Heavy-tailed branching random walks on multidimensional lattices. A moment approach

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We study a continuous-time branching random walk (BRW) on the lattice $\mathbb{Z}^d$, $d \in \mathbb{N}$, with a single source of branching, that is the lattice point where the birth and death of particles can occur. The random walk is assumed to be spatially homogeneous, symmetric and irreducible but, in contrast to the majority of previous investigations, the random walk transition intensities $a(x,y)$ decrease as $|y - x|^{-(d+\alpha)}$ for $|y - x| \to \infty$, where $\alpha \in (0,2)$, that leads to an infinite variance of the random walk jumps. The mechanism of the birth and death of particles at the source is governed by a continuous-time Markov branching process. The source intensity is characterized by a certain parameter β. We calculate the long-time asymptotic behaviour for all integer moments for the number of particles at each lattice point and for the total population size. With respect to the parameter β, a non-trivial critical point $\beta_c > 0$ is found for every $d \geqslant 1$. In particular, if $\beta > \beta_c$ the evolutionary operator generated a behaviour of the first moment for the number of particles has a positive eigenvalue. The existence of a positive eigenvalue yields an exponential growth in t of the particle numbers in the case $\beta > \beta_c$ called *supercritical*. Classification of the BRW treated as *subcritical* ($\beta < \beta_c$) or *critical* ($\beta = \beta_c$) for the heavy-tailed random walk jumps is more complicated than for a random walk with a finite variance of jumps. We study the asymptotic behaviour of all integer moments of a number of particles at any point $y \in \mathbb{Z}^d$ and of the particle population on $\mathbb{Z}^d$ according to the ratio d/α.

1. Introduction

We study a continuous-time branching random walk (BRW) on the lattice $\mathbb{Z}^d$ with a single *source* of branching, that is the lattice point where the birth and death of particles can occur. Our goal is to justify the classification of the number of particles growth rate in the case of the so-called BRWs with heavy tails, previously presented in [**20**].

Informally, the process can be described as follows. Suppose that initially there is a single particle at a point $x \in \mathbb{Z}^d$, which then performs a random walk on $\mathbb{Z}^d$

until reaching the source $x_0 = 0$. At the origin, the particle waits an exponentially distributed time and then jumps to another lattice point distinct from the origin or dies or gives a random number of offspring. In case no branching has occurred, the particle continues its walk until the next return to the source, and so on. Each of the newborn particles evolves according to the same rule, independently of the others. The simplest model with one source of branching was apparently first introduced in [**24**] and after that reconsidered in a more general setting in a series of publications, see, e.g. [**2**, **3**]. Similarly, in [**13**, **28**, **29**] there were considered BRWs with finitely many sources $x_1, \ldots, x_N$. The BRW models are of principal interest, especially in cases when the branching environment is spatially inhomogeneous and the phase space $\mathbb{Z}^d$ is unbounded, see, e.g. [**24**].

The formation of the theory of BRWs as a separate discipline goes back to seminal works on branching processes with diffusion of particles, see, e.g., [**21**] and references in [**22**]. Depending on what assumptions are made regarding the space in which the walking takes place (bounded, as is in [**21**] or unbounded, as is in [**12**, **24**]), the time in which the particle system 'lives' (is it continuous [**7**, **12**] or discrete [**9**]), the nature of the random walk (symmetric, a mainly considered case, asymmetric [**6**, **28**] or periodic in space [**16**]), the probabilistic properties of walking and branching (the variance of jumps is finite or infinite, see, e.g., [**10**, **19**]), etc. the task setting and research objectives are significantly changing. The methods used to study the BRWs also vary greatly. Therefore, for example, the technique of analysis of the branching processes with motion of particles on continuous spaces (e.g. $\mathbb{R}^d$), as a rule, cannot be used unchanged in the analysis of the BRWs on discrete spaces (e.g. $\mathbb{Z}^d$). There is a fairly wide range of studies based on purely probabilistic approaches in which BRWs can be reduced to the Bellman–Harris branching processes (see, e.g. [**6**, **23**]). In studies motivated by the problems of statistical physics an approach is more often used that goes back to the description of the evolution of systems using the Kolmogorov equations, see, e.g., [**2**, **14**]. In such studies, the main research focus is shifted to the use of functional–analytical methods and the application of the theory of differential equations in infinite dimensional spaces. The current study is adjacent to the second direction.

In statistical physics for processes in random media, it is well known that inhomogeneity of the environment plays a key role in the formation of anomalous properties of transport processes which is conventionally expressed in terms of the so-called intermittency concept. For the detailed discussion of the concept of intermittency of random fields, and in particular the concept of 'strong centres' of the field generation, see, e.g. [**4**] and references therein. In this context, the model of BRW with a single source of branching (or, more generally, with finitely many sources) may be viewed as taking into account principal 'perturbation' terms. On the other hand, the non-compactness of the space destroys the pure-point spectrum of the operator associated with the process [**3**, **24**], so that the conventional spectral methods are no more readily applicable [**25**].

In this paper, we assume that the branching mechanism at the source is governed by a continuous-time Markov branching process. It is also assumed that the underlying random walk is spatially homogeneous, symmetric and irreducible but, in contrast to the majority of previous investigations, the random walk transition

intensities $a(x, y)$ decrease as $|y - x|^{-(d+\alpha)}$ for $|y - x| \to \infty$, where $\alpha \in (0, 2)$, that leads to an infinite variance of the random walk jumps. For such a BRW (called a heavy-tailed BRW) the precise asymptotic behaviour of all integer moments of the number of particles at each lattice point and on the entire lattice is remained unexplored in the case when the positive pure-point spectrum of the operator is empty. Different models of heavy-tailed random walks without branching of particles were investigated by many authors, see, e.g. [5] and references therein. The random walk transition probabilities $p(t, x, y)$ for symmetric heavy-tailed random walks were investigated in [1] under an appropriate regularity condition on the tails of the jump distributions for $|y - x| + t \to \infty$. The results for the random walk transition probabilities $p(t, x, y)$ without any regularity conditions, but for fixed space coordinates were obtained in [18]. Therefore, the main goal of the current study is to find the long-time asymptotic behaviour of all integer moments for the number of particles at each lattice point and for the total population size, for the case of a heavy-tailed BRW with one source of branching. A BRW with the sources of branching at every lattice point is considered, e.g. in [10]. Some applications of the theory of heavy-tailed processes can be found, e.g. in [30]. Below, we achieve the goal by studying the chain of the backward differential-difference equations for the moments. The difference operator involved in each moment equation is of the form $\mathcal{H}_\beta := \mathcal{A} + \beta \delta_0(\cdot)$, where $\mathcal{A}$ is the random walk generator, and the delta-term reflects the presence of the branching source. The spectral properties of the operator $\mathcal{H}_\beta$ play an essential role in the problem. With respect to a certain parameter β characterizing intensity of the source, a non-trivial critical point $\beta_c > 0$ is found for every $d \geqslant 1$ (unlike a BRW with a finite variance of jumps for which $\beta_c > 0$ only for $d \geqslant 3$) see, e.g. [27], which is related to the existence of a positive eigenvalue λ_0 of the operator $\mathcal{H}_\beta$. The probabilistic meaning of the parameter λ_0 is that it determines the rate of the process exponential growth [13], and hence the case $\beta > \beta_c$ corresponds to the supercritical regime.

We study how the asymptotic behaviour, as $t \to \infty$, of the moments of the particle population and of the number of particles at the point $y \in \mathbb{Z}^d$ depends on the parameters of walking, branching, and the ratio d/α. The various combinations of these parameters allow classifying the BRW as subcritical, critical or supercritical. For the case of heavy-tailed random walk jumps, this classification is turned out to be more complex than for a random walk with a finite variance of jumps [20]. It is noteworthy that, for a similar process with branching and motion on $\mathbb{R}^d$, in [12] it was established the 'persistence' criteria based on the ratio between d/α and $1/\beta$, where α and β are the parameters of motion and branching, respectively.

The structure of the work is as follows. In § 2, a formal description of a BRW with one branching source and main previous results are reminded. In § 3, we investigate the structure of the non-negative discrete spectrum of a BRW, lemma 3.3, and find the limits of the mean of the local number of particles, theorem 3.4. In § 4, theorem 4.3 about asymptotics of all moments of particle number for a critical BRW is obtained. Finally, in § 5, we study the asymptotic behaviour of all moments of particle number for a subcritical BRW, theorem 5.3.

