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Conjugacy of half-linear second-order differential equations

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Focal point and conjugacy criteria for the half-linear second-order differential equation

$$(|y'|^{p-1}\operatorname{sgn} y')' + c(t)|y|^{p-1}\operatorname{sgn} y = 0, \qquad p > 1$$

are obtained using the generalized Riccati transformation. An oscillation criterion is given in case when the function $c(t)$ is periodic.

1. Introduction

In this paper we investigate oscillatory properties of the half-linear second-order differential equation

$$[\Phi(y')]' + c(t)\Phi(y) = 0, \tag{1.1}$$

where $c : \mathbb{R} \to \mathbb{R}$ is a piecewise continuous function and $\Phi(s) := |s|^{p-1}\operatorname{sgn} s = |s|^{p-2}s$ with $p > 1$. Another notation for the function $\Phi(s)$ is $s^{(p-1)*}$ (see [3]).

Considerable effort has been made over the years to extend oscillation theory of the linear equation

$$y'' + c(t)y = 0 \tag{1.2}$$

(which is the special case $p = 2$ of (1.1)) to half-linear equation (1.1), see [4, 6–10, 18, 20]. This was motivated, among others, by the study of nonlinear boundary value problems associated with the equation

$$[\Phi(y')]' + f(t, y) = 0, \tag{1.3}$$

see [12, 13, 19] and references therein. Roughly speaking, the more we know about the half-linear equation (1.1), the more we may state about the nonlinear boundary value problems associated with (1.3).

The aim of this paper is to investigate conditions on the function $c(t)$ in (1.1) which guarantee that this equation possesses a non-trivial solution with at least

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two zeros in $\mathbb{R}$, the so-called *conjugacy criteria*. Using a suitable transformation, we extend our results to the more general half-linear equation

$$[r(t)\Phi(y')]' + c(t)\Phi(y) = 0, \tag{1.4}$$

where $r : \mathbb{R} \to \mathbb{R}$ is a positive function.

Concerning the linear case ($p = 2$), conjugacy criteria for (1.2) and the more general equation

$$(r(t)y')' + c(t)y = 0 \tag{1.5}$$

have been investigated in several papers (see [1, 2, 11, 15, 16, 20]). In [21] Tipler proved that under the condition $\int_{-\infty}^{\infty} c(t)\,\mathrm{d}t > 0$, linear differential equation (1.2) is conjugate in $\mathbb{R}$. Recently, Peña [17] has shown that the same condition is also sufficient for conjugacy in $\mathbb{R}$ of half-linear equation (1.1). Moreover, he proved a more general criterion, namely that (1.4) is conjugate in an interval $I = (a, b)$ provided

$$\int_a r^{1/(1-p)}(t)\,\mathrm{d}t = \infty = \int^b r^{1/(1-p)}(t)\,\mathrm{d}t \quad \text{and} \quad \int_a^b c(t)\,\mathrm{d}t > 0.$$

This criterion reduces to the result of Müller-Pfeiffer [16] in case $p = 2$ which was proved using the variational method. In [1,2] the first named author has proved that (1.5) is conjugate in (a, b) provided there exist constants $\varepsilon_1, \varepsilon_2 > 0$ and $t_0 \in (a, b)$ such that

$$\varepsilon_1 \int_{t_0}^b r^{-1}(x) \exp\biggl\{2\int_{t_0}^x r^{-1}(t)\biggl[\int_{t_0}^t c(s)\,\mathrm{d}s - \varepsilon_1\biggr]\mathrm{d}t\biggr\}\,\mathrm{d}x > \frac{\pi}{2}, \tag{1.6$_1$}$$

$$\varepsilon_2 \int_a^{t_0} r^{-1}(x) \exp\biggl\{2\int_{t_0}^x r^{-1}(t)\biggl[\int_{t_0}^t p(s)\,\mathrm{d}s + \varepsilon_2\biggr]\mathrm{d}t\biggr\}\,\mathrm{d}x > \frac{\pi}{2} \tag{1.6$_2$}$$

and as a consequence of this criterion, differential equation (1.2) is conjugate in $\mathbb{R}$ provided

