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INCOME DISTRIBUTION BY AGE GROUP AND PRODUCTIVE BUBBLES

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The aim of this paper is to study the role of the distribution of income by age group on the existence of speculative bubbles. A crucial question is whether this distribution may promote a bubble associated to a larger level of capital, that is a productive bubble. We address these issues in an overlapping generations model where agents live three periods and productive investment done in the first period of life is an illiquid investment whose return occurs in the following two periods. A bubble is a liquid speculative investment that facilitates intertemporal consumption smoothing. We show that the distribution of income by age group determines both the existence and the effect of bubbles on aggregate production. We also show that fiscal policy, by changing the distribution of income, may facilitate or prevent the existence of bubbles and may also modify the effect that bubbles have on aggregate production.

Keywords: Bubble, Efficiency, Income Distribution, Overlapping Generations

1. INTRODUCTION

Individuals have heterogeneous savings behaviors over the life cycle. This suggests that the population size of each generation may affect the asset market and is a determinant of the asset price. This has been studied by Abel (2001) and Geanakoplos et al. (2004), among others, who have shown that the relative size of the different age groups affects the price of the assets.

We adopt a complementary view taking into account that the distribution of income by age group is an important determinant of the aggregate savings. Accordingly, we examine how the distribution of income by age group affects the asset market. Interestingly, cross-country differences in this distribution are very large. Table 1 shows a cross-country comparison of the distribution of income by age group when we consider three age groups: young, middle-aged, and old.[1] This table shows that middle-aged individuals generally obtain the largest fraction of

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TABLE 1. Income distribution by age group

	Young	Middle-aged	Old
Austria	0.36	0.40	0.24
Belgium	0.38	0.40	0.22
Czech Republic	0.43	0.37	0.20
Denmark	0.36	0.42	0.22
Finland	0.35	0.42	0.23
France	0.33	0.39	0.28
Germany	0.36	0.40	0.24
Greece	0.37	0.36	0.27
Hungary	0.39	0.37	0.24
Italy	0.34	0.38	0.28
Netherlands	0.37	0.41	0.22
Norway	0.38	0.40	0.22
Poland	0.41	0.39	0.20
Portugal	0.37	0.37	0.26
Spain	0.39	0.37	0.24
Sweden	0.36	0.40	0.24
United Kingdom	0.40	0.38	0.22
United States	0.36	0.45	0.19

Note: The second column is the fraction of income obtained by young individuals, the third column is the fraction of income obtained by middle-aged individuals and the last column is the fraction of income obtained by old individuals. The data sources used to obtain these fractions are mentioned in Appendix A.4.

total income, whereas the old individuals obtain the smallest fraction. However, beyond this common feature, there are large cross-country differences in the distribution of income by age group. For example, the minimum value of the fraction of total income obtained by the young individuals is 33%, whereas the maximum value is 43%. These cross-country differences are even larger if we consider the fraction of total income obtained by the old individuals. The maximum value of this fraction is 28%, whereas its minimum value is only 19%.

We are interested in the interplay between income distribution by age group and the value of assets without fundamental value, that is, bubbles. Indeed, the literature has already shown that the existence of bubbles depends on the savings decisions over the life cycle. In particular, Tirole (1985) shows that bubbles arise when the equilibrium of an overlapping generations (OLGs) model is dynamically inefficient.[2] This form of inefficiency is explained by imperfections that force individuals to use productive capital to postpone consumption. In this case, they overaccumulate capital and, hence, the equilibrium is dynamically inefficient. Tirole (1985) shows that, in this situation, individuals may use an asset without fundamental value to postpone consumption. Therefore, when the equilibrium without bubble is dynamically inefficient, an equilibrium with bubbles may also exist.[3] These bubbles reduce the stock of productive capital and also

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gross domestic product (GDP). However, more recently, Caballero et al. (2006) and Martin and Ventura (2012) provide convincing evidence showing that bubbles arise during economic booms. Obviously, this evidence suggests that GDP should be larger in the equilibrium with bubbles. To explain this evidence, we refer to the concept of productive bubbles, defined as bubbles that facilitate a larger accumulation of productive capital. Therefore, we can distinguish between unproductive bubbles, which arise when the equilibrium without bubbles is dynamically inefficient, and productive bubbles, which may arise when the equilibrium without bubbles is dynamically efficient.

The purpose of this paper is to contribute to the aforementioned literature by showing how the distribution of income by age group affects dynamic efficiency of the bubbleless equilibrium and the existence of productive bubbles. To this end, we extend the OLG model with agents who live three periods studied in Raurich and Seegmuller (2019) by assuming that individuals work in the first two periods of life. As a consequence, labor income is distributed between young and middle-aged individuals. We will highlight that this assumption plays a crucial role for our results. In this model, the distribution of labor income between young and middle-aged individuals and the distribution of capital income between middle-aged and old individuals determine the distribution of the total income by age group. We show that the model can generate the income distributions displayed in Table 1.

