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# Motohashi's fourth moment identity for non-archimedean test functions and applications

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# Motohashi's fourth moment identity for non-archimedean test functions and applications

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## Abstract

Motohashi established an explicit identity between the fourth moment of the Riemann zeta function weighted by some test function and a spectral cubic moment of automorphic $L$-functions. By an entirely different method, we prove a generalization of this formula to a fourth moment of Dirichlet $L$-functions modulo $q$ weighted by a non-archimedean test function. This establishes a new reciprocity formula. As an application, we obtain sharp upper bounds for the fourth moment twisted by the square of a Dirichlet polynomial of length $q^{1/4}$. An auxiliary result of independent interest is a sharp upper bound for a certain sixth moment for automorphic $L$-functions, which we also use to improve the best known subconvexity bounds for automorphic $L$-functions in the level aspect.

## 1. Introduction

### 1.1 A reciprocity formula

A landmark result in the theory of $L$-functions, because of both its structural beauty and its applications, is Motohashi's identity for the fourth moment of the Riemann zeta function [Mot97, Theorem 4.2]: if $F$ is a sufficiently nice test function, then

$$\int_{\mathbb{R}} |\zeta(1/2 + it)|^4 F(t)\, dt \tag{1.1}$$

is equal to an explicit main term plus a cubic moment of the shape

$$\sum_j L(1/2, \psi_j)^3 \check{F}(t_j) + \text{similar holomorphic and Eisenstein contribution,} \tag{1.2}$$

where the sum runs over Maaß forms $\psi_j$ with spectral parameter $t_j$ for the group $\mathrm{SL}_2(\mathbb{Z})$ and $\check{F}$ is a certain integral transform of $F$ given explicitly in terms of hypergeometric functions. This is an important instance of a formula between two different families of $L$-functions that may be regarded as a 'reciprocity formula'. Choosing the test function $F$ appropriately, it can be used, for instance, to prove sharp upper bounds for the fourth moment of the Riemann zeta function on the critical line in short intervals $t \in [T, T + T^{2/3}]$. Motohashi's formula can also be inverted to some extent; Ivić [Ivi01] used this to obtain Weyl-type subconvexity bounds for the $L$-values $L(1/2, \psi_j)$.

Motohashi's proof starts by opening the four zeta values as Dirichlet series and integrating over $t$, which, after a change of variables, gives a Dirichlet series containing a shifted convolution problem

$$\tau(n)\tau(n+h). \tag{1.3}$$

A spectral decomposition then yields the spectral cubic moment.

A very different strategy was suggested by Michel and Venkatesh [MV10, §4.5]: we interpret (1.1) as a second moment of $L$-functions associated with an Eisenstein series $E$ and choose $F$ as the corresponding local $L$-factors at infinity. Denoting the completed $L$-functions by $\Lambda(s, E)$, we have by Hecke's integral representation and Parseval's theorem (ignoring convergence)

$$\int_{\mathbb{R}} |\Lambda(1/2 + it, E)|^2 \, dt \approx \int_0^\infty |E(iy)|^2 \frac{dy}{y}.$$

Decomposing spectrally (and suppressing the continuous spectrum for notational simplicity), using Rankin–Selberg theory and Hecke's integral representation again, this 'equals'

$$\int_0^\infty \sum_j \langle |E|^2, \psi_j \rangle \psi_j(iy) \frac{dy}{y} \approx \int_0^\infty \sum_j \Lambda(1/2, \psi_j \times E) \psi_j(iy) \frac{dy}{y} \approx \sum_j \Lambda(1/2, \psi_j)^3.$$

This very beautiful idea comes with two technical challenges: first, none of the integrals converge and some regularization is necessary; and second, while this works very nicely for the special test function $F(t) = |L_\infty(1/2 + it, E)|^2$, it is not easy to spell out what happens for general test functions $F$. (See, however, recent work by Nelson [Nel19] on this approach.)

In this paper we offer yet another proof of Motohashi's identity, which has the advantage of working nicely in greater generality. The set-up we are interested in is as follows. For $q \in \mathbb{N}$, consider

$$\sum_{\chi \,(\mathrm{mod}\, q)} F_{\mathrm{fin}}(\chi) \int_{\mathbb{R}} |L(1/2 + it, \chi)|^4 F_\infty(t) \, dt$$

for some function $F_{\mathrm{fin}}$. This can be seen as the proper adèlic analogue of (1.1), twisting $\zeta(s)$ by the complete family of GL(1) characters $n^{it}\chi(n)$. By elementary Fourier analysis, every $F_{\mathrm{fin}}$ is a linear combination of character values. With applications in mind, we consider test functions of the shape

$$F_{\mathrm{fin}}(\chi) = \chi(a)\overline{\chi}(b) \tag{1.4}$$

for some integers $a, b \in \mathbb{N}$. It is straightforward to include a character average in Motohashi's proof, which essentially results in a shifted convolution problem (1.3) where $h$ is divisible by $q$. It is much less straightforward to include a general test function (1.4), because then the shifted convolution problem becomes a sum over $\tau(n)\tau(m)$ subject to the condition $an \equiv bm \pmod{q}$. The difficulty of such an extension (with sufficient control on $a, b$) was already observed in [DFI94, p. 210]. A heuristic argument based on a different strategy that we sketch in §1.4 suggests that we should expect something like

$$\sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)\overline{\chi}(b) \int_{\mathbb{R}} |L(1/2+it, \chi)|^4 F(t) \, dt \rightsquigarrow \frac{q^{1/2}}{a^{1/2}} \sum_{\psi_j \text{ of level } ab} \lambda_j(q)\lambda_j(b) L(1/2, \psi_j)^3 \check{F}(t_j). \tag{1.5}$$

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This indicates that the period integral approach will not be straightforward to extend because at the very least some non-trivial combinatorics in the Hecke algebra (see [Zac19b] for how this could look in a slightly different situation) must happen in order to generate the Hecke eigenvalues on the right-hand side. As an aside, the left-hand side of (1.5) is symmetric in $a, b$ as long as $F$ is even, while this is not directly apparent on the right-hand side (which is only written in suggestive terms anyway). It follows from (2.3) that $\lambda_j(b) = \pm b^{-1/2}$, which to some extent restores the symmetry.

We will present a proof of (1.5) in the spirit of recent reciprocity formulae of the first and third authors [BK19a, BK19b] that deal with this more general set-up without essential structural difficulties. We proceed to describe our first main result in detail. Let $a, b, q \in \mathbb{N}$, $s, u, v \in \mathbb{C}$, $F$ be an even holomorphic function that is Schwartz class on fixed vertical lines, and $f$ an automorphic form for $\mathrm{SL}_2(\mathbb{Z})$ that is either cuspidal or the standard Eisenstein series $(d/ds)E(z, s)|_{s=1/2}$. We denote its Hecke eigenvalues by $\lambda(n)$, so that $\lambda(n) = \tau(n) := \sum_{ab=n} 1$ if $f$ is Eisenstein. We define

$$\mathcal{T}_{a,b,q}(s, u, v) := \sum_{\substack{\chi \,(\mathrm{mod}\ q) \\ \chi \text{ primitive}}} \chi(a)\overline{\chi}(b) \int_{(0)} L(s+z, \chi) L(u+z, \chi) L(v-z, f \times \overline{\chi}) F(z) \frac{dz}{2\pi i}, \quad (1.6)$$

where the integration is over the vertical line $\Re z = 0$. We may assume without loss of generality that $(a, b) = (ab, q) = 1$. It is convenient to assume that $F$ is divisible by $(1-u)(v-1)^2 \prod_{j=1}^{50}(j-s)$. A typical function we have in mind is

$$F(z) = e^{z^2}(z^2 - (1-u)^2)^2(z^2 - (v-1)^2)^2 \prod_{j=1}^{50}(z^2 - (j-s)^2), \quad (1.7)$$

which is positive for $\Re z = 0$, $s = u = v = 1/2$. To get a nice-looking formula, we also need to include non-primitive characters, and for simplicity we assume that $q$ is prime. For a suitable correction polynomial $P_q(s, u, v, z)$ defined explicitly in (3.2) below and satisfying

$$P_q(s, u, v, z) \ll 1 + |\lambda(q)|, \quad \Re s, \Re u, \Re v \geqslant 1/2, \quad \Re z = 0, \quad (1.8)$$

we define the analogue for the trivial character

$$\mathcal{T}_q^{\mathrm{triv}}(s, u, v) := \int_{(0)} \zeta(s+z)\zeta(u+z) L(v-z, f) P_q(s, u, v, z) F(z) \frac{dz}{2\pi i}. \quad (1.9)$$

Note that our assumptions on $F$ imply that the integrand is holomorphic (since the poles of $\zeta$ are cancelled) and that we can shift the $z$-contour in any way we want.

On the spectral side, we define

$$\mathcal{M}_{a,b,q}^{\mathrm{Maaß}}(s, u, v) := \sum_{A|ab} \sum_{\psi \in \mathcal{B}^*(A)} \Theta_{a,b,q}^{\mathrm{Maaß}}(s, u, v, \psi) \frac{L\left(\frac{s+u-1+2v}{2}, \psi\right) L\left(\frac{1-s+u}{2}, f \times \psi\right)}{L(1, \mathrm{Ad}^2 \psi)}, \quad (1.10)$$

where $\mathcal{B}^*(A)$ denotes an orthonormal Hecke basis of Maaß *new*forms of level $A$ and $\Theta_{a,b,q}^{\mathrm{Maaß}}(s, u, v, \psi)$ is a (complicated, but) completely explicit expression defined in (3.21) that satisfies

$$\Theta_{a,b,q}^{\mathrm{Maaß}}(s, u, v, \psi) \ll_{s,u,v,F,\varepsilon} q^{1/2} A^{-1/2}(1 + |\lambda_\psi(q)|)(1 + |t_\psi|)^{-30}(abq)^\varepsilon \quad (1.11)$$

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for $\Re s, \Re u, \Re v = 1/2$ and $a \asymp b$. Similarly, we define

$$
\mathcal{M}_{a,b,q}^{\mathrm{hol}}(s,u,v) := \sum_{A|ab} \sum_{k \in 2\mathbb{N}} \sum_{\psi \in \mathcal{B}_k^*(A)} \Theta_{a,b,q}^{\mathrm{hol}}(s,u,v,\psi) \frac{L\left(\frac{s+u-1+2v}{2}, \psi\right) L\left(\frac{1-s+u}{2}, f \times \psi\right)}{L(1, \mathrm{Ad}^2 \psi)},
$$

where $\mathcal{B}_k^*(A)$ denotes an orthonormal Hecke basis of holomorphic newforms of weight $k$ and level $A$ and $\Theta_{a,b,q}^{\mathrm{hol}}(s,u,v,\psi)$ satisfies the analogous bound

$$
\Theta_{a,b,q}^{\mathrm{hol}}(s,u,v,\psi) \ll_{s,u,v,F,\varepsilon} q^{1/2} A^{-1/2} k^{-30} (abq)^\varepsilon \tag{1.12}
$$

for $\Re s, \Re u, \Re v = 1/2$ and $a \asymp b$. For simplicity, we assume that $a, b$ are square-free, so that the Eisenstein spectrum is parametrized by $\tau(ab)$ cusps. We define (initially in $\Re(s+u+2v) > 3$ and $\Re(u-s) > 1$)

$$
\mathcal{M}_{a,b,q}^{\mathrm{Eis}}(s,u,v)
$$
$$
:= \int_{\mathbb{R}} \Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t) \frac{\zeta\left(\frac{s+u-1+2v}{2} + it\right) \zeta\left(\frac{s+u-1+2v}{2} - it\right) L\left(\frac{1-s+u}{2} + it, f\right) L\left(\frac{1-s+u}{2} - it, f\right)}{\zeta(1+2it)\zeta(1-2it)} \frac{dt}{2\pi},
$$

where $\Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t)$ is defined in (3.26) and satisfies

$$
\Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t) \ll_{s,u,v,\varepsilon,F} (abq)^\varepsilon q^{1/2} (ab)^{\theta-1/2} (1+|t|)^{-30} \tag{1.13}
$$

for $\Re s = \Re v = \Re u = 1/2$, $t \in \mathbb{R}$, where $\theta \leqslant 7/64$ is an admissible exponent for the Ramanujan conjecture for the fixed form $f$ (in particular, $\theta = 0$ if $f$ is holomorphic or Eisenstein). The expressions $\mathcal{M}_{a,b,q}^*(s,u,v)$ for $* \in \{\mathrm{Maaß, hol}\}$ are obviously holomorphic in $\Re s, \Re u, \Re v \geqslant 1/2$. By contour shifts it is not hard to see that $\mathcal{M}_{a,b,q}^{\mathrm{Eis}}(s,u,v)$ has meromorphic continuation to this region; see § 3.9. In the intersection of $\Re s, \Re u, \Re v \geqslant 1/2$ with $\Re(s+u+2v) < 3$, $\Re(u-s) < 1$ the meromorphic continuation is given explicitly by the same term plus an additional polar term, defined in (3.27). We define

$$
\mathcal{M}_{a,b,q}(s,u,v) := \mathcal{M}_{a,b,q}^{\mathrm{Maaß}}(s,u,v) + \mathcal{M}_{a,b,q}^{\mathrm{hol}}(s,u,v) + \mathcal{M}_{a,b,q}^{\mathrm{Eis}}(s,u,v).
$$

We are now ready to state the reciprocity formula to which we have already alluded.

THEOREM 1. *Let* $q, a, b \in \mathbb{N}$, *$q$ be prime,* $(ab,q) = (a,b) = 1$, *$a, b$ be square-free,* $a \asymp b$. *Let* $1/2 \leqslant \Re s, \Re u, \Re v < 3/4$ *and* $\Re s \leqslant \Re u$. *Suppose that $F$ is holomorphic, Schwartz class on vertical lines, and divisible by* $(1-u)(v-1)^2 \prod_{j=1}^{50}(j-s)$. *Then*

$$
\mathcal{T}_{a,b,q}(s,u,v) + \mathcal{T}_q^{\mathrm{triv}}(s,u,v) = \mathcal{P}_{a,b,q}(s,u,v) + \mathcal{M}_{a,b,q}(s,u,v), \tag{1.14}
$$

*where the 'main term'* $\mathcal{P}_{a,b,q}(s,u,v)$ *is defined in (3.28) and satisfies*

$$
\mathcal{P}_{a,b,q}(s,u,v) \ll_{s,u,v,\varepsilon,F} q(ab)^{-1/2+\theta} (abq)^\varepsilon \tag{1.15}
$$

*for* $\Re s = \Re u = \Re v = 1/2$, $a \asymp b$, *where $\theta$ is an admissible exponent for the Ramanujan conjecture for $f$.*

We emphasize that even though $\mathcal{M}_{a,b,q}(s,u,v)$ depends on $q$, it only involves the spectrum of level $ab$. This is the ultimate reason for the specific design of the term $\mathcal{T}_q^{\mathrm{triv}}(s,u,v)$. In this sense, our formula is a clean reciprocity formula, where the pair (level, arithmetic of weight function) on the Dirichlet side is $(q,ab)$ and on the spectral side is $(ab,q)$.

Theorem 1 contains a number of simplifying assumptions, most of which can be removed without any structural difficulties at the cost of more technical work. If $q$ is not prime, we need slightly more complicated correction terms for non-primitive characters. The assumption that $F$ has zeros at $1 - u, v - 1, j - s, 1 \leqslant j \leqslant 50$, can be relaxed considerably, and probably entirely removed, but it saves us from computing several polar terms and buys us convergence without any trickery. The regularity assumptions on $F$ can also be relaxed. The assumption that $a$ is square-free is only to keep the formulae a little simpler. The assumption that $b$ is square-free is slightly more serious and enables us to use the Kuznetsov formula in a version that involves only Fourier expansions at infinity. For arbitrary $b$, one can use the analysis of Kıral and Young [KY19, Lemma 2.5 and Theorem 3.4] instead. As mentioned before, the assumption $(ab, q) = (a, b) = 1$ is without loss of generality, and if $a$ and $b$ are not of the same order of magnitude, our bounds may deteriorate by $(\max(b, a)/\min(b, a))^{O(1)}$ (this is unavoidable; see the sketch in § 1.4).

The spectral side (1.2) of Motohashi's original formula goes deeper in the spectrum (i.e. the support of $\check{F}$ is larger) the more complicated the test function $F$ is (e.g. in terms of oscillation). Our formula features a similar phenomenon for the non-archimedean test function, except that the spectral support now increases, in some sense orthogonally, in terms of the level instead of the spectral parameter.

## 1.2 A sixth moment

In practice, we want to estimate the right-hand side of (1.5) for large $q$ and somewhat large $a, b$, and a possible problem could be the occurrence of $\lambda_j(q)$ in (1.11) for Maaß forms $\psi_j$ for which the Ramanujan conjecture is not known. The factor $\lambda_j(b)$ is not a problem, since $b$ divides the level; see (2.3). A trivial bound on $\lambda_j(q)$, however, may invoke an undesirable factor of $q^\theta$ due to our limited knowledge of the Ramanujan conjecture. In order to avoid this, one may try to use the extra average over the forms of level $ab$ and apply the Cauchy–Schwarz inequality.[1] This is successful if there is an additional average over $a, b$, and to this end we will prove the following sixth moment bound, which is of independent interest.

Theorem 2. Let $Q, T \geqslant 1$, and for $q \in \mathbb{N}$, let $\mathcal{B}^*(q)$ denote an orthonormal basis of Hecke–Maaß newforms $\psi$ of level $q$ having spectral parameter $t_\psi$. Then

$$\sum_{q \leqslant Q} \sum_{\substack{\psi \in \mathcal{B}^*(q) \\ |t_\psi| \leqslant T}} \frac{|L(1/2, \psi)|^6}{L(1, \mathrm{Ad}^2\psi)} \ll_\varepsilon (QT)^\varepsilon T^8 Q^2.$$

The emphasis here is on the $Q$-aspect, which is sharp up to the presence of $Q^\varepsilon$; the $T$-aspect only needs to be polynomial. For comparison, it is classical, although technically difficult, to understand the fourth moment for an individual large level $q$. Our result is easier because we have an additional average over $q$ (which, however, is spectrally not easy to exploit), but also harder because we study a higher moment. Any spectral method will have to complete the discrete spectral sum to an entire spectral expression including Eisenstein series, and already in the fourth moment one of the biggest obstacles is the fact that the additional continuous contribution is quite large in the level aspect. It is not surprising that this becomes even worse for the sixth moment, and here the artificially added Eisenstein term exceeds the targeted bound by a substantial power of $Q$.

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[1] Depending on the application, it may also be convenient to use Hölder's inequality with suitable exponents; see, for example, [BM15].

An overview of the method of proof and how the various technical and conceptual issues are addressed will be given in § 1.4. We present an immediate application of Theorem 2.

