$L(\mathbf{z}) = 1 + \sum_{|\mathbf{k}|\geqslant 1} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}, \tag{1.1}$$

where $c_{\mathbf{k}} \in \mathbb{C}$, $|\mathbf{k}| \geqslant 1$, be an FMPS at $\mathbf{z} = \mathbf{0}$. It is obvious that the set $\mathbb{L}$ of all FMPS at $\mathbf{z} = \mathbf{0}$ forms a ring with unity respect to the operations addition and multiplication of series.

Let $R(\mathbf{z})$ be multivariate function holomorphic in a neighbourhood of the origin ($\mathbf{z} = \mathbf{0}$). Let the mapping $\Lambda : R(\mathbf{z}) \to \Lambda(R)$ associate with $R(\mathbf{z})$ its Taylor expansion in a neighbourhood of the origin.

A sequence $\{R_n(\mathbf{z})\}$ of multivariate functions holomorphic at the origin is said to correspond at $\mathbf{z} = \mathbf{0}$ to an FMPS $L(\mathbf{z})$ if

$$\lim_{n\to\infty} \lambda(L - \Lambda(R_n)) = \infty,$$

where λ is the function defined as follows: $\lambda : \mathbb{L} \to \mathbb{Z}_{\geqslant 0} \cup \{\infty\}$; if $L(\mathbf{z}) \equiv 0$ then $\lambda(L) = \infty$; if $L(\mathbf{z}) \not\equiv 0$ then $\lambda(L) = m$, where m is the smallest degree of homogeneous terms for which $c_{\mathbf{k}} \neq 0$, that is $m = |\mathbf{k}|$.

If $\{R_n(\mathbf{z})\}$ corresponds at $\mathbf{z} = \mathbf{0}$ to an FMPS $L(\mathbf{z})$, then the order of correspondence of $R_n(\mathbf{z})$ is defined to be

$$\nu_n = \lambda(L - \Lambda(R_n)).$$

By the definition of λ, the series $L(\mathbf{z})$ and $\Lambda(R_n)$ agree for all homogeneous terms up to and including degree $(\nu_n - 1)$.

Let $i(0) = 0$ and $\mathcal{I}_0 = \{0\}$. Let us introduce the following sets of multiindices

$$\mathcal{I}_k = \{i(k) : \ i(k) = (i_1, i_2, \ldots, i_k), \ 1 \leqslant i_p \leqslant i_{p-1}, \ 1 \leqslant p \leqslant k, \ i_0 = N\} \quad \text{for } n \geqslant 1.$$

Let $\langle \{a_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{Z}_{>0}}, \{b_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{Z}_{\geqslant 0}} \rangle$ denote the ordered pair of sequences of complex numbers with $a_{i(k)} \neq 0$ for all $i(k) \in \mathcal{I}_k$, $k \geqslant 1$, and if for $k \geqslant 1$ there exist a multiindex $i(k) \in \mathcal{I}_k$ such that $b_{i(k)} = 0$, than $b_{i(k-1)j} \neq 0$ for $1 \leqslant j \leqslant i_{k-1}$ and $j \neq i_k$. The sequence $\{f_k\}$ is defined as follows: $f_0 = b_0$,

$$f_1 = b_0 + \sum_{i_1=1}^{N} \frac{a_{i(1)}}{b_{i(1)}}, \text{ etc., } f_k = b_0 + \sum_{i_1=1}^{N} \frac{a_{i(1)}}{b_{i(1)}} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)}}{b_{i(2)}} + \cdots + \sum_{i_k=1}^{i_{k-1}} \frac{a_{i(k)}}{b_{i(k)}}, \text{ etc.}$$

The ordered pair $\langle\langle\{a_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{Z}_{>0}}, \{b_{i(k)}\}_{i(k)\in\mathcal{I}_k,\, k\in\mathbb{Z}_{\geqslant 0}}\rangle, \{f_k\}_{k\in\mathbb{Z}_{\geqslant 0}}\rangle$ is the branched continued fraction with independent variables denoted by the symbol

$$b_0 + \sum_{i_1=1}^{N} \frac{a_{i(1)}}{b_{i(1)}} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)}}{b_{i(2)}} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)}}{b_{i(3)}} + \cdots. \tag{1.2}$$

The numbers $a_{i(k)}$ and $b_{i(k)}$ are called the elements of the branched continued fraction with independent variables, the relation $a_{i(k)}/b_{i(k)}$ is called the kth partial quotient and the value f_k is called the kth approximant.

Let $(i_1, i_2, \ldots, i_k, \ldots)$ be a fixed infinite multiindex such that $1 \leqslant i_k \leqslant i_{k-1}$ for $k \geqslant 1$, where $i_0 = N$. The continued fraction

$$\frac{a_{i_1}}{b_{i_1}} + \frac{a_{i_1,i_2}}{b_{i_1,i_2}} + \frac{a_{i_1,i_2,i_3}}{b_{i_1,i_2,i_3}} + \cdots$$

is called the $(i_1, i_2, \ldots, i_k, \ldots)$-branch of the branched continued fraction with independent variables (1.2).

A branched continued fraction with independent variables of the form

$$1 + \sum_{i_1=1}^{N} \frac{a_{i(1)} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{a_{i(2)} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{a_{i(3)} z_{i_3}}{1} + \cdots, \tag{1.3}$$

where the coefficients $a_{i(k)} \in \mathbb{C} \setminus \{0\}$ for all $i(k) \in \mathcal{I}_k$ and $k \geqslant 1$, is called the multidimensional regular C-fraction with independent variables.

A multidimensional regular C-fraction with independent variables is said to correspond at $\mathbf{z} = \mathbf{0}$ to an FMPS $L(\mathbf{z})$ if its sequence of approximants corresponds to $L(\mathbf{z})$.

We study here the correspondence between FMPS (1.1) and multidimensional regular C-fraction with independent variables (1.3). As a result the algorithm for the expansion of the given FMPS (1.1) into the corresponding multidimensional regular C-fraction with independent variables (1.3) is constructed and the conditions for the existence of such an algorithm are established in theorem 2.1. As an application, we include in § 3 some numerical experiments.

2. Algorithm

Let $N \geqslant 2$, $e_0 = (0, 0, \ldots, 0)$ and $e_k = (\delta_{k,1}, \delta_{k,2}, \ldots, \delta_{k,N})$ be a multiindex, where $1 \leqslant k \leqslant N$, $\delta_{i,j}$ is the Kronecker delta. Let us introduce the following sets of multiindices $\mathcal{E}_0 = \{e_0\}$ and for $k \geqslant 1$

$$\mathcal{E}_k = \{e_{i(k)} : \ e_{i(k)} = e_{i_1,i_2,\ldots,i_k} = e_{i_1} + e_{i_2} + \cdots + e_{i_k},\ i(k) \in \mathcal{I}_k\},$$

and the mapping $\varphi : \mathcal{I}_k \to \mathcal{E}_k$, such that $\varphi(i(k)) = e_{i(k)}$ for all $i(k) \in \mathcal{I}_k$, $k \geqslant 1$. It can be shown that the mapping φ is bijective.

Let $b_{e_{i(k)}} = a_{i(k)}$ for all $e_{i(k)} \in \mathcal{E}_k$, $i(k) \in \mathcal{I}_k$ and $k \geqslant 1$. Then we write multidimensional regular C-fraction with independent variables (1.3) in the form

$$1 + \sum_{i_1=1}^{N} \frac{b_{e_{i(1)}} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{b_{e_{i(2)}} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{b_{e_{i(3)}} z_{i_3}}{1} + \cdots, \tag{2.1}$$

where $b_{e_{i(k)}} \in \mathbb{C} \setminus \{0\}$ for all $e_{i(k)} \in \mathcal{E}_k$ and $k \geqslant 1$.

We shall construct and analyse the algorithm for the expansion of the given FMPS (1.1) into the corresponding multidimensional regular C-fraction with independent variables (2.1). The process of constructing of multidimensional regular C-fraction with independent variables (2.1) will be shown step by step.