2. Model and previous results

We consider a stochastic process for a system of particles when every particle can walk on the lattice points and give offsprings or die at the origin (called the *source of branching*).

The underlying random walk is determined by a matrix of transition intensities $A = (a(x, y))_{x,y \in \mathbb{Z}^d}$, where $a(x, y) \geqslant 0$ for $y \neq x$, $-\infty < a(x, x) < 0$ and $\sum_{y \in \mathbb{Z}^d} a(x, y) = 0$. By the axiomatics from [11], the corresponding transition probability $p(h, x, y)$ of a particle movement from point x to point y over a short period of time h, satisfies the equations

$$p(h, x, y) = a(x, y)h + o(h) \qquad \text{for } y \neq x,$$
$$p(h, x, x) = 1 + a(x, x)h + o(h).$$

Such relations imply that the Kolmogorov's backward equations hold

$$\frac{\mathrm{d}p(t, x, y)}{\mathrm{d}t} = \sum_{x' \in \mathbb{Z}^d} a(x, x')p(t, x', y), \quad p(0, x, y) = \delta_y(x), \qquad (2.1)$$

where $\delta_y(\cdot)$ is the discrete Kronecker delta-function on $\mathbb{Z}^d$.

We consider a symmetric random walk with $a(x, y) = a(y, x)$. Then, due to the sufficient condition of the boundedness of linear operators given in [28, lemma 1], the expression

$$(\mathcal{A}u)(x) := \sum_{x' \in \mathbb{Z}^d} a(x, x')u(x'), \quad u \in \ell^p(\mathbb{Z}^d), \qquad (2.2)$$

defines the bounded linear operator $\mathcal{A} : \ell^p(\mathbb{Z}^d) \to \ell^p(\mathbb{Z}^d)$ for any $p \in [1, \infty]$. Note that the operator $\mathcal{A}$ is self-adjoint in $\ell^2(\mathbb{Z}^d)$. Now we can rewrite equations (2.1) as differential equations in Banach spaces

$$\frac{\mathrm{d}p(t, x, y)}{\mathrm{d}t} = (\mathcal{A}p(t, \cdot, y))(x), \quad p(0, x, y) = \delta_y(x). \qquad (2.3)$$

We assume that $a(x, y) = a(0, y - x)$ which can be treated as a spatial homogeneity of the random walk. Then the values of $a(x, y)$ can be expressed by a function of one argument as follows: $a(x, y) = a(y - x)$, where $a(z) := a(0, z)$, $z \in \mathbb{Z}^d$. We assume also that the random walk is irreducible which means that for every $z \in \mathbb{Z}^d$ there exists a set of vectors $z_1, \ldots, z_k \in \mathbb{Z}^d$ such that $z = \sum_{i=1}^k z_i$ and $a(z_i) \neq 0$ for $1 \leqslant i \leqslant k$.

Finally, for the random walk, we assume that for all $z \in \mathbb{Z}^d$ with sufficiently large norm the following asymptotic relation holds

$$a(z) \sim \frac{H\left(z/|z|\right)}{|z|^{d+\alpha}}, \quad \alpha \in (0, 2), \qquad (2.4)$$

where $H(\cdot)$ is a continuous positive symmetric function on the unit sphere $\mathbb{S}^{d-1}$. This implies that the variance of jumps of the random walk

$$\sigma^2 = \sum_{z \neq 0} |z|^2 \frac{a(z)}{-a(0)}$$

becomes infinite (see, e.g. [**27**] for a more detailed discussion of the related preliminaries) which means that the random walk under consideration has heavy tails.

The solution of the Cauchy problem (2.3) has a well-known Gaussian approximation, see, e.g. [**11**], for the case of a finite variance of jumps

$$p(t, x, y) \sim \gamma_d t^{-d/2}, \qquad t \to \infty,$$

with $\gamma_d > 0$. In [**18**, theorem 2], it was proved that under assumption (2.4) the probability $p(t, x, y)$ has the following asymptotics:

$$p(t, x, y) \sim h_{\alpha, d} t^{-d/\alpha}, \qquad t \to \infty, \tag{2.5}$$

where $h_{\alpha, d} > 0$. Moreover,

$$p(t, 0, 0) - p(t, x, 0) \sim \frac{\tilde{\gamma}_{d, \alpha}(x)}{t^{(d+2)/\alpha}}, \qquad t \to \infty, \tag{2.6}$$

where $\tilde{\gamma}_{d, \alpha}(x) > 0$, see [**19**, theorem 1].

Now, let us turn to assumptions concerning the branching component of the process. We assume that the branching process at the origin is covered by the continuous-time Markov branching process having the following infinitesimal generation function

$$f(u) = \sum_{n=0}^{\infty} b_n u^n, \quad 0 \geqslant u \geqslant 1,$$

where $b_n \geqslant 0$ for $n \neq 1$, $b_1 < 0$ and $\sum_{n=0}^{\infty} b_n = 0$. Assume also $\beta^{(r)} := f^{(r)}(1) < \infty$ for every $r \in \mathbb{N}$. Let us denote *intensity of the source* by

$$\beta = f'(1) = \sum_{n=1}^{\infty} n b_n = (-b_1) \left(\sum_{n \neq 1} n \frac{b_n}{(-b_1)} - 1 \right),$$

where the last sum is the mean number of offsprings. According to the imposed assumptions, the probability of producing $n \neq 1$ particles from a single one has the following form

$$p_*(h, n) = b_n h + o(h), \qquad h \to 0,$$

where the value $n = 0$ means that the particle died.

Now we combine the walking and branching components of the process. At the origin (where the source of branching is located), it is possible that during a short period of time h the particle jumps to a point $y \neq 0$ with the transition probability $p(h, 0, y)$, or produces $n \neq 1$ particles with the probability $p_*(h, n)$, where $n = 0$ means the particle death. Otherwise the particle remains at the origin during the period of time h with the probability

$$p_{\mathrm{brw}}(h, x, x) = 1 + a(0)h + \delta_0(x) b_1 h + o(h), \qquad h \to 0.$$

In what follows, the main object of interest will be the long-time asymptotic behaviour of the local particle number at an arbitrary point $\mu_t(y)$, $y \in \mathbb{Z}^d$, and

the total population size $\mu_t = \sum_{y \in \mathbb{Z}^d} \mu_t(y)$. We will investigate the asymptotic behaviour of the moments $m_n(t, x, y) = \mathsf{E}_x \mu_t^n(y)$ and $m_n(t, x) = \mathsf{E}_x \mu_t^n$, where E_x is the expectation under condition that there was only one particle in the system at the initial moment of time, and it located at the point $x \in \mathbb{Z}^d$.

Introduce the linear operator

$$\mathcal{H}_\beta = \mathcal{A} + \beta \Delta_0 \tag{2.7}$$

on $\ell^p(\mathbb{Z}^d)$, $p \in [1, \infty]$, where the operator $\mathcal{A}$ is defined by (2.2), and the operator Δ_0 is defined by $(\Delta_0 u)(x) := \delta_0(x)u(x)$, where $u \in \ell^p(\mathbb{Z}^d)$. Then the equations for the first moments are as follows, see, e.g. [25, theorem 1.3.1],

$$\frac{\mathrm{d}m_1(t, x)}{\mathrm{d}t} = (\mathcal{H}_\beta m_1(t, \cdot))(x), \qquad m_1(0, x) \equiv 1, \tag{2.8}$$

$$\frac{\mathrm{d}m_1(t, x, y)}{\mathrm{d}t} = (\mathcal{H}_\beta m_1(t, \cdot, y))(x), \quad m_1(0, x, y) = \delta_y(x). \tag{2.9}$$

We would like to point out that equations (2.8) and (2.9) were obtained in [25] for the case of a finite variance of jumps, however their derivation and the final forms remain unchanged for the case of heavy tails as the differential and integral equations below.