COROLLARY 3. *Let $q$ be prime. Then*

$$\sum_{\psi \in \mathcal{B}^*(q)} L(1/2, \psi)^5 e^{-t_\psi^2} \ll_\varepsilon q^{1+\theta/3+\varepsilon}$$

*for every $\varepsilon > 0$. In particular, for $\psi \in \mathcal{B}^*(q)$, we have*

$$L(1/2, \psi) \ll_{t_\psi, \varepsilon} q^{1/5+\theta/15+\varepsilon} \ll q^{0.2073}.$$

This improves the $\theta$-dependence of the fifth moment bound in [BK19b, Theorem 3] and provides the numerical subconvexity record for $L(1/2, \psi)$ in the level aspect (the previous exponent being 0.217 from [BK19b, Theorem 4]; cf. [KY17]).

**1.3 Fourth moments twisted by Dirichlet polynomials**
For many applications, in particular with respect to the amplification, mollification, or resonance method, one wishes to augment moment results on $L$-functions by inserting well-chosen Dirichlet polynomials – ideally as long as possible – that in effect often act as additional fractional moments. This is classical for the Riemann zeta function, where Watt [Wat95], building on work of Deshouillers and Iwaniec [DI82], proved

$$\int_0^T |\zeta(1/2+it)|^4 \left| \sum_{m \leqslant M} a_m m^{it} \right|^2 dt \ll \|\mathbf{a}\|_\infty^2 (MT)^{1+\varepsilon}$$

for $M \leqslant T^{1/4}$ and an arbitrary sequence $\mathbf{a} = (a_m)_{1 \leqslant m \leqslant M}$. This can be turned into an asymptotic formula; see [Mot96, HY10, BBLR16]. Versions for Dirichlet $L$-functions with conductors sufficiently small with respect to $T$ can be found in [HWW04], along with applications to primes in arithmetic progressions and short intervals. We also mention [FI92] as an interesting variation if the sequence is factorizable in the ring of multiplicative functions.

As an application of Theorems 1 and 2, we will prove the following analogous sharp upper bound for a fourth moment of Dirichlet $L$-functions twisted by the square of a Dirichlet polynomial of length up to $q^{1/4}$.

THEOREM 4. *Let $q$ be a prime, $1 \leqslant M \leqslant q^{1/4}$, and $\{a(m)\}_{1 \leqslant m \leqslant M}$ a sequence of complex numbers supported on square-free numbers. Then*

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left| \sum_{m \leqslant M} a(m)\chi(m) \right|^2 |L(1/2, \chi)|^4 \ll_\varepsilon \|\mathbf{a}\|_\infty^2 (Mq)^{1+\varepsilon}.$$

To get a feeling for the strength of the result, we mention that it implies trivially the Burgess bound $L(1/2, \chi) \ll_\varepsilon q^{3/16+\varepsilon}$ for every non-trivial character modulo $q$. The reader may wonder to what extent this upper bound can be turned into an asymptotic formula, but interestingly this is a much harder problem than in the case of the Riemann zeta function. The reason is that a $\delta$-mass at the point $1/2$ is not a proper test function. On a technical level, the $t$-integral with a holomorphic test function and the freedom to shift its contour is crucial to establish convergence throughout the argument, even in the special case $M = 1$. Therefore, a corresponding asymptotic formula can be achieved if an additional $t$-average (essentially of constant length) is included, but for the central point individually, one has to use other methods (see, for example, [Hou16, Zac19a]) that yield much weaker results. See [Wat08] for a slightly more general result, which, however, is not independent of bounds towards the Ramanujan conjecture.

## 1.4 Heuristics

We conclude this introduction with a heuristic argument supporting relation (1.5) and the bound in Theorem 2 as well as some additional comments. This section is not intended to provide proofs, but may serve as a roadmap.

We start with (1.5). For the sake of argument, we will use approximate functional equations, although our proof works with Dirichlet series in the region of absolute convergence and continues meromorphically only at the very end (the great advantage of this is that we do not have to deal with a root number term, and so we will ignore this term also in the present sketch). For simplicity, we will also ignore the $t$-average whose purpose is to achieve convergence, as well as all 'main terms' that arise in the course of the computation. We have

$$\sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)\overline{\chi}(b)|L(1/2,\chi)|^4$$

$$\approx \sum_{\chi \,(\mathrm{mod}\, q)} \chi(a)\overline{\chi}(b) \sum_{n,m,r_1,r_2 \asymp q^{1/2}} \frac{\chi(nm)\overline{\chi}(r_1 r_2)}{(nmr_1 r_2)^{1/2}} \approx \sum_{\substack{n,m,r_1,r_2 \asymp q^{1/2} \\ anm \equiv br_1 r_2 \,(\mathrm{mod}\, q)}} 1.$$

Rather than solving a shifted convolution problem, we take an asymmetric approach and apply Poisson summation only in one variable, say $n$. This gives

$$\frac{1}{q} \sum_{n,m,r_1,r_2 \asymp q^{1/2}} e\left(\frac{\overline{a}bn\overline{m}r_1 r_2}{q}\right).$$

Suppose that $a \asymp b$. Then $bnr_1 r_2 \asymp amq$, so we can apply the additive reciprocity formula

$$e\left(\frac{n\overline{d}}{c}\right) = e\left(-\frac{n\overline{c}}{d}\right)e\left(\frac{n}{cd}\right) \tag{1.16}$$

to obtain

$$\frac{1}{q} \sum_{n,m,r_1,r_2 \asymp q^{1/2}} e\left(\frac{\overline{q}bnr_1 r_2}{am}\right).$$

Applying Poisson summation in $n, r_1, r_2$, this gives roughly

$$\frac{1}{a^2} \sum_{m \asymp q^{1/2}} \sum_{n,r_1,r_2 \asymp a} S(qr_1 r_2 \overline{b}, n, am).$$

If we assume for simplicity that $b$ is prime and coprime to $aqmr_1 r_2$ (this is where the assumption '$b$ square-free' in Theorem 1 is used), then $S(qr_1 r_2, bn, abm) = -S(qr_1 r_2 \overline{b}, n, am)$ by twisted multiplicativity. For the Kloosterman sum on the left-hand side, we are in the 'Linnik range' $\sqrt{qr_1 r_2 bn} \asymp abm$, and an application of the Kuznetsov formula yields the right-hand side of (1.5).

A back-of-the-envelope computation for Theorem 2 looks as follows. By an approximate functional equation, we have roughly

$$L(1/2,\psi)^6 \approx \sum_{n,m \ll Q^{3/2}} \frac{\tau_3(n)\tau_3(m)\lambda_\psi(n)\lambda_\psi(m)}{(nm)^{1/2}}$$

for $\psi \in \mathcal{B}^*(q)$, $q \asymp Q$, where for simplicity we regard $T$ as fixed; here $\tau_3(n) := \sum_{abc=n} 1$. Summing $\psi \in \mathcal{B}^*(q)$ and $q \asymp Q$ by the Kuznetsov formula, the diagonal term is of size $Q^2$ and the off-diagonal term looks roughly like

$$Q \sum_{q \asymp Q} \sum_{n,m \ll Q^{3/2}} \frac{\tau_3(n)\tau_3(m)}{(nm)^{1/2}} \sum_{c \ll Q^{1/2}} \frac{S(n,m,qc)}{qc}. \tag{1.17}$$

The key idea is to switch the roles of $q$ and $c$ and to apply the Kuznetsov formula backwards, but this time viewed as a spectral summation formula of level $c$. This switching principle is well known from sieve theory; here we apply it in an automorphic context, to some extent similar in spirit to [DI82]. We obtain roughly

$$Q^{1/2} \sum_{c \ll Q^{1/2}} \sum_{\psi \in \mathcal{B}^*(c)} \sum_{n,m \ll Q^{3/2}} \frac{\tau_3(n)\tau_3(m)\lambda_\psi(n)\lambda_\psi(m)}{(nm)^{1/2}}.$$

Applying Voronoi summation on the long $n,m$-sum, we may hope to get complete square root cancellation, obtaining the final bound $Q^{3/2}$ for the off-diagonal contribution.

Apart from neglecting oldforms, whose presence is technically challenging, this heuristic argument has an important deficiency: it ignores the continuous spectrum that needs to be added artificially before applying the Kuznetsov formula, and this contribution is of size $Q^{5/2}$ and substantially exceeds our target bound. In particular, it is impossible to estimate (1.17) by $Q^{3/2}$ as indicated, as we know in advance that it is of size $Q^{5/2}$. This dilemma of a gigantic continuous spectrum contribution is well known to experts and was first encountered in [DFI02], where the contribution was carefully computed and matched with another main term that occurred at a different stage of the argument. In [BHM07], the problem was solved by introducing additional zeros in the Mellin transform of the weight function in the approximate functional equation. Unfortunately, this loses positivity (and therefore many convenient simplifications), and it is also a very technical task to find the initial zeros at the end of the argument where they are needed to make a certain main term disappear. In the present situation, we argue differently and find a rather soft way to match two Eisenstein terms without actually computing them.

## 2. Preliminaries

### 2.1 Hecke theory
We generally denote Hecke eigenvalues, with or without subscript, by $\lambda(n)$. For newforms of level $N$, we will often use the multiplicativity relation

$$\lambda(nm) = \sum_{\substack{d|(m,n) \\ (d,N)=1}} \mu(d)\lambda(m/d)\lambda(n/d). \tag{2.1}$$

We have the general upper bound

$$\lambda(n) \ll_\varepsilon n^{\theta+\varepsilon}. \tag{2.2}$$

For a newform of level $N = N_1 N_2$ and trivial central character with $N_1$ square-free, $(N_1, N_2) = 1$, and some $n \mid N_1$, we have

$$|\lambda(n)| = n^{-1/2} \tag{2.3}$$

via [Ogg69, Theorem 2].

### 2.2 Functional equation for the Hurwitz zeta function
For $\alpha \in \mathbb{R}$, $\Re s > 1$, let

$$\zeta(s,\alpha) := \sum_{n+\alpha>0} (n+\alpha)^{-s}$$

denote the Hurwitz zeta function. It has meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$ of residue 1 and satisfies the functional equation

$$\zeta(s,\alpha) = \sum_{\pm} G^{\mp}(1-s)\zeta^{(\pm\alpha)}(1-s), \tag{2.4}$$

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where

$$G^{\pm}(s) = (2\pi)^{-s}\Gamma(s)\exp(\pm i\pi s/2)$$

and $\zeta^{(\alpha)}(s)$ is (the meromorphic continuation of) $\sum_n e(\alpha n)n^{-s}$. For $\alpha \in \mathbb{Q}$, this is a reformulation of Poisson summation in residue classes.

### 2.3 Functional equation for twisted automorphic *L*-functions

For $\alpha \in \mathbb{R}$, $\Re s > 1$, let

$$L(s, \alpha, f) := \sum_n \lambda_f(n)e(\alpha n)n^{-s}, \tag{2.5}$$

where, as before, $f$ is a Hecke eigenform of the group $\mathrm{SL}_2(\mathbb{Z})$, either Maaß with spectral parameter $t$ and parity $\epsilon \in \{\pm 1\}$, or holomorphic of weight $k$, or the standard Eisenstein series with $\lambda_f(n) = \tau(n)$. If $\alpha = a/c \in \mathbb{Q}$ with $(a,c) = 1$, this $L$-function has meromorphic continuation to all $s \in \mathbb{C}$ with a double pole at $s = 1$ with Laurent expansion

$$\frac{1}{c}\left(\frac{1}{(s-1)^2} + \frac{2\gamma - 2\log c}{s-1} + O(1)\right) \tag{2.6}$$

if $f$ is Eisenstein; note that this is independent of $a$. The twisted $L$-function satisfies the functional equation (see, for example, [HM06, §2.4])

$$L(s, a/c, f) = \sum_{\pm} G_f^{\mp}(1-s)c^{1-2s}L(s, \pm\overline{a}/c, f), \tag{2.7}$$

where

$$G_f^+(s) = i^k(2\pi)^{1-2s}\frac{\Gamma(s + (k-1)/2)}{\Gamma(1 - s + (k-1)/2)}, \quad G_f^-(s) = 0$$

if $f$ is holomorphic of weight $k$ and

$$G_f^{\pm}(s) = \epsilon^{(1\mp 1)/2}\frac{\Gamma(\tfrac{1}{2}(s+it))\Gamma(\tfrac{1}{2}(s-it))}{\Gamma(\tfrac{1}{2}(1-s+it))\Gamma(\tfrac{1}{2}(1-s-it))} \mp \frac{\Gamma(\tfrac{1}{2}(1+s+it))\Gamma(\tfrac{1}{2}(1+s-it))}{\Gamma(\tfrac{1}{2}(2-s+it))\Gamma(\tfrac{1}{2}(2-s-it))} \tag{2.8}$$

if $f$ is Maaß with spectral parameter $t$ and parity $\epsilon \in \{\pm 1\}$. This also holds for $f$ equal to the standard Eisenstein series with $t = 0$ and $\epsilon = 1$.

### 2.4 Fourier coefficients

We quote from [BK19b, §3] and refer to this source for more details and references. The cuspidal spectrum is parametrized by pairs $(\psi, M)$ of $\Gamma_0(N)$-normalized newforms $\psi$ of level $N_0 \mid N$ and integers $M \mid N/N_0$. The corresponding Fourier coefficients are

$$\rho_{\psi,M,N}(n) = \frac{1}{L(1, \mathrm{Ad}^2\psi)^{1/2}(N\nu(N))^{1/2}}\prod_{p|N_0}\left(1 - \frac{1}{p^2}\right)^{1/2}\sum_{d|(M,n)}\xi_\psi(M,d)\frac{d}{M^{1/2}}\lambda_\psi(n/d) \tag{2.9}$$

for $n \in \mathbb{N}$, where $\nu(N) = \prod_{p|N}(1 + 1/p)$ and the multiplicative function $\xi_\psi$ is defined in [BK19b, (3.10)] and satisfies in particular

$$\xi_\psi(p, p) = \left(1 - \frac{\lambda_\psi(p)^2}{p(1 + 1/p)^2}\right)^{-1/2}, \quad \xi_\psi(p, 1) = \frac{-\lambda_\psi(p)}{1 + 1/p^2}, \quad \xi_\psi(1, 1) = 1 \tag{2.10}$$

for $p \nmid N_0$ and in general

$$\xi_\psi(M, d) \ll_\varepsilon M^\varepsilon (M/d)^\theta. \tag{2.11}$$

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For $-n \in \mathbb{N}$ we have $\rho_{\psi,M,N}(n) = \epsilon_\psi \rho_{\psi,M,N}(-n)$ if $\psi$ is Maaß of parity $\epsilon_\psi \in \{\pm 1\}$ and $\rho_{\psi,M,N}(n) = 0$ if $\psi$ is holomorphic.

If $N$ is square-free, the Fourier coefficients of Eisenstein series of level $N$ are easy to describe. They are parametrized by divisors $v \mid N$ and a continuous parameter $s = 1/2 + it$. The corresponding Fourier coefficients are given by (see, for example, [CI00, (3.25)])

$$\rho_{v,N}(n,t) = \frac{C(v,M,t)}{(Nv)^{1/2}\zeta^{(N)}(1+2it)} \sum_{b|v} \sum_{\gamma|N/v} \mu(b\gamma)b\left(\frac{b}{\gamma}\right)^{it} \eta\left(\frac{|n|}{b\gamma},t\right), \qquad (2.12)$$

where $\eta(n,t) = \sum_{d_1 d_2 = n}(d_1/d_2)^{it}$ for $n \in \mathbb{N}$ (and 0 otherwise) and $|C(v,N,t)| = 1$. For general $N$, we follow [BK19b, §3] and parametrize unitary Eisenstein series of $\Gamma_0(N)$ by a continuous parameter $s = 1/2 + it$ together with pairs $(\chi, M)$, where $\chi$ is a primitive Dirichlet character of conductor $c_\chi$ and $M \in \mathbb{N}$ satisfies $c_\chi^2 \mid M \mid N$. Note that the role of $M$ is different than in the cuspidal case. We write

$$\tilde{\mathfrak{n}}_N(M) = \left( \prod_{\substack{p|N \\ p\nmid(M,N/M)}} \frac{p}{p+1} \prod_{p|(M,N/M)} \frac{p-1}{p+1} \right)^{1/2},$$
$$M = c_\chi M_1 M_2, \quad \text{where } (M_2, c_\chi) = 1, \quad M_1 \mid c_\chi^\infty,$$

so that $c_\chi \mid M_1$ and $(M_1, M_2) = 1$. The Fourier coefficients of the Eisenstein series attached to the data $(N, M, t, \chi)$ are

$$\rho_{\chi,M,N}(n,t) = \frac{\tilde{C}(\chi,M,t)|n|^{it}}{(N\nu(N))^{1/2}\tilde{\mathfrak{n}}_N(M)L^{(N)}(1+2it,\chi^2)}\left(\frac{M_1}{M_2}\right)^{1/2}$$
$$\times \sum_{\delta|M_2} \delta\mu\left(\frac{M_2}{\delta}\right)\overline{\chi}(\delta) \sum_{\substack{cM_1\delta f = n \\ (c,N/M)=1}} \frac{\chi(c)}{c^{2it}}\overline{\chi}(f), \qquad (2.13)$$

where $|\tilde{C}(\chi, M, t)| = 1$.