Step 1: Let $c_{e_{i_1}} \neq 0$ for $2 \leqslant i_1 \leqslant N$. Then $L(\mathbf{z})$ can be written

$$L(\mathbf{z}) = P_{e_0}(z_1) + \sum_{i_1=2}^{N} c_{e_{i_1}} z_{i_1} R_{e_{i_1}}(\mathbf{z}),$$

where

$$P_{e_0}(z_1) = 1 + \sum_{n=1}^{\infty} c_{ne_1} z_1^n, \; R_{e_{i_1}}(\mathbf{z}) = \sum_{\substack{\mathbf{k} \geqslant \mathbf{0} \\ k_j=0,\; i_1+1 \leqslant j \leqslant N}} \frac{c_{\mathbf{k}+e_{i_1}}}{c_{e_{i_1}}} \mathbf{z}^{\mathbf{k}}.$$

Step 2: Let $H_{e_1}(n) \neq 0$ and $H_{2e_1}(n) \neq 0$ for $n \geqslant 1$, where

$$H_{le_1}(n) = \begin{vmatrix} c_{le_1} & c_{(l+1)e_1} & \cdots & c_{(l+n-1)e_1} \\ c_{(l+1)e_1} & c_{(l+2)e_1} & \cdots & c_{(l+n)e_1} \\ \dots & \dots & \dots & \dots \\ c_{(l+n-1)e_1} & c_{(l+n)e_1} & \cdots & c_{(l+2n-2)e_1} \end{vmatrix}, \; l = 1, 2, \tag{2.2}$$

(we note that there $H_{e_1}(n)$ and $H_{2e_1}(n)$ are the Hankel determinants (of dimension n) associated with the formal power series $P_{e_0}(z_1)$). Then according to theorem 7.2 [**11**, pp. 223–226] we have

$$1 + \sum_{n=1}^{\infty} c_{ne_1} z_1^n \sim 1 + \frac{b_{e_1} z_1}{1} + \frac{b_{2e_1} z_1}{1} + \frac{b_{3e_1} z_1}{1} + \cdots = F_{e_0}(z_1),$$

where the symbol '$\sim$' means the correspondence between $P_{e_0}(z_1)$ and $F_{e_0}(z_1)$ (at the origin), and where $b_{e_1} = H_{e_1}(1)$ and for $n \geqslant 1$

$$b_{2ne_1} = -\frac{H_{e_1}(n-1) H_{2e_1}(n)}{H_{e_1}(n) H_{2e_1}(n-1)}, \; b_{(2n+1)e_1} = -\frac{H_{e_1}(n+1) H_{2e_1}(n-1)}{H_{e_1}(n) H_{2e_1}(n)},$$

where $H_{e_1}(0) = H_{2e_1}(0) = 1$. Thus we can write

$$L(\mathbf{z}) \sim F_{e_0}(z_1) + \sum_{i_1=2}^{N} c_{e_{i_1}} z_{i_1} R_{e_{i_1}}(\mathbf{z}).$$

Step 3: Let $H_{e_{i_1}}(n) \neq 0$ and $H_{2e_{i_1}}(n) \neq 0$ for $2 \leqslant i_1 \leqslant N$ and $n \geqslant 1$, where $H_{e_{i_1}}(n)$, $H_{2e_{i_1}}(n)$ defined by analogy (2.2). Then according to theorem 7.2 [**11**, pp.

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223–226] we have for each $2 \leqslant i_1 \leqslant N$

$$\sum_{n=1}^{\infty} c_{ne_{i_1}} z_{i_1}^n \sim \frac{b'_{e_{i_1}} z_{i_1}}{1} + \frac{b'_{e_{2i_1}} z_{i_1}}{1} + \frac{b'_{e_{3i_1}} z_{i_1}}{1} + \cdots,$$

where $b'_{e_{i_1}} = H_{e_{i_1}}(1)$ and for $n \geqslant 1$

$$b'_{2ne_{i_1}} = -\frac{H_{e_{i_1}}(n-1)H_{2e_{i_1}}(n)}{H_{e_{i_1}}(n)H_{2e_{i_1}}(n-1)},\ b'_{(2n+1)e_{i_1}} = -\frac{H_{e_{i_1}}(n+1)H_{2e_{i_1}}(n-1)}{H_{e_{i_1}}(n)H_{2e_{i_1}}(n)},$$

where $H_{e_{i_1}}(0) = H_{2e_{i_1}}(0) = 1$. Since $c_{e_{i_1}} = b'_{e_{i_1}}$, $2 \leqslant i_1 \leqslant N$, then we put $b_{e_{i_1}} = b'_{e_{i_1}}$, $2 \leqslant i_1 \leqslant N$. Thus

$$L(\mathbf{z}) \sim F_{e_0}(z_1) + \sum_{i_1=2}^{N} b_{e_{i_1}} z_{i_1} R_{e_{i_1}}(\mathbf{z}).$$

Step 4: For each $2 \leqslant i_1 \leqslant N$, let

$$R'_{e_{i_1}}(\mathbf{z}) = \sum_{\substack{|\mathbf{k}|\geqslant 0 \\ k_j=0,\ i_1+1\leqslant j\leqslant N}} c_{\mathbf{k}}^{e_{i_1}} \mathbf{z}^{\mathbf{k}} \tag{2.3}$$

be reciprocal to FMPS $R_{e_{i_1}}(\mathbf{z})$. It is known that the coefficients $c_{\mathbf{k}}^{e_{i_1}}$, $k_j = 0$, $i_1 + 1 \leqslant j \leqslant N$, $|\mathbf{k}| \geqslant 1$, of FMPS (2.3) are uniquely determined by a recurrent formula

$$c_{\mathbf{k}}^{e_{i_1}} = -\sum_{|\mathbf{r}|=1}^{|\mathbf{k}|} c_{\mathbf{k}-\mathbf{r}}^{e_{i_1}} \frac{c_{\mathbf{r}+e_{i_1}}}{c_{e_{i_1}}}, \tag{2.4}$$

where $c_{\mathbf{0}}^{e_{i_1}} = 1$, moreover, $c_{\mathbf{k}}^{e_{i_1}} = 0$, if here exists an index j such that $1 \leqslant j \leqslant N$ and that $k_j < 0$. Thus we can write

$$L(\mathbf{z}) \sim F_{e_0}(z_1) + \sum_{i_1=2}^{N} \frac{b_{e_{i_1}} z_{i_1}}{R'_{e_{i_1}}(\mathbf{z})}.$$

The next construction of the multidimensional regular C-fraction with independent variables (2.1) will be carried out using the ideas laid out in Steps 1–4.

We apply Step 1 to each FMPS $R'_{e_{i_1}}(\mathbf{z})$, where $2 \leqslant i_1 \leqslant N$. By condition $c_{e_{i_2}}^{e_{i_1}} \neq 0$ for $2 \leqslant i_2 \leqslant i_1$ and $2 \leqslant i_1 \leqslant N$ we write for each $2 \leqslant i_1 \leqslant N$

$$R'_{e_{i_1}}(\mathbf{z}) = P_{e_{i_1}}(z_1) + \sum_{i_2=2}^{i_1} c_{e_{i_2}}^{e_{i_1}} z_{i_2} R_{e_{i(2)}}(\mathbf{z}),$$

where

$$P_{e_{i_1}}(z_1) = 1 + \sum_{n=1}^{\infty} c_{ne_1}^{e_{i_1}} z_1^n,\ R_{e_{i(2)}}(\mathbf{z}) = \sum_{\substack{|\mathbf{k}|\geqslant 0 \\ k_i=0,\ i_2+1\leqslant i\leqslant N}} \frac{c_{\mathbf{k}+e_{i_2}}^{e_{i_1}}}{c_{e_{i_2}}^{e_{i_1}}} \mathbf{z}^{\mathbf{k}}.$$

Thus

$$L(\mathbf{z}) \sim F_{e_0}(z_1) + \sum_{i_1=2}^{N} \frac{b_{e_{i_1}} z_{i_1}}{P_{e_{i_1}}(z_1)} + \sum_{i_2=2}^{i_1} c_{e_{i_2}}^{e_{i_1}} z_{i_2} R_{e_{i(2)}}(\mathbf{z}).$$