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t (t - s)c(s)\,\mathrm{d}s > 0, \qquad \lim_{t\to-\infty} \frac{1}{t}\int_t^0 (t - s)c(s)\,\mathrm{d}s > 0. \tag{1.7}$$

Note that both (1.6) and (1.7) are actually focal point criteria. Indeed, if $y(t)$ is a solution of (1.5) satisfying the initial conditions $y(t_0) = 1, y'(t_0) = 0$, condition (1.6$_1$) implies that this solution has a zero (so-called *right focal point* of $t = t_0$) in (t_0, b) and (1.6$_2$) implies that this solution has a zero in (a, t_0) (so-called *left focal point* of $t = t_0$). Conjugacy and focal point criteria for (1.2) of different a kind than those given by (1.6), (1.7) are established in [11].

Our paper is organized as follows. In the next section we recall basic properties of solutions of (1.4) and we also formulate the main results of the paper: focal point and conjugacy criteria for (1.1) which extend a recently established results of Peña [17]. These criteria seem to give new results even in the linear case $p = 2$. In the third section we investigate conjugacy (and hence oscillation) of (1.1) with a periodic coefficient $c(t)$ and the last section is devoted to some comments and remarks concerning statements of the previous sections and their possible extensions.

2. Focal point and conjugacy criteria

The transformation of the variables

$$x = \int_0^t [r(s)]^{1-q}\,\mathrm{d}s, \qquad z(x) = y(t), \qquad \frac{1}{p} + \frac{1}{q} = 1 \tag{2.1}$$

transforms (1.4) into the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[\Phi\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)\right] + C(x)\Phi(z) = 0,$$

where $C(x) = [r(t)]^{q-1}c(t)$ and $t = t(x)$ is the inverse function of $x = x(t)$ given by (2.1). Clearly, the new differential equation obtained by transformation (2.1) is again of the form (1.1) hence we may restrict ourselves to the differential equations (1.1).

As basic references concerning qualitative properties of (1.1) or (1.4) are usually regarded the papers [3] and [14]. In these papers, the existence, uniqueness and continuabilty up to infinity of a solution of (1.1) is established (see [3, p. 161]) and Sturmian-type statements for zeros of solutions of half-linear differential equations are proved [14, theorem 1]. In particular, it was shown that for any $y_0, y_1 \in \mathbb{R}$ with $y_0^2 + y_1^2 > 0$ there exists unique solution of (1.1) satisfying the initial conditions $y(t_0) = y_0$, $y'(t_0) = y_1$ for any $t_0 \in \mathbb{R}$. Moreover, it was proved that if t_1 and t_2 are two consecutive zeros of a non-trivial solution $y(t)$ of (1.1), then any other solution, which is not a constant multiple of $y(t)$, has exactly one zero in (t_1, t_2). An important consequence of this fact is that every solution of (1.1) is either oscillatory or non-oscillatory.

Let $y(t)$ be a solution of (1.4) such that $y(t) \neq 0$ on some interval (a, b) and let

$$w(t) = -\frac{r(t)\Phi(y'(t))}{\Phi(y(t))}, \tag{2.2}$$

then $w(t)$ is a solution of the generalized Riccati equation

$$w' = c(t) + (p-1)r^{1-q}(t)|w|^q, \tag{2.3}$$

where q is the conjugate number of p, i.e. the same as in (2.1). This relation between solutions of (1.4) and (2.3) is a very useful tool for investigation of oscillatory properties of half-linear equations (see, for example, [4]).

A crucial role in our investigation of conjugacy of (1.1) is played by the following focal point criterion. This result seems to be new even in the linear case $p = 2$ (compare (1.7)).