## 2.5 The Kuznetsov formula
For $x > 0$, we define the integral kernels

$$\mathcal{J}^+(x,t) := \frac{\pi i}{\sinh(\pi t)}(J_{2it}(4\pi x) - J_{-2it}(4\pi x)),$$
$$\mathcal{J}^-(x,t) := \frac{\pi i}{\sinh(\pi t)}(I_{2it}(4\pi x) - I_{-2it}(4\pi x)) = 4\cosh(\pi t)K_{2it}(4\pi x),$$
$$\mathcal{J}^{\mathrm{hol}}(x,k) := 2\pi i^k J_{k-1}(4\pi x) = \mathcal{J}^+(x,(k-1)/(2i)), \quad k \in 2\mathbb{N}.$$

If $H \in C^3((0,\infty))$ satisfies $x^j H^{(j)}(x) \ll \min(x, x^{-3/2})$ for $0 \leqslant j \leqslant 3$, we define

$$\mathcal{L}^\diamond H = \int_0^\infty \mathcal{J}^\diamond(x,.)H(x)\frac{dx}{x}$$

for $\diamond \in \{+, -, \mathrm{hol}\}$, and for $n, m, N \in \mathbb{N}$, we have

$$\sum_{N|c} \frac{S(\pm n, m, c)}{c} H\left(\frac{\sqrt{nm}}{c}\right)$$
$$= \mathcal{A}_N^{\mathrm{Maaß}}(\pm n, m; \mathcal{L}^\pm H) + \mathcal{A}_N^{\mathrm{Eis}}(\pm n, m; \mathcal{L}^\pm H) + \mathcal{A}_N^{\mathrm{hol}}(\pm n, m; \mathcal{L}^{\mathrm{hol}} H), \qquad (2.14)$$

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where

$$\mathcal{A}_N^{\mathrm{Maa\beta}}(n,m;h) := \sum_{N_0 M | N} \sum_{\psi \in \mathcal{B}^*(N_0)} \rho_{\psi,M,N}(n) \overline{\rho_{\psi,M,N}(m)} h(t_\psi),$$

$$\mathcal{A}_N^{\mathrm{Eis}}(n,m;h) := \sum_{v|N} \int_{\mathbb{R}} \rho_{v,N}(n,t) \overline{\rho_{v,N}(m,t)} h(t) \frac{dt}{2\pi} \quad (N \text{ square-free}),$$

$$\mathcal{A}_N^{\mathrm{Eis}}(n,m;h) := \sum_{c_\chi^2 | M | N} \int_{\mathbb{R}} \rho_{\chi,M,N}(n,t) \overline{\rho_{\chi,M,N}(m,t)} h(t) \frac{dt}{2\pi} \quad (\text{in general}),$$

$$\mathcal{A}_N^{\mathrm{hol}}(n,m;h) := \sum_{N_0 M | N} \sum_{\psi \in \mathcal{B}_{\mathrm{hol}}^*(N_0)} \rho_{\psi,M,N}(n) \overline{\rho_{\psi,M,N}(m)} h(k_\psi).$$

(2.15)

(Recall that $\psi \in \mathcal{B}^*(N_0)$ and $\psi \in \mathcal{B}_{\mathrm{hol}}^*(N_0)$ are $\Gamma_0(N)$-normalized.) Conversely, if $h$ is holomorphic in an $\varepsilon$-neighbourhood of $|\Im t| \leqslant 1/2$ and satisfies $h(t) \ll (1 + |t|)^{-2-\delta}$ in this region for some $\delta > 0$, then for $n, m \in \mathbb{N}$, we have [BK19b, (3.14)]

$$\mathcal{A}_N^{\mathrm{Maa\beta}}(n,m;h) + \mathcal{A}_N^{\mathrm{Eis}}(n,m;h) = \delta_{n,m} \int_{-\infty}^{\infty} h(t) \frac{t \tanh(\pi t)\, dt}{2\pi^2} + \sum_{N|c} \frac{S(n,m,c)}{c} \mathscr{K} h\left(\frac{\sqrt{nm}}{c}\right),$$

(2.16)

where

$$\mathscr{K} h(x) = \int_{-\infty}^{\infty} \mathcal{J}^+(x,t) h(t) t \tanh(\pi t) \frac{dt}{2\pi^2} = \frac{i}{\pi} \int_{-\infty}^{\infty} \frac{J_{2it}(4\pi x)}{\cosh(\pi t)} h(t) t\, dt. \qquad (2.17)$$

## 2.6 Integral transforms

We generalize (2.17) slightly and define for $s \in \mathbb{C}$ the transform $\mathscr{K}_s h$ by $\mathscr{K}_s h(x) := x^s \mathscr{K} h(x)$.

LEMMA 1. *Let $s \in \mathbb{C}$ with $\Re s < -10$, and suppose that $h$ is holomorphic in $|\Im t| < (-\Re s + 15)/2$, satisfying $h(t) \ll (1 + |t|)^{-10}$ and having zeros at $\pm i(2n-1)/2$, $n \in \mathbb{N}$, in this region. Then $H := \mathscr{K}_s h$ satisfies the assumptions of (2.14), that is, $x^j H^{(j)}(x) \ll_s \min(x, x^{-3/2})$ for $0 \leqslant j \leqslant 3$.*

*Proof.* We record the formula [GR07, (8.411.10)]

$$\frac{J_{2it}(4\pi x)}{\cosh(\pi t)} = \frac{(2\pi x)^{2it}}{\sqrt{\pi} \Gamma(1/2 + 2it) \cosh(\pi t)} \int_{-1}^{1} (1 - y^2)^{2it - 1/2} \cos(4\pi xy)\, dy \qquad (2.18)$$

for $\Re(2it) > -1/2$, $x > 0$. In particular,

$$\frac{d^j}{dx^j} \frac{J_{2it}(4\pi x)}{\cosh(\pi t)} \ll_{j,A,\varepsilon} \left(\frac{x}{1+|t|}\right)^{\Re(2it)} (1 + |t|/x)^j, \quad -1/2 + \varepsilon \leqslant \Re(2it) < A, \qquad (2.19)$$

for $j, A \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, $\varepsilon > 0$. Let $s \in \mathbb{C}$ with $\Re s < -10$ be fixed. For $x \geqslant 1$ and $0 \leqslant j \leqslant 3$, we obtain

$$x^j H^{(j)}(x) \ll x^{\Re s + j} \int_{-\infty}^{\infty} \left(1 + \frac{|t|}{x}\right)^j |h(t)t|\, dt \ll x^{\Re s + j} \ll x^{-3/2},$$

and for $x < 1$, we shift the contour to $\Re(2it) = -\Re s + 10$ (not passing any pole by our assumption on $h$), getting

$$x^j H^{(j)}(x) \ll x^{\Re s} \int_{-\infty}^{\infty} \left(\frac{x}{1+|t|}\right)^{-\Re s + 10} \left(1 + \frac{|t|}{x}\right)^j \left| h\left(t - \frac{i}{2}(10 - \Re s)\right) t \right| dt \ll x. \qquad \square$$

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LEMMA 2. *Let $s \in \mathbb{C}$ with $\Re s < 1$, and suppose that $h$ is even and holomorphic in $|\Im t| < (-\Re s + 15)/2$, satisfying $h(t) \ll e^{-|t|}$ and having zeros at $\pm i(2n-1)/2$, $n \in \mathbb{N}$, in this region.*

(a) *The transform $\mathscr{L}^+ \mathscr{K}_s h$, defined for $\Re s < -10$ by Lemma 1, has analytic continuation to $\Re s < 1$, and we have the Sears–Titchmarsh inversion formula $\mathscr{L}^+ \mathscr{K}_0 h = h$.*

(b) *We have the uniform bounds*

$$\mathscr{L}^+ \mathscr{K}_s h(t) \ll_{\Re s} \int_{-\infty}^{\infty} \left| h\left(\tau - \frac{i}{2}(1 - \Re s)\right) \right| (1 + |\tau|) \prod_{\pm} \left( 1 + \left| \frac{1}{2}\Im s - \tau \pm \Re t \right| \right)^{-1 + \Re s} d\tau$$

*for $t \in \mathbb{R} \cup [-i\theta, i\theta]$ and $\Re s < 1$, and*

$$\mathscr{L}^{\mathrm{hol}} \mathscr{K}_s h(k) \ll_{\Re s} \int_{-\infty}^{\infty} \left| h\left(\tau - \frac{i}{2}\max(2 - k - \Re s, 0)\right) \right| (1 + |\tau|) \prod_{\pm} \left( \left| \frac{1}{2}\Im s \pm \tau \right| + k \right)^{-1 + \Re s} d\tau$$

*for $k \in 2\mathbb{N}$, where the implied constants depend only on $\Re s$ (but not on $t, k, h, \Im s$).*

*Proof.* For $\Re s < -10$, we have, by definition,

$$\mathscr{K}_s h(x) = x^s \frac{i}{\pi} \int_{\Re(i\tau)=(10-\Re s)/2} \frac{J_{2i\tau}(4\pi x)}{\cosh(\pi\tau)} h(\tau) \tau \, d\tau,$$

and we have an absolutely convergent double integral

$$\mathscr{L}^+ \mathscr{K}_s h(t) = -\int_0^{\infty} \frac{1}{\sinh(\pi t)} (J_{2it}(4\pi x) - J_{-2it}(4\pi x)) x^s \int_{\Re(i\tau)=(10-\Re s)/2} \frac{J_{2i\tau}(4\pi x)}{\cosh(\pi\tau)} h(\tau) \tau \, d\tau \frac{dx}{x}.$$

To see the absolute convergence, we use (2.19) with $j = 0$ to bound $J_{\pm 2it}(x) \ll_t \min(x^{2\theta}, x^{-2\theta})$ for $t \in \mathbb{R} \cup [-i\theta, i\theta]$, and we combine (2.19) with the bound $J_\nu(x) \ll x^{-1/2}$ for $x \gg |\nu|^2$ (which follows from the asymptotic formula [GR07, (8.451.1)]) to bound

$$J_{2i\tau}(x) \ll \min(x^{-\Re s + 10}, x^{-1/2}(1 + |\tau|)^{-\Re s + 11}).$$

We can compute the $x$-integral explicitly using [GR07, (6.574.2)], getting

$$\mathscr{L}^+ \mathscr{K}_s h(t) = \int_{\Re(i\tau)=(10-\Re s)/2} \frac{\Gamma(1-s)\cos(i\pi\tau + \pi s/2)h(\tau)\tau}{(2\pi)^s \pi i \cosh(\pi\tau)} \prod_{\pm} \frac{\Gamma\left(\frac{s}{2} + i\tau \pm it\right)}{\Gamma\left(1 - \frac{s}{2} + i\tau \pm it\right)} d\tau. \quad (2.20)$$

Here we can put any $s$ with $\Re s < 1$ in the integrand (and also shift the contour to, say, $\Re(i\tau) = 5$), in particular $s = 0$, so that

$$\mathscr{L}^+ \mathscr{K}_0 h(t) = \int_{\Re(i\tau)=5} \frac{h(\tau)\tau}{t^2 - \tau^2} \frac{d\tau}{\pi i}.$$

The integrand is odd, so the integral equals half the sum of the two residues at $\tau = \pm t$ and part (a) of the lemma follows. To prove part (b) for $\mathscr{L}^+ \mathscr{K}_s h(t)$, we shift the $\tau$-contour to $\Re(i\tau) = (1 - \Re(s))/2$ and estimate trivially in (2.20) using Stirling's formula. For $\mathscr{L}^{\mathrm{hol}} \mathscr{K}_s h(t)$, we have the similar expression

$$\mathscr{L}^{\mathrm{hol}} \mathscr{K}_s h(k) = i^{k+1} \int_{\Re(i\tau)=a} \frac{\Gamma(1-s)(2\pi)^{-s}\Gamma\left(\frac{k-1}{2} + \frac{s}{2} + i\tau\right)}{\Gamma\left(\frac{k+1}{2} - \frac{s}{2} - i\tau\right)\Gamma\left(\frac{3-k}{2} - \frac{s}{2} + i\tau\right)\Gamma\left(\frac{k+1}{2} - \frac{s}{2} + i\tau\right)} \frac{h(\tau)\tau}{\cosh(\pi\tau)} d\tau$$

for any $a \in \mathbb{R}$ satisfying $k - 1 + \Re s + 2a > 0$, say $a = \max((2 - k - \Re s)/2, 0)$. We can rewrite this as

$$\int_{\Re(i\tau)=a} \frac{\Gamma(1-s)\cos(i\pi\tau + \frac{1}{2}\pi s)h(\tau)\tau}{(2\pi)^s \pi i \cosh(\pi\tau)} \prod_{\pm} \frac{\Gamma\left(\frac{k-1}{2} + \frac{s}{2} \pm i\tau\right)}{\Gamma\left(\frac{k+1}{2} - \frac{s}{2} \pm i\tau\right)} d\tau.$$

The desired bound follows now from $\Gamma(z+w)/\Gamma(z) \ll_w (1 + |z|)^w$ for $w \in \mathbb{R}$ and $|z|$ sufficiently large; see, for example, [GR07, (8.328.2)]. $\qquad\square$

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## 3. Proof of Theorem 1

### 3.1 The set-up

We recall the definitions (1.6) and (1.9) for a prime $q$ and integers $a, b$ satisfying $(ab, q) = (a, b) = 1$, $a \asymp b$, and an even holomorphic test function $F$ that is rapidly decaying on vertical lines and is divisible by $(1-u)(v-1)^2 \prod_{j=1}^{50}(j-s)$. Initially we assume

$$2 < \Re s, \quad \Re v < 3, \quad 10 < \Re u < 11. \tag{3.1}$$

In this section all implicit constants may depend on $s, u, v, \varepsilon$ and $F$. Additional dependencies will be mentioned. We proceed to define the correction polynomial $P_q(s, u, v, z)$, and to this end we define three auxiliary quantities

$$\mathcal{T}_q^{(1)}(s, u, v) := \int_{\mathbb{R}} \zeta^{(q)}(s+z) \zeta^{(q)}(u+z) L^{(q)}(v-z, f) F(z) \frac{dz}{2\pi i},$$

$$\mathcal{T}_q^{(2)}(s, u, v) := q^{1-s-v} \int_{\mathbb{R}} \left( \lambda(q) - \frac{1}{q^{v-z}} \right) \zeta(s+z) \zeta^{(q)}(u+z) L(v-z, f) F(z) \frac{dt}{2\pi i},$$

$$\mathcal{T}_q^{(3)}(s, u, v) := \int_{\mathbb{R}} (q^{2-s-u-2v} - q^{1-u-2v+z}) \zeta(s+z) \zeta(u+z) L(v-z, f) F(z) \frac{dz}{2\pi i}.$$

Here $\zeta^{(q)}(s) := \zeta(s) \prod_{p|q}(1 - p^{-s})$ and $L^{(q)}(s, f) = L(s, f) \prod_{p|q}(1 - \lambda_f(p)p^{-s} + p^{-2s})$ are the usual $L$-functions with the Euler factors dividing $q$ omitted. We define

$$\mathcal{T}_q^{\mathrm{triv}}(s, u, v) := \sum_{j=1}^{3} \mathcal{T}_q^{(j)}(s, u, v),$$

so that (1.9) holds with

$$P_q(s, u, v, z) = \left( 1 - \frac{1}{q^{s+z}} \right) \left( 1 - \frac{1}{q^{u+z}} \right) \left( 1 - \frac{\lambda(q)}{q^{v-z}} + \frac{1}{q^{2v-2z}} \right)$$
$$+ q^{1-s-v} \left( \lambda(q) - \frac{1}{q^{v-z}} \right) \left( 1 - \frac{1}{q^{u+z}} \right) + q^{2-s-u-2v} - q^{1-u-2v+z}. \tag{3.2}$$

It is easy to see that this satisfies (1.8). In the range (3.1), we can open the Dirichlet series and obtain

$$\mathcal{T}_{a,b,q}(s, u, v) + \mathcal{T}_q^{(1)}(s, u, v) = q \int_{(0)} F(z) \sum_{\substack{(nmr,q)=1 \\ anm \equiv br \,(\mathrm{mod}\, q)}} \frac{\lambda(r)}{n^{s+z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i},$$

$$\mathcal{T}_q^{(2)}(s, u, v) = q^{1-s-v} \int_{(0)} F(z) \left( \lambda(q) - \frac{1}{q^{v-z}} \right) \sum_{n, r, (m,q)=1} \frac{\lambda(r)}{n^{s+z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i},$$

$$= q \int_{(0)} F(z) \sum_{n, r, (m,q)=1} \frac{\lambda(qr)}{(qn)^{s+z}m^{u+z}(qr)^{v-z}} \frac{dz}{2\pi i},$$

$$\mathcal{T}_q^{(3)}(s, u, v) = q^{2-s-u-2v} \int_{(0)} F(z)(1 - q^{s+z-1}) \sum_{n, r, m} \frac{\lambda(r)}{n^{s+z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i}.$$

We conclude that

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$$\tilde{\mathcal{T}}_{a,b,q}(s,u,v) := \mathcal{T}_{a,b,q}(s,u,v) + \mathcal{T}_q^{(1)}(s,u,v) + \mathcal{T}_q^{(2)}(s,u,v)$$

$$= q \int_{(0)} F(z) \sum_{\substack{(m,q)=1 \\ n\equiv\overline{am}br \,(\mathrm{mod}\ q)}} \frac{\lambda(r)}{n^{s+z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i}.$$

Eventually the term $\mathcal{T}_q^{(3)}(s,u,v)$ will remove the last coprimality condition $(m,q)=1$, but this has to wait until the end of argument. Until then, we transform $\tilde{\mathcal{T}}_{a,b,q}(s,u,v)$ and $\mathcal{T}_q^{(3)}(s,u,v)$ in a parallel fashion.

### 3.2 Poisson summation

We write the $n$-sum in terms of the Hurwitz zeta function with $\alpha = \overline{am}br/q$ and shift the $z$-contour to the left to $\Re z = -4$. Our assumption on $F$ implies that the potential pole at $z = 1-s$ is cancelled. The $m, r$-sums are still absolutely convergent, and we apply the functional equation (2.4), getting

$$\tilde{\mathcal{T}}_{a,b,q}(s,u,v) = \sum_{\pm} \int_{(-4)} F(z)G^{\pm}(1-s-z)q^{1-s-z} \sum_{n,r,(m,q)=1} \frac{\lambda(r)e(\mp\overline{a}brn\overline{m}/q)}{n^{1-s-z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i} \qquad (3.3)$$

and

$$\mathcal{T}_q^{(3)}(s,u,v) = q^{2-s-u-2v} \int_{(-4)} F(z)(1-q^{s+z-1})G^{\pm}(1-s-z) \sum_{n,r,m} \frac{\lambda(r)}{n^{1-s-z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i}$$

$$= q^{2-s-u-2v} \int_{(-4)} F(z)G^{\pm}(1-s-z) \sum_{(n,q)=1,r,m} \frac{\lambda(r)}{n^{1-s-z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i}. \qquad (3.4)$$

### 3.3 Reciprocity

For $\alpha \in \mathbb{R} \setminus \{0\}$, we recall the absolutely convergent Mellin integral

$$e(\alpha) = \int_{\mathcal{C}} G^{\mathrm{sgn}(\alpha)}(s)|\alpha|^{-w} \frac{dw}{2\pi i},$$

where $\mathcal{C}$ is the contour

$$\mathcal{C} = (-\tfrac{3}{5} - i\infty, -\tfrac{3}{5} - i] \cup [-\tfrac{3}{5} - i, \tfrac{1}{10}] \cup [\tfrac{1}{10}, -\tfrac{3}{5} + i] \cup [-\tfrac{3}{5} + i, -\tfrac{3}{5} + i\infty).$$

In (3.3), we insert

$$1 = e\left(\pm\frac{brn}{amq}\right) \int_{\mathcal{C}} G^{\pm}(w)\left(\frac{brn}{amq}\right)^{-w} \frac{dw}{2\pi i} \qquad (3.5)$$

and apply the additive reciprocity formula (1.16). This gives the absolutely convergent expression

$$\tilde{\mathcal{T}}_{a,b,q}(s,u,v) = \sum_{\pm} \int_{(-4)} \int_{\mathcal{C}} F(z)\left(\frac{b}{a}\right)^{-w} G^{\pm}(w)G^{\pm}(1-s-z)q^{1-s-z+w}$$

$$\times \sum_{n,(m,q)=1,r} \frac{\lambda(r)e(\pm\overline{q}brn/(am))}{n^{1-s-z+w}m^{u+z-w}r^{v-z+w}} \frac{dw}{2\pi i} \frac{dz}{2\pi i}.$$