Now apply Step 2 to each formal power series $P_{e_{i_1}}(z_1)$, where $2 \leqslant i_1 \leqslant N$. Let $H_{e_1}^{e_{i_1}}(n) \neq 0$ and $H_{2e_1}^{e_{i_1}}(n) \neq 0$ for $2 \leqslant i_1 \leqslant N$ and $n \geqslant 1$, where

$$H_{le_1}^{e_{i_1}}(n) = \begin{vmatrix} c_{le_1}^{e_{i_1}} & c_{(l+1)e_1}^{e_{i_1}} & \cdots & c_{(l+n-1)e_1}^{e_{i_1}} \\ c_{(l+1)e_1}^{e_{i_1}} & c_{(l+2)e_1}^{e_{i_1}} & \cdots & c_{(l+n)e_1}^{e_{i_1}} \\ \cdots & \cdots & \cdots & \cdots \\ c_{(l+n-1)e_1}^{e_{i_1}} & c_{(l+n)e_1}^{e_{i_1}} & \cdots & c_{(l+2n-2)e_1}^{e_{i_1}} \end{vmatrix}, \ l = 1, 2. \tag{2.5}$$

Then according to theorem 7.2 [11, pp. 223–226] we have for each $2 \leqslant i_1 \leqslant N$

$$1 + \sum_{n=1}^{\infty} c_{ne_1}^{e_{i_1}} z_1^n \sim 1 + \frac{b_{e_{i_1}+e_1} z_1}{1} \underset{+}{} \frac{b_{e_{i_1}+2e_1} z_1}{1} \underset{+}{} \frac{b_{e_{i_1}+3e_1} z_1}{1} \underset{+}{} \cdots = F_{e_{i_1}}(z_1),$$

where $b_{e_{i_1}+e_1} = H_{e_1}^{e_{i_1}}(1)$ and for $n \geqslant 1$

$$b_{e_{i_1}+2ne_1} = -\frac{H_{e_1}^{e_{i_1}}(n-1) H_{2e_1}^{e_{i_1}}(n)}{H_{e_1}^{e_{i_1}}(n) H_{2e_1}^{e_{i_1}}(n-1)}, \ b_{e_{i_1}+(2n+1)e_1} = -\frac{H_{e_1}^{e_{i_1}}(n+1) H_{2e_1}^{e_{i_1}}(n-1)}{H_{e_1}^{e_{i_1}}(n) H_{2e_1}^{e_{i_1}}(n)},$$

where $H_{e_1}^{e_{i_1}}(0) = H_{2e_1}^{e_{i_1}}(0) = 1$. Thus we can write

$$L(\mathbf{z}) \sim F_{e_0}(z_1) + \sum_{i_1=2}^{N} \frac{b_{e_{i_1}} z_{i_1}}{F_{e_{i_1}}(z_1)} + \sum_{i_2=2}^{i_1} c_{e_{i_2}}^{e_{i_1}} z_{i_1} R_{e_{i(2)}}(\mathbf{z}).$$

Next we apply Step 3. Let $H_{e_{i_2}}^{e_{i_1}}(n) \neq 0$ and $H_{2e_{i_2}}^{e_{i_1}}(n) \neq 0$ for $2 \leqslant i_2 \leqslant i_1 - 1$, $2 \leqslant i_1 \leqslant N$ and $n \geqslant 1$, where $H_{e_{i_2}}^{e_{i_1}}(n)$, $H_{2e_{i_2}}^{e_{i_1}}(n)$ defined by analogy (2.5). Then according to theorem 7.2 [11, pp. 223–226] we have for each $2 \leqslant i_2 \leqslant i_1 - 1$ and $2 \leqslant i_1 \leqslant N$

$$\sum_{n=1}^{\infty} c_{ne_{i_2}}^{e_{i_1}} z_{i_2}^n \sim \frac{b'_{e_{i_1}+e_{i_2}} z_{i_2}}{1} \underset{+}{} \frac{b'_{e_{i_1}+2e_{i_2}} z_{i_2}}{1} \underset{+}{} \frac{b'_{e_{i_1}+3e_{i_2}} z_{i_2}}{1} \underset{+}{} \cdots,$$

where $b'_{e_{i(2)}} = H_{e_{i_2}}^{e_{i_1}}(1)$ and for $n \geqslant 1$

$$b'_{e_{i_1}+2ne_{i_2}} = -\frac{H_{e_{i_2}}^{e_{i_1}}(n-1) H_{2e_{i_2}}^{e_{i_1}}(n)}{H_{e_{i_2}}^{e_{i_1}}(n) H_{2e_{i_2}}^{e_{i_1}}(n-1)}, \ b'_{e_{i_1}+(2n+1)e_{i_2}} = -\frac{H_{e_{i_2}}^{e_{i_1}}(n+1) H_{2e_{i_2}}^{e_{i_1}}(n-1)}{H_{e_{i_2}}^{e_{i_1}}(n) H_{2e_{i_2}}^{e_{i_1}}(n)},$$

where $H_{e_{i_2}}^{e_{i_1}}(0) = H_{2e_{i_2}}^{e_{i_1}}(0) = 1$. Since

$$c_{e_{i_2}}^{e_{i_1}} = -\frac{c_{e_{i(2)}}}{c_{e_{i_1}}} = b'_{e_{i(2)}}, \ c_{e_{i_1}}^{e_{i_1}} = -\frac{c_{2e_{i_1}}}{c_{e_{i_1}}} = b'_{2e_{i_1}}, \ 2 \leqslant i_2 \leqslant i_1 - 1, \ 2 \leqslant i_1 \leqslant N,$$

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than we put $b_{e_{i(2)}} = b'_{e_{i(2)}}$, $b_{2e_{i_1}} = b'_{2e_{i_1}}$, $2 \leqslant i_2 \leqslant i_1 - 1$, $2 \leqslant i_1 \leqslant N$. Thus

$$L(\mathbf{z}) \sim F_{e_0}(z_1) + \sum_{i_1=2}^{N} \frac{b_{e_{i_1}} z_{i_1}}{F_{e_{i_1}}(z_1) + \sum_{i_2=2}^{i_1} b_{e_{i(2)}} z_{i_2} R_{e_{i(2)}}(\mathbf{z})}.$$

At last, applying Step 4 to each FMPS $R_{e_{i(2)}}(\mathbf{z})$, where $2 \leqslant i_2 \leqslant i_1$ and $2 \leqslant i_1 \leqslant N$, we can write

$$L(\mathbf{z}) \sim F_{e_0}(z_1) + \sum_{i_1=2}^{N} \frac{b_{e_{i_1}} z_{i_1}}{F_{e_{i_1}}(z_1) + \sum_{i_2=2}^{i_1} \frac{b_{e_{i(2)}} z_{i_2}}{R'_{e_{i(2)}}(\mathbf{z})}},$$

where $R'_{e_{i(2)}}(\mathbf{z})$ be reciprocal to FMPS $R_{e_{i(2)}}(\mathbf{z})$.

Further construction of the multidimensional regular C-fraction with independent variables (2.1) consists in the gradual application of Steps 1–4 to all FMPS that are in the denominators of the ending partial quotients of the finite branches of the branched continued fraction with independent variables. As a result, computing the coefficients $c_{\mathbf{k}}^{e_{i_1}}$, $|\mathbf{k}| \geqslant 1$, $k_j = 0$, $i_1 + 1 \leqslant j \leqslant N$, $2 \leqslant i_1 \leqslant N$, by a recurrent formula (2.4) and the coefficients $c_{\mathbf{k}}^{e_{i(k)}}$, $|\mathbf{k}| \geqslant 1$, $k_j = 0$, $i_k + 1 \leqslant j \leqslant N$, $k \geqslant 2$, $2 \leqslant i_p \leqslant i_{p-1}$, $1 \leqslant p \leqslant k$, by a recurrent formula

$$c_{\mathbf{k}}^{e_{i(k)}} = -\sum_{|\mathbf{r}|=1}^{|\mathbf{k}|} c_{\mathbf{k}-\mathbf{r}}^{e_{i(k)}} \frac{c_{\mathbf{r}+e_{i_k}}^{e_{i(k-1)}}}{c_{e_{i_k}}^{e_{i(k-1)}}}, \tag{2.6}$$

where $c_{\mathbf{0}}^{e_{i(k)}} = 1$, moreover, $c_{\mathbf{k}}^{e_{i(k)}} = 0$, if here exists an index j such that $1 \leqslant j \leqslant N$ and that $k_j < 0$, provided that for $1 \leqslant i_1 \leqslant N$ and $n \geqslant 1$