Using the notation $\{m_i(t)\}_{i=1}^{n-1}$ for the sets of functions

$$\{m_i(t, x, y)\}_{i=1}^{n-1} \quad \text{and} \quad \{m_i(t, x)\}_{i=1}^{n-1},$$

we can write the following differential equations for the higher-order moments, $n \geqslant 2$

$$\frac{\mathrm{d}m_n(t)}{\mathrm{d}t} = (\mathcal{H}_\beta m_n(t))(x) + (\Delta_0 g_n(m_1(t), \ldots, m_{n-1}(t)))(x),$$

where

$$g_n(m_1, m_2, \ldots, m_{n-1}) = \sum_{r=2}^{n} \frac{\beta^{(r)}}{r!} \sum_{\substack{i_1, \ldots, i_r > 0 \\ i_1 + \cdots + i_r = n}} \frac{n!}{i_1! \cdots i_r!} m_{i_1} \cdots m_{i_r}. \tag{2.10}$$

By analogue with the scheme of deriving the integral equations from the differential equations from [25], it is possible to derive the integral equations for the first moments

$$m_1(t, x, y) = p(t, x, y) + \beta \int_0^t p(t - s, x, 0) m_1(s, 0, y) \mathrm{d}s, \tag{2.11}$$

$$m_1(t, x, y) = p(t, x, y) + \beta \int_0^t m_1(s, x, 0) p(t - s, 0, y) \mathrm{d}s, \tag{2.12}$$

$$m_1(t, x) = 1 + \beta \int_0^t m_1(s, x, 0) \mathrm{d}s, \tag{2.13}$$

and the integral equations for the higher-order moments, $n \geqslant 2$, in forms

$$m_n(t, x, y) = m_1(t, x, y)$$
$$+ \int_0^t m_1(t - s, x, 0) g_n(m_1(s, 0, y), \ldots, m_{n-1}(s, 0, y)) \mathrm{d}s, \qquad (2.14)$$

$$m_n(t, x) = m_1(t, x) + \int_0^t m_1(t - s, x, 0) g_n(m_1(s, 0), \ldots, m_{n-1}(s, 0)) \mathrm{d}s. \quad (2.15)$$

To investigate the solutions of such equations, the Laplace transform

$$G_\lambda(x, y) = \int_0^\infty \mathrm{e}^{-\lambda t} p(t, x, y) \mathrm{d}t, \qquad \lambda \geqslant 0, \qquad (2.16)$$

of transition probability can be used. The function $G_\lambda(x, y)$, which is conventionally called the Green function of the transition probabilities, can be expressed, see [25], also as

$$G_\lambda(x, y) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{\mathrm{e}^{i\langle \theta, y-x \rangle}}{\lambda - \phi(\theta)} \mathrm{d}\theta,$$

where

$$\phi(\theta) = \sum_{z \in \mathbb{Z}^d} a(z) \mathrm{e}^{i\langle \theta, z \rangle}, \qquad \theta \in [-\pi, \pi]^d,$$

is the Fourier transform of the transition intensity $a(z)$.

3. Spectral analysis of the evolution operator

Due to equations (2.8) and (2.9), the spectrum of the operator $\mathcal{H}_\beta$ defined by (2.7) determines the asymptotic behaviour of the first moment of particle number.

By [25, lemma 3.1.3], the proof of which does not depend on the conditions for variance of jumps, the number λ is an eigenvalue of the operator $\mathcal{H}_\beta$ with an eigenvector $f \in \ell^2(\mathbb{Z}^d)$ if and only if the following conditions are satisfied

$$f(0) \neq 0, \quad \beta \neq 0,$$

$$\int_{[-\pi, \pi]^d} |\lambda - \phi(\theta)|^{-2} \mathrm{d}\theta < \infty, \qquad (3.1)$$

$$\beta I_0(\lambda) = 1, \qquad (3.2)$$

where

$$I_x(\lambda) = \frac{1}{(2\pi)^d} \int_{[-\pi, \pi]^d} \frac{\mathrm{e}^{-i\langle \theta, x \rangle}}{\lambda - \phi(\theta)} \mathrm{d}\theta, \qquad x \in \mathbb{Z}^d, \qquad (3.3)$$

(note that $I_x(\lambda) \equiv G_\lambda(x, 0)$, when $\lambda \geqslant 0$).

Moreover, the eigenvector f corresponding to the eigenvalue λ is defined by the equality

$$f(x) = \beta f(0) I_x(\lambda), \quad x \in \mathbb{Z}^d,$$

and therefore each eigenvalue of the operator $\mathcal{H}_\beta$ is simple.

In the next lemma, we answer the question which combinations of the parameters d, α and β imply that the value $\lambda = 0$ is an eigenvalue of the operator $\mathcal{H}_\beta$.

LEMMA 3.1. *Let $\lambda = 0$. If $d/\alpha \in (1/2, 2]$, $d \in \mathbb{N}$, then condition* (3.1) *is not valid, if $d/\alpha \in (2, \infty)$, $d \in \mathbb{N}$, it is valid.*

Proof. By theorem 5 from [15] about the asymptotics as $\theta \to 0$ of the function

$$\sum_{z \in \mathbb{Z}^d \setminus \{0\}} a_z (1 - \cos\langle z, \theta \rangle), \quad \theta \in \mathbb{R}^d,$$

where $a_z \|z\|^{d+\alpha} \to 1$ for $\|z\| \to \infty$, $\alpha \in (0, 2]$, and $\langle \cdot, \cdot \rangle$ is the standard inner product and $\| \cdot \|$ is the max-norm on $\mathbb{R}^d$, we have

$$c|\theta|^\alpha \leqslant |\phi(\theta)| \leqslant C|\theta|^\alpha$$

in a sufficiently small neighbourhood of zero, where $c, C > 0$ are some constants. Then condition (3.1) for $\lambda = 0$ is equivalent to the condition

$$\int_{[-\pi,\pi]^d} |\phi(\theta)|^{-2} \mathrm{d}\theta < \infty$$

which (passing to the generalized polar coordinates) is equivalent to

$$\int_0^\rho r^{d-1-2\alpha} \mathrm{d}r < \infty$$

for some small $\rho > 0$. Taking into account that the last condition is valid only for $d/\alpha \in (2, \infty)$ and hence is not valid for $d/\alpha \in (1/2, 2]$, we finalize the proof of the lemma. $\qquad\square$

To determine the values of $\lambda \geqslant 0$ at which equation (3.2) is solvable, we investigate the properties of the function $I_0(\lambda)$.

LEMMA 3.2. *For $\lambda > 0$, the function $I_0(\lambda)$ in* (3.3) *is defined, continuous, strictly decreases and positive, wherein*

$$\lim_{\lambda \to \infty} I_0(\lambda) = 0 \qquad \text{for } d/\alpha \in (1/2, \infty),$$

$$\lim_{\lambda \to 0} I_0(\lambda) = +\infty \qquad \text{for } d/\alpha \in (1/2, 1],$$

$$\lim_{\lambda \to 0} I_0(\lambda) = G_0(0,0) < \infty \quad \text{for } d/\alpha \in (1, \infty),$$

where $\alpha \in (0, 2)$, $d \in \mathbb{N}$.

Proof. The proof is based on the study of convergence domain of the integral

$$\int_{[-\pi,\pi]^d} |\lambda - \phi(\theta)|^{-1} \mathrm{d}\theta$$

according to a scheme similar to that of lemma 3.1, and on continuity and monotonicity in λ of function $(\lambda - \phi(\theta))^{-1}$. $\qquad\square$

As follows from lemma 3.1, by virtue of the continuity and strict decreasing of the function $I_0(\lambda)$, the spectrum of the operator $\mathcal{H}_\beta$ may contain no more than one positive eigenvalue of unit multiplicity. Furthermore, the set of possible values of the intensity of the source β can be divided by a certain threshold value β_c in such a way that for $\beta < \beta_c$ there will be no positive eigenvalues, and for $\beta > \beta_c$, there is a unique positive eigenvalue of unit multiplicity. In this sense, the value β_c of the intensity of the source can be called the *critical* value.

Let us introduce a classification of the BRW with one source of branching which depends on the growth behaviour of the population size. As will be explained in §3, for certain dimension d and certain value of the parameter α of the underlying random walk, there exist a critical value β_c of the intensity of the source such that the exponential growth of the mean population size is possible only for $\beta > \beta_c$. This is a consequence of the fact that an isolated positive eigenvalue in the spectrum of the operator $\mathcal{H}_\beta$ exists only for $\beta > \beta_c$ by definition of β_c. In view of this, we will call a BRW *supercritical* if $\beta > \beta_c$, *critical* if $\beta = \beta_c$, and *subcritical* if $\beta < \beta_c$.

For the case of finite variance of jumps the asymptotics of all the moments $m_n(t, x, y)$, $m_n(t, x)$ was found in the book [**25**]. The asymptotics of all the moments $m_n(t, x, y)$, $m_n(t, x)$ with no restrictions on the variance of jumps for a supercritical BRW with arbitrary finite number of branching sources was found in [**13**].