THEOREM 2.1. *Suppose that the function* $c(t) \not\equiv 0$ *in* $(0, \infty)$ *and there exist constants* $\alpha \in (-1/p, p-2]$ *and* $T \geqslant 0$ *such that*

$$\int_0^t s^\alpha \left(\int_0^s c(\tau)\,\mathrm{d}\tau\right) \mathrm{d}s \geqslant 0 \quad \textit{for } t \geqslant T. \tag{2.4}$$

Then the solution $y(t)$ *of (1.1) satisfying the initial conditions* $y(0) = 1$, $y'(0) \leqslant 0$ *has a zero in* $(0, \infty)$.

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Clearly, in theorem 2.1 the starting point $t_0 = 0$ can be shifted to any other value $t_0 \in \mathbb{R}$ if the condition (2.4) would be modified to

$$\int_{t_0}^{t} (s - t_0)^{\alpha} \left(\int_{t_0}^{s} c(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s \geqslant 0 \quad \text{for } t \geqslant T \geqslant t_0.$$

A similar statement can be formulated on the interval $(-\infty, t_0)$, too.

Proof. Suppose the contrary. Then the solution $y(t)$ of (1.1) has no zero on $(0, \infty)$, i.e. $y(t) > 0$. Let the function $w(t)$ be defined by (2.2) with $r(t) \equiv 1$. Since $w(0) \geqslant 0$, we have by (2.3)

$$\begin{aligned} w(t) &= w(0) + \int_0^t c(s)\, \mathrm{d}s + (p-1) \int_0^t |w(s)|^q\, \mathrm{d}s \\ &\geqslant \int_0^t c(s)\, \mathrm{d}s + (p-1) \int_0^t |w(s)|^q\, \mathrm{d}s \end{aligned} \tag{2.5}$$

and

$$\int_0^t s^{\alpha} w(s)\, \mathrm{d}s \geqslant \int_0^t \left(s^{\alpha} \int_0^s c(\tau)\, \mathrm{d}\tau \right) \mathrm{d}s + S(t),$$

where

$$S(t) = (p-1) \int_0^t s^{\alpha} \left(\int_0^s |w(\tau)|^q\, \mathrm{d}\tau \right) \mathrm{d}s.$$

Then

$$S'(t) = (p-1) t^{\alpha} \int_0^t |w(\tau)|^q\, \mathrm{d}\tau \geqslant 0 \quad \text{for } t \geqslant 0 \tag{2.6}$$

and according to (2.4)

$$S(t) \leqslant \int_0^t s^{\alpha} w(s)\, \mathrm{d}s \qquad \text{for } t \geqslant T.$$

By Hölder inequality we have

$$\begin{aligned} \int_0^t s^{\alpha} w(s)\, \mathrm{d}s &\leqslant \left[\int_0^t s^{p\alpha}\, \mathrm{d}s \right]^{1/p} \left[\int_0^t |w(s)|^q\, \mathrm{d}s \right]^{1/q} \\ &= \left[\frac{t^{1+p\alpha}}{1+p\alpha} \right]^{1/p} \left[\int_0^t |w(s)|^q\, \mathrm{d}s \right]^{1/q} \end{aligned}$$

hence

$$\frac{t^{(1+p\alpha)(q/p)}}{(1+p\alpha)(q/p)} \int_0^t |w(s)|^q\, \mathrm{d}s \geqslant S^q(t).$$

Here we need the relation $S(t) > 0$ for sufficiently large t. By (2.6) $S(t)$ is non-decreasing function of t and $S(0) = 0$. The relation $S(t) = 0$ for all $t \geqslant 0$ would imply that $w(t) \equiv 0$, consequently by (2.2) $y'(t) \equiv 0$ for $t \geqslant 0$. But this may happen (see the corresponding lines in [5, p. 495]) only if $c(t) \equiv 0$, which case has been excluded. Hence we may suppose that T is already chosen so large that the inequality $S(t) > 0$ holds for $t \geqslant T$.