$$H_{e_{i_1}}(n) \neq 0,\ H_{2e_{i_1}}(n) \neq 0, \tag{2.7}$$

where

$$H_{le_{i_1}}(n) = \begin{vmatrix} c_{le_{i_1}} & c_{(l+1)e_{i_1}} & \cdots & c_{(l+n-1)e_{i_1}} \\ c_{(l+1)e_{i_1}} & c_{(l+2)e_{i_1}} & \cdots & c_{(l+n)e_{i_1}} \\ \cdots & \cdots & \cdots & \cdots \\ c_{(l+n-1)e_{i_1}} & c_{(l+n)e_{i_1}} & \cdots & c_{(l+2n-2)e_{i_1}} \end{vmatrix},\ l = 1, 2,$$

and for $1 \leqslant i_{k+1} \leqslant i_k - 1$, $2 \leqslant i_p \leqslant i_{p-1}$, $1 \leqslant p \leqslant k$, $k \geqslant 1$ and $n \geqslant 1$

$$H_{e_{i_{k+1}}}^{e_{i(k)}}(n) \neq 0,\ H_{2e_{i_{k+1}}}^{e_{i(k)}}(n) \neq 0, \tag{2.8}$$

where

$$H_{le_{i_{k+1}}}^{e_{i(k)}}(n) = \begin{vmatrix} c_{le_{i_{k+1}}}^{e_{i(k)}} & c_{(l+1)e_{i_{k+1}}}^{e_{i(k)}} & \cdots & c_{(l+n-1)e_{i_{k+1}}}^{e_{i(k)}} \\ c_{(l+1)e_{i_{k+1}}}^{e_{i(k)}} & c_{(l+2)e_{i_{k+1}}}^{e_{i(k)}} & \cdots & c_{(l+n)e_{i_{k+1}}}^{e_{i(k)}} \\ \cdots & \cdots & \cdots & \cdots \\ c_{(l+n-1)e_{i_{k+1}}}^{e_{i(k)}} & c_{(l+n)e_{i_{k+1}}}^{e_{i(k)}} & \cdots & c_{(l+2n-2)e_{i_{k+1}}}^{e_{i(k)}} \end{vmatrix},\ l = 1, 2,$$

for FMPS (1.1) we obtain multidimensional regular C-fraction with independent variables (2.1), where the $b_{e_{i(k)}}$ for all $e_{i(k)} \in \mathcal{E}_k$ and $k \geqslant 1$ defined by the following formulas

$$b_{e_{i_1}} = H_{e_{i_1}}(1),\ b_{2ne_{i_1}} = -\frac{H_{e_{i_1}}(n-1)H_{2e_{i_1}}(n)}{H_{e_{i_1}}(n)H_{2e_{i_1}}(n-1)}, \tag{2.9a}$$

$$b_{(2n+1)e_{i_1}} = -\frac{H_{e_{i_1}}(n+1)H_{2e_{i_1}}(n-1)}{H_{e_{i_1}}(n)H_{2e_{i_1}}(n)}, \tag{2.9b}$$

where $1 \leqslant i_1 \leqslant N$, $n \geqslant 1$, $H_{e_{i_1}}(0) = H_{2e_{i_1}}(0) = 1$, and

$$b_{e_{i(k+1)}} = H^{e_{i(k)}}_{e_{i_{k+1}}}(1),\ b_{e_{i(k)}+2ne_{i_{k+1}}} = -\frac{H^{e_{i(k)}}_{e_{i_{k+1}}}(n-1)H^{e_{i(k)}}_{2e_{i_{k+1}}}(n)}{H^{e_{i(k)}}_{e_{i_{k+1}}}(n)H^{e_{i(k)}}_{2e_{i_{k+1}}}(n-1)}, \tag{2.10a}$$

$$b_{e_{i(k)}+(2n+1)e_{i_{k+1}}} = -\frac{H^{e_{i(k)}}_{e_{i_{k+1}}}(n+1)H^{e_{i(k)}}_{2e_{i_{k+1}}}(n-1)}{H^{e_{i(k)}}_{e_{i_{k+1}}}(n)H^{e_{i(k)}}_{2e_{i_{k+1}}}(n)}, \tag{2.10b}$$

where $2 \leqslant i_p \leqslant i_{p-1}$, $1 \leqslant p \leqslant k$, $1 \leqslant i_{k+1} \leqslant i_k - 1$, $k \geqslant 1$, $n \geqslant 1$, and $H^{e_{i(k)}}_{e_{i_{k+1}}}(0) = H^{e_{i(k)}}_{2e_{i_{k+1}}}(0) = 1$.

Thus, we constructed the recurrent algorithm for computing the coefficients of the multidimensional regular C-fraction with independent variables (2.1) in terms of the corresponding FMPS (1.1).

We show that the constructed multidimensional regular C-fraction with independent variables (2.1) corresponds at $\mathbf{z} = \mathbf{0}$ to the FMPS (1.1). Let $\{f_n(\mathbf{z})\}$ be a sequence of approximants of the multidimensional regular C-fraction with independent variables (2.1). Using relations (2.4), (2.6) and (2.9a), (2.9b), (2.10a), (2.10b) we curtail the $f_n(\mathbf{z})$ for $n \geqslant 1$.

For $n = 1$ we have

$$f_1(\mathbf{z}) = 1 + \sum_{i_1=1}^{N} b_{e_{i_1}} z_{i_1} = 1 + \sum_{i_1=1}^{N} c_{e_{i_1}} z_{i_1}.$$

Since

$$1 + \sum_{i_1=1}^{N} c_{e_{i_1}} z_{i_1} - \left(1 + \sum_{|\mathbf{k}| \geqslant 1} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}\right) = -\sum_{|\mathbf{k}| \geqslant 2} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}},$$

then $f_1(\mathbf{z}) \sim L(\mathbf{z})$ and the order of correspondence is $\nu_1 = 2$.

Now, according to the higher described algorithm for e_0 and for all $e_{i(k)}$ such that $2 \leqslant i_p \leqslant i_{p-1}$, $1 \leqslant p \leqslant k$ and $k \geqslant 1$ the continued fraction

$$F_{e_{i(k)}}(z_1) = 1 + \frac{b_{e_{i(k)}+e_1} z_1}{1} \underset{+}{} \frac{b_{e_{i(k)}+2e_1} z_1}{1} \underset{+}{} \frac{b_{e_{i(k)}+3e_1} z_1}{1} \underset{+}{} \cdots,$$

corresponds at the origin to the formal power series $P_{e_{i(k)}}(z_1)$ and the order of correspondence is $\nu_n = n + 1$. From this it follows that for e_0 and for each $e_{i(k)}$ such that $2 \leqslant i_p \leqslant i_{p-1}$, $1 \leqslant p \leqslant k$, $k \geqslant 1$ and for $n \geqslant 2$ the finite continued fraction

$$F^{(n)}_{e_{i(k)}}(z_1) = 1 + \frac{b_{e_{i(k)}+1e_1} z_1}{1} \underset{+}{} \frac{b_{e_{i(k)}+2e_1} z_1}{1} \underset{+}{} \cdots \underset{+}{} \frac{b_{e_{i(k)}+ne_1} z_1}{1}$$

has formal power series expansion

$$P^{(n)}_{e_{i(k)}}(z_1) = 1 + \sum_{l=1}^{n} c^{e_{i(k)}}_{le_1} z_1^l + O(z_1^{n+1}),$$

where $c^{e_0}_{le_1} = c_{le_1}$ for $1 \leqslant l \leqslant n$, $n \geqslant 2$; $O(z_1^p)$ is a symbolic mark for some formal power series, whose minimal degree of terms is not less than p, $p \geqslant 3$.