We temporarily straighten the $\mathcal{C}$-contour to $\Re w = -3/5$, picking up the polar term

$$\sum_{\pm} \int_{(-4)} F(z)G^{\pm}(1-s-z)q^{1-s-z} \sum_{n,(m,q)=1,r} \frac{\lambda(r)e(\pm\overline{q}brn/(am))}{n^{1-s-z}m^{u+z}r^{v-z}} \frac{dz}{2\pi i}.$$

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In the remaining double integral, we change variables $w \mapsto w + z$ (so that $\Re z = -4$, $\Re w = 17/5$), exchange the two integrals, and in the inner $z$-integral we bend the contour to the right to the contour

$$\mathcal{C}(w) = \mathcal{C} - w = (-\tfrac{3}{5} - w - i\infty, -\tfrac{3}{5} - w - i] \cup [-\tfrac{3}{5} - w - i, \tfrac{1}{10} - w]$$
$$\cup [\tfrac{1}{10} - w, -\tfrac{3}{5} - w + i] \cup [-\tfrac{3}{5} - w + i, -\tfrac{3}{5} - w + i\infty)$$

consisting of a union of straight lines. This picks up a polar term

$$-\sum_{\pm} \int_{(17/5)} F(-w) G^{\pm}(1 - s + w) q^{1-s+w} \sum_{n,(m,q)=1,r} \frac{\lambda(r) e(\pm\overline{q} brn/(am))}{n^{1-s+w} m^{u-w} r^{v+w}} \frac{dw}{2\pi i},$$

which cancels the previous one. This shows that

$$\tilde{\mathcal{T}}_{a,b,q}(s, u, v) = \sum_{\pm} \int_{(17/5)} \Phi^{\pm}_{a,b,s}(w) q^{1-s+w} \sum_{n,(m,q)=1,r} \frac{\lambda(r) e(\pm\overline{q} brn/(am))}{n^{1-s+w} m^{u-w} r^{v+w}} \left(\frac{b}{a}\right)^{-w} \frac{dw}{2\pi i}, \quad (3.6)$$

where

$$\Phi^{\pm}_{a,b,s}(w) = \int_{\mathcal{C}(w)} F(z) \left(\frac{b}{a}\right)^{-z} G^{\pm}(w + z) G^{\pm}(1 - s - z) \frac{dz}{2\pi i}$$
$$= (2\pi)^{s-w-1} e^{\pm i\pi(w+1-s)/2} \int_{\mathcal{C}(w)} F(z) \left(\frac{b}{a}\right)^{-z} \Gamma(w + z) \Gamma(1 - s - z) \frac{dz}{2\pi i}.$$

Here we can straighten the contour and shift it to the far left to $\Re z = -A$, say. This gives a sum of polar terms of the shape

$$p_n(w) := \frac{(-1)^n}{n!} (2\pi)^{s-w-1} e^{\pm i\pi(w+1-s)/2} F(-w - n) \left(\frac{b}{a}\right)^{w+n} \Gamma(1 - s + w + n), \quad n \in \mathbb{N}_0,$$

and a remaining integral that is holomorphic in the half plane $\Re w > -A$ and bounded by $\ll_{\Re w, A} (b/a)^A (1 + |w|)^{\Re w - A - 1/2}$. Since $F(1 - s) = \cdots = F(50 - s) = 0$, we conclude that $\Phi^{\pm}_{a,b,s}$ is

$$\text{holomorphic in } |\Re w| < 48 \text{ and satisfies } \Phi^{\pm}_{a,b,s}(w) \ll (1 + |w|)^{-100} \quad (3.7)$$

as long as $a \asymp b$ and $0 < \Re s < 3$. We also observe that

$$\sum_{\pm} \Phi^{\pm}_{a,b,s}(s) = 0. \quad (3.8)$$

Inserting

$$1 = e\left(\pm\frac{brn}{am}\right) \int_{\mathcal{C}} G^{\pm}(w) \left(\frac{brn}{am}\right)^{-w} \frac{dw}{2\pi i}$$

into (3.4) (similarly to (3.5), but with different variables), we obtain in the same way

$$\mathcal{T}^{(3)}_q(s, u, v) = \sum_{\pm} \int_{(17/5)} \Phi^{\pm}_{a,b,s}(w) q^{2-s-u-2v} \sum_{m,(n,q)=1,r} \frac{\lambda(r) e(\pm brn/(am))}{n^{1-s+w} m^{u-w} r^{v+w}} \left(\frac{b}{a}\right)^{-w} \frac{dw}{2\pi i}.$$

(The expression is still independent of $a, b$, even though the right-hand side seems to depend on $a, b$.)

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where the polar term $\mathcal{P}_{a,b,q}^{(1)}(s,u,v)$ vanishes unless $f$ is Eisenstein, in which case it equals

$$\mathcal{P}_{a,b,q}^{(1)}(s,u,v) = \operatorname*{Res}_{w=1-v} \sum_{\pm} \sum_{\sigma\in\{\pm\}} \Phi_{a,b,s}^{\pm}(w)G^{-\sigma}(s-w)q^{1-s+w}$$

$$\times \sum_{\alpha_1\alpha_2=a} \sum_{\beta_1\beta_2=b} \zeta^{(\alpha_2\beta_2 q)}(1+u-s)$$

$$\times \sum_{\beta_1|n} \sum_{(m_2,q\beta_2)=1} \frac{a^{s-1}\beta_1 r_{\alpha_2 m_2}(n/\beta_1)}{(\beta_1 m_2)^{u-s+1}n^{s-w}b^w} L\left(v+w,\frac{*}{\alpha_2 m_2},f\right), \qquad (3.12)$$

where $r_c(n)$ denotes the Ramanujan sum. (Recall that by (2.6) the residue is independent of the numerator $*$ in the twist of the $L$-function.) Similarly,

$$\mathcal{T}_q^{(3)}(s,u,v) = \sum_{\pm} \sum_{\sigma,\tau\in\{\pm\}} \int_{(-4)} \Phi_{a,b,s}^{\pm}(w)G^{-\sigma}(s-w)q^{1-w-u-2v}$$

$$\times \sum_{\alpha_1\alpha_2=a} \sum_{\beta_1\beta_2=b} \zeta^{(\alpha_2\beta_2 q)}(1+u-s)G_f^{-\tau}(1-v-w)$$

$$\times \sum_{r,n} \sum_{(m_2,\beta_2)=1} \sum_{\substack{\nu\,(\mathrm{mod}\ q\alpha_2\beta_1 m_2)\\ (\nu,\alpha_2 m_2 q)=1}} e\left(\frac{\pm\tau\overline{\beta_2}\nu r}{\alpha_2 m_2}\right)$$

$$\times \frac{\lambda(r)e(\sigma n\nu/(\alpha_2\beta_1 m_2 q))}{n^{s-w}(\beta_1 m_2)^{u-s+1}r^{1-v-w}}\frac{a^{s-1}}{b^w}(\alpha_2 m_2)^{1-2v-2w}\frac{dw}{2\pi i} + \mathcal{P}_{a,b,q}^{(2)}(s,u,v), \quad (3.13)$$

where

$$\mathcal{P}_{a,b,q}^{(2)}(s,u,v) = \operatorname*{Res}_{w=1-v} \sum_{\pm} \sum_{\sigma\in\{\pm\}} \Phi_{a,b,s}^{\pm}(w)G^{-\sigma}(s-w)q^{1-w-u-2v}$$

$$\times \sum_{\alpha_1\alpha_2=a} \sum_{\beta_1\beta_2=b} \zeta^{(\alpha_2\beta_2 q)}(1+u-s)$$

$$\times \sum_{\beta_1|n} \sum_{(m_2,\beta_2)=1} \frac{a^{s-1}\beta_1 r_{\alpha_2 m_2 q}(n/\beta_1)}{(\beta_1 m_2)^{u-s+1}n^{s-w}b^w} L\left(v+w,\frac{*}{\alpha_2 m_2},f\right).$$

We will compute the two polar terms in a moment, but we observe already at this point that now the time has come to combine the two main terms. Indeed, the main term in (3.13) simply counteracts the condition $(m_2,q)=1$ of the main term in (3.11) and supplies the missing terms $q\mid m$. Combining the two, we see that

$$\tilde{\mathcal{T}}_{a,b,q}(s,u,v) + \mathcal{T}_q^{(3)}(s,u,v) = \mathcal{T}_{a,b,q}^*(s,u,v) + \sum_{j=1}^2 \mathcal{P}_{a,b,q}^{(j)}(s,u,v),$$

where

$$\mathcal{T}_{a,b,q}^*(s,u,v) = \sum_{\pm} \sum_{\sigma,\tau\in\{\pm\}} \int_{(-4)} \Phi_{a,b,s}^{\pm}(w)G^{-\sigma}(s-w)G_f^{-\tau}(1-v-w)q^{1-w-u-2v}$$

$$\times \sum_{\alpha_1\alpha_2=a} \sum_{\beta_1\beta_2=b} \zeta^{(\alpha_2\beta_2 q)}(1+u-s)\frac{\alpha_1^{s-1}\alpha_2^{s-2v-2w}}{\beta_1^{u-s+1+w}\beta_2^w}$$

$$\times \sum_{r,n} \sum_{(m_2,\beta_2)=1} \sum_{\substack{\nu\,(\mathrm{mod}\ \alpha_2\beta_1 m_2)\\ (\nu,\alpha_2 m_2)=1}} e\left(\frac{\pm\tau q\overline{\beta_2}\nu r}{\alpha_2 m_2}\right)\frac{\lambda(r)e(\sigma n\nu/(\alpha_2\beta_1 m_2))}{n^{s-w}m_2^{u-s+2v+2w}r^{1-v-w}}\frac{dw}{2\pi i}. \quad (3.14)$$

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### 3.6 Computation of polar terms

In this subsection we compute $\mathcal{P}_{a,b,q}^{(j)}(s,u,v)$ for $j = 1,2$. We consider first

$$\sum_{\alpha_1\alpha_2=a} \sum_{\beta_1\beta_2=b} \zeta^{(\alpha_2\beta_2 q)}(1+u-s) \sum_n \sum_{(m_2,q\beta_2)=1} \frac{a^{s-1}\beta_1 r_{\alpha_2 m_2}(n)}{(\beta_1 m_2)^{u-s+1}(\beta_1 n)^{s-w}b^w} L\left(v+w, \frac{*}{\alpha_2 m_2}, f\right)$$

corresponding to the last four sums in (3.12) for $w$ in a neighbourhood of $1-v$. Substituting

$$r_{\alpha_2 m_2}(n) = \sum_{\substack{d_1 d_2 = \alpha_2 m \\ d_1 \mid n}} d_1 \mu(d_2),$$

we obtain

$$\zeta(s-w) \sum_{\alpha_1\alpha_2=a} \sum_{\beta_1\beta_2=b} \zeta^{(\alpha_2\beta_2 q)}(1+u-s)\frac{a^{s-1}}{\beta_1^u \beta_2^w}$$

$$\times \sum_{(m_2,q\beta_2)=1} \sum_{d_1 d_2 = \alpha_2 m_2} \frac{\mu(d_2)}{d_1^{s-w-1}} \frac{1}{m_2^{u-s+1}} L\left(v+w, \frac{*}{\alpha_2 m_2}, f\right)$$

$$= \zeta(s-w) \sum_{\alpha_1\alpha_2=a} \sum_{\beta_1\beta_2=b} \zeta^{(\alpha_2\beta_2 q)}(1+u-s)\frac{a^{s-1}}{\beta_1^u \beta_2^w}$$

$$\times \sum_{\substack{\alpha_2 \mid d_1 d_2 \\ (d_1 d_2/\alpha_2, q\beta_2)=1}} \frac{\mu(d_2)}{d_1^{s-w-1}} \frac{\alpha_2^{u-s+1}}{(d_1 d_2)^{u-s+1}} L\left(v+w, \frac{*}{d_1 d_2}, f\right).$$

We write $(d_1, \alpha_2) = A_1$, $A_1 A_2 = \alpha_2$, $A_2 \mid d_2$, eventually getting

$$\mathcal{P}_{a,b,q}^{(1)}(s,u,v) = \operatorname*{Res}_{w=1-v} \sum_{\pm} \sum_{\sigma\in\{\pm\}} \Phi_{a,b,s}^{\pm}(w) G^{-\sigma}(s-w) q^{1-s+w} \zeta(s-w)$$

$$\times \sum_{\alpha_1 A_1 A_2=a} \sum_{\beta_1\beta_2=b} \frac{\zeta^{(A_1 A_2\beta_2 q)}(1+u-s)\mu(A_2)A_2^w}{\beta_1^u \beta_2^w (\alpha_1 A_2)^{1-s}}$$

$$\times \sum_{(d_1 d_2, qA_2\beta_2)=1} \frac{\mu(d_2)L(v+w, */A_1 A_2 d_1 d_2, f)}{d_1^{u-w} d_2^{u-s+1}}. \qquad (3.15)$$

By (2.6), this is a linear combination of

$$\sum_{\alpha_1 A_1 A_2=a} \sum_{\beta_1\beta_2=b} \Phi_{a,b,q}^{\pm}(1-v) G^{-\sigma}(s+v-1) q^{2-v-s} \zeta(s+v-1)$$

$$\times \frac{\zeta^{(A_1 A_2\beta_2 q)}(1-s+u)\zeta^{(qA_2\beta_2)}(u+v)}{\zeta^{(qA_2\beta_2)}(2+u-s)\alpha_1^{1-s} A_2^{1+v-s} A_1\beta_1^u \beta_2^{1-v}}$$

and derivatives thereof. The same computation shows that

$$\mathcal{P}_{a,b,q}^{(2)}(s,u,v) = \operatorname*{Res}_{w=1-v} \sum_{\pm} \sum_{\sigma\in\{\pm\}} \Phi_{a,b,s}^{\pm}(w) G^{-\sigma}(s-w) q^{1-w-u-2v} \zeta(s-w)$$

$$\times \sum_{q_1 q_2=q} \frac{1}{q_1^{s-w-1}} \sum_{\alpha_1 A_1 A_2=a} \sum_{\beta_1\beta_2=b} \frac{\zeta^{(A_1 A_2\beta_2 q)}(1-s+u)\mu(A_2 q_2)A_1^w}{\beta_1^u \beta_2^w (\alpha_1 A_2)^{1-s}}$$

$$\times \sum_{\substack{(d_1, A_2\beta_2 q_2)=1 \\ (d_2, A_2\beta_2)=1}} \frac{\mu(d_2)L(v+w, */A_1 A_2 d_1 d_2, f)}{d_1^{u-w} d_2^{u-s+1}}, \qquad (3.16)$$

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which is a linear combination of

$$
\sum_{q_1 q_2 = q} \sum_{\alpha_1 A_1 A_2 = a} \sum_{\beta_1 \beta_2 = b} \Phi_{a,b,q}^{\pm}(1-v) G^{-\sigma}(s+v-1)\zeta(s+v-1)
$$
$$
\times \frac{q_1^{2-s-u-2v}}{q_2^{u+v}} \frac{\zeta^{(A_1 A_2 \beta_2 q)}(1-s+u)\zeta^{(A_2 \beta_2 q_2)}(u+v-1)}{\zeta^{(A_2 \beta_2)}(2+u-s)\alpha_1^{1-s} A_2^{2-s} A_1^v \beta_1^u \beta_2^{1-v}}.
$$

and derivatives thereof.

### 3.7 Application of the Kuznetsov formula

We return to (3.14) and recognize the $\nu$-sum as a Kloosterman sum. More precisely, the $\nu$-sum vanishes unless $\beta_1 \mid n$, so that the second and third lines of (3.14) equal

$$
\sum_{\alpha_1 \alpha_2 = a} \sum_{\beta_1 \beta_2 = b} \zeta^{(\alpha_2 \beta_2 q)}(1+u-s)\frac{\alpha_1^{s-1}\alpha_2^{s-2v-2w}}{\beta_1^u \beta_2^w} \sum_{r,n} \sum_{(m_2,\beta_2)=1} \frac{\lambda(r)S(\pm \tau q \overline{\beta_2} r, \sigma n, \alpha_2 m_2)}{n^{s-w} m_2^{u-s+2v+2w} r^{1-v-w}}. \quad (3.17)
$$

For $(\beta_2, q\alpha_2 m) = 1$, we have, by the twisted multiplicativity of Kloosterman sums,

$$
S(\pm \tau q r, \sigma \beta_2 n, \beta_2 \alpha_2 m) = r_{\beta_2}(r) S(\pm \tau q r \overline{\beta_2}, \sigma n, \alpha_2 m).
$$

At this point, we use the fact that $b$ is square-free; in particular, the Ramanujan sum $r_{\beta_2}(r)$ does not vanish. Write $B_1 = (\beta_2, r)$, $B_2 B_1 = \beta_2$, $r = r' B_1$, $(r', B_2) = 1$. Then

$$
S(\pm \tau q r \overline{\beta_2}, \sigma n, \alpha_2 m) = \frac{1}{r_{\beta_2}(r)} S(\pm \tau q r' B_1, \sigma B_1 B_2 n, B_1 B_2 \alpha_2 m)
$$
$$
= \frac{\phi(B_1)}{\phi(B_1)\mu(B_2)} S(\pm \tau q r', \sigma B_2 n, B_2 \alpha_2 m),
$$

so that (3.17) is equal to

$$
\sum_{\alpha_1 \alpha_2 = a} \sum_{\beta_1 B_1 B_2 = b} \frac{\zeta^{(\alpha_2 B_1 B_2 q)}(1+u-s)\mu(B_2)\alpha_1^{s-1}\alpha_2^{s-2v-2w}}{\beta_1^u B_2^w B_1^{1-v}}
$$
$$
\times \sum_{(r,B_2)=1,n} \sum_{(m_2,\beta_2)=1} \frac{\lambda(r B_1)S(\pm \tau q r, \sigma B_2 n, B_2 \alpha_2 m_2)}{n^{s-w} m_2^{u-s+2v+2w} r^{1-v-w}}.
$$

Here we can drop the condition $(m_2, B_2) = 1$, since otherwise the Kloosterman sum vanishes (since $(r, B_2) = 1$). We remove the remaining condition $(m_2, B_1) = 1$ by Möbius inversion, getting

$$
\sum_{\alpha_1 \alpha_2 = a} \sum_{\beta_1 B_3 B_4 B_2 = b} \frac{\zeta^{(\alpha_2 B_2 B_3 B_4 q)}(1+u-s)\mu(B_2)\mu(B_3)\alpha_1^{s-1}\alpha_2^{s-2v-2w}}{\beta_1^u B_2^w B_3^{1+u-s+v+2w} B_4^{1-v}}
$$
$$
\times \sum_{(r,B_2)=1,n} \sum_{m_2} \frac{\lambda(r B_3 B_4)S(\pm \tau q r, \sigma B_2 n, B_2 B_3 \alpha_2 m_2)}{n^{s-w} m_2^{u-s+2v+2w} r^{1-v-w}}.
$$