In the model, productive investment is done by young individuals, and it is an illiquid investment whose return occurs in the following two periods of life. The bubble is a liquid investment that facilitates intertemporal consumption smoothing. Note that this model introduces an important distinction between young and middle-aged individuals. The former invest in productive capital, whereas the later only invest in financial assets to smooth consumption. This distinction introduces heterogeneity across individuals that, as shown in Martin and Ventura (2012) and Raurich and Seegmuller (2019), is necessary to have productive bubbles. Therefore, bubbles can be either productive or unproductive.

We first show that if a large part of the labor income is earned by middle-aged individuals and a large part of the capital income is earned by old individuals then neither the young nor the middle-aged individuals are interested in holding the speculative asset in order to postpone consumption. In this case, an equilibrium with bubbles does not exist.

In addition to its existence, we also study how the distribution of income by age group affects whether a bubble is productive or not. On the one hand, we show that if a large fraction of the labor income is earned by the young individuals and a large fraction of the capital income is earned by the middle-aged individuals, households overaccumulate capital to postpone consumption. In this case, the equilibrium without bubbles is dynamically inefficient. As in Tirole (1985), an equilibrium with bubbles exists, but these bubbles are unproductive because they are aimed to postpone consumption.

On the other hand, we show that bubbles can be productive in two different cases: when the income obtained by the middle-aged individuals is sufficiently large and when it is sufficiently small.[4] In the first case, young individuals are short sellers of the bubble, which is used to transfer consumption from the middle-aged period to the other two periods of life. This transfer reduces the cost of investment, in terms of marginal utility, and hence, young individuals increase productive investment. This explains that the bubble is productive. In the second case, the middle-aged individuals obtain a small fraction of income and the bubble is used to transfer consumption from the young and the old periods of life to the middle-aged period. In this case, middle-aged households are short sellers of the bubble. As a consequence, the marginal utility of consumption of the middle-aged individuals decreases, which increases the relative benefit, in terms of marginal utility, of the investment in the productive asset. This explains that the bubble is productive in this second case. At this point, it is important to highlight that this last mechanism is different from the existing literature where the bubble is productive only when it provides liquidities to the young investor.

The distribution of income by age group is largely modified by capital and labor income taxes. Fiscal policy, by changing the distribution of income, may facilitate or prevent the existence of productive bubbles. We also show that the effects of fiscal policy crucially depend on the distribution of income. We illustrate numerically this conclusion showing that, for the distributions of income in the US and several European economies, the effect on production of the same fiscal policy may be substantially different in these countries.

The paper is organized as follows. Section 2 presents the model. Section 3 studies the equilibrium without bubbles and characterizes dynamic efficiency. Section 4 studies the equilibrium with bubbles and obtains the distribution of income by age group for which bubbles exist and are productive. Section 5 discusses the effect of fiscal policy on productive capital. Section 6 concludes the paper. Some technical details are relegated to an Appendix.

2. MODEL

Consider an OLG economy with agents who live three periods. In period t, the economy is populated by N_t young individuals. Let $n = N_t/N_{t-1} > 0$ be the constant ratio between the number of young and middle-aged individuals in period t. The utility of an individual born in period t is

$$\ln c_{1,t} + \beta \ln c_{2,t+1} + \beta^2 \ln c_{3,t+2}, \tag{1}$$

where $c_{1,t}$ is the consumption when young, $c_{2,t+1}$ is the consumption in the middle age, $c_{3,t+2}$ is the consumption when old, and $\beta \in (0, 1)$ is the subjective discount rate.

Young individuals work and obtain an after-tax labor income $(1 - \tau_w)\, \xi_1 w_t$ that they use to consume $c_{1,t}$ and invest in both a speculative asset, $b_{1,t}$, and a non-speculative asset, a_{t+1}. The wage per efficiency unit is w_t, $\xi_1 > 0$ measures the

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efficiency units of a young worker, and $\tau_w \in (0, 1)$ is the tax rate on labor income. We assume that only the young individuals can invest in the non-speculative asset, which is an illiquid investment that provides returns in the following two periods of life. In the second period of life, agents also work and obtain an after-tax labor income $(1 - \tau_w)\,\xi_2 w_{t+1}$, where $\xi_2 > 0$ measures the efficiency units of a middle-aged worker. Middle-aged workers also obtain capital income from the return on the non-speculative asset that after taxes is $(1 - \tau_k)\,\phi_1 q_{t+1}$. The return of one unit of productive capital is q_{t+1}, ϕ_1 are the units of productive capital that middle-aged individuals obtain from one unit of investment, and $\tau_k \in (0, 1)$ is the capital income tax rate. Finally, they sell the speculative asset and obtain $R_{t+1}b_{1,t}$.[5] The return from selling the bubble, R_{t+1}, is the growth rate of the price of the bubble. The income obtained by middle-aged individuals is used to consume, $c_{2,t+1}$, and invest in speculative assets, $b_{2,t+1}$. In the last period of life, individuals are retired and, hence, they do not obtain labor income. They sell the speculative asset, $R_{t+2}b_{2,t+1}$, and they obtain $(1 - \tau_k)\,\phi_2 q_{t+2}$ from the return after taxes on the non-speculative asset, where ϕ_2 are the units of productive capital that old individuals obtain from one unit of investment done in the first period of life. Old individuals consume $c_{3,t+2}$. It follows that the budget constraints of the young, middle-aged, and old individuals are, respectively,