Than, for $n = 2$ we can write

$$\begin{aligned} f_2(\mathbf{z}) &= F^{(2)}_{e_0}(z_1) + \sum_{i_1=2}^{N} \frac{b_{e_{i_1}} z_{i_1}}{1} \underset{+}{} \sum_{i_2=1}^{i_1} b_{e_{i(2)}} z_{i_2} \\ &= P^{(2)}_{e_0}(z_1) + \sum_{i_1=2}^{N} \frac{c_{e_{i_1}} z_{i_1}}{1} \underset{+}{} \sum_{i_2=1}^{i_1} c^{e_{i_1}}_{e_{i_2}} z_{i_2} \\ &= P^{(2)}_{e_0}(z_1) + \sum_{i_1=2}^{N} c_{e_{i_1}} z_{i_1} \left(1 + \sum_{i_2=1}^{i_1} \frac{c_{e_{i(2)}}}{c_{e_{i_1}}} z_{i_2} + O(\mathbf{z}^2) \right) \\ &= 1 + \sum_{i_1=1}^{N} c_{e_{i_1}} z_{i_1} + \sum_{i_1=1}^{N} z_{i_1} \left(\sum_{i_2=1}^{i_1} c_{e_{i(2)}} z_{i_2} \right) + O(\mathbf{z}^3) \\ &= 1 + \sum_{|\mathbf{k}|=1}^{2} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} + O(\mathbf{z}^3), \end{aligned}$$

where $O(\mathbf{z}^p)$ is a symbolic mark for some FMPS, whose minimal degree of homogeneous terms is not less than p, $p \geqslant 2$. Since

$$1 + \sum_{|\mathbf{k}|=1}^{2} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} + O(\mathbf{z}^3) - \left(1 + \sum_{|\mathbf{k}| \geqslant 1} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} \right) = O'(\mathbf{z}^3),$$

where $O'(\mathbf{z}^p)$ is a symbolic mark for some FMPS, whose minimal degree of homogeneous terms is not less than p, $p \geqslant 3$, then $f_2(\mathbf{z}) \sim L(\mathbf{z})$ and $\nu_2 = 3$.

Next, let n be an arbitrary natural number such that $n \geqslant 3$. Then we get

$$\begin{aligned} f_n(\mathbf{z}) = F^{(n)}_{e_0}(z_1) &+ \sum_{i_1=2}^{N} \frac{b_{e_{i_1}} z_{i_1}}{F^{(n-1)}_{e_{i_1}}(z_1)} \underset{+}{} \cdots \underset{+}{} \sum_{i_{n-2}=2}^{i_{n-3}} \frac{b_{e_{i(n-2)}} z_{i_{n-2}}}{F^{(2)}_{e_{i(n-2)}}(z_1)} \\ &+ \sum_{i_{n-1}=2}^{i_{n-2}} \frac{b_{e_{i(n-1)}} z_{i_{n-1}}}{1} \underset{+}{} \sum_{i_n=1}^{i_{n-1}} b_{e_{i(n)}} z_{i_n} \\ = P^{(n)}_{e_0}(z_1) &+ \sum_{i_1=2}^{N} \frac{c_{e_{i_1}} z_{i_1}}{P^{(n-1)}_{e_{i_1}}(z_1)} \underset{+}{} \cdots \underset{+}{} \sum_{i_{n-2}=2}^{i_{n-3}} \frac{c^{e_{i(n-3)}}_{e_{i_{n-2}}} z_{i_{n-2}}}{P^{(2)}_{e_{i(n-2)}}(z_1)} \end{aligned}$$

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$$+\sum_{i_{n-1}=2}^{i_{n-2}} \frac{c_{e_{i_{n-1}}}^{e_{i(n-2)}} z_{i_{n-1}}}{1} + \sum_{i_n=1}^{i_{n-1}} c_{e_{i_n}}^{e_{i(n-1)}} z_{i_n}$$

$$= P_{e_0}^{(n)}(z_1) + \sum_{i_1=2}^{N} \frac{c_{e_{i_1}} z_{i_1}}{P_{e_{i_1}}^{(n-1)}(z_1)} + \cdots + \sum_{i_{n-3}=2}^{i_{n-4}} \frac{c_{e_{i_{n-3}}}^{e_{i(n-4)}} z_{i_{n-3}}}{P_{e_{i(n-3)}}^{(3)}(z_1)}$$

$$+ \sum_{i_{n-2}=2}^{i_{n-3}} \frac{c_{e_{i_{n-2}}}^{e_{i(n-3)}} z_{i_{n-2}}}{P_{e_{i(n-2)}}^{(2)}(z_1)} + \sum_{i_{n-1}=2}^{i_{n-2}} c_{e_{i_{n-1}}}^{e_{i(n-2)}} z_{i_{n-1}}$$

$$\times \left(1 + \sum_{i_n=1}^{i_{n-1}} \frac{c_{e_{i_{n-1},i_n}}^{e_{i(n-2)}}}{c_{e_{i_{n-1}}}^{e_{i(n-2)}}} z_{i_n} + O(\mathbf{z}^2)\right)$$

$$= P_{e_0}^{(n)}(z_1) + \sum_{i_1=2}^{N} \frac{c_{e_{i_1}} z_{i_1}}{P_{e_{i_1}}^{(n-1)}(z_1)} + \cdots + \sum_{i_{n-3}=2}^{i_{n-4}} \frac{c_{e_{i_{n-3}}}^{e_{i(n-4)}} z_{i_{n-3}}}{P_{e_{i(n-3)}}^{(3)}(z_1)} + \sum_{i_{n-2}=2}^{i_{n-3}} \frac{c_{e_{i_{n-2}}}^{e_{i(n-3)}} z_{i_{n-2}}}{1}$$

$$+ \left(\sum_{i_{n-1}=1}^{i_{n-2}} c_{e_{i_{n-1}}}^{e_{i(n-2)}} z_{i_{n-1}} + \sum_{i_{n-1}=1}^{i_{n-2}} z_{i_{n-1}} \left(\sum_{i_n=1}^{i_{n-1}} c_{e_{i_{n-1},i_n}}^{e_{i(n-2)}} z_{i_n}\right) + O(\mathbf{z}^3)\right)$$

Continuing this process on the final step we obtain

$$f_n(\mathbf{z}) = P_{e_0}^{(n)}(z_1) + \sum_{i_1=2}^{N} \frac{c_{e_{i_1}} z_{i_1}}{P_{e_{i_1}}^{(n-1)}(z_1)}$$

$$+ \sum_{i_2=2}^{i_1} c_{e_{i_2}}^{e_{i_1}} z_{i_2} \left(1 + \sum_{i_3=1}^{i_2} \frac{c_{e_{i_2,i_3}}^{e_{i_1}}}{c_{e_{i_2}}^{e_{i_1}}} z_{i_3} + \sum_{i_3=1}^{i_2} z_{i_3} \left(\sum_{i_4=1}^{i_3} \frac{c_{e_{i_2,i_3,i_4}}^{e_{i_1}}}{c_{e_{i_2}}^{e_{i_1}}} z_{i_4}\right)\right.$$

$$\left. + \ldots + \sum_{i_3=1}^{i_2} z_{i_3} \left(\ldots \left(\sum_{i_{n-1}=1}^{i_{n-2}} z_{i_{n-1}} \left(\sum_{i_n=1}^{i_{n-1}} \frac{c_{e_{i(n)}-e_{i_1}}^{e_{i_1}}}{c_{e_{i_2}}^{e_{i_1}}} z_{i_n}\right) \ldots\right)\right) + O(\mathbf{z}^{n-1})\right)$$

$$= P_{e_0}^{(n)}(z_1) + \sum_{i_1=2}^{N} \frac{c_{e_{i_1}} z_{i_1}}{1}$$

$$+ \left(\sum_{i_2=1}^{i_1} c_{e_{i_2}}^{e_{i_1}} z_{i_2} + \sum_{i_2=1}^{i_1} z_{i_2} \left(\sum_{i_3=1}^{i_2} c_{e_{i_2,i_3}}^{e_{i_1}} z_{i_3}\right) + \ldots + \sum_{i_2=1}^{i_1} z_{i_2}\right.$$

$$\left. \times \left(\sum_{i_3=1}^{i_2} z_{i_3} \left(\ldots \left(\sum_{i_{n-1}=1}^{i_{n-2}} z_{i_{n-1}} \left(\sum_{i_n=1}^{i_{n-1}} c_{e_{i(n)}-e_{i_1}}^{e_{i_1}} z_{i_n}\right) \ldots\right)\right)\right) + O(\mathbf{z}^n)\right).$$