Now we summarize the results about the presence of non-negative eigenvalues in the spectrum of the operator $\mathcal{H}_\beta$.

LEMMA 3.3. *For the BRW on $\mathbb{Z}^d$, $d \in \mathbb{N}$, satisfying (2.4), the following statements about β_c and non-negative eigenvalues of $\mathcal{H}_\beta$ are valid*

  (i) *if $d/\alpha \in (1/2, 1]$, then $\beta_c = 0$; for $\beta \leqslant \beta_c$ the operator $\mathcal{H}_\beta$ has no non-negative eigenvalues; for $\beta > \beta_c$ the operator $\mathcal{H}_\beta$ has an eigenvalue $\lambda > 0$ of unit multiplicity that is a solution of equation (3.2);*

  (ii) *if $d/\alpha \in (1, 2]$, then $\beta_c = G_0^{-1}(0, 0) > 0$; for $\beta \leqslant \beta_c$ the operator $\mathcal{H}_\beta$ has no non-negative eigenvalues; for $\beta > \beta_c$ the operator $\mathcal{H}_\beta$ has an eigenvalue $\lambda > 0$ of unit multiplicity that is a solution of equation (3.2);*

  (iii) *if $d/\alpha \in (2, \infty)$, then $\beta_c = G_0^{-1}(0, 0) > 0$; for $\beta < \beta_c$ the operator $\mathcal{H}_\beta$ has no non-negative eigenvalues; for $\beta \geqslant \beta_c$ the operator $\mathcal{H}_\beta$ has an eigenvalue λ of unit multiplicity that is a solution of equation (3.2), wherein $\lambda = 0$ for $\beta = \beta_c$ and $\lambda > 0$ for $\beta > \beta_c$.*

Proof. To determine when conditions (3.1) and (3.2) from [**25**, lemma 3.1.3] about eigenvalues of the operator $\mathcal{H}_\beta$ are satisfied, we use lemmas 3.1 and 3.2. Then condition (3.1) is valid only for $d/\alpha \in (2, \infty)$ wherein condition (3.1) is valid for $\lambda > 0$ if $\beta > \beta_c$ and for $\lambda = 0$ if $\beta = \beta_c$. The equality $\beta_c = G_0^{-1}(0, 0)$, when $G_0^{-1}(0, 0) < \infty$, is a consequence of equation (3.2). □

Now we can evaluate the limiting behaviour as $t \to \infty$ of the average number of particles $m_1(t, x, y)$ at the every point $y \in \mathbb{Z}^d$ when the process starts from x. For

every $\lambda \in \mathbb{C}$ and $x, y \in \mathbb{Z}^d$, when $G_\lambda(x, 0)$, $G_\lambda(0, y)$ are defined and finite, denote

$$c(\lambda, x, y) := \frac{G_\lambda(x, 0)G_\lambda(0, y)}{\|G_\lambda(x, 0)\|_{\ell^2(\mathbb{Z}^d)}^2}.$$

THEOREM 3.4. *Let $m_1(t, x, y)$ be the solution of the Cauchy problem* (2.9). *Then for the BRW on $\mathbb{Z}^d$, $d \in \mathbb{N}$, satisfying* (2.4), *and every $y \in \mathbb{Z}^d$, the following statements are valid*

(i) *if $\beta > \beta_c$ and $d/\alpha \in (1/2, \infty)$ then*

$$\lim_{t \to \infty} m_1(t, x, y)\mathrm{e}^{-\lambda t}c^{-1}(\lambda, x, y) = 1, \qquad (3.4)$$

   where λ is the solution of equation (3.2);

(ii) *if $\beta = \beta_c$ and $d/\alpha \in (2, \infty)$ then*

$$\lim_{t \to \infty} m_1(t, x, y)c^{-1}(0, x, y) = 1, \qquad (3.5)$$

   and $m_1(t, x, y)$ is monotonically non-increasing;

(iii) *if $\beta \leqslant \beta_c$ and $d/\alpha \in (1/2, 2]$ or if $\beta < \beta_c$ and $d/\alpha \in (2, \infty)$ then*

$$\lim_{t \to \infty} m_1(t, x, y) = 0. \qquad (3.6)$$

   Moreover, the function $m_1(t, x, y)$ converges to zero as $t \to \infty$ uniformly in $x \in \mathbb{Z}^d$, and it is monotonically non-increasing.

Proof. By lemma 3.3 the operator $\mathcal{H}_\beta$ has a simple non-negative eigenvalue λ. Then applying lemmas 3.3.2, 3.3.3 and 3.3.4 from [**25**] (about asymptotics of a solution of the Cauchy problem (2.9) with a bounded self-adjoint operator) we readily obtain limits (3.4)–(3.6).

By analogue with [**25**, lemma 3.3.5], if the spectrum $\mathcal{H}_\beta$ has no positive eigenvalues, then the function $m_1(t, x, y)$ is monotonically non-increasing in t. □

4. Critical BRW

Let us find the asymptotics as $t \to \infty$ of the mean number of particles $m_1(t, x, y)$ at the point $y \in \mathbb{Z}^d$ when the process started from $x \in \mathbb{Z}^d$ for the cases of recurrent random walks.

THEOREM 4.1. *Let $\beta = \beta_c$. Then for every $y \in \mathbb{Z}^d$, $d \in \mathbb{N}$, the solution $m_1(t, x, y)$, $x \in \mathbb{Z}^d$, of the Cauchy problem* (2.9) *under $d/\alpha \in (1/2, 1]$ satisfies the relation*

$$m_1(t, x, y) \sim h_{\alpha, d}t^{-1/\alpha}, \qquad t \to \infty,$$

where $h_{\alpha, d}$ is defined in (2.5).

Proof. By statement (i) from lemma 3.3, we have $\beta_c = 0$. Then the Cauchy problem (2.9) is equivalent to the Cauchy problem for transition probabilities (2.3). Therefore, the solution $m_1(t, x, y)$ has the same asymptotics (2.5). □

Now we find the asymptotics as $t \to \infty$ of the function $m_1(t, x, y)$ for the cases of transient random walks.

THEOREM 4.2. *Let $\beta = \beta_c$. Then for every $y \in \mathbb{Z}^d$, $d \in \mathbb{N}$, the solution $m_1(t, x, y)$, $x \in \mathbb{Z}^d$, of the Cauchy problem (2.9) under $d/\alpha \in (1, \infty)$ satisfies the relation*

$$m_1(t, x, y) \sim C_1(x, y) u_1(t), \qquad t \to \infty,$$

where the functions $C_1(x, y)$ and $u_1(t)$ are as follows:

$$C_1(x, y) = G_0(x, 0) G_0(0, y) \gamma_{d,\alpha}^{-1} \Gamma^{-1}(d/\alpha - 1), \quad u_1(t) = t^{d/\alpha - 2} \quad \text{for } d/\alpha \in (1, 2),$$
$$C_1(x, y) = G_0(x, 0) G_0(0, y) \gamma_{d,\alpha}^{-1}, \qquad\qquad u_1(t) = (\ln t)^{-1} \quad \text{for } d/\alpha = 2,$$
$$C_1(x, y) = G_0(x, 0) G_0(0, y) \gamma_{d,\alpha}^{-1}, \qquad\qquad u_1(t) = 1 \qquad \text{for } d/\alpha \in (2, \infty),$$

and $\gamma_{d,\alpha}$ are positive constants.

Proof. We find the asymptotics of $m_1(t, x, y)$ as $t \to \infty$ by evaluating the asymptotics of its Laplace transform $\widehat{m}_1(\lambda, x, y)$ as $\lambda \to 0$, using the Tauberian theorems (see [**8**, Ch. XIII]). For $\beta \leqslant \beta_c$ by [**25**, lemma 5.1.3], the Laplace transform of the solution $m_1(t, x, y)$ of the Cauchy problem (2.9) for $\operatorname{Re} \lambda > 0$ is well defined and can be represented in the following form

$$\widehat{m}_1(\lambda, x, y) = \frac{\beta G_\lambda(0, y) G_\lambda(x, 0)}{1 - \beta G_\lambda(0, 0)} + G_\lambda(x, y),$$

where $G_\lambda(x, y)$ is defined in (2.16).