$$c_{1,t} + a_{t+1} + b_{1,t} = (1 - \tau_w)\,\xi_1 w_t, \tag{2}$$

$$c_{2,t+1} + b_{2,t+1} = (1 - \tau_w)\,\xi_2 w_{t+1} + (1 - \tau_k)\,q_{t+1}\phi_1 a_{t+1} + R_{t+1}b_{1,t}, \tag{3}$$

$$c_{3,t+2} = R_{t+2}b_{2,t+1} + (1 - \tau_k)\,q_{t+2}\phi_2 a_{t+1}. \tag{4}$$

The speculative asset is short sell when either $b_{1,t} < 0$ or $b_{2,t+1} < 0$. In such a case, we assume that there is no default on reimbursement, especially at the old age. It can be justified by considering that $b_{i,t} < 0$ corresponds to loan contracts with some financial institution, which are enforceable through binding legal commitments. We can also argue that if $b_{2,t+1} < 0$, the speculative asset is collateralized by income at the old age, meaning that $R_{t+2}b_{2,t+1} \geqslant -(1 - \tau_k)\,q_{t+2}\phi_2 a_{t+1}$. Such a constraint is never binding, since the consumption $c_{3,t+2}$ is always strictly positive.

We note first that the investment in the non-speculative asset only when young is a simplifying assumption aimed to introduce a relevant difference in the productivity of the investment decisions of the different age groups. In fact, it is a reasonable assumption once this productive investment is considered as investment in education or investment in new companies. These forms of productive investment clearly decline as individuals get older. We also note that the return on productive investment depends on whether the investment has been done one or two periods before. This is a consequence of assuming that the productivity of capital depends on the period in which investment has been done. This is formalized through a simple form of vintage capital. This second assumption is introduced to generate the distribution of capital income between middle-aged and old individuals. Similarly, the difference in the efficiency units of labor between young and middle-aged individuals is introduced to generate the distribution of labor income between these two groups of individuals. The joint distribution of labor and capital income will be used in our analysis to determine the distribution of total income by age group.

We assume that government revenues are used to finance a useless government spending, G_t. Thus, an increase in the tax rates will cause a variation in this government spending that will not affect individual's decisions, as government spending is assumed to be useless. The government budget constraint is

$$\tau_w(\xi_1 w_t N_t + \xi_2 w_t N_{t-1}) + \tau_k\,(q_t\phi_1 a_t N_{t-1} + q_t\phi_2 a_{t-1} N_{t-2}) = G_t.$$

Technology is characterized by the following aggregate production function:

$$Y_t = AK_t^{\alpha} L_t^{1-\alpha}, \text{ with } A > 0 \text{ and } \alpha \in (0, 1),$$

where Y_t is the aggregate production, L_t is the total amount of efficiency units of labor, and K_t is the stock of productive capital in the economy. Using $k_t \equiv K_t/L_t$, $Y_t/L_t = Ak_t^{\alpha}$, and competitive factor prices satisfy

$$w_t = (1-\alpha)\,Ak_t^{\alpha}, \tag{5}$$

and

$$q_t = \alpha Ak_t^{\alpha-1}. \tag{6}$$

We complete the characterization of the model with the market-clearing conditions for capital, labor, and the speculative asset. The market-clearing condition for capital is

$$K_t = N_{t-1}\phi_1 a_t + N_{t-2}\phi_2 a_{t-1},$$

where $\phi_1 a_t$ and $\phi_2 a_{t-1}$ measure, respectively, the units of productive capital owned by middle-aged and old individuals. The market-clearing condition for efficiency units of labor is

$$L_t = N_t\xi_1 + N_{t-1}\xi_2,$$

where ξ_1 and ξ_2 measure, respectively, the efficiency units of labor provided by young and middle-aged workers. We use these two market-clearing conditions to define the fraction of productive capital owned by the middle-aged individuals:

$$\Omega_t = \frac{n\phi_1 a_t}{n\phi_1 a_t + \phi_2 a_{t-1}}, \tag{7}$$

and the fraction of efficiency units of employment provided by the young individuals:

$$\Sigma = \frac{