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From this we have

$$\begin{aligned}
f_n(\mathbf{z}) &= P_{e_0}^{(n)}(z_1) + \sum_{i_1=2}^{N} c_{e_{i_1}} z_{i_1} \left(1 + \sum_{i_2=1}^{i_1} \frac{c_{e_{i(2)}}}{c_{e_{i_1}}} z_{i_2} + \sum_{i_2=1}^{i_1} z_{i_2} \left(\sum_{i_3=1}^{i_2} \frac{c_{e_{i(3)}}}{c_{e_{i_1}}} z_{i_3}\right)\right. \\
&\quad \left. + \ldots + \sum_{i_2=1}^{i_1} z_{i_2} \left(\ldots \left(\sum_{i_{n-1}=1}^{i_{n-2}} z_{i_{n-1}} \left(\sum_{i_n=1}^{i_{n-1}} \frac{c_{e_{i(n)}}}{c_{e_{i_1}}} z_{i_n}\right) \ldots\right)\right) + O(\mathbf{z}^n)\right) \\
&= 1 + \sum_{i_1=1}^{N} c_{e_{i_1}} z_{i_1} + \sum_{i_1=1}^{N} z_{i_1} \left(\sum_{i_2=1}^{i_1} c_{e_{i(2)}} z_{i_2}\right) + \ldots + \sum_{i_1=1}^{N} z_{i_1} \\
&\quad \times \left(\sum_{i_2=1}^{i_1} z_{i_2} \left(\ldots \left(\sum_{i_{n-1}=1}^{i_{n-2}} z_{i_{n-1}} \left(\sum_{i_n=1}^{i_{n-1}} c_{e_{i(n)}} z_{i_n}\right) \ldots\right)\right)\right) + O(\mathbf{z}^{n+1}) \\
&= 1 + \sum_{|\mathbf{k}|=1}^{n} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} + O(\mathbf{z}^{n+1}).
\end{aligned}$$

Since

$$1 + \sum_{|\mathbf{k}|=1}^{n} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}} + O(\mathbf{z}^{n+1}) - \left(1 + \sum_{|\mathbf{k}|\geqslant 1} c_{\mathbf{k}} \mathbf{z}^{\mathbf{k}}\right) = O'(\mathbf{z}^{n+1}),$$

then $f_n(\mathbf{z}) \sim L(\mathbf{z})$ and the order of correspondence is $\nu_n = n+1$.

Finally, from arbitrary n it follows that $f_n(\mathbf{z}) \sim L(\mathbf{z})$ for $n \geqslant 1$, and that the order of correspondence is $\nu_n = n+1$. From this it follows, that the series $\Lambda(f_n)$ and $L(\mathbf{z})$ agree for all homogeneous terms up to and including degree n. Since

$$\lim_{n\to+\infty} \nu_n = \lim_{n\to+\infty} n+1 = +\infty,$$

then the multidimensional regular C-fraction with independent variables (2.1) corresponds at $\mathbf{z} = \mathbf{0}$ to the FMPS (1.1).

Hence, we prove the following theorem:

THEOREM 2.1. *The multidimensional regular C-fraction with independent variables (1.3) corresponds at $\mathbf{z} = \mathbf{0}$ to the given FMPS (1.1) if and only if the conditions (2.7) for $1 \leqslant i_1 \leqslant N$, $n \geqslant 1$, and the conditions (2.8) for $1 \leqslant i_{k+1} \leqslant i_k - 1$, $2 \leqslant i_p \leqslant i_{p-1}$, $1 \leqslant p \leqslant k$, $k \geqslant 1$, $n \geqslant 1$, are satisfied.*

3. Examples

We shall consider now a few examples.

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Example 3.1. The function

$$f(z_1, z_2, z_3) = e^{z_1+z_2+z_3}$$

has a formal triple power series at origin given by

$$L(z_1, z_2, z_3) = \sum_{k=0}^{\infty} \frac{(z_1 + z_2 + z_3)^k}{k!}.$$

Applying recurrent algorithm constructed in § 2, we obtain:

Step 1.1: We have

$$L(z_1, z_2, z_3) = \sum_{k=0}^{\infty} \frac{z_1^k}{k!} + z_2 \left(1 + z_1 + \frac{z_2}{2} + \frac{z_1^2}{2} + \frac{z_1 z_2}{2} + \frac{z_2^2}{6} + \cdots\right)$$
$$+ z_3 \left(1 + z_1 + z_2 + \frac{z_3}{2} + \frac{(z_1 + z_2)^2}{2} + \frac{(z_1 + z_2) z_3}{2} + \frac{z_3^2}{6} + \cdots\right).$$

Steps 1.2 and **1.3**: By formulas (2.9a) and (2.9b) we obtain

$$b_{1,0,0} = b_{0,1,0} = b_{0,0,1} = 1,$$
$$b_{2k,0,0} = b_{0,2k,0} = b_{0,0,2k} = \frac{1}{2 - 4k},\ k \geqslant 1,$$
$$b_{2k+1,0,0} = b_{0,2k+1,0} = b_{0,0,2k+1} = \frac{1}{4k + 2},\ k \geqslant 1.$$

Thus

$$L(z_1, z_2, z_3) \sim K(z_1) + z_2 \left(1 + z_1 + \frac{z_2}{2} + \frac{z_1^2}{2} + \frac{z_1 z_2}{2} + \frac{z_2^2}{6} + \cdots\right)$$
$$+ z_3 \left(1 + z_1 + z_2 + \frac{z_3}{2} + \frac{(z_1 + z_2)^2}{2} + \frac{(z_1 + z_2) z_3}{2} + \frac{z_3^2}{6} + \cdots\right),$$

where

$$K(z_1) = 1 + \frac{z_1}{1} \underset{+}{} \frac{a_2 z_1}{1} \underset{+}{} \frac{a_3 z_1}{1} \underset{+}{} \cdots,\ a_{2k} = \frac{1}{2 - 4k},\ a_{2k+1} = \frac{1}{2 + 4k},\ k \geqslant 1. \tag{3.1}$$

Step 1.4: By a recurrent formula (2.4) we obtain

$$L(z_1, z_2, z_3) \sim K(z_1) + \frac{z_2}{1 - z_1 - \frac{z_2}{2} + \frac{z_1^2}{2} + \frac{z_1 z_2}{2} + \frac{z_2^2}{12} + \cdots}$$
$$+ \frac{z_3}{1 - z_1 - z_2 - \frac{z_3}{2} + \frac{(z_1 + z_2)^2}{2} + \frac{(z_1 + z_2) z_3}{2} + \frac{z_3^2}{12} + \cdots}.$$

Step 2.1: We have

$$L(z_1,z_2,z_3)\sim K(z_1)+\cfrac{z_2}{\displaystyle\sum_{k_1=0}^{\infty}\frac{(-z_1)^{k_1}}{k_1!}-\frac{z_2}{2}\left(1-z_1-\frac{z_2}{6}+\frac{z_1^2}{2}+\frac{z_1z_2}{6}+\cdots\right)}$$
$$+\cfrac{z_3}{\displaystyle\sum_{k_1=0}^{\infty}\frac{(-z_1)^{k_1}}{k_1!}-z_2\left(1-z_1-\frac{z_2}{2}+\cdots\right)-\frac{z_3}{2}\left(1-z_1-z_2-\frac{z_3}{6}+\cdots\right)}.$$

Steps 2.2 and **2.3**: By formulas (2.10a) and (2.10b) we obtain

$$b_{1,1,0}=b_{1,0,1}=b_{0,1,1}=-1,$$
$$b_{2k,1,0}=b_{2k,0,1}=b_{0,2k,1}=-\frac{1}{2-4k},\ k\geqslant 1,$$
$$b_{2k+1,1,0}=b_{2k+1,0,1}=b_{0,2k+1,1}=-\frac{1}{2+4k},\ k\geqslant 1.$$

Thus

$$L(z_1,z_2,z_3)\sim K(z_1)+\cfrac{z_2}{K(-z_1)-\frac{z_2}{2}\left(1-z_1-\frac{z_2}{6}+\frac{z_1^2}{2}+\frac{z_1z_2}{6}+\cdots\right)}$$
$$+\cfrac{z_3}{K(-z_1)-z_2\left(1-z_1-\frac{z_2}{2}+\cdots\right)-\frac{z_3}{2}\left(1-z_1-z_2-\frac{z_3}{6}+\cdots\right)}.$$