At first, we consider the case $x = y = 0$, then we get

$$\widehat{m}_1(\lambda, 0, 0) = \frac{G_\lambda(0, 0)}{1 - \beta_c G_\lambda(0, 0)} = \frac{G_0(0, 0) G_\lambda(0, 0)}{G_\lambda(0, 0) - G_\lambda(0, 0)}.$$

Applying the asymptotics of $G_\lambda(0, 0)$ from [**29**, theorem 1] we obtain that $\widehat{m}_1(\lambda, 0, 0)$ is asymptotically equivalent to $G_0^2(0, 0) \gamma_{d,\alpha}^{-1} f(\lambda)$ as $\lambda \to 0$, where

$$f(\lambda) = \lambda^{-d/\alpha + 1}, \qquad \gamma_{d,\alpha} = \Gamma(2 - d/\alpha) h_{\alpha,d} \qquad\qquad \text{for } d/\alpha \in (1, 2),$$
$$f(\lambda) = (\lambda \ln(1/\lambda))^{-1}, \quad \gamma_{d,\alpha} = -h_{\alpha,d} \qquad\qquad\qquad \text{for } d/\alpha = 2,$$
$$f(\lambda) = \lambda^{-1}, \qquad\qquad \gamma_{d,\alpha} = \int_0^\infty \left(\int_t^\infty p(s, 0, 0) \mathrm{d}s \right) \mathrm{d}t \quad \text{for } d/\alpha \in (2, \infty),$$

and the constants $h_{\alpha,d}$ are in (2.5).

Under the condition $\beta = \beta_c$ the function $m_1(t, 0, 0)$ is monotone by theorem 3.4. Therefore, we can apply the Tauberian theorems (see [**8**, Ch. XIII]) and find that,

as $t \to \infty$, the following relations are valid

$$m_1(t,0,0) \sim G_0^2(0,0)\gamma_{d,\alpha}^{-1}(\ln t)^{-1} \qquad \text{for } d/\alpha \in (1,2),$$

$$m_1(t,0,0) \sim G_0^2(0,0)\gamma_{d,\alpha}^{-1}\Gamma^{-1}(d/\alpha - 1)t^{d/\alpha-2} \quad \text{for } d/\alpha = 2,$$

$$m_1(t,0,0) \sim G_0^2(0,0)\gamma_{d,\alpha}^{-1} \qquad \text{for } d/\alpha \in (2,\infty).$$

To find the asymptotics of $m_1(t,x,0)$, we express it by the integral equation (2.11), and after that we use statement (e) from lemma 2 for convolutions in [**26**]. Then it remains only to find the asymptotics of $m_1(t,x,y)$, and now we represent it by the integral equation (2.12), apply the same lemma for convolutions and the fact that $\beta = \beta_c = 1/G_0(0,0)$ from lemma 3.3. Theorem 4.2 is proved. $\qquad \square$

Now, we investigate the asymptotic behaviour as $t \to \infty$ of the mean of population size $m_1(t,x)$ when the process started from point $x \in \mathbb{Z}^d$.

THEOREM 4.3. *Let $\beta = \beta_c$. Then the solution $m_1(t,x)$, $x \in \mathbb{Z}^d$, $d \in \mathbb{N}$, of the Cauchy problem (2.8) under $d/\alpha \in (1/2,\infty)$ satisfies the relation*

$$m_1(t,x) \sim C_1(x)v_1(t), \qquad t \to \infty,$$

where the functions $C_1(x)$ and $v_1(t)$ are as follows

$$C_1(x) = 1, \qquad\qquad v_1(t) = 1 \qquad \textit{for } d/\alpha \in (1/2,1],$$

$$C_1(x) = G_0(x,0)\gamma_{d,\alpha}^{-1}\Gamma^{-1}(d/\alpha), \quad v_1(t) = t^{d/\alpha-1} \quad \textit{for } d/\alpha \in (1,2),$$

$$C_1(x) = G_0(x,0)\gamma_{d,\alpha}^{-1}, \qquad v_1(t) = t\ln^{-1} t \quad \textit{for } d/\alpha = 2,$$

$$C_1(x) = G_0(x,0)\gamma_{d,\alpha}^{-1}, \qquad v_1(t) = t \qquad \textit{for } d/\alpha \in (2,\infty),$$

and $\gamma_{d,\alpha}$ are positive constants.

Proof. For the case $d/\alpha \in (1/2,1]$, we have $\beta = \beta_c = 0$ by (i) in lemma 3.3. Then equation (2.13) implies that $m_1(t,x) \equiv 1$.

For the case $d/\alpha \in (1,\infty)$, we find the asymptotics of $m_1(t,x)$ as $t \to \infty$ by evaluating the asymptotics of its Laplace transform $\widehat{m}_1(\lambda,x)$ as $\lambda \to 0$ using the Tauberian theorems (see [**8**, Ch. XIII]). For $\beta \leqslant \beta_c$ by [**25**, lemma 5.1.4], the Laplace transform of the solution $m_1(t,x)$ of the Cauchy problem (2.8) for $\text{Re}\lambda > 0$ is defined and can be expressed in the form

$$\widehat{m}_1(\lambda,x) = \frac{1 - \beta(G_\lambda(0,0) - G_\lambda(x,0))}{\lambda(1 - \beta G_\lambda(0,0))}. \tag{4.1}$$

Herewith, we have $\beta = \beta_c = G_0^{-1}(0,0)$ by (ii), (iii) in lemma 3.3. Then by asymptotic relations for $G_\lambda(0,0)$ from [**29**, theorem 1], we get that the asymptotics of

$\widehat{m}_1(\lambda, x)$ as $\lambda \to 0$ is $G_0(x, 0)\gamma_{d,\alpha}^{-1}f(\lambda)$, where

$$
\begin{aligned}
f(\lambda) &= \lambda^{-d/\alpha} && \text{for } d/\alpha \in (1/2, 1), \\
f(\lambda) &= \lambda^{-2}\ln^{-1}(1/\lambda) && \text{for } d/\alpha = 2, \\
f(\lambda) &= \lambda^{-2} && \text{for } d/\alpha \in (2, \infty).
\end{aligned}
$$

As is seen from the integral equation (2.13), the function $m_1(t, x)$ is monotone. Then applying the Tauberian theorem (see [**8**, Ch. XIII]) to $\widehat{m}_1(\lambda, x)$ we find the asymptotics of $m_1(t, x)$ indicated in the statement of the theorem. $\qquad\square$

Let us complete the study of the asymptotic behaviour as $t \to \infty$ of the moments of particle number for the critical BRW.

THEOREM 4.4. *Let $\beta = \beta_c$. Then for the BRW on $\mathbb{Z}^d$, $d \in \mathbb{N}$, under condition (2.4), and every $x, y \in \mathbb{Z}^d$, $n \in \mathbb{N}$, the following asymptotic relations hold*

$$m_n(t, x, y) \sim C_n(x, y)u_n(t), \quad m_n(t, x) \sim C_n(x)v_n(t), \qquad t \to \infty,$$

where $C_n(x, y)$, $C_n(x)$ are positive constants, and the functions $u_n(t)$, $v_n(t)$ are as follows:

$$
\begin{aligned}
u_n(t) &= t^{-1/\alpha}, & v_n(t) &= t^{(1-1/\alpha)(n-1)} && \text{for } d/\alpha \in (1/2, 1), \\
u_n(t) &= t^{-1}, & v_n(t) &= (\ln t)^{n-1} && \text{for } d/\alpha = 1, \\
u_n(t) &= t^{d/\alpha-2}, & v_n(t) &= t^{(d/\alpha-1)(2n-1)} && \text{for } d/\alpha \in (1, 3/2), \\
u_n(t) &= t^{-1/2}(\ln t)^{n-1}, & v_n(t) &= t^{n-1/2} && \text{for } d/\alpha = 3/2, \\
u_n(t) &= t^{(d/\alpha-2)(2n-1)+n-1}, & v_n(t) &= t^{(d/\alpha-1)(2n-1)} && \text{for } d/\alpha \in (3/2, 2), \\
u_n(t) &= t^{n-1}(\ln t)^{1-2n}, & v_n(t) &= t^{2n-1}(\ln t)^{-2n+1} && \text{for } d/\alpha = 2, \\
u_n(t) &= t^{n-1}, & v_n(t) &= t^{2n-1} && \text{for } d/\alpha \in (2, \infty).
\end{aligned}
$$

Proof. We will find the asymptotics of the moments by induction using formulas (2.14), (2.15) and (2.10).