Denote $\beta = \alpha - (1 + p\alpha)(q/p)$ and $K = (p-1)(1+p\alpha)(q/p) > 0$. Then by (2.6) the last inequality yields $S'S^{-q} \geqslant Kt^{\beta}$. Integrating this inequality from T to t, we get

$$\frac{1}{q-1} S^{1-q}(T) > \frac{1}{q-1}[S^{1-q}(T) - S^{1-q}(t)] \geqslant K \int_T^t s^{\beta}\,\mathrm{d}s,$$

where the integral on the right-hand side tends to ∞ as $t \to \infty$ because easy computation shows that $\alpha \leqslant p-2$ implies $\beta \geqslant -1$. This contradiction proves that $y(t)$ must have a positive zero. □

Using the just established focal point criterion, we can prove the following conjugacy criterion for (1.1).

THEOREM 2.2. *Suppose that $c(t) \not\equiv 0$ both in $(-\infty, 0)$ and $(0, \infty)$ and there exist constants $\alpha_1, \alpha_2 \in (-1/p, p-2]$ and $T_1, T_2 \in \mathbb{R}$, $T_1 < 0 < T_2$, such that*

$$\left.\begin{aligned} \int_t^0 |s|^{\alpha_1} \int_s^0 c(\tau)\,\mathrm{d}\tau \geqslant 0, \quad t \leqslant T_1, \\ \int_0^t s^{\alpha_2} \int_0^s c(\tau)\,\mathrm{d}\tau \geqslant 0, \quad t \geqslant T_2. \end{aligned}\right\} \tag{2.7}$$

Then differential equation (1.1) is conjugate in $\mathbb{R}$, more precisely, there exists a solution of (1.1) having at least one positive and one negative zero.

Proof. The statement follows immediately from theorem 2.1 since by this theorem the solution $y(t)$ given by $y(0) = 1$, $y'(0) = 0$ has a positive zero. Using the same argument as in theorem 2.1 and the second condition in (2.7) we can show the existence of a negative zero. □

3. Equations with periodic coefficient

THEOREM 3.1. *If the coefficient $c(t)$ in (1.1) is a periodic function with the period ω, $c(t) \not\equiv 0$, and*

$$\int_0^{\omega} c(t)\,\mathrm{d}t \geqslant 0,$$

then (1.1) is oscillatory both at $t = \infty$ and at $t = -\infty$.

Proof. To prove oscillation of (1.1) it is sufficient to find a solution of this equation with at least two zeros. Indeed, periodicity of the function c implies that if $y(t)$ is a solution of (1.1), then $y(t \pm \omega)$ is a solution as well and hence any solution with two zeros has actually infinitely many of them, tending both to ∞ and $-\infty$.

Theorem 3.1 is clearly true if $c(t)$ is a positive constant function. So we have to consider the cases when $c(t)$ is not constant. Also it is sufficient to deal with the cases when $\int_0^{\omega} c(t)\,\mathrm{d}t = 0$ because otherwise we can define

$$c_0 = \frac{1}{\omega} \int_0^{\omega} c(t)\,\mathrm{d}t > 0 \quad \text{and} \quad \tilde{c}(t) = c(t) - c_0.$$

Clearly, we have $c(t) > \tilde{c}(t)$. If we prove (1.1) to be oscillatory with $\tilde{c}(t)$, then by the Sturm theory the differential equation (1.1) with $c(t)$ is also oscillatory.

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Now let

$$C(t)=\int_0^t c(s)\,\mathrm{d}s.$$

This is a continuous periodic function with period ω. Let γ and δ be defined by

$$C(\delta)=\max_{0\leqslant t\leqslant\omega} C(t),\qquad C(\gamma)=\min_{\delta\leqslant t\leqslant\delta+\omega} C(t).$$

Then $0\leqslant\delta<\gamma<\delta+\omega$ and

$$\int_\gamma^t c(s)\,\mathrm{d}s\geqslant 0,\qquad \int_t^\delta c(s)\,\mathrm{d}s\geqslant 0\quad\text{for } t\in\mathbb{R}.$$