Step 2.4: By a recurrent formula (2.6) we obtain

$$L(z_1,z_2,z_3)\sim K(z_1)+\cfrac{z_2}{K(-z_1)+\cfrac{-z_2/2}{1+z_1+\frac{z_2}{6}+\frac{z_1^2}{2}-\frac{z_1z_2}{6}+\frac{z_2^3}{36}+\cdots}}$$
$$+\cfrac{z_3}{K(-z_1)+\cfrac{-z_2}{1+z_1+\frac{z_2}{2}+\ldots}+\cfrac{-z_3/2}{1+z_1+z_2+\frac{z_3}{6}+\cdots}}.$$

And so on, in the end we will get the corresponding three-dimensional regular C-fraction with independent variables of the form

$$K(z_1)+\cfrac{z_2}{K(-z_1)+\cfrac{a_2z_2}{K(z_1)+\cfrac{a_3z_2}{K(-z_1)+\ddots}}}$$
$$+\cfrac{z_3}{K(-z_1)+\cfrac{-z_2}{K(z_1)+\cfrac{-a_2z_2}{K(-z_1)+\ddots}}+\cfrac{a_2z_3}{K(z_1)+\cfrac{z_2}{K(-z_1)+\ddots}+\cfrac{a_3z_3}{K(-z_1)+\ddots}}},$$

where the $K(z_1)$ and the a_k for $k\geqslant 2$ are defined by (3.1).

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We also note that it can be shown that the function

$$f(\mathbf{z}) = e^{z_1+z_2+\ldots+z_N}$$

has a corresponding multidimensional regular C-fraction with independent variables (2.1), where for $1 \leqslant i_1 \leqslant N$ and $n \geqslant 1$

$$b_{e_{i_1}} = 1,\ b_{2ne_{i_1}} = \frac{1}{2-4n},\ b_{(2n+1)e_{i_1}} = \frac{1}{2+4n}$$

and for $2 \leqslant i_p \leqslant i_{p-1}$, $1 \leqslant p \leqslant k$, $1 \leqslant i_{k+1} \leqslant i_k - 1$, $k \geqslant 1$ and $n \geqslant 1$

$$b_{e_{i(k+1)}} = (-1)^k,\ b_{e_{i(k)}+2ne_{i_{k+1}}} = \frac{(-1)^k}{2-4n},\ b_{e_{i(k)}+(2n+1)e_{i_{k+1}}} = \frac{(-1)^k}{2+4n}.$$

Example 3.2. The function

$$f(\mathbf{z}) = (1 + z_1 + z_2 + \cdots + z_N)^{1/2}$$

has an FMPS at $\mathbf{z} = \mathbf{0}$ given by

$$L(\mathbf{z}) = \sum_{k=0}^{\infty} \frac{(-1/2)_k}{k!} \left(-\sum_{l=1}^{N} z_l\right)^k,$$

where $(a)_k$ is the Pochhammer symbol, that is $(a)_k = a(a+1)(a+2)\ldots(a+k-1)$ for $k \geqslant 1$ and $(a)_0 = 1$. Applying recurrent algorithm constructed in § 2, we obtain the corresponding multidimensional regular C-fraction with independent variables (1.3), where $a_{i(1)} = 1/2$, $a_{i(k+1)} = (1/2)^{1+\delta_{i_k,i_{k+1}}}$ for all $i(k) \in \mathcal{I}_k$ and $k \geqslant 1$.

Example 3.3. The function

$$F(\alpha,\beta,\gamma;z_1,z_2,z_3) = \frac{(1+z_1)^{-\alpha}(1+z_2(1+z_1)^{2\alpha})^{-\beta}}{(1+z_3(1+z_1)^{2\alpha}(1+z_2(1+z_1)^{2\alpha})^{2\beta})^{\gamma}}$$

has a formal triple power series at origin given by

$$L(z_1,z_2,z_3) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{k!}(-z_1)^k \sum_{l=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-2\alpha)_k}{k!}(-z_1)^k\right)^{2l} \frac{(\beta)_l}{l!}(-z_2)^l$$
$$\times \sum_{r=0}^{\infty} \left(\sum_{l=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-2\alpha)_k}{k!}(-z_1)^k\right)^{2l+1} \frac{(-2\beta)_l}{l!}(-z_2)^l\right)^{2r} \frac{(\gamma)_r}{r!}(-z_3)^r,$$

where $\alpha,\beta,\gamma \notin \mathbb{Z}$. Applying recurrent algorithm constructed in § 2, we obtain the corresponding three-dimensional regular C-fraction with independent variables

$$1 + \sum_{i_1=1}^{3} \frac{b_{e_{i(1)}} z_{i_1}}{1} + \sum_{i_2=1}^{i_1} \frac{b_{e_{i(2)}} z_{i_2}}{1} + \sum_{i_3=1}^{i_2} \frac{b_{e_{i(3)}} z_{i_3}}{1} + \cdots,$$

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where

$$b_{e_1+pe_2+qe_3} = -\alpha,\ b_{e_2+qe_3} = -\beta,\ b_{e_3} = -\gamma,\ p \geqslant 0,\ q \geqslant 0,$$

$$b_{ne_1+pe_2+qe_3} = \frac{[n/2]([n/2]+(-1)^n\alpha)}{(n-1)n},\ n \geqslant 2,\ p \geqslant 0,\ q \geqslant 0,$$

$$b_{pe_2+qe_3} = \frac{[p/2]([p/2]+(-1)^p\beta)}{(p-1)p},\ p \geqslant 2,\ q \geqslant 0,$$

$$b_{qe_3} = \frac{[q/2]([q/2]+(-1)^q\gamma)}{(q-1)q},\ q \geqslant 2,$$

(here '[.]' means an integer part of the number).

Example 3.4. The special case of the functions considered in examples 3.2 and 3.3, is the function

$$f(z_1, z_2, z_3) = F(-1/2, -1/2, -1/2; z_1, z_2, z_3) = (1 + z_1 + z_2 + z_3)^{1/2}$$

which has a formal triple power series at origin given by

$$L(z_1, z_2, z_3) = \sum_{k=0}^{\infty} \frac{(-1/2)_k}{k!}(-z_1 - z_2 - z_3)^k. \tag{3.2}$$

Thus by example 3.2 the $f(z_1, z_2, z_3)$ has the corresponding three-dimensional regular C-fraction with independent variables of the form

$$1 + \cfrac{z_1/2}{1 + \cfrac{z_1/4}{1 + \cfrac{z_1/4}{1 + \ddots}}} + \cfrac{z_2/2}{1 + \cfrac{z_1/2}{1 + \cfrac{z_1/4}{1+\ddots}} + \cfrac{z_2/4}{1 + \cfrac{z_1/2}{1+\ddots} + \cfrac{z_2/4}{1+\ddots}}}$$
$$+ \cfrac{z_3/2}{1 + \cfrac{z_1/2}{1 + \cfrac{z_1/4}{1+\ddots}} + \cfrac{z_2/2}{1 + \cfrac{z_1/2}{1+\ddots} + \cfrac{z_2/4}{1+\ddots}} + \cfrac{z_3/4}{1 + \cfrac{z_1/2}{1+\ddots} + \cfrac{z_2/2}{1+\ddots} + \cfrac{z_3/4}{1+\ddots}}}. \tag{3.3}$$

The results of computation of the function $f(z_1, z_2, z_3)$ and its approximations $f_n(z_1, z_2, z_3)$, $1 \leqslant n \leqslant 10$, for different values of z_1, z_2 and z_3 are given in table 1.