First we prove the assertions of the theorem for the local moments $m_n(t, x, y)$, $n \geqslant 1$. Denote the convolution

$$W_n(t) := \int_0^t m_1(t-s, x, 0)g_n(m_1(s, 0, y), \dots, m_{n-1}(s, 0, y))\mathrm{d}s.$$

By theorems 4.1 and 4.2, we have the asymptotics of $m_1(t, x, 0)$. Then by definition (2.10), we find the asymptotics of $g_n(m_1(s, 0, y), \dots, m_{n-1}(s, 0, y))$, when $n = 2$. Now we can find the asymptotics of convolution $W_n(t)$, $n = 2$, by lemma 2 for convolutions from [**26**]. To find the asymptotics of $m_n(t, x, y)$, $n = 2$, it remains only to express one by formula (2.14) and compare the growth rate of $m_1(t, x, y)$ and $W_n(t)$, $n = 2$. Following this scheme for $n > 2$, we obtain statements of the theorem for $m_n(t, x, y)$, $n > 2$.

Using a similar reasoning scheme and applying theorem 4.3, we find asymptotics of the moments $m_n(t, x)$, $n \geqslant 2$. □

Thus, by proving theorem 4.4, we thereby proved the main assertion from [**20**] for the critical case $\beta = \beta_c$, which was only announced in [**20**]. Moreover, in this section we have got an explicit form of $C_1(x, y)$ and $C_1(x)$.

5. Subcritical BRW

Let us find the asymptotic behaviour as $t \to \infty$ of the mean of particle population size $m_1(t, x)$ when the process started from point $x \in \mathbb{Z}^d$.

THEOREM 5.1. *Let $\beta < \beta_c$. Then the solution $m_1(t, x)$, $x \in \mathbb{Z}^d$, $d \in \mathbb{N}$, of the Cauchy problem* (2.8) *under $d/\alpha \in (1/2, \infty)$ satisfies the relation*

$$m_1(t, x) \sim C_1(x)v_1(t), \qquad t \to \infty,$$

where the functions $C_1(x)$ and $v_1(t)$ are as follows:

$$C_1(x) = -(\beta\gamma_{1,\alpha}\Gamma(1/\alpha))^{-1}, \qquad v_1(t) = t^{1/\alpha-1} \quad \text{for } d/\alpha \in (1/2, 1),$$

$$C_1(x) = -(\beta\gamma_{1,1})^{-1}, \qquad v_1(t) = \ln^{-1} t \quad \text{for } d/\alpha = 1,$$

$$C_1(x) = \frac{1 - \beta(G_0(0,0) - G_0(x,0))}{1 - \beta G_0(0,0)}, \quad v_1(t) \equiv 1 \qquad \text{for } d/\alpha \in (1, \infty),$$

and $\gamma_{1,\alpha}$ are positive constants.

Proof. To find the asymptotics of the function $m_1(t, x)$, we will use the Tauberian theorems (see [**8**, Ch. XIII]). By [**25**, lemma 5.1.4], which remains valid in the case of heavy tails (2.4), the Laplace transform of the solution $m_1(t, x)$ of the Cauchy problem (2.8) has the representation by the Green's function (4.1) for λ with $\mathrm{Re}\lambda > l$ where l is some non-negative number.

The asymptotics of $G_\lambda(0,0)$ as $\lambda \to 0$ was found in [**29**, theorem 1]. By inspection of its proof, we can observe that the first term of the asymptotics of $G_\lambda(x,y)$ with arbitrary $x, y \in \mathbb{Z}^d$ has the same form, because by (2.5) the asymptotic representation of the transition probability $p(t, x, y)$ is $h_{\alpha,d}/t^{d/\alpha}$, and it does not depend on the lattice points x, y. Then the next relations, as $\lambda \to 0$, follow

$$\widehat{m}_1(\lambda, x) \sim -(\beta\gamma_{1,\alpha})^{-1}\lambda^{-1/\alpha} \quad \text{for } d/\alpha \in (1/2, 1),$$

$$\widehat{m}_1(\lambda, x) \sim (\beta\gamma_{1,1})^{-1}(\lambda \ln \lambda)^{-1} \quad \text{for } d/\alpha = 1,$$

where the constants $\gamma_{d,\alpha} > 0$ were defined in [**29**, theorem 1].

Due to the monotonicity of $m_1(t, x)$ by (2.13), we can apply the Tauberian theorem for densities (see [**8**, Ch. XIII, theorem 4]) and get the following asymptotic

relations as $t \to \infty$

$$m_1(t,x) \sim -(\beta\gamma_{1,\alpha}\Gamma(1/\alpha))^{-1}t^{(1-\alpha)/\alpha} \quad \text{for } d/\alpha \in (1/2,1),$$
$$m_1(t,x) \sim -(\beta\gamma_{1,1})^{-1}(\ln t)^{-1} \quad\quad \text{for } d/\alpha = 1.$$

Let us consider the remaining case $d/\alpha \in (1,\infty)$, in which for every $x,y \in \mathbb{Z}^d$ the value of $G_0(x,y)$ is finite. Here by (4.1) we have

$$\widehat{m}_1(\lambda,x) \sim \frac{1 - \beta(G_0(0,0) - G_0(x,0))}{\lambda(1 - \beta G_0(0,0))}, \quad \lambda \to 0,$$

and therefore, by the Tauberian theorem for densities (see [**8**, Ch. XIII, theorem 4]), the asymptotics of the mean size of particle population is

$$m_1(t,x) \sim \frac{1 - \beta(G_0(0,0) - G_0(x,0))}{1 - \beta G_0(0,0)}, \quad t \to \infty.$$

Theorem 5.1 is proved. $\qquad\square$

Now we can find the asymptotics as $t \to \infty$ of mean of particle number $m_1(t,x,y)$ at every point $y \in \mathbb{Z}^d$ when process started from $x \in \mathbb{Z}^d$.

THEOREM 5.2. *Let $\beta < \beta_c$. Then for the BRW on $\mathbb{Z}^d$, $d \in \mathbb{N}$, satisfying condition (2.4), with representations for the first moments*

$$m_1(t,0,0) \sim C_1(0,0)u_1(t), \quad m_1(t,x) \sim C_1(x)v_1(t), \quad t \to \infty,$$

for every $x,y \in \mathbb{Z}^d$ the following asymptotic relation hold

$$m_1(t,x,y) \sim C_1(x,y)u_1(t), \quad t \to \infty, \tag{5.1}$$

where

$$C_1(x,y) = C_1(0,0)g(x)g(y), \qquad\qquad u_1(t) = t^{1/\alpha - 2} \qquad \text{for } d/\alpha \in (1/2,1),$$
$$C_1(x,y) = C_1(0,0)g(x)g(y), \qquad\qquad u_1(t) = t^{-1}\ln^{-2} t \quad \text{for } d/\alpha = 1,$$
$$C_1(x,y) = (C_1(x) + \beta G_0(y,0)C_1(0))h_{\alpha,d} \quad u_1(t) = t^{-d/\alpha} \qquad\quad \text{for } d/\alpha \in (1,\infty),$$
$$\qquad\quad + \beta^2 C_1(0,0)G_0(x,0)G_0(y,0),$$

and

$$g(x) := 1 - \beta\int_0^\infty (p(t,0,0) - p(t,x,0))\mathrm{d}s, \tag{5.2}$$

and $h_{\alpha,d}$ is defined in (2.5).

Proof. First, we find the asymptotics of $m_1(t,x,0)$. Substituting $y = 0$ into the integral equation (2.11), we obtain

$$m_1(t,x,0) = p(t,x,0) + \beta\int_0^t p(t-s,x,0)m_1(s,0,0)\mathrm{d}s, \tag{5.3}$$

where the asymptotics of $p(t,x,0)$ is known by relation (2.5) and the asymptotics of $m_1(t,0,0)$ is known by [**17**, theorem 3].