Now, by theorem 2.2 and the remark given below theorem 2.1, the solution of (1.1) given by the initial condition $y(\delta)=1$, $y'(\delta)=0$ has a zero in (∞,δ). Indeed, $C(t)\not\equiv 0$ and

$$\int_t^\delta |s-\delta|^\alpha\biggl(\int_t^\delta c(\tau)\,\mathrm{d}\tau\biggr)\,\mathrm{d}s\leqslant 0\quad\text{for } t\leqslant\delta,$$

with any $\alpha\in(-1/p,p-2]$. Now we have to show that this solution must have a zero also on (δ,∞). We proceed by indirect way, suppose that $y(t)>0$ for $t\geqslant\delta$. Consider the function $w(t)$ given by (2.2) (again with $r(t)\equiv 1$) on $[\delta,\infty)$ and its differential equation (2.3). Then by integration we have

$$w(t+\omega)-w(t)=(p-1)\int_t^{t+\omega}|w(s)|^q\,\mathrm{d}s\quad(t\geqslant\beta)\tag{3.1}$$

hence

$$w(t+\omega)>w(t).$$

Consider now the sequence $w(\gamma)$, $w(\gamma+\omega)$, $w(\gamma+2\omega)$, $\dots$. By theorem 2.1 and by our indirect assumption on the solution $y(t)$, this sequence consists of negative terms:

$$w(\gamma)<w(\gamma+\omega)<w(\gamma+2\omega)<\cdots<0.$$

Indeed, if $w(\gamma+k\omega)\geqslant 0$ for some $k\in\mathbb{N}$, then by theorem 2.1 the solution $y(t)$ would have a zero in $(\gamma+k\omega,\infty)$. Hence $\lim_{k\to\infty}w(\gamma+k\omega)\leqslant 0$, consequently by (3.1)

$$w(\gamma)+(p-1)\int_\gamma^\infty|w(s)|^q\,\mathrm{d}s\leqslant 0,$$

i.e. the integral $\int_\gamma^\infty|w(s)|^q\,\mathrm{d}s$ is convergent.

This implies by (2.3) that

$$w(t)=w(\gamma)+\int_\gamma^t c(s)\,\mathrm{d}s+(p-1)\int_\gamma^t|w(s)|^q\,\mathrm{d}s$$

and the function $w(t)$ is bounded. Again by (2.3) we find that $w'(t)$ is also bounded, say, $|w'(t)|<L$. Then

$$\biggl|\frac{|w(t_2)|^{q+1}-|w(t_1)|^{q+1}}{q+1}\biggr|=\biggl|\int_{t_1}^{t_2}w'(s)|w(s)|^q\operatorname{sgn}w(s)\,\mathrm{d}s\biggr|\leqslant L\int_{t_1}^{t_2}|w(s)|^q\,\mathrm{d}s,$$

$\gamma<t_1<t_2$, hence $\lim_{t\to\infty}|w(t)|^{q+1}$ exists. Clearly, we have $\lim_{t\to\infty}w(t)=0$.

On the other hand, $w(\delta)=0$, and by (3.1) we have $\lim_{k\to\infty}w(\delta+k\omega)>0$ and this contradicts the fact that $\lim_{t\to\infty}w(t)=0$. $\square$

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4. Remarks

(i) Conjugacy criterion (2.7) is formulated for equation (1.1), however, using transformation (2.1), one may formulate the result also for the more general differential equation (1.4). By this reformulation theorem 2.2 reads as follows.