Rearranging, we obtain the final expression

$$
\sum_{\pm} \sum_{\sigma,\tau \in \{\pm\}} q^{(3-s-u-2v)/2} \sum_{\alpha_1 \alpha_2 = a} \sum_{\beta_1 B_3 B_4 B_2 = b} \frac{\mu(B_2)\mu(B_3)\alpha_1^{s-1}\alpha_2^u \zeta^{(\alpha_2 B_2 B_3 B_4 q)}(1-s+u)}{\beta_1^u B_2^{(s-u-1-2v)/2} B_3^{1-v} B_4^{1-v}}
$$
$$
\times \sum_{(r,B_2)=1,n} \frac{\lambda(r B_3 B_4)}{n^{(s+u-1+2v)/2} r^{(1-s+u)/2}} \sum_{B_2 B_3 \alpha_2 \mid m_2} \frac{S(\pm \sigma \tau q r, B_2 n, m_2)}{m_2} \Psi_{a,b,s,u,v}^{\pm,\sigma,\tau}\left(\frac{\sqrt{q r \overline{B_2} n}}{m_2}\right),
$$
$$(3.18)$$

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for $\mathcal{T}^*_{a,b,q}(s,u,v)$, where

$$
\begin{aligned}
\Psi(x) = \Psi^{\pm,\sigma,\tau}_{a,b,s,u,v}(x) &= x^{u-s+2v-1} \int_{(-4)} \Phi^{\pm}_{a,b,s}(w) G_f^{-\tau}(1-v-w) G^{-\sigma}(s-w) x^{2w} \frac{dw}{2\pi i} \\
&= \int_{(0)} \Phi^{\pm}_{a,b,s}\left(\frac{1+s-u-2v-w}{2}\right) G_f^{-\tau}\left(\frac{1-s+u+w}{2}\right) G^{-\sigma}\left(\frac{s+u+2v-1+w}{2}\right) \\
&\quad \times x^{-w}\frac{dw}{2\pi i}.
\end{aligned}
$$

In the region (3.1), the integrand is holomorphic in $2\theta - 8 < \Re w < 34$ (recalling (2.8) and (3.7)) and rapidly decaying on vertical lines; in particular, the assumption $x^j \Psi^{(j)}(x) \ll \min(x, x^{-3/2})$ for $0 \leqslant j \leqslant 3$ of the Kuznetsov formula (2.14) is satisfied. By (2.14), the $m_2$-sum equals

$$
\mathcal{A}^{\text{Maaß}}_B(\epsilon qr, B_2r; \mathscr{L}^\epsilon \Psi) + \mathcal{A}^{\text{Eis}}_B(\epsilon qr, B_2r; \mathscr{L}^\epsilon \Psi) + \mathcal{A}^{\text{hol}}_B(\epsilon qr, B_2r; \mathscr{L}^{\text{hol}} \Psi)
$$

with $B = B_2 B_3 \alpha_2$ and $\epsilon = \pm\sigma\tau$. In the larger region

$$
1/2 \leqslant \Re s, \Re v \leqslant 3, \quad \Re s \leqslant \Re u \leqslant 11, \tag{3.19}
$$

the integrand of $\Psi$ is holomorphic in $2\theta - 1 < \Re w < 32$ (and meromorphic in $|\Re w| < 32$) and rapidly decaying on vertical lines. By [BK19b, Lemma 3a] and (3.7), we conclude that, uniformly in this region,

$$
\mathscr{L}^{\pm}\Psi(t) \ll (1+|t|)^{-30}, \quad \mathscr{L}^{\text{hol}}\Psi(k) \ll k^{-30} \tag{3.20}
$$

as long as $a \asymp b$ for $t \in \mathbb{R} \cup [-7i/64, 7i/64]$ and $k \in 2\mathbb{N}$. Recall again that $\Psi$ depends on $s, u, v$, which are currently restricted to (3.1).

### 3.8 The cuspidal contribution
We start with the analysis of the Maaß spectrum. Inserting definitions (2.15) and (2.9) and using the notation and conventions of §2.4, we obtain

$$
\begin{aligned}
\mathcal{A}^{\text{Maaß}}_B(\epsilon qr, B_2r; \mathscr{L}^\epsilon \Psi) &= \sum_{B_0|B} \sum_{\psi \in \mathcal{B}^*(B_0)} \sum_{M|(B/B_0)} \rho_{\psi,M,B}(\epsilon qr)\rho_{\psi,M,B}(B_2n)\mathscr{L}^\epsilon\Psi(t_\psi) \\
&= \sum_{B_0|B} \sum_{\psi \in \mathcal{B}^*(B_0)} \epsilon_\psi^{(1-\epsilon)/2} \sum_{M|(B/B_0)} \frac{\prod_{p|B_0}(1-p^{-2})}{L(1, \text{Ad}^2\psi)B\nu(B)} \\
&\quad \times \sum_{d_1,d_2|M} \xi_\psi(M,d_1)\xi_\psi(M,d_2)\frac{d_1 d_2}{M}\lambda_\psi\left(\frac{qr}{d_1}\right)\lambda_\psi\left(\frac{B_2 n}{d_2}\right)\mathscr{L}^\epsilon\Psi(t_\psi).
\end{aligned}
$$

Summing over $n$ and $r$ as in (3.18), we obtain

$$
\begin{aligned}
&\sum_{B_0|B} \sum_{\psi \in \mathcal{B}^*(B_0)} \epsilon_\psi^{(1-\epsilon)/2} \sum_{M|(B/B_0)} \frac{\prod_{p|B_0}(1-p^{-2})}{L(1, \text{Ad}^2\psi)B\nu(B)} \sum_{d_1,d_2|M} \xi_\psi(M,d_1)\xi_\psi(M,d_2)\frac{d_1 d_2}{M} \\
&\quad \times \sum_{(r,B_2)=1} \frac{\lambda(rB_3 B_4)\lambda_\psi(qr/d_1)}{r^{(1-s+u)/2}} \sum_n \frac{\lambda_\psi(B_2 n/d_2)}{n^{(s+u-1+2v)/2}}\mathscr{L}^\epsilon\Psi(t_\psi).
\end{aligned}
$$

Since $(q,B) = 1$, we have $(d_1, q) = 1$, and so by (2.1), the $r$-sum equals

$$\frac{\delta_{(d_1,B_2)=1}}{d_1^{(1-s+u)/2}} \sum_{(r,B_2)=1} \frac{\lambda(rd_1B_3B_4)\lambda_\psi(qr)}{r^{(1-s+u)/2}}$$

$$= \frac{\delta_{(d_1,B_2)=1}}{d_1^{(1-s+u)/2}} \sum_{\substack{\delta_1|d_1B_3B_4 \\ (\delta_1,B_0)=1}} \frac{\mu(\delta_1)\lambda(d_1B_3B_4/\delta_1)}{\delta_1^{(1-s+u)/2}} \sum_{\delta_2|q} \frac{\mu(\delta_2)\lambda_\psi(q/\delta)}{\delta_2^{(1-s+u)/2}} \frac{L^{(B_2)}(f\times\psi,(1-s+u)/2)}{\zeta^{(B_2)}(1-s+u)}.$$

Similarly, the $n$-sum equals

$$\left(\frac{(B_2,d_2)}{d_2}\right)^{(s+u-1+2v)/2} \sum_n \frac{\lambda_\psi(B_2n/(d_2,B_2))}{n^{(s+u-1+2v)/2}}$$

$$= \left(\frac{(B_2,d_2)}{d_2}\right)^{(s+u-1+2v)/2} \sum_{\substack{B^*|B_2/(d_2,B_2) \\ (B^*,B_0)=1}} \frac{\mu(B^*)\lambda_\psi(B_2/((d_2,B_2)B^*))}{(B^*)^{(s+u-1+2v)/2}} L(\psi,(s+u-1+2v)/2).$$

Putting everything together, the Maaß contribution to (3.18) equals

$$\sum_{A|ab} \sum_{\psi\in\mathcal{B}^*(A)} \Theta_{a,b,q}^{\mathrm{Maaß}}(s,u,v,\psi) \frac{L\left(\frac{s+u-1+2v}{2},\psi\right)L\left(\frac{1-s+u}{2},f\times\psi\right)}{L(1,\mathrm{Ad}^2\psi)},$$

where

$$\Theta_{a,b,q}^{\mathrm{Maaß}}(s,u,v,\psi)$$

$$:= \sum_\pm \sum_{\sigma,\tau\in\{\pm\}} \epsilon_\psi^{(1\mp\sigma\tau)/2} q^{(3-s-u-2v)/2} \sum_{\alpha_1\alpha_2=a} \sum_{\substack{\beta_1B_3B_4B_2=b \\ A|B_2B_3\alpha_2}} \frac{\mu(B_2)\mu(B_3)\alpha_1^{s-1}\alpha_2^u}{\beta_1^u B_2^{(s-u-1-2v)/2}B_3^{1-v}B_4^{1-v}}$$

$$\times \sum_{M|(B_2B_3\alpha_2/A)} \sum_{\substack{d_1,d_2|M \\ (d_1,B_2)=1}} \frac{\xi_\psi(M,d_1)\xi_\psi(M,d_2)d_1d_2((B_2,d_2)/d_2)^{(s+u-1+2v)/2}}{Md_1^{(1-s+u)/2}L_{B_2}(f\times\psi,(1-s+u)/2)\zeta_{\alpha_2B_3B_4q}(1-s+u)}$$

$$\times \sum_{\substack{\delta_1|d_1B_3B_4 \\ (\delta_1,A)=1}} \frac{\mu(\delta_1)\lambda(d_1B_3B_4/\delta_1)}{\delta_1^{(1-s+u)/2}} \sum_{\delta_2|q} \frac{\mu(\delta_2)\lambda_\psi(q/\delta)}{\delta_2^{(1-s+u)/2}}$$

$$\times \sum_{\substack{B^*|(B_2/(d_2,B_2)) \\ (B^*,A)=1}} \frac{\mu(B^*)\lambda_\psi(B_2/((d_2,B_2)B^*))}{(B^*)^{(s+u-1+2v)/2}} \mathscr{L}^{\pm\sigma\tau} \Psi_{a,b,s,u,v}^{\pm,\sigma,\tau}(t_\psi) \qquad (3.21)$$

for $\psi\in\mathcal{B}^*(A)$ of spectral parameter $t_\psi$ and parity $\epsilon_\psi$. Clearly this expression is holomorphic in the region (3.19). We proceed to confirm the bound (1.11) for $\Re s = \Re u = \Re v = 1/2$. This requires a little more than a trivial bound of (3.21). The critical variable is $B_2$. In order to get enough saving, we need to exploit some cancellation. To this end, we write $M = M_1M_2$, where $(M_1,B_2)=1$ and $M_2\mid B_2$. (Recall that $ab$ is square-free.) Since $(d_1,B_2)=1$, we have $d_1\mid M_1$, and we write $d_2 = d_2'd_2''$ with $d_2'\mid M_1$, $d_2''\mid M_2$. In this way, the $M_2$-sum becomes

$$\sum_{d_2''|M_2|(B_2/(A,B_2))} \frac{\xi_\psi(M_2,1)\xi_\psi(M_2,d_2'')d_2''((B_2,d_2'')/d_2'')^{(s+u-1+2v)/2}}{M_2}$$

$$\times \sum_{\substack{B^*|(B_2/(d_2'',B_2)) \\ (B^*,A)=1}} \frac{\mu(B^*)\lambda_\psi(B_2/((d_2'',B_2)B^*))}{(B^*)^{(s+u-1+2v)/2}}. \qquad (3.22)$$

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If $B_2 \mid A$, this is equal to $\lambda_\psi(B_2) \ll B_2^{-1/2}$ by (2.3). If $B_2 \nmid A$, this is equal to

$$\prod_{p \mid B_2} \left( \left( \lambda_\psi(p) - \frac{1}{p^{(s+u-1+2v)/2}} \right) \left( 1 + \frac{\xi_\psi(p,1)^2}{p} \right) + \xi_\psi(p,1)\xi_\psi(p,p) \right).$$

By (2.10), the leading term $\lambda_\psi(p)$ cancels (to first-order approximation), and each $p$-factor in the preceding display is bounded by $p^{-1/2} + p^{3\theta-1} \ll p^{-1/2}$ for $\Re s = \Re u = \Re v = 1/2$. Hence in all cases the $M_2$-sum is $\ll B_2^{-1/2+\varepsilon}$. Combining with (3.20) and (2.11), we obtain

$$\Theta_{a,b,q}^{\mathrm{Maa\ss}}(s,u,v,\psi) \ll \frac{(abq)^\varepsilon q^{1/2}}{(1+|t_\psi|)^{30}} \sum_{\substack{\alpha_1\alpha_2=a \\ A \mid B_2 B_3 \alpha_2}} \sum_{\beta_1 B_3 B_4 B_2 = b} \sum_{\substack{d_1,d_2 \mid M \mid (B_2 B_3 \alpha_2/A) \\ (M,B_2)=1}} \frac{d_1^{1/2} d_2^{1/2-\theta}(1+|\lambda_\psi(q)|)}{\alpha_2^{1/2} B_3^{3/2-\theta} M^{1-2\theta} B_2^{1/2}}$$

for $\Re s = \Re u = \Re v = 1/2$. This is increasing in $d_1, d_2$, and the result is increasing in $M$, so that one easily confirms (1.11).

The same formula holds for the holomorphic contribution to (3.18), except that the transform $\mathscr{L}^{\pm\sigma\tau}\Psi_{a,b,s,u,v}^{\pm,\sigma,\tau}(t_\psi)$ has to be replaced with $\mathscr{L}^{\mathrm{hol}}\Psi_{a,b,s,u,v}^{\pm,\sigma,\tau}(k_\psi)$ and $\epsilon_\psi = 0$ if $\pm\sigma\tau = -1$. The corresponding bound (1.12) is even simpler to obtain because $\theta = 0$ in the holomorphic case.

### 3.9 The Eisenstein contribution
By (2.12), we have

$$\mathcal{A}_{B_2 B_3 \alpha_2}^{\mathrm{Eis}}(\epsilon qr, B_2 r; \mathscr{L}^\epsilon \Psi)$$
$$= \int_{\mathbb{R}} \frac{1}{B|\zeta^{(B_2 B_3 \alpha_2)}(1+2it)|^2} \sum_{v \mid B_2 B_3 \alpha_2} \frac{1}{v} \sum_{b_1, b_2 \mid v} \sum_{\gamma_1, \gamma_2 \mid B_2 B_3 \alpha_2/v}$$
$$\times \mu(b_1\gamma_1)\mu(b_2\gamma_2)b_1 b_2 \left( \frac{b_1\gamma_2}{b_2\gamma_1} \right)^{it} \eta\left( \frac{qr}{b_1\gamma_1}, t \right) \eta\left( \frac{B_2 n}{b_2\gamma_2}, -t \right) \mathscr{L}^\epsilon \Psi(t) \frac{dt}{2\pi}. \tag{3.23}$$

We saw in the previous subsection that the $B_2$-variable was the most critical variable, and we finally used the strong bound (2.3) to get a sufficient saving. We do not have a direct analogue of this bound in the Eisenstein case when we make the choice of basis arising from (2.12), but luckily we can obtain additional cancellation by summing non-trivially over the cusps $v$. This again requires some subtle manipulations.

Since $(B_2, B_3\alpha_2) = 1$, we write $v = v_1 v_2$ with $v_1 \mid B_3\alpha_2$, $v_2 \mid B_3\alpha_2$, $b_j = b_j' b_j''$, where $b_j' \mid v_1$, $b_j'' \mid v_2$, and $\gamma_j = \gamma_j'\gamma_j''$, where $\gamma_j' \mid B_2/v_1$, $\gamma_j'' \mid B_3\alpha_2/v_2$. The key observation is that $(qr, B_2) = 1$ in our application, so that $b_1' = \gamma_1' = 1$. In this way, we can recast the previous $v, b_1, b_2, \gamma_1, \gamma_2$-sum as

$$\sum_{\substack{b_2' \mid v_1 \mid B_2 \\ b_1'', b_2'' \mid v_2 \mid B_3\alpha_2}} \sum_{\gamma_2' \mid (B_2/v_1)} \sum_{\gamma_1'', \gamma_2'' \mid (B_3\alpha_2/v_2)}$$
$$\times \frac{\mu(b_1''\gamma_1'')\mu(b_2'b_2''\gamma_2'\gamma_2'')b_1'' b_2' b_2''}{v_1 v_2} \left( \frac{b_1''\gamma_2'\gamma_2''}{b_2'b_2''\gamma_1''} \right)^{it} \eta\left( \frac{qr}{b_1''\gamma_1''}, t \right) \eta\left( \frac{B_2 n}{b_2'b_2''\gamma_2'\gamma_2''}, -t \right).$$

We consider only the $B_2$-part,

$$\sum_{b_2' \mid v_1 \mid B_2} \sum_{\gamma_2' \mid (B_2/v_1)} \frac{1}{v_1} \mu(b_2'\gamma_2')b_2' \left( \frac{\gamma_2'}{b_2'} \right)^{it} \eta\left( \frac{B_2 n}{b_2'b_2''\gamma_2'\gamma_2''}, -t \right), \tag{3.24}$$

for fixed $b_2'', \gamma_2''$, where we parametrize $v_1 = b_2' b^*$, $B_2/v_1 = \gamma_2' \gamma^*$, getting

$$\sum_{b_2' b^* \gamma_2' \gamma^* = B_2} \frac{1}{b^*} \mu(b_2' \gamma_2') \left( \frac{\gamma_2'}{b_2'} \right)^{it} \eta\left( b^* \gamma^* \frac{n}{b_2'' \gamma_2''}, -t \right).$$

We must have $b_2'' \gamma_2'' \mid n$, so we write $n = b_2'' \gamma_2'' n^*$. Applying the Hecke relation (2.1) for $\eta(n, t)$, we obtain

$$\sum_{\substack{b_2' b^* \gamma_2' \gamma^* = B_2 \\ \delta \mid (n^*, b^* \gamma^*)}} \mu(\delta) \frac{1}{b^*} \mu(b_2' \gamma_2') \left( \frac{\gamma_2'}{b_2'} \right)^{it} \eta\left( \frac{b^* \gamma^*}{\delta}, -t \right) \eta\left( \frac{n^*}{\delta}, -t \right).$$

We parametrize $\delta = \delta_1 \delta_2$, $b^* = \delta_1 b_0$, $\gamma^* = \delta_2 \gamma_0$, so that the previous line is equal to

$$\sum_{\substack{b_2' \delta_1 b_0 \gamma_2' \delta_2 \gamma_0 = B_2 \\ \delta_1 \delta_2 \mid n^*}} \mu(\delta_1 \delta_2) \frac{1}{\delta_1 b} \mu(b_2' \gamma_2') \left( \frac{\gamma_2'}{b_2'} \right)^{it} \eta(b_0, -t) \eta(\gamma_0, -t) \eta\left( \frac{n^*}{\delta_1 \delta_2}, -t \right).$$