Note, that for $d/\alpha \in (1/2, 1]$ due to lemma 2 for convolutions from [**26**], the asymptotics of the integral $\int_0^t p(t-s,x,0)m_1(s,0,0)\mathrm{d}s$ has the same principal term up to a constant as the asymptotics of $p(t,x,0)$. Then the sum of the coefficients on the right-hand side of equation (5.3) satisfies the limit equalities

$$\lim_{t\to\infty}\left(1+\beta\int_0^t m_1(s,0,0)\mathrm{d}s\right)=\lim_{t\to\infty}m_1(t,0)=0,$$

due to representation of the $m_1(t,0)$ by the integral equation (2.13) and the limit of $m_1(t,0)$ by theorem 5.1. But for $d/\alpha \in (1,\infty)$, not all coefficients at principal terms of the asymptotics of the right-hand side of (5.3) tend to zero. For this reason, further we consider the case $d/\alpha \in (1/2, 1]$, when the BRW is recurrent, and the case $d/\alpha \in (1,\infty)$, when the BRW is transient, separately.

For $d/\alpha \in (1/2, 1]$, by the integral equation (2.11) and symmetry of the random walk, we have

$$m_1(t,x,0)-m_1(t,0,0)=f(t,0,x)+\beta\int_0^t f(t-s,0,x)m_1(s,0,0)\mathrm{d}s, \qquad (5.4)$$

where

$$f(t,z_1,z_2):=p(t,z_1,z_2)-p(t,z_1,0)$$

for any $z_1, z_2 \in \mathbb{Z}^d$. To find the asymptotics of $f(t,0,x)$, we use the symmetry of the underlying random walk and apply the statement about asymptotics of $p(t,0,0)-p(t,x,0)$ from (2.6). The asymptotics of $m_1(t,0,0)$ is known by [**17**, theorem 3]. To find the asymptotics of the integral in (5.4), it remains to use lemma 2 for convolutions from [**26**]. Then we obtain the equality

$$\int_0^t f(t-s,0,x)m_1(s,0,0)\mathrm{d}s=(1+\alpha_1(t))m_1(t,0,0)\int_0^\infty f(s,0,x)\mathrm{d}s,$$

for some $\alpha_1(t)\to 0$, $t\to\infty$. Finally, from equation (5.4), the asymptotic relation follows:

$$m_1(t,x,0)\sim g(x)m_1(t,0,0), \qquad t\to\infty, \qquad (5.5)$$

where $g(x)$ is defined in (5.2).

By the integral equation (2.12) and symmetry of random walk, we can write also

$$m_1(t,x,y)-m_1(t,x,0)=f(t,x,y)+\beta\int_0^t f(t-s,0,y)m_1(s,x,0)\mathrm{d}s. \qquad (5.6)$$

Properties of underlying random walks provide the equality $f(t,x,y)=-(p(t,0,0)-p(t,x-y,0))+(p(t,0,0)-p(t,x,0))$, and then by the asymptotic relation (2.6) we have

$$f(t,x,y)\sim(\widehat{\gamma}_{d,\alpha}(x)-\widehat{\gamma}_{d,\alpha}(x-y))t^{-(d+2)/\alpha}, \qquad t\to\infty. \qquad (5.7)$$

We now use relations (5.5) and (5.7) to find the asymptotics of the integral in (5.6) by lemma 2 for convolutions from [**26**]. Then by relation (5.6), it follows that

$$m_1(t,x,y)\sim g(x)g(y)m_1(t,0,0), \qquad t\to\infty,$$

where asymptotics of $m_1(t,0,0)$ is found in [**17**, theorem 3].

For the case $d/\alpha \in (1, \infty)$, by the integral equation (2.11) we express $m_1(t, x, 0)$. We use the asymptotics of $p(t, x, y)$ in (2.5) and the asymptotics of $m_1(t, 0, 0)$ in [**17**, theorem 3] to find the asymptotics of the integral by lemma 2 for convolutions from [**26**]. Then in view of theorem 5.1, the following equality holds

$$m_1(t, x, 0) = \left(1 + (1 + a_{d,\alpha}(t, x, 0))\beta \int_0^\infty m(s, 0, 0) \mathrm{d}s\right) p(t, x, 0)$$
$$+ \left((1 + b_{d,\alpha}(t, 0, 0))\beta \int_0^\infty p(s, x, 0) \mathrm{d}s\right) m_1(t, 0, 0),$$

where $a_{d,\alpha}(t, x, 0) \to 0$, $b_{d,\alpha}(t, 0, 0) \to 0$ as $t \to \infty$ for each $x \in \mathbb{Z}^d$. Then by (2.13)

$$m_1(t, x, 0) \sim m_1(t, 0)p(t, x, 0) + \beta G_0(x, 0)m_1(t, 0, 0), \qquad t \to \infty.$$

In the same way, from the integral equation (2.12) we derive

$$m_1(t, x, y) = p(t, x, y) + \left((1 + a_{d,\alpha}(t, 0, y))\beta \int_0^\infty m_1(s, x, 0) \mathrm{d}s\right) p(t, 0, y)$$
$$+ \left((1 + b_{d,\alpha}(t, x, 0))\beta \int_0^\infty p(s, 0, y) \mathrm{d}s\right) m_1(t, x, 0),$$

where $a_{d,\alpha}(t, 0, y) \to 0$, $b_{d,\alpha}(t, x, 0) \to 0$ as $t \to \infty$ for each $x, y \in \mathbb{Z}^d$. Then, due to the independence of the first term of the asymptotics $p(t, x, y)$ from the coordinates $x, y \in \mathbb{Z}^d$ and by equality (2.13), the asymptotic equality holds

$$m_1(t, x, y) \sim m_1(t, x)h_{\alpha,d} t^{-d/\alpha} + \beta G_0(y, 0)m_1(t, x, 0), \qquad t \to \infty.$$

From here we derive the asymptotic representation (5.1) for $m_1(t, x, y)$. Theorem 5.2 is proved. $\qquad \square$

Analysis of the first-order moments $m_1(t, x)$ and $m_1(t, x, y)$ was fulfilled in theorems 5.1 and 5.2. In the next theorem we extend the related results, for the subcritical BRW, to the case of higher-order moments $m_n(t, x)$ and $m_n(t, x, y)$, $n \geqslant 2$.

THEOREM 5.3. *Let $\beta < \beta_c$. Then for the BRW on $\mathbb{Z}^d$, $d \in \mathbb{N}$, satisfying condition (2.4) for every $n \geqslant 1$ and every $x, y \in \mathbb{Z}^d$ the asymptotic relations hold*

$$m_n(t, x, y) \sim C_n(x, y)u_1(t), \quad t \to \infty, \tag{5.8}$$
$$m_n(t, x) \sim C_n(x)v_1(t), \qquad t \to \infty. \tag{5.9}$$

Here $C_1(x, y)$, $u_1(t)$ and $C_1(x)$, $v_1(t)$ are obtained in theorem 5.2, while for $n \geqslant 2$ we have

$$C_n(x, y) = C_1(x, y) + C_1(x, 0) \int_0^\infty g_n(m_1(s, 0, y), \ldots, m_{n-1}(s, 0, y)) \mathrm{d}s,$$

where for the function $g_n(m_1(t), \ldots, m_{n-1}(t))$ defined in (2.10), with the notation $m_i(t) \equiv m_i(t, 0, y)$, $i \in \mathbb{N}$, the following asymptotic relation holds

$$g_n(m_1(t), m_2(t), \ldots, m_{n-1}(t)) \sim \frac{\beta^{(2)}}{2} \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} m_i(t) m_{n-i}(t), \quad t \to \infty.$$
(5.10)

Besides, for $n \geqslant 2$ we also have

$$C_n(x) = C_1(x), \qquad\qquad\qquad \text{for } d/\alpha \in (1/2, 1],$$

$$C_n(x) = C_1(x) + \chi_n(x) \int_0^\infty m_1(s, x, 0) \mathrm{d}s, \qquad \text{for } d/\alpha \in (1, \infty),$$

where

$$\chi_n(x) = \frac{\beta^{(2)}}{2} \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} C_i(x) C_{n-i}(x).$$
(5.11)

Proof. We use the notation $m_i(t)$, $i \in \mathbb{N}$, to formulate statements that are valid simultaneously for $m_i(t, x, y)$ and $m_i(t, x)$. The proof is by induction on $n \geqslant 2$ and consists in alternate derivation of asymptotics of the functions $g_n(m_1(t), \ldots, m_{n-1}(t))$ and of the moments $m_n(t)$ by the integral equations for moments (2.14) and (2.15).

For $n = 2$ by definition (2.10), we have $g_2(m_1(t)) = \beta^{(2)} m_1^2(t)$. Then relation (5.10) under $n = 2$ follows.