THEOREM 4.1. *Let $t_0 \in (a,b)$ and suppose that $c(t) \not\equiv 0$ in both intervals (a,t_0) and (t_0,b). Further suppose that there exist constants $T_1, T_2 \in (a,b)$, $T_1 < t_0 < T_2$ and $\alpha_1, \alpha_2 \in (-1/p, p-2]$ such that*

$$\int_a^{t_0} r^{1-q}(t)\,\mathrm{d}t = \infty = \int_{t_0}^b r^{1-q}(t)\,\mathrm{d}t \tag{4.1}$$

and

$$\left.\begin{aligned}\int_t^{t_0}\biggl[\int_s^{t_0} r^{1-q}(\tau)\,\mathrm{d}\tau\biggr]^{\alpha_1} r^{1-q}(s)\biggl(\int_s^{t_0} c(\tau)\,\mathrm{d}\tau\biggr)\mathrm{d}s \geqslant 0, \quad t\in(a,T_1),\\ \int_{t_0}^{t}\biggl[\int_{t_0}^{s} r^{1-q}(\tau)\,\mathrm{d}\tau\biggr]^{\alpha_2} r^{1-q}(s)\biggl(\int_{t_0}^{s} c(\tau)\,\mathrm{d}\tau\biggr)\mathrm{d}s \geqslant 0, \quad t\in(T_2,b).\end{aligned}\right\} \tag{4.2}$$

Then the solution $y(t)$ of (1.4) given by the initial condition $y(t_0) = 1$, $y'(t_0) = 0$ has at least two zeros in (a,b): one in (a,t_0), another one in (t_0,b).

(ii) Observe that condition (4.1) implies that $y_0(t) = 1$ is the unique (up to multiplication by a non-zero real constant) solution of the one-term equation

$$(r(t)\Phi(y'))' = 0, \tag{4.3}$$

which is non-zero in the whole interval (a,b). Indeed, the solution space of (4.3) is the two-dimensional linear space with the basis $y_1 = 1$, $y_2 = \int_{t_0}^t r^{1-q}(s)\,\mathrm{d}s$, $t_0 \in (a,b)$. If (4.1) holds, then obviously y_1 is the only solution (again up to multiplication) without zero in (a,b). On the other hand, if one of the integrals in (4.1) is convergent, say $\int_a r^{1-q}(t)\,\mathrm{d}t < \infty$, then another non-zero solution of (4.3) is $y = \int_a^t r^{1-q}(s)\,\mathrm{d}s$.

Motivated by theorem 4.1, equation (1.4) is said to be *1-special* in (a,b) if there exists exactly one solution of this equation (up to multiplication) without zero point in (a,b). Using this terminology, when (1.4) viewed as a perturbation of (4.3), theorem 4.1 may be interpreted as follows. If (4.3) is 1-special in (a,b) and the function c is 'slightly positive in (a,b)' (in the sense (4.2)), then equation (1.4) is conjugate in this interval, i.e. a 'slightly positive perturbation' of 1-general equation (4.3) makes the perturbed equation (1.4) be conjugate in (a,b).

The importance of 1-special *linear* equations for the investigation of conjugacy of (1.5) was emphasized in [2]. In particular, it was shown that if (1.5) is 1-special in (a,b), y_0 is the its only solution without zero in this interval, and $\tilde{c}$ is a continuous function such that

$$\int_a^b \tilde{c}(t)y_0^2(t)\,\mathrm{d}t \geqslant 0, \quad \tilde{c}(t) \not\equiv 0 \text{ in } (a,b),$$

then the perturbed equation

$$(r(t)y')' + (c(t) + \tilde{c}(t))y = 0$$

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is conjugate in (a, b). It is the subject of the present investigation whether a similar 'perturbation principle' holds also for general two-term half-linear equations (1.4).

(iii) Principal results of the paper are proved using the Riccati technique. In the linear case $p = 2$, another useful tool for investigation of oscillation and conjugacy (non-oscillation and disconjugacy) is the variational principle based on the relation between positivity of the quadratic functional

$$\int_a^b [r(t)y'^2 - c(t)y^2]\,\mathrm{d}t$$

and disconjugacy of (1.5) on $[a, b]$. A similar relation between positivity of the 'p-degree functional'

$$\int_a^b [r(t)|u'|^p - c(t)|u|^p]\,\mathrm{d}t$$

and disconjugacy of (1.4) was established in a recent paper [8]. It seems that this idea may be used for investigation of half-linear equations in the same way as in the linear case. Also this problem is a subject of the present investigation.

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