The key point is now that, by Möbius inversion, the $b_2', \gamma_2', \gamma_0$-sum disappears, so that (3.24) is equal to

$$\sum_{\substack{\delta_1 b_0 \delta_2 = B_2 \\ \delta_1 \delta_2 \mid n^*}} \mu(\delta_1 \delta_2) \frac{1}{\delta_1 b} \eta(b_0, -t) \eta\left( \frac{n^*}{\delta_1 \delta_2}, -t \right),$$

and hence (3.23) is equal to

$$\int_{\mathbb{R}} \sum_{\substack{b_1, b_2 \mid v \mid B_3 \alpha_2 \\ b_2 \gamma_2 \mid n}} \sum_{\gamma_1, \gamma_2 \mid B_3 \alpha_2 / v} \frac{\mu(b_1 \gamma_1) \mu(b_2 \gamma_2) b_1 b_2}{v B_2 B_3 \alpha_2 |\zeta^{(B_2 B_3 \alpha_2)}(1 + 2it)|^2} \left( \frac{b_1 \gamma_2}{b_2 \gamma_1} \right)^{it} \eta\left( \frac{qr}{b_1 \gamma_1}, t \right)$$

$$\times \sum_{\substack{\delta_1 b_0 \delta_2 = B_2 \\ \delta_1 \delta_2 \mid n}} \frac{\mu(\delta_1 \delta_2) \eta(b_0, -t)}{\delta_1 b_0} \eta\left( \frac{n}{b_2 \gamma_2 \delta_1 \delta_2}, -t \right) \mathscr{L}^\epsilon \Psi(t) \frac{dt}{2\pi}.$$

After this manoeuvre, we are now in shape to sum over $r$ and $n$ as in (3.18). This gives

$$\int_{\mathbb{R}} \sum_{b_1, b_2 \mid v \mid B_3 \alpha_2} \sum_{\gamma_1, \gamma_2 \mid (B_3 \alpha_2 / v)} \frac{\mu(b_1 \gamma_1) \mu(b_2 \gamma_2) b_1 b_2}{v B_2 B_3 \alpha_2 |\zeta^{(B_2 B_3 \alpha_2)}(1 + 2it)|^2} \left( \frac{b_1 \gamma_2}{b_2 \gamma_1} \right)^{it} \sum_{\delta_1 b_0 \delta_2 = B_2} \frac{\mu(\delta_1 \delta_2) \eta(b_0, -t)}{\delta_1 b_0}$$

$$\times \sum_{(r, B_2) = 1} \frac{\lambda(r b_1 \gamma_1 B_3 B_4) \eta(qr, t)}{(b_1 \gamma_1 r)^{(1 - s + u)/2}} \sum_n \frac{\eta(n, -t)}{(b_2 \gamma_2 \delta_1 \delta_2 n)^{(s + u - 1 + 2v)/2}} \mathscr{L}^{\pm \sigma \tau} \Psi(t) \frac{dt}{2\pi}.$$

The $n$-sum can be easily evaluated in terms of the Riemann zeta function. The $r$-sum requires multiple applications of (2.1). Checking local factors, we confirm that, for $(B, q) = (B_2, Bq) = 1$, $B$ square-free, and $q$ prime, we have

$$\sum_{(B_2, r) = 1} \frac{\lambda(rB) \eta(qr, t)}{r^z} = \frac{\eta(q, t) - \lambda(q) q^{-z}}{1 - q^{-2z}} \prod_{p \mid B} \frac{\lambda(p) - \eta(p, t) p^{-z}}{1 - p^{-2z}} \sum_{(B_2, r) = 1} \frac{\lambda(r) \eta(r, t)}{r^z}.$$

Putting everything together, the Eisenstein contribution to (3.18) is equal to

$$\int_{\mathbb{R}} \Theta_{a,b,q}^{\mathrm{Eis}}(s, u, v, t) \frac{\zeta\left( \frac{s + u - 1 + 2v}{2} + it \right) L\left( \frac{s + u - 1 + 2v}{2} - it \right) \zeta\left( \frac{1 - s + u}{2} + it, f \right) L\left( \frac{1 - s + u}{2} - it, f \right)}{\zeta(1 + 2it) \zeta(1 - 2it)} \frac{dt}{2\pi},$$

$$(3.25)$$

where

$$\Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t) := \sum_{\pm} \sum_{\sigma,\tau\in\{\pm\}} \sum_{\alpha_1\alpha_2=a} \sum_{\beta_1 B_3 B_4 B_2=b} \frac{q^{(3-s-u-2v)/2}\mu(B_2)\mu(B_3)\alpha_1^{s-1}\alpha_2^u}{\beta_1^u B_2^{(s-u-1-2v)/2} B_3^{1-v} B_4^{1-v}}$$

$$\times \sum_{b_1,b_2|vB_3\alpha_2} \sum_{\gamma_1,\gamma_2|(B_3\alpha_2/v)} \frac{\mu(b_1\gamma_1)\mu(b_2\gamma_2)b_1 b_2}{vB_2 B_3\alpha_2|\zeta^{(B_2 B_3\alpha_2)}(1+2it)|^2} \left(\frac{b_1\gamma_2}{b_2\gamma_1}\right)^{it}\frac{\eta(q,t)-\lambda(q)q^{-(1-s+u)/2}}{1-q^{-(1-s+u)}}$$

$$\times \sum_{\delta_1 b_0\delta_2=B_2} \frac{\mu(\delta_1\delta_2)\eta(b_0,-t)}{\delta_1 b_0(b_1\gamma_1)^{(1-s+u)/2}(b_2\gamma_2\delta_1\delta_2)^{(s+u-1+2v)/2}} \prod_{p|b_1\gamma_1 B_3 B_4} \frac{\lambda(p)-\eta(p,t)p^{-(1-s+u)/2}}{1-p^{-(1-s+u)}}$$

$$\times \frac{\zeta_{B_2 B_3\alpha_2}(1+2it)\zeta_{B_2 B_3\alpha_2}(1-2it)}{\zeta_{\alpha_2 B_3 B_4 q}(1-s+u)L_{B_2}\left(\frac{1-s+u}{2}+it,f\right)L_{B_2}\left(\frac{1-s+u}{2}-it,f\right)} \mathscr{L}^{\pm\sigma\tau}\Psi_{a,b,s,u,v}^{\pm,\sigma,\tau}(t). \tag{3.26}$$

The term (3.25) is clearly holomorphic in the range (3.1) and it can easily be extended as long as $\Re(u-s) > 1$ and $\Re(s+u+2v) > 3$. To pass these two hyperplanes, we observe that the presence of the Riemann zeta function in the numerator contributes residues, and so we apply the argument of [BK19b, Lemma 16] to show that the meromorphic continuation of (3.25) in the region $\Re(u-s) < 1$ and $\Re(s+u+2v) < 3$ is given by the same expression plus the polar term

$$\mathcal{P}_{a,b,q}^{(3)}(s,u,v) := \sum_{\substack{t=\pm i(1+s-u)/2 \\ t=\pm i(3-s-u-2v)/2}} \operatorname*{Res} \ (\pm i) \frac{\Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t)\zeta\left(\frac{s+u-1+2v}{2}+it\right)}{\zeta(1+2it)\zeta(1-2it)}$$

$$\times \zeta\left(\frac{s+u-1+2v}{2}-it\right) L\left(\frac{1-s+u}{2}+it,f\right)L\left(\frac{1-s+u}{2}-it,f\right). \tag{3.27}$$

A trivial estimation confirms (1.13) for the term on the right-hand side of (3.25) with $\Re s = \Re u = \Re v = 1/2$, $t\in\mathbb{R}$, $a\asymp b$ (which differs from the meromorphic continuation of (3.25) to this region by (3.27)).

It remains to meromorphically continue and bound the joint polar term

$$\mathcal{P}_{a,b,q}(s,u,v) := \sum_{j=1}^{3} \mathcal{P}_{a,b,q}^{(j)}(s,u,v), \tag{3.28}$$

where we recall (3.15) and (3.16) for $j=1,2$. In these cases, it is easily seen that $\mathcal{P}_{a,b,q}^{(j)}(s,u,v)$ continues meromorphically to a neighbourhood of (3.19), and for $1/2-\varepsilon < \Re s = \Re u = \Re v < 1/2+\varepsilon$, $a\asymp b$, we have the bound

$$|\mathcal{P}_{a,b,q}^{(1)}(s,u,v)| + |\mathcal{P}_{a,b,q}^{(2)}(s,u,v)| \ll q(ab)^{-1/2}(abq)^{\varepsilon}$$

away from poles. The treatment of $\mathcal{P}_{a,b,q}^{(3)}(s,u,v)$ requires slightly more effort, because we need to analyze $\Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t)$ for $|\Im t| \leqslant 1/2$. The meromorphic continuation of $\mathscr{L}^{\pm\sigma\tau}\Psi_{a,b,s,u,v}^{\pm,\sigma,\tau}(t)$ with at most finitely many poles (and hence of $\Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t)$) to that region follows from [BK19b, Lemma 3b]. Again, a trivial upper bound yields

$$\Theta_{a,b,q}^{\mathrm{Eis}}(s,u,v,t) \ll (abq)^{\varepsilon}\frac{q}{(ab)^{1/2-\theta}}$$

for fixed $s,u,v,t$ with $1/2-\varepsilon < \Re s = \Re u = \Re v < 1/2+\varepsilon$, $|\Im t| < 1/2+\varepsilon$, $a\asymp b$ away from possible poles, so that also

$$\mathcal{P}_{a,b,q}^{(3)}(s,u,v) \ll (abq)^{\varepsilon}\frac{q}{(ab)^{1/2-\theta}}$$

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in the region $1/2 - \varepsilon < \Re s = \Re u = \Re v < 1/2 + \varepsilon$, away from possible poles. We have established (1.14) as an equality of meromorphic functions, but since all terms except possibly $\mathcal{P}_{a,b,q}(s,u,v)$ are holomorphic for $\Re s = \Re u = \Re v = 1/2$, $\mathcal{P}_{a,b,q}(s,u,v)$ must also be holomorphic for $\Re s = \Re u = \Re v = 1/2$, and the general bound (1.15) then follows by Cauchy's integral theorem in the same way as at the end of [BK19b, § 10].

*Remark* 1. We briefly discuss an alternate approach to dealing with the Eisenstein contribution suggested by the referee. Instead of inserting the expression (2.12) for the Fourier coefficients of the Eisenstein series into (3.23), we could use an identity akin to the expression (2.9) for the Fourier coefficients of cuspidal newforms. Such an identity follows from the work of Young [You19, § 8.5]. With this identity in hand, the treatment of the Eisenstein contribution can be done via the same process as for the cuspidal contribution, as in § 3.8. A key difference is that when we arrive at an expression of the form (3.22), we cannot have $B_2 \mid A$, because there are no Eisenstein newforms of level $N$ with trivial central character such that $p \parallel N$ for some prime $p$. Thus instead of the bound $\lambda_\psi(B_2) \ll B_2^{-1/2}$ when $B_2 \mid A$ by (2.3), we have the stronger bound $\lambda_\psi(B_2) = 0$, which gives us the requisite savings.

## 4. Proof of Theorem 2

### 4.1 Initial manipulations

Let $P = TQ$. By 'negligible', we mean a quantity that is $O(P^{-100})$. By a dyadic decomposition, we may replace the conditions $q \leqslant Q$, $|t_\psi| \leqslant T$ with $\frac{1}{2}Q \leqslant q \leqslant Q$, $\frac{1}{2}T \leqslant t_\psi \leqslant T$ or $t_\psi \in [0,1] \cup [-i\theta, i\theta]$ where in the last case we formally put $T = 1$.

Let $E_3$ denote the standard minimal Eisenstein series for $\mathrm{SL}_3(\mathbb{Z})$ with Fourier coefficients

$$A(n,m) = \sum_{d \mid (n,m)} \mu(d)\tau_3(n/d)\tau_3(m/d).$$

Then, for $\psi \in \mathcal{B}^*(q)$ with $|t_\psi| \leqslant T$, we have

$$L(s,\psi)^3 = L(s, \psi \times E_3) = \sum_n \sum_{(m,q)=1} \frac{A(n,m)\lambda_\psi(n)}{n^s m^{2s}} = P_q(s) \sum_{n,m} \frac{A(n,m)\lambda_\psi(n)}{n^s m^{2s}}$$

for $\Re s > 1$, where

$$P_q(s) = \prod_{p \mid q} \left(1 - \frac{\lambda_\psi(p)}{p^s}\right)^{-3} \left(1 - \frac{\lambda_\psi(p)}{p^s} + \frac{1}{p^{2s}}\right)^3$$

is holomorphic and uniformly bounded in $\Re s \geqslant 1/2$. By a standard approximate functional equation, we have

$$|L(1/2, \psi)^3| \leqslant 2 \left| \sum_{n,m} \frac{A(n,m)\lambda_\psi(n)}{n^{1/2}m} V_\psi\left(\frac{nm^2}{q^{3/2}}\right) \right|,$$

where

$$V_\psi(y) = \frac{1}{2\pi i} \int_{(2)} P(\tfrac{1}{2} + u)$$
$$\times \frac{\Gamma(\frac{1}{2}(\frac{1}{2} + u + \epsilon_\psi + it_\psi))^3 \Gamma(\frac{1}{2}(\frac{1}{2} + u + \epsilon_\psi - it_\psi))^3}{\Gamma(\frac{1}{2}(\frac{1}{2} + \epsilon_\psi + it_\psi))^3 \Gamma(\frac{1}{2}(\frac{1}{2} + \epsilon_\psi - it_\psi))^3} \pi^{-3u} e^{u^2} y^{-u} \frac{du}{u}.$$

Shifting the contour to the far right, we see that $V_\psi(y)$ is negligible if $y \geqslant T^3 P^\varepsilon$. Remembering this, we shift the contour to $\Re u = \varepsilon$. There we may truncate the integral at $|\Im u| \leqslant P^\varepsilon$ at the cost of a negligible error. Applying a smooth dyadic decomposition, we have shown that

$$L(1/2, \psi)^3 \ll_\varepsilon P^\varepsilon \int_{-P^\varepsilon}^{P^\varepsilon} \sum_{2^\nu = N \leqslant Q^{3/2} T^3 P^\varepsilon} \left| \sum_{n,m} \frac{A(n,m)\lambda_\psi(n)}{(nm^2)^{1/2+iv}} V\left(\frac{nm^2}{N}\right) \right| dv,$$

where $V$ has support in $[1, 2]$, is independent of $\psi$, and satisfies $V^{(j)}(y) \ll_j 1$ for all $j \in \mathbb{N}_0$. Multiplying two such expressions together and using the Cauchy–Schwarz inequality, we obtain

$$L(1/2, \psi)^6$$
$$\ll_\varepsilon P^\varepsilon \max_{|v| \leqslant P^\varepsilon} \max_{N \leqslant Q^{3/2} T^3 P^\varepsilon} \sum_{n_1, n_2, m_1, m_2} \frac{A(n_1, m_1) A(n_2, m_2) \lambda_\psi(n_1) \lambda_\psi(n_2)}{(n_1 m_1^2)^{1/2+iv} (n_2 m_2^2)^{1/2-iv}} V\left(\frac{n_1 m_1^2}{N}\right) \overline{V\left(\frac{n_2 m_2^2}{N}\right)}.$$

For $\psi \in \mathcal{B}^*(q)$, we have

$$\frac{\lambda_\psi(n)\lambda_\psi(m)}{L(1, \mathrm{Ad}^2 \psi)} = q \prod_{p|q} \left(1 - \frac{1}{p}\right)^{-1} \rho_{\psi,1,q}(n) \rho_{\psi,1,q}(m)$$

by (2.9). For the purpose of Theorem 2, it therefore suffices to bound

$$\mathcal{S}_v(Q, T, N) := \sum_q W\left(\frac{q}{Q}\right) Q \sum_{\psi \in \mathcal{B}^*(q)} h_T(t_\psi)$$
$$\times \sum_{n_1, n_2, m_1, m_2} \frac{A(n_1, m_1) A(n_2, m_2) \rho_{\psi,1,q}(n_1) \rho_{\psi,1,q}(n_2)}{(n_1 m_1^2)^{1/2+iv} (n_2 m_2^2)^{1/2-iv}} V\left(\frac{n_1 m_1^2}{N}\right) \overline{V\left(\frac{n_2 m_2^2}{N}\right)},$$
(4.1)

where $N \leqslant Q^{3/2} T^3 P^\varepsilon$, $|v| \leqslant P^\varepsilon$, and

$$h_T(t) = e^{-(t/T)^2} \prod_{n=1}^{\lfloor \varepsilon^{-1} \rfloor} \left(\frac{1}{T^2}\left(t^2 + \frac{(2n-1)^2}{4}\right)\right).$$

Note that this function satisfies the assumptions of Lemmas 1 and 2.

## 4.2 The Eisenstein contribution associated with the trivial character

The $\psi$-sum in (4.1) can be evaluated by the Kuznetsov formula (2.16). To this end, we need to add, using positivity, the contribution from the oldforms and the continuous spectrum. As mentioned in the introduction, this manoeuvre is costly, and we single out the contribution of the continuous spectrum associated with the trivial character:

$$\mathcal{S}_v^*(Q, T, N) := \sum_q W\left(\frac{q}{Q}\right) Q \sum_{M|q} \int_\mathbb{R} h_T(t)$$
$$\times \sum_{n_1, n_2, m_1, m_2} \frac{A(n_1, m_1) A(n_2, m_2) \rho_{\mathrm{triv},M,q}(n_1, t) \overline{\rho_{\mathrm{triv},M,q}(n_2, t)}}{(n_1 m_1^2)^{1/2+iv} (n_2 m_2^2)^{1/2-iv}} V\left(\frac{n_1 m_1^2}{N}\right) \overline{V\left(\frac{n_2 m_2^2}{N}\right)} \frac{dt}{2\pi},$$

which we rewrite in more compact form as

$$\int_{(2)} \int_{(2)} \int_{(1)} \int_\mathbb{R} \widehat{V}(z_1) \widehat{V}(z_2) \widehat{W}(s) Q^{1+s} N^{z_1+z_2}$$
$$\times \frac{\mathcal{D}_t(s, 1/2 + iv + z_1, 1/2 - iv + z_2; 0, 0)}{|\zeta(1+2it)|^2} h_T(t) \frac{dt}{2\pi} \frac{ds\, dz_1\, dz_2}{(2\pi i)^3},$$
(4.2)

where

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$$\mathcal{D}_t(\mathbf{s}, \mathbf{z}_1, \mathbf{z}_2; \mathbf{w}_1, \mathbf{w}_2)$$
$$= |\zeta(1+2it)|^2 \sum_{q, n_1, n_2, m_1, m_2} \sum_{M|q} \frac{A(n_1, m_1)A(n_2, m_2)\rho_{\mathrm{triv}, M, q}(n_1, t)\overline{\rho_{\mathrm{triv}, M, q}(n_2, t)}}{q^{\mathbf{s}}(n_1 m_1^2)^{\mathbf{z}_1}(n_2 m_2^2)^{\mathbf{z}_2}m_1^{2\mathbf{w}_1}m_2^{2\mathbf{w}_2}}.$$

Recalling the definition (cf. (2.13))

$$|\zeta(1+2it)|^2\rho_{\mathrm{triv}, M, q}(n_1, t)\overline{\rho_{\mathrm{triv}, M, q}(n_2, t)}$$
$$= \frac{|\zeta^{(q)}(1+2it)|^2(n_1/n_2)^{it}}{q\nu(q)\tilde{\mathfrak{n}}_q(M)^2} \sum_{\delta_1, \delta_2 | M} \frac{\delta_1 \delta_2 \mu(M/\delta_1)\mu(M/\delta_2)}{M} \sum_{\substack{c_1\delta_1 f_1 = n_1 \\ (c_1, q/M)=1}} \sum_{\substack{c_2\delta_2 f_2 = n_2 \\ (c_2, q/M)=1}} \left(\frac{c_2}{c_1}\right)^{2it},$$

we see that (for $t \in \mathbb{R}$) the series $\mathcal{D}_t(\mathbf{s}, \mathbf{z}_1, \mathbf{z}_2; \mathbf{w}_1, \mathbf{w}_2)$ is absolutely convergent in $\Re\mathbf{s} > 0$, $\Re\mathbf{z}_1$, $\Re\mathbf{z}_2 > 1$, $\Re(\mathbf{z}_1 + \mathbf{w}_1)$, $\Re(\mathbf{z}_2 + \mathbf{w}_2) > 1/2$, and admits an Euler product of the shape

$$\prod_p \left( 1 + \frac{1}{p^{\mathbf{s}+1}} + \sum_{j=1,2} \sum_{\pm} \frac{3}{p^{\mathbf{z}_j \pm it}} \right.$$
$$\left. + O\left( \frac{1}{p^{2\min(\Re\mathbf{z}_1, \Re\mathbf{z}_2, \Re(\mathbf{z}_1+\mathbf{w}_1), \Re(\mathbf{z}_2+\mathbf{w}_2))}} + \frac{1}{p^{\Re\mathbf{s}+\min(\Re\mathbf{z}_1, 1)+\min(\Re\mathbf{z}_2, 1)}} \right) \right),$$

where the bounds in the error term hold uniformly in

$$\Re\mathbf{z}_1, \quad \Re\mathbf{z}_2, \quad \Re(\mathbf{z}_1 + \mathbf{w}_1), \quad \Re(\mathbf{z}_2 + \mathbf{w}_2), \quad \Re\mathbf{s} + \min(\Re\mathbf{z}_1, 1) + \min(\Re\mathbf{z}_2, 1) > 0.$$

In particular, we have

$$\mathcal{D}_t(\mathbf{s}, \mathbf{z}_1, \mathbf{z}_2; \mathbf{w}_1, \mathbf{w}_2) = \zeta(\mathbf{s}+1)\zeta(\mathbf{z}_1+it)^3\zeta(\mathbf{z}_1-it)^3\zeta(\mathbf{z}_2+it)^3\zeta(\mathbf{z}_2-it)^3\mathcal{E}_t(\mathbf{s}, \mathbf{z}_1, \mathbf{z}_2; \mathbf{w}_1, \mathbf{w}_2), \quad (4.3)$$

where $\mathcal{E}_t(\mathbf{s}, \mathbf{z}_1, \mathbf{z}_2; \mathbf{w}_1, \mathbf{w}_2)$ is holomorphic and uniformly bounded in

$$\Re\mathbf{z}_1, \ \Re\mathbf{z}_2, \ \Re(\mathbf{z}_1 + \mathbf{w}_1), \ \Re(\mathbf{z}_2 + \mathbf{w}_2) \geqslant 1/2 + \varepsilon, \quad \Re\mathbf{s} + \min(\Re\mathbf{z}_1, 1) + \min(\Re\mathbf{z}_2, 1) \geqslant 1 + \varepsilon, \quad (4.4)$$

as long as $\Im t = 0$. Hence in (4.2), we may shift the contours to $\Re s = -1 + \varepsilon$ (picking up a residue at $s = 0$), and in the remaining integral we shift the $z_1, z_2$-contours to $\Re z_1 = \Re z_2 = 1/2 + \varepsilon$, getting

$$\mathcal{S}_v^*(Q, T, N) = \widehat{W}(0)Q \int_{(2)} \int_{(2)} \int_{\mathbb{R}} \prod_{\pm} \frac{\zeta(1/2 + iv + z_1 \pm it)^3\zeta(1/2 - iv + z_2 \pm it)^3}{\zeta(1 \pm 2it)}$$
$$\times \mathcal{E}_t(0, 1/2 + iv + z_1, 1/2 - iv + z_2; 0, 0)\widehat{V}(z_1)\widehat{V}(z_2)N^{z_1+z_2}h_T(t)\frac{dz_1\, dz_2}{(2\pi i)^2} \frac{dt}{2\pi} + O(TNP^\varepsilon).$$
$$(4.5)$$

### 4.3 Applying the Kuznetsov formula twice

By the Kuznetsov formula (and positivity), we obtain

$$\mathcal{S}_v(Q, T, N) + \mathcal{S}_v^*(Q, T, N)$$
$$\leqslant Q \sum_q W\left(\frac{q}{Q}\right) \sum_{n_1, n_2, m_1, m_2} \frac{A(n_1, m_1)A(n_2, m_2)}{(n_1 m_1^2)^{1/2+iv}(n_2 m_2^2)^{1/2-iv}} V\left(\frac{n_1 m_1^2}{N}\right) \overline{V\left(\frac{n_2 m_2^2}{N}\right)}$$
$$\times \left( \delta_{n_1, n_2} \int_{-\infty}^{\infty} h_T(t) \frac{t \tanh(\pi t)\, dt}{2\pi^2} + \sum_c \frac{S(n_1, n_2, qc)}{qc} \mathscr{K} h_T\left(\frac{\sqrt{n_1 n_2}}{qc}\right) \right).$$

The diagonal term is easy to deal with and is trivially bounded by

$$O_\varepsilon(P^\varepsilon Q^2 T^2). \tag{4.6}$$

By Mellin inversion, we can recast the off-diagonal term as

$$\int_{(-12)} Q^{s+1}\widehat{W}(s) \sum_{n_1,n_2,m_1,m_2} \frac{A(n_1,m_1)A(n_2,m_2)}{(n_1 m_1^2)^{1/2+iv}(n_2 m_2^2)^{1/2-iv}(n_1 n_2)^{s/2}} V\left(\frac{n_1 m_1^2}{N}\right)\overline{V\left(\frac{n_2 m_2^2}{N}\right)}$$
$$\times \sum_{c,q} c^s \frac{S(n_1,n_2,qc)}{qc}\mathscr{K}_s h_T\left(\frac{\sqrt{n_1 n_2}}{qc}\right)\frac{ds}{2\pi i}$$

with $\mathscr{K}_s h_T(x) = x^s \mathscr{K} h_T(x)$ as in §2.6. Applying the Kuznetsov formula immediately in the other direction (which we may do by Lemma 1, but this time the summation is over $q$ instead of $c$), we obtain by (2.14) that the previous expression is equal to

$$\int_{(-12)} Q^{s+1}\widehat{W}(s) \sum_{n_1,n_2,m_1,m_2} \frac{A(n_1,m_1)A(n_2,m_2)}{(n_1 m_1^2)^{1/2+iv}(n_2 m_2^2)^{1/2-iv}(n_1 n_2)^{s/2}} V\left(\frac{n_1 m_1^2}{N}\right)\overline{V\left(\frac{n_2 m_2^2}{N}\right)}$$
$$\times \sum_c c^s (\mathcal{A}_c^{\text{Maaß}}(n_1,n_2;\mathscr{L}^+\mathscr{K}_s h_T) + \mathcal{A}_c^{\text{Eis}}(n_1,n_2;\mathscr{L}^+\mathscr{K}_s h_T) + \mathcal{A}_c^{\text{hol}}(n_1,n_2;\mathscr{L}^{\text{hol}}\mathscr{K}_s h_T))\frac{ds}{2\pi i}. \tag{4.7}$$

Lemma 2(b) implies that $\mathscr{L}^+\mathscr{K}_s h_T(t)$ has analytic continuation to $\Re s < 1$, and we proceed to derive a uniform bound. If $|t| \geqslant 10|\Im s|$ (so that $t \pm \frac{1}{2}|\Im s| \asymp t$), we have

$$\mathscr{L}^+\mathscr{K}_s h_T(t) \ll \int_{\mathbb{R}} \frac{e^{-|\tau|/T}(1+|\tau|)}{(1+|t|+|\tau|)^{2-2\Re s}}\,d\tau + \frac{e^{-|t|/T}(1+|t|)}{(1+|t|)^{1-\Re s}}\int_0^{1+|t|} \frac{1}{(1+|\tau|)^{1-\Re s}}\,d\tau$$
$$\ll \frac{T^2}{(1+|t|)^{2-2\Re s}} + e^{-|t|/T}((1+|t|)^{\Re s} + (1+|t|)^{2\Re s}) \ll \frac{T^{2+\max(0,-\Re s)}}{(1+|t|)^{2-2\Re s}}.$$

If $|t| \leqslant 10|\Im s|$, we have trivially $\mathscr{L}^+\mathscr{K}_s h_T(t) \ll T^2$, so that altogether we obtain the uniform bound

$$\mathscr{L}^+\mathscr{K}_s h_T(t) \ll_{\Re s} (1+|\Im s|)^{2-2\Re s}\frac{T^{2+\max(0,-\Re s)}}{(1+|t|)^{2-2\Re s}}. \tag{4.8}$$

The problematic expression in (4.7) is the part of $\mathcal{A}_c^{\text{Eis}}(n_1,n_2;\mathscr{L}^+\mathscr{K}_s h_T)$ that is associated with the trivial character. We spell this out explicitly as

$$\mathcal{S}_v^{**}(Q,T,N) = \int_{(20)}\int_{(20)}\int_{(-12)}\int_{\mathbb{R}} \widehat{V}(z_1)\widehat{V}(z_2)\widehat{W}(s)Q^{1+s}N^{z_1+z_2}$$
$$\times \frac{\mathcal{D}_t(-s, 1/2+iv+s/2+z_1, 1/2-iv+s/2+z_2; -s/2, -s/2)}{|\zeta(1+2it)|^2}$$
$$\times \mathscr{L}^+\mathscr{K}_s h_T(t)\frac{dt}{2\pi}\frac{ds\,dz_1\,dz_2}{(2\pi i)^3}.$$

Shifting the $s$-contour to the far left and simultaneously the $z_1, z_2$-contours to $\Re z_1 = \frac{1}{2}(1 - \Re s) + \varepsilon$, we see from (4.8) that the $t$-integral is negligible for $|t| \geqslant \sqrt{NT/Q}P^\varepsilon$. In particular, we may truncate at $|t| \leqslant (T + \sqrt{NT/Q})P^\varepsilon$.

Next we shift the $s$-contour to $\Re s = \varepsilon$, past the pole at $s = 0$. By Lemma 2(a), the residue matches exactly the main term in (4.5) except for the truncation of the $t$-integral, but by the

rapid decay of $\mathscr{L}^+\mathscr{K}_0 h_T = h_T$ for $|t| \geqslant T$, we may reinsert the tail at the cost of a negligible error.

To estimate the remaining integral, we shift the $z_1, z_2$-contours to left, past the triple poles at $z_1 = 1/2 - iv - s/2 \pm it$, $z_2 = 1/2 - iv - s/2 \pm it$ to $\Re z_1, \Re z_2 = \varepsilon$. Thus we need to bound the contributions from the remaining integral and the two residues. The remaining multiple integral contains a $t$-integral that can be bounded by

$$\ll_\varepsilon \int_{|t| \leqslant (T+\sqrt{NT/Q})P^\varepsilon} \left| \zeta\left(\frac{1}{2} + \varepsilon + it + i\tau\right) \right|^{12} \frac{T^2}{(1+|t|)^{2-\varepsilon}}\, dt \ll_\varepsilon T^2(1+|\tau|)^2 P^\varepsilon,$$

where $\tau = \pm v + \Im z_j + \frac{1}{2}\Im s_j$ and we used Heath-Brown's twelfth moment bound [Hea78]. Thus the total contribution of the remaining integral is $O_\varepsilon(QT^2P^\varepsilon)$. It remains to deal with the two residues. Here the rapid decay of $\widehat{W}$ and $\widehat{V}_{1,2}$ and their derivatives at $z = 1/2 \pm iv - s/2 \pm it$ makes the $t$-integral rapidly convergent regardless of the real part of $s$, so we may shift the contour to $\Re s = 1 - \varepsilon$ (so that $\Re z_j = \varepsilon$), getting a contribution of $O_\varepsilon(Q^2T^2P^\varepsilon)$.

Combining (4.6) and the error term in (4.5) with the previous two error terms, we have accomplished so far the bound

$$\mathcal{S}_v(Q,T,N) \ll_\varepsilon P^\varepsilon(Q^2T^2 + NT)$$
$$+ \left| \int_{(-12)} Q^{s+1}\widehat{W}(s) \sum_{n_1,n_2,m_1,m_2} \frac{A(n_1,m_1)A(n_2,m_2)}{(n_1 m_1^2)^{1/2+iv}(n_2 m_2^2)^{1/2-iv}(n_1 n_2)^{s/2}} V\left(\frac{n_1 m_1^2}{N}\right)\overline{V\left(\frac{n_2 m_2^2}{N}\right)} \right.$$
$$\left. \times \sum_c c^s (\mathcal{A}_c^{\text{Maaß}}(n_1,n_2;\mathscr{L}^+\mathscr{K}_s h_T) + \mathcal{A}_c^{\text{Eis},*}(n_1,n_2;\mathscr{L}^+\mathscr{K}_s h_T) + \mathcal{A}_c^{\text{hol}}(n_1,n_2;\mathscr{L}^{\text{hol}}\mathscr{K}_s h_T)) \frac{ds}{2\pi i} \right|,$$
$$(4.9)$$

where $\mathcal{A}_c^{\text{Eis},*}$ denotes the contribution of level-$c$ Eisenstein series without the trivial character.

## 4.4 The endgame

We consider the Maaß contribution in (4.9) given by

$$\int_{(-12)} Q^{s+1}\widehat{W}(s) \sum_c c^s \sum_{c_0 M | c} \sum_{\psi \in \mathcal{B}^*(c_0)} \sum_{n_1,n_2,m_1,m_2} \frac{A(n_1,m_1)A(n_2,m_2)\rho_{\psi,M,N}(n)\overline{\rho_{\psi,M,N}(m)}}{(n_1 m_1^2)^{1/2+iv}(n_2 m_2^2)^{1/2-iv}(n_1 n_2)^{s/2}}$$
$$\times V\left(\frac{n_1 m_1^2}{N}\right)\overline{V\left(\frac{n_2 m_2^2}{N}\right)} \mathscr{L}^+\mathscr{K}_s h_{\text{T}}(t_\psi) \frac{ds}{2\pi i}.$$

Shifting the $s$-contour to the far left, we see that we can truncate both the $c$-sum and the $\psi$-sum at $c(1+|t_\psi|^2) \leqslant P^\varepsilon NT/Q$ at the cost of a negligible error (recall (4.8) and the rapid decay of $\widehat{V}$). Having done this, we shift the $s$-contour back to $\Re s = 0$. By Mellin inversion, we obtain

$$\int_{(0)} Q^{s+1}\widehat{W}(s) \sum_c \sum_{\substack{c_0 | c \\ c(1+|t_\psi|^2) \leqslant P^\varepsilon NT/Q}} \sum_{\psi \in \mathcal{B}^*(c_0)} \frac{c^s}{c\nu(c)} \prod_{p|c_0}(1-p^{-2}) \int_{(\varepsilon)} \int_{(\varepsilon)} N^{z_1+z_2}\widehat{V}(z_1)\widehat{V}(z_2)$$
$$\times \frac{\mathcal{D}_{\psi,c}^{\text{Maaß}}(1/2 + iv + s/2 + z_1, 1/2 - iv + s/2 + z_2, -s/2, -s/2)}{L(1, \text{Ad}^2\psi)}$$
$$\times \mathscr{L}^+\mathscr{K}_s h_{\text{T}}(t_\psi) \frac{dz_1\, dz_2}{(2\pi i)^2} \frac{ds}{2\pi i},$$
$$(4.10)$$

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where (recalling the notation in (2.9))

$$
\begin{aligned}
&\mathcal{D}_{\psi,c}^{\mathrm{Maa\beta}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{w}_1, \mathbf{w}_2) \\
&= \frac{L(1, \mathrm{Ad}^2\psi)c\nu(c)}{\prod_{p|c_0}(1 - p^{-2})} \sum_{M|(c/c_0)} \sum_{n_1,n_2,m_1,m_2} \frac{A(n_1,m_1)A(n_2,m_2)\rho_{\psi,M,c}(n)\overline{\rho_{\psi,M,c}(m)}}{(n_1 m_1^2)^{\mathbf{z}_1}(n_2 m_2^2)^{\mathbf{z}_2} m_1^{2\mathbf{w}_1} m_2^{2\mathbf{w}_2}}
\end{aligned}
$$

for $\psi \in \mathcal{B}^*(c_0)$ with $c_0 \mid c$. Using (2.9) and (2.11) (with $\theta \leqslant 1/2$), we see as in (4.3) that

$$
\mathcal{D}_{\psi,c}^{\mathrm{Maa\beta}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{w}_1, \mathbf{w}_2) = L(\mathbf{z}_1, \psi)^3 L(\mathbf{z}_2, \psi)^3 \mathcal{E}_{\psi,c}^{\mathrm{Maa\beta}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{w}_1, \mathbf{w}_2), \tag{4.11}
$$

where

$$
\mathcal{E}_{\psi,c}^{\mathrm{Maa\beta}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{w}_1, \mathbf{w}_2) \ll_\varepsilon c^\varepsilon
$$

uniformly in $\Re\mathbf{z}_1, \Re\mathbf{z}_2, \Re(\mathbf{z}_1 + \mathbf{w}_1), \Re(\mathbf{z}_2 + \mathbf{w}_2) \geqslant 1/2 + \varepsilon$. The convexity bound for $L(\mathbf{z}, \psi)$ is

$$
L(\mathbf{z}, \psi) \ll_\varepsilon (c_0(1 + |t_\psi| + |\Im\mathbf{z}|)^2)^{1/4+\varepsilon}, \quad \Re\mathbf{z} \geqslant 1/2.
$$

We can afford to use the convexity bound on four of the six $L$-functions in (4.11). We may then truncate the $s, z_1, z_2$-contours at height $P^\varepsilon$, and after a trivial estimation, we bound (4.10) by

$$
\ll_\varepsilon P^\varepsilon Q \frac{NT}{Q} \max_{|\xi| \leqslant P^\varepsilon} \sum_{\substack{c \\ c(1+|t_\psi|^2) \leqslant P^\varepsilon NT/Q}} \sum_{\psi \in \mathcal{B}^*(c)} \frac{1}{c} \left| L\left(\frac{1}{2} + \varepsilon + i\xi, \psi\right) \right|^2 \frac{T^2}{(1 + |t_\psi|^2)}. \tag{4.12}
$$

It is an easy exercise with the Kuznetsov formula or the spectral large sieve to obtain a Lindelöf on average bound for the second moment, which can safely be left to the reader: the length of the approximate functional equation in each factor is $O_\varepsilon(P^\varepsilon c^{1/2}(1 + |t_\psi|))$, so the Kloosterman term in the Kuznetsov formula is essentially invisible. Thus by Weyl's law, the total contribution of the previous expression is

$$
\ll_\varepsilon P^\varepsilon N^2 T^2 Q^{-1} \ll_\varepsilon P^\varepsilon Q^2 T^8
$$

for $N \leqslant Q^{3/2} T^3 P^\varepsilon$, and this majorizes all preceding error terms.

The contribution of $\mathcal{A}_c^{\mathrm{hol}}(n_1, n_2; \mathscr{L}^{\mathrm{hol}} \mathscr{K}_s h_T)$ can be bounded in same way using the analogous bound for $\mathscr{L}^{\mathrm{hol}} \mathscr{K}_s h_T$ in Lemma 2(b).

Finally, for the contribution $\mathcal{A}_c^{\mathrm{Eis},*}(n_1, n_2; \mathscr{L}^+ \mathscr{K}_s h_T)$, we observe that after removing the trivial character, the analogously defined function

$$
\begin{aligned}
&\mathcal{D}_{(\chi,t),c}^{\mathrm{Eis}}(\mathbf{z}_1, \mathbf{z}_2, \mathbf{w}_1, \mathbf{w}_2) \\
&= |L(1 + 2it, \chi^2)|^2 c\nu(c) \sum_{c_\chi^2|M|c} \sum_{n_1,n_2,m_1,m_2} \frac{A(n_1,m_1)A(n_2,m_2)\rho_{\chi,M,c}(n)\overline{\rho_{\chi,M,c}(m)}}{(n_1 m_1^2)^{\mathbf{z}_1}(n_2 m_2^2)^{\mathbf{z}_2} m_1^{2\mathbf{w}_1} m_2^{2\mathbf{w}_2}}
\end{aligned}
$$

is pole-free in $\Re\mathbf{z}_1, \Re\mathbf{z}_2, \Re(\mathbf{z}_1 + \mathbf{w}_1), \Re(\mathbf{z}_2 + \mathbf{w}_2) \geqslant 1/2 + \varepsilon$ since $\chi$ is primitive of conductor $> 1$, and it can be approximated by $L(\mathbf{z}_1+it, \chi)^3 L(\mathbf{z}_1-it, \overline{\chi})^3 L(\mathbf{z}_2+it, \chi)^3 L(\mathbf{z}_2-it, \overline{\chi})^3$ in this region up to a holomorphic factor bounded by $O_\varepsilon(c^\varepsilon)$. Here we can even afford to apply the convexity bound for all 12 Dirichlet $L$-functions. The quantity corresponding to (4.12) is then

$$
P^\varepsilon Q \left(\frac{NT}{Q}\right)^{3/2} \sum_c \int_{c(1+|t|)^2 \leqslant P^\varepsilon NT/Q} \frac{\#\{\chi : c_\chi^2 \mid c\}T^2}{c(1 + |t|^2)} \, dt \ll_\varepsilon P^\varepsilon Q \left(\frac{NT}{Q}\right)^{3/2} \ll_\varepsilon P^\varepsilon Q^{7/4} T^6.
$$

This completes the proof of Theorem 2.

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## 5. Applications

It is now an easy task to prove Corollary 3 and Theorem 4. For both applications, we need the following auxiliary result.

Lemma 3. *Let* $T \geqslant 1$, $N, q \in \mathbb{N}$, $(N, q) = 1$. *Then*

$$\sum_{\psi \in \mathcal{B}^*(N)} \frac{\lambda_\psi(q)^2}{L(1, \mathrm{Ad}^2 \psi)} e^{-(t_\psi/T)^2} \ll_\varepsilon (NTq)^\varepsilon (T^2 N + q^{1/2}).$$

This is a simple application of the Kuznetsov formula and Weil's bounds for Kloosterman sums; see, for example, [BM15, Lemma 12] or its ancestor [Mot97, Lemma 2.4].

### 5.1 Proof of Corollary 3

From [BK19b, § 12.1], we quote

$$\sum_{\psi \in \mathcal{B}^*(q)} L(1/2, \psi)^5 e^{-t_\psi^2} \ll_\varepsilon q^\varepsilon \max_{|\tau| \leqslant (\log q)^2} \sum_{\ell \leqslant q^{1/2+\varepsilon}} \frac{1}{\ell^{1/2}} \left| \sum_{f \in \mathcal{B}^*(q)} \frac{L(1/2, f)^4}{L(1, \mathrm{Ad}^2 f)} \lambda_f(\ell) h_\tau(t_f) \right| + q^{-10}, \quad (5.1)$$

where

$$h_\tau(t_f) = \frac{L_\infty(1/2 + \varepsilon + i\tau, f)}{L_\infty(1/2, f)} \frac{G_f(\varepsilon + i\tau)}{G_f(0)} e^{-t_f^2} (1 + |t_f|)^\varepsilon$$

with

$$G_f(s) = \prod_{j=0}^{1000} \prod_{\epsilon_1, \epsilon_2 \in \{\pm 1\}} \left( \frac{1}{2} + \epsilon_1 s + i\epsilon_2 t_f + j \right).$$

This is an application of a carefully designed approximate functional equation. Now the formula two displays below [BK19b, (11.4)], together with [BK19b, Lemma 1], shows that for $q$ prime,

$$\frac{\phi(q)}{q^2} \sum_{f \in \mathcal{B}^*(q)} \frac{L(1/2, f)^4}{L(1, \mathrm{Ad}^2 f)} \lambda_f(\ell) h_\tau(t_f) = \sum_{ab=\ell} \mathcal{M}_{q,a}^+(1/2, 1/2, \mathfrak{h}_\tau) \left( \frac{a}{b} \right)^{1/2} + O(\ell^{\theta+\varepsilon} q^{-1})$$

with $\mathfrak{h}_\tau = (h_\tau, 0)$ in the notation of [BK19b, (1.3), (1.7)]. Here the error term also includes the oldforms of level 1. On the other hand, [BK19b, (11.4)] states that

$$\sum_{ab=\ell} \mathcal{M}_{q,a}^+(1/2, 1/2, \mathfrak{h}_\tau) \left( \frac{a}{b} \right)^{1/2} \ll_\varepsilon (\ell q)^\varepsilon \sum_{ab=\ell} \left( \frac{a}{b} \right)^{1/2} \left( \frac{1}{a} + \frac{1}{q} + \sum_\pm \left| \mathcal{M}_{a,q}^\pm(1/2, 1/2; \mathcal{T}_{1/2,1/2}^\pm \mathfrak{h}_\tau) \right| \right)$$

uniformly in $|\tau| \leqslant (\log q)^2$, and the analysis of [BK19b, § 11] shows

$$\mathcal{M}_{a,q}^\pm(1/2, 1/2; \mathcal{T}_{1/2,1/2}^\pm \mathfrak{h}_\tau) \ll_\varepsilon (aq)^\varepsilon \left( \frac{1}{q^{1/2}} + \frac{1}{aq^{1/2}} \sum_{a_0|a} \sum_{f \in \mathcal{B}^*(a_0)} \frac{|L(1/2, f)|^4}{L(1, \mathrm{Ad}^2 f)} \frac{|\Lambda_f(q, 1/2)|}{(1 + |t_f|)^{15}} \right)$$

(again uniformly in $|\tau| \leqslant (\log q)^2$) where $\Lambda_f(q, 1/2) := \lambda_f(q) - q^{-1/2}$ for $q$ prime. Combining these estimates, we obtain

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$$\frac{\phi(q)}{q^2} \sum_{f\in\mathcal{B}^*(q)} \frac{L(1/2,f)^4}{L(1,\mathrm{Ad}^2 f)} \lambda_f(\ell) h_\tau(t_f)$$

$$\ll_\varepsilon (\ell q)^\varepsilon \left( \frac{1}{\ell^{1/2}} + \frac{\ell^{1/2}}{q^{1/2}} + \frac{1}{(\ell q)^{1/2}} \sum_{a_0|\ell} \sum_{f\in\mathcal{B}^*(a_0)} \frac{|L(1/2,f)|^4}{L(1,\mathrm{Ad}^2\psi)} \frac{(1+|\lambda_f(q)|)}{(1+|t_f|)^{15}} \right).$$

Substituting back into (5.1), this yields

$$\sum_{\psi\in\mathcal{B}^*(q)} L(1/2,\psi)^5 e^{-t_\psi^2} \ll_\varepsilon q^{1+\varepsilon} \sum_{\ell\leqslant q^{1/2+\varepsilon}} \left( \frac{1}{\ell} + \frac{1}{q^{1/2}} + \frac{1}{\ell q^{1/2}} \sum_{a|\ell} \sum_{f\in\mathcal{B}^*(a)} \frac{|L(1/2,f)|^4}{L(1,\mathrm{Ad}^2\psi)} \frac{(1+|\lambda_f(q)|)}{(1+|t_f|)^{15}} \right)$$

$$\ll_\varepsilon q^{1+\varepsilon} + q^{1/2+\varepsilon} \sum_{a\leqslant q^{1/2+\varepsilon}} \frac{1}{a} \sum_{f\in\mathcal{B}^*(a)} \frac{|L(1/2,f)|^4}{L(1,\mathrm{Ad}^2\psi)} \frac{(1+|\lambda_f(q)|)}{(1+|t_f|)^{15}}$$

$$\ll_\varepsilon q^{1+\varepsilon} + q^{1/2+\varepsilon} \max_{A\leqslant q^{1/2+\varepsilon}} \frac{1}{A} \sum_{a\asymp A} \sum_{f\in\mathcal{B}^*(a)} \frac{|L(1/2,f)|^4}{L(1,\mathrm{Ad}^2\psi)} \frac{(1+|\lambda_f(q)|)}{(1+|t_f|)^{15}}.$$

So far this is essentially a restatement of the analysis in [BK19b], but now we insert an additional application of Hölder's inequality. In this way, we obtain

$$\sum_{\psi\in\mathcal{B}^*(q)} L(1/2,\psi)^5 e^{-t_\psi^2}$$

$$\ll_\varepsilon q^{1+\varepsilon} + q^{1/2+\varepsilon} \max_{A\leqslant q^{1/2+\varepsilon}} \frac{1}{A} \left( \sum_{a\asymp A} \sum_{f\in\mathcal{B}^*(a)} \frac{|L(1/2,f)|^6}{L(1,\mathrm{Ad}^2\psi)(1+|t_f|)^{15}} \right)^{2/3}$$

$$\times \left( \sum_{a\asymp A} \sum_{f\in\mathcal{B}^*(a)} \frac{(1+|\lambda_f(q)|)^3}{L(1,\mathrm{Ad}^2\psi)(1+|t_f|)^{15}} \right)^{1/3}.$$

By Theorem 2 and Lemma 3, we obtain

$$\sum_{\psi\in\mathcal{B}^*(q)} L(1/2,\psi)^5 e^{-t_\psi^2} \ll_\varepsilon q^\varepsilon \left( q + q^{1/2} \max_{A\leqslant q^{1/2+\varepsilon}} \frac{1}{A} A^{4/3}(A^2+q^{1/2})^{1/3} q^{\theta/3} \right) \ll_\varepsilon q^{1+\theta/3+\varepsilon}.$$

### 5.2 Proof of Theorem 4

By a dyadic decomposition, we can replace the summation condition $m\leqslant M$ by $m\asymp M$. Let us also assume without loss of generality that $\|\mathbf{a}\|_\infty \leqslant 1$. In order to apply Theorem 1, we would like to bound $L(1/2,\chi)$ by a small integral over the imaginary axis. This can be done by a standard argument based on the functional equation and the residue theorem (which seems to have been first applied by Heath-Brown [Hea78, Lemma 3]) as follows. Fix $0<\varepsilon<1/10$ and suppose that $\chi$ is a primitive character modulo $q$. We have

$$L(1/2,\chi)^4 = \int_{(\varepsilon)} L(1/2+s,\chi)^4 \frac{e^{2s^2}}{s} \frac{ds}{2\pi i} + \int_{(\varepsilon)} L(1/2-s,\chi)^4 \frac{e^{2s^2}}{s} \frac{ds}{2\pi i}$$

$$= \int_{(\varepsilon)} L(1/2+s,\chi)^4 f(s) \frac{ds}{2\pi i},$$

where

$$f(s) = \left( 1 + \frac{\Gamma(\frac{1}{2}(\frac{1}{2}+s+\mathfrak{a}))^4}{\Gamma(\frac{1}{2}(\frac{1}{2}-s+\mathfrak{a}))^4} \left( \frac{q}{\pi} \right)^{4s} \right) \frac{e^{2s^2}}{s}$$

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with $\mathfrak{a} = 0$ if $\chi$ is even and $\mathfrak{a} = 1$ if $\chi$ is odd. Applying the same argument again, we have

$$L(1/2 + s, \chi)^4 = \int_{(-\varepsilon)} L(1/2 + s + u, \chi)^4 g_s(u) \frac{ds}{2\pi i},$$

where

$$g_s(u) = -\left( 1 + \frac{\Gamma(\frac{1}{2}(\frac{1}{2} + s + u + \mathfrak{a}))^4}{\Gamma(\frac{1}{2}(\frac{1}{2} + s - u + \mathfrak{a}))^4} \left( \frac{q}{\pi} \right)^{4u} \right) \frac{e^{2u^2}}{u}.$$

Inserting and changing variables, we obtain

$$L(1/2, \chi)^4 = \int_{(0)} L(1/2 + v, \chi)^4 h(v) \frac{dv}{2\pi i},$$

where

$$h(v) = \int_{(\varepsilon)} g_s(v - s) f(s) \frac{ds}{2\pi i} \ll e^{-|v|^2} q^{4\varepsilon}.$$

Now choosing $F$ as in (1.7), we get

$$|L(1/2, \chi)|^4 \ll_\varepsilon q^\varepsilon \int_{(0)} L(1/2 + z, \chi)^2 L(1/2 - z, \overline{\chi})^2 F(z) \frac{dz}{2\pi i}.$$

Opening the square, we have

$$\sum_{\chi \,(\mathrm{mod}\ q)} \left| \sum_{m \asymp M} a(m) \chi(m) \right|^2 |L(1/2, \chi)|^4 \ll_\varepsilon M^2 + q^\varepsilon \sum_d \sum_{\substack{m_1, m_2 \asymp M/d \\ (m_1, m_2) = 1}} |\mathcal{T}_{m_1, m_2, q}(1/2, 1/2, 1/2)|$$

with the notation as in (1.6) in the special case where $f$ is the standard Eisenstein series with $\theta = 0$. By Theorem 1 and (1.8) we have

$$\sum_{\chi \,(\mathrm{mod}\ q)} \left| \sum_{m \asymp M} a(m) \chi(m) \right|^2 |L(1/2, \chi)|^4$$
$$\ll_\varepsilon M^2 + (Mq)^{1+\varepsilon} + q^\varepsilon \sum_d \sum_{\substack{m_1, m_2 \asymp M/d \\ (m_1, m_2) = 1}} \sum_{* \in \{\mathrm{Maa\ss}, \mathrm{hol}, \mathrm{Eis}\}} |\mathcal{M}^*_{m_1, m_2, q}(1/2, 1/2, 1/2)|.$$

We only deal with the Maaß case; the other two cases are similar but easier. By (1.10) and (1.11), we have

$$\sum_d \sum_{\substack{m_1, m_2 \asymp M/d \\ (m_1, m_2) = 1}} |\mathcal{M}^{\mathrm{Maa\ss}}_{m_1, m_2, q}(1/2, 1/2, 1/2)|$$
$$\ll_\varepsilon (qM)^\varepsilon \sum_d \sum_{\substack{m_1, m_2 \asymp M/d \\ (m_1, m_2) = 1}} \sum_{N | m_1 m_2} \sum_{\psi \in \mathcal{B}^*(N)} \frac{q^{1/2}}{N^{1/2}} \frac{(1 + |\lambda_\psi(q)|)}{(1 + |t_\psi|)^{30}} \frac{L(1/2, \psi)^3}{L(1, \mathrm{Ad}^2 \psi)}.$$

We drop the condition $(m_1, m_2) = 1$ and write $m_1 m_2 = m = NK$, obtaining by the standard divisor bound that the previous display is bounded by

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$$\ll_\varepsilon (qM)^\varepsilon \sum_d \sum_{NK \asymp M^2/d^2} \frac{q^{1/2}}{N^{1/2}} \sum_{\psi \in \mathcal{B}^*(N)} \frac{(1+|\lambda_\psi(q)|)}{(1+|t_\psi|)^{30}} \frac{L(1/2,\psi)^3}{L(1,\mathrm{Ad}^2\psi)}$$

$$\ll_\varepsilon (qM)^\varepsilon \sum_d \sum_{N \ll M^2/d^2} \frac{q^{1/2}M^2}{d^2 N^{3/2}} \sum_{\psi \in \mathcal{B}^*(N)} \frac{(1+|\lambda_\psi(q)|)}{(1+|t_\psi|)^{30}} \frac{L(1/2,\psi)^3}{L(1,\mathrm{Ad}^2\psi)}$$

$$\ll_\varepsilon (q^{1/2}M^2)^{1+\varepsilon} \max_{\mathcal{N} \ll M^2} \frac{1}{\mathcal{N}^{3/2}} \sum_{N \asymp \mathcal{N}} \sum_{\psi \in \mathcal{B}^*(N)} \frac{(1+|\lambda_\psi(q)|)}{(1+|t_\psi|)^{30}} \frac{L(1/2,\psi)^3}{L(1,\mathrm{Ad}^2\psi)}.$$

By the Cauchy–Schwarz inequality, Theorem 2, and Lemma 3, this is

$$\ll_\varepsilon (q^{1/2}M^2)^{1+\varepsilon} \max_{\mathcal{N} \ll M^2} \frac{1}{\mathcal{N}^{3/2}}(\mathcal{N}+q^{1/4}\mathcal{N}^{1/2})\mathcal{N} \ll_\varepsilon (q^{1/2}M^2)^{1+\varepsilon}(M+q^{1/4}).$$

For $M \leqslant q^{1/4}$, we obtain altogether

$$\sum_{\chi \,(\mathrm{mod}\, q)} \left| \sum_{m \asymp M} a(m)\chi(m) \right|^2 |L(1/2,\chi)|^4 \ll_\varepsilon (Mq)^{1+\varepsilon},$$

as desired.

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