This makes it possible to find the asymptotics of $m_2(t, x, y)$ and $m_2(t, x)$ using the integral equations for moments (2.14) and (2.15). To find asymptotics of convolutions in these equations, we write down the asymptotics of $u_1(t)$, $u_1^2(t)$ and $v_1^2(t)$ from theorems 5.2 and 5.1, that is

$$u_1(t) = t^{1/\alpha - 2}, \qquad u_1^2(t) = t^{2/\alpha - 4}, \qquad v_1^2(t) = t^{2/\alpha - 2} \quad \text{for } d/\alpha \in (1/2, 1),$$

$$u_1(t) = t^{-1} \ln^{-2} t, \quad u_1^2(t) = t^{-2} \ln^{-4} t, \quad v_1^2(t) = \ln^{-2} t \quad \text{for } d/\alpha = 1,$$

$$u_1(t) = t^{-d/\alpha}, \qquad u_1^2(t) = t^{-2d/\alpha}, \qquad v_1^2(t) = 1 \qquad \text{for } d/\alpha \in (1, \infty),$$

and apply lemma 2 for convolutions from [26], namely, statements (k), (n) to $m_2(t, x, y)$, and (d), (e) to $m_2(t, x)$ for cases $d/\alpha = 1$ and $d/\alpha \in (1/2, 1) \cup (1, \infty)$ respectively. Then the following equalities hold

$$\int_0^t m_1(t - s, x, 0) g_2(m_1(s, 0, y)) \mathrm{d}s$$

$$= (1 + a_{d,\alpha}(t)) m_1(t, x, 0) \int_0^\infty g_2(m_1(s, 0, y)) \mathrm{d}s,$$
(5.12)

$$\int_0^t m_1(t - s, x, 0) g_2(m_1(s, 0)) \mathrm{d}s$$

$$= (1 + b_{d,\alpha}(t)) g_2(m_1(t, 0)) \int_0^\infty m_1(s, x, 0) \mathrm{d}s,$$
(5.13)

where $a_{d,\alpha}(t) \to 0$, $b_{d,\alpha}(t) \to 0$, as $t \to 0$. Hence, by the integral equations (2.14) and (2.15), we obtain the following asymptotic equalities

$$m_2(t,x,y) \sim m_1(t,x,y) + m_1(t,x,0) \int_0^\infty g_2(m_1(s,0,y)) \mathrm{d}s, \quad t \to \infty,$$

$$m_2(t,x) \sim m_1(t,x) + g_2(m_1(t,0)) \int_0^\infty m_1(s,x,0) \mathrm{d}s, \qquad t \to \infty.$$

Now we rewrite the asymptotics of the functions above in terms of $C_1(x,y)$, $u_1(t)$, $C_1(x)$, $v_1(t)$ from theorems 5.1 and 5.2, namely

$$g_2(m_1(t,x,y)) \sim \beta^{(2)} C_1^2(x,y) u_1^2(t), \qquad\qquad\qquad t \to \infty,$$

$$m_2(t,x,y) \sim \left(C_1(x,y) + C_1(x,0) \int_0^\infty g_2(m_1(s,0,y)) \mathrm{d}s \right) u_1(t), \quad t \to \infty,$$

$$g_2(m_1(t,x)) \sim \beta^{(2)} C_1^2(x) v_1^2(t), \qquad\qquad\qquad t \to \infty.$$

Then for function $m_2(t,x)$, we have under $d/\alpha \in (1/2, 1]$ the asymptotics

$$m_2(t,x) \sim C_1(x) v_1(t), \qquad t \to \infty,$$

and, under $d/\alpha \in (1, \infty)$, the asymptotics

$$m_2(t,x) \sim \left(C_1(x) + \beta^{(2)} C_1^2(0) \int_0^\infty m_1(s,x,0) \mathrm{d}s \right) v_1(t), \qquad t \to \infty.$$

Continuing the same reasoning, by alternate applying the formula in definition (2.10) and the integral equations (2.14), (2.15), we will obtain the asymptotic representations for $g_n(m_1(t), \ldots, m_{n-1}(t))$, $n \geqslant 2$, as polynomials of $u_1(t)$, when $m_n(t)$ is $m_n(t,x,y)$, and as polynomials of $v_1(t)$, when $m_n(t)$ is $m_n(t,x)$. Also we will obtain asymptotic representations of $m_n(t,x,y)$ and $m_n(t,x)$ as the product of some constant on $u_1(t)$ and $v_1(t)$ respectively.

Assume that for some $n \geqslant 2$ the asymptotic relations (5.10), (5.8) and (5.9) hold. Then, in particular, we have

$$m_{n-1}(t,x,y) \sim C_{n-1}(x,y) u_1(t), \quad m_{n-1}(t,x) \sim C_{n-1}(x) v_1(t), \qquad t \to \infty.$$

Let us prove that these relations are also valid for $n+1$.

Rewrite formula (5.10) in the form

$$g_n(m_1(t,x,y), \ldots, m_{n-1}(t,x,y)) \sim \frac{\beta^{(2)}}{2} \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} C_i(x,y) C_{n-i}(x,y) u_1^2(t),$$

$$g_n(m_1(t,x), \ldots, m_{n-1}(t,x)) \sim \frac{\beta^{(2)}}{2} \sum_{i=1}^{n-1} \frac{n!}{i!(n-i)!} C_i(x) C_{n-i}(x) v_1^2(t).$$

Due to the power-logarithmic behaviour of the functions $u_1(t)$ and $v_1(t)$, and by analogue with the scheme of obtaining (5.12) and (5.13), we derive by lemma 2 for

convolutions from [**26**], as $t \to \infty$, the following asymptotic representations

$$\int_0^t m_1(t-s,x,0)g_n(m_1(t,0,y),\ldots,m_{n-1}(t,0,y))\mathrm{d}s$$

$$= (1+a_{d,\alpha,n}(t))m_1(t,x,0)\int_0^\infty g_2(m_1(s,0,y),\ldots,m_{n-1}(s,0,y))\mathrm{d}s,$$

$$\int_0^t m_1(t-s,x,0)g_n(m_1(s,0),\ldots,m_{n-1}(s,0))\mathrm{d}s$$

$$= (1+b_{d,\alpha,n}(t))g_n(m_1(t,0),\ldots,m_{n-1}(t,0))\int_0^\infty m_1(s,x,0)\mathrm{d}s,$$

where $a_{d,\alpha,n}(t) \to 0$, $b_{d,\alpha,n}(t) \to 0$ as $t \to \infty$. By analogue with the scheme of obtaining the asymptotic relations for $m_2(t,x,y)$ and $m_2(t,x)$ above and equations (2.14) and (2.15), as $t \to \infty$, for $d/\alpha \in (1/2,\infty)$, it is possible to obtain the following relations

$$m_n(t,x,y) \sim \left(C_1(x,y) + C_1(x,0)\int_0^\infty g_n(m_1(s,0,y),\ldots,m_{n-1}(s,0,y))\mathrm{d}s\right) u_1(t),$$

as well as

$$m_n(t,x) \sim C_1(x)v_1(t) \qquad\qquad\qquad \text{for } d/\alpha \in (1/2,1],$$

$$m_n(t,x) \sim \left(C_1(x) + \chi_0(x)\int_0^\infty m_1(s,x,0)\mathrm{d}s\right)v_1(t) \qquad \text{for } d/\alpha \in (1,\infty),$$

where $\chi_0(x)$ is defined in (5.11).

Then the representation

$$g_{n+1}(m_1(t),\ldots,m_n(t)) = \frac{\beta^{(2)}}{2}\sum_{i=1}^n \frac{(n+1)!}{i!(n+1-i)!}m_i(t)m_{n+1-i}(t)$$

$$+ \sum_{r=3}^{n+1}\frac{\beta^{(r)}}{r!}\sum_{\substack{i_1,\ldots,i_r>0,\\i_1+\cdots+i_r=n+1}}\frac{(n+1)!}{i_1!\cdots i_r!}m_{i_1}(t)\cdots m_{i_r}(t),$$

justifies the validity of relation (5.10) and simultaneously the validity of relations (5.8) and (5.9), which completes the proof of the theorem. $\qquad\square$

By proving theorem 5.3, we have provided also complete proofs for the subcritical case $\beta < \beta_c$ of the theorem from [**20**] and obtained an explicit form of the appropriate constants.

Therefore, in §§ 4 and 5 the classification of the number of particles growth rate for the critical and subcritical BRWs with heavy tails, previously announced in [**20**], has been obtained.

Acknowledgements

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