Analysis of the results of computation shows that the truncation error bounds $\Delta_{f_n}(z_1, z_2, z_3) = |f(z_1, z_2, z_3) - f_n(z_1, z_2, z_3)|$ for the function $f(z_1, z_2, z_3)$ decrease with increase in the index n and, at points close to zero, the approximation is the best:

$$\Delta_{f_{10}}(1, 2, 3) = 1.49436 \times 10^{-6},$$

$$\Delta_{f_{10}}(4, 4, 4) = 4.11958 \times 10^{-4},$$

$$\Delta_{f_{10}}(-0.1, -1, 1) = 2.22045 \times 10^{-16},$$

$$\Delta_{f_{10}}(0.001, 0.001, 0.1) = 2.22045 \times 10^{-16}.$$

Table 1. *Values of* $f(z_1, z_2, z_3)$ *and* $f_n(z_1, z_2, z_3)$ *from algorithm for* $f(z_1, z_2, z_3) = (1 + z_1 + z_2 + z_3)^{1/2}$

(z_1, z_2, z_3)	(−0.1, −1,1)	(1,2,3)	(4,4,4)	(0.001,0.001,0.1)
$F(z_1, z_2, z_3)$	0.948683298	2.645751311	3.605551275	1.049761878
$f_1(z_1, z_2, z_3)$	0.950000000	4.000000000	7.000000000	1.051000000
$f_2(z_1, z_2, z_3)$	0.948717949	2.361538462	2.833333333	1.049732444
$f_3(z_1, z_2, z_3)$	0.948684211	2.726698392	3.972222222	1.049762578
$f_4(z_1, z_2, z_3)$	0.948683322	2.626946223	3.475984556	1.049761861
$f_5(z_1, z_2, z_3)$	0.948683299	2.649989938	3.656642379	1.049761878
$f_6(z_1, z_2, z_3)$	0.948683298	2.644842099	3.586261429	1.049761878
$f_7(z_1, z_2, z_3)$	0.948683298	2.64594046	3.612953179	1.049761878
$f_8(z_1, z_2, z_3)$	0.948683298	2.645712938	3.602728839	1.049761878
$f_9(z_1, z_2, z_3)$	0.948683298	2.645758944	3.606630066	1.049761878
$f_{10}(z_1, z_2, z_3)$	0.948683298	2.645749817	3.605139318	1.049761878

According to theorem 7.2 [**11**, pp. 223–226] the formal triple power series (3.2) has a corresponding regular C-fraction of the form

$$1 + \frac{(z_1 + z_2 + z_3)/2}{1} \underset{+}{} \frac{(z_1 + z_2 + z_3)/4}{1} \underset{+}{} \frac{(z_1 + z_2 + z_3)/4}{1} \underset{+}{} \cdots . \tag{3.4}$$

Let $g_n(z_1, z_2, z_3)$ be the nth approximant of (3.4), $n \geqslant 1$, and let $s_n(z_1, z_2, z_3)$ be the nth partial sum of (3.2), $n \geqslant 1$. We give the truncation error bounds $\Delta_{f_n}(z_1, z_2, z_3)$, $\Delta_{g_n}(z_1, z_2, z_3) = |f(z_1, z_2, z_3) - f_n(z_1, z_2, z_3)|$ and $\Delta_{s_n}(z_1, z_2, z_3) = |f(z_1, z_2, z_3) - s_n(z_1, z_2, z_3)|$ at point close to zero in table 2. From this it can be seen that the truncation error bounds decrease with increase in the index n. The approximation of $f(z_1, z_2, z_3)$ by the nth approximants $f_n(z_1, z_2, z_3)$ of (3.3) is better then by the nth approximants $g_n(z_1, z_2, z_3)$ of (3.4) and the nth partial sum $s_n(z_1, z_2, z_3)$ of (3.2).

Example 3.5. Now, we consider the following function

$$f(z_1, z_2) = \frac{1 + \ln(1 + z_2 e^{2z_1})}{e^{z_1}}$$

which has a formal double power series at origin given by

$$\sum_{k=0}^{\infty} \frac{(-z_1)^k}{k!} \left(1 - \sum_{l=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(2z_1)^k}{k!}\right)^{l+1} \frac{(-z_2)^{l+1}}{l+1}\right).$$

Applying recurrent algorithm constructed in § 2, we obtain the corresponding two-dimensional regular C-fraction with independent variables

$$1 + \cfrac{a_{10}z_1}{1 + \cfrac{a_{20}z_1}{1 + \cfrac{a_{30}z_1}{1 + \ddots}}} + \cfrac{a_{01}z_2}{1 + \cfrac{a_{11}z_1}{1 + \cfrac{a_{21}z_1}{1 + \ddots}} + \cfrac{a_{02}z_2}{1 + \cfrac{a_{12}z_1}{1 + \ddots} + \cfrac{a_{03}z_2}{1 + \ddots}}}, \tag{3.5}$$

Table 2. *Truncation error bounds for* $f(z_1, z_2, z_3) = (1 + z_1 + z_2 + z_3)^{1/2}$

n	$\Delta_{f_n}(0.1, 0.1, 0.1)$	$\Delta_{g_n}(0.1, 0.1, 0.1)$	$\Delta_{s_n}(0.1, 0.1, 0.1)$
1	9.8246×10^{-3}	9.8246×10^{-3}	9.8246×10^{-3}
2	4.3886×10^{-4}	6.4054×10^{-3}	1.4254×10^{-3}
3	1.6466×10^{-5}	4.1966×10^{-5}	2.6207×10^{-4}
4	5.5317×10^{-7}	2.7486×10^{-6}	5.4331×10^{-5}
5	1.7283×10^{-8}	1.8003×10^{-7}	1.2114×10^{-5}
6	5.1314×10^{-10}	1.1791×10^{-8}	2.8362×10^{-6}
7	1.4672×10^{-11}	7.7229×10^{-10}	6.8774×10^{-7}
8	4.0790×10^{-13}	5.0583×10^{-11}	1.7123×10^{-7}
9	1.1102×10^{-14}	3.3129×10^{-12}	4.3516×10^{-8}
10	2.2204×10^{-16}	2.1694×10^{-13}	1.1243×10^{-8}

Table 3. *Truncation error bounds for* $f(z_1, z_2) = (1 + \ln(1 + z_2 e^{2z_1}))/e^{z_1}$

n	$\Delta_{f_n}(0.1, 0.1)$	$\Delta_{f_n}(-0.01, 0.1)$	$\Delta_{f_n}(-1, 2)$	$\Delta_{f_n}(-1.5, -1.5)$
1	9.1089×10^{-3}	5.5016×10^{-3}	6.3057×10^{-1}	3.1338
2	9.1615×10^{-4}	1.0851×10^{-4}	2.9723×10^{-1}	2.0090
3	2.8325×10^{-5}	2.6530×10^{-6}	3.6099×10^{-2}	3.9116×10^{-1}
4	2.9834×10^{-7}	5.7997×10^{-8}	3.6805×10^{-3}	2.8478×10^{-2}
5	1.3227×10^{-8}	1.4324×10^{-9}	1.0550×10^{-3}	9.5613×10^{-3}
6	3.5439×10^{-10}	3.1747×10^{-11}	1.6880×10^{-5}	1.8260×10^{-4}
7	1.1725×10^{-11}	7.7649×10^{-13}	7.4420×10^{-6}	1.1861×10^{-4}
8	2.9377×10^{-13}	1.7319×10^{-14}	2.1980×10^{-7}	8.6896×10^{-7}
9	9.5479×10^{-15}	4.4409×10^{-16}	2.8629×10^{-8}	8.4861×10^{-7}
10	2.2204×10^{-16}	2.2204×10^{-16}	9.4623×10^{-10}	1.9619×10^{-8}

where $a_{1l} = -1$, $a_{2k,l} = 1/(4k-2)$, $a_{2k+1,l} = -1/(4k+2)$, $a_{01} = 1$, $a_{0,2l} = l/(4l-2)$, $a_{0,2l+1} = l/(4l+2)$, $k \geqslant 0$, $l \geqslant 1$. The results of computation of the truncation error bounds $\Delta_{f_n}(z_1, z_2) = |f(z_1, z_2) - f_n(z_1, z_2)|$, where $f_n(z_1, z_2)$ is an nth approximant of (3.5), different values of z_1 and z_2 are given in table 3. From this it can be seen that the truncation error bounds for the function $f(z_1, z_2)$ decrease with increase in the index n and, at points close to zero, the approximation is the best (like in the previous example).

4. Conclusion

Numerical experiments illustrate the efficiency of the constructed algorithm for approximation of multivariable functions, which are represented by FMPS. Examples 3.2 and 3.4 show, on the one hand, the method of accelerate convergence of regular C-fraction and, on the other, the method of computing square roots of some complex numbers by multidimensional regular C-fraction with independent variables.

The question of the class of multivariable functions which are represented by multidimensional regular C-fraction with independent variables remains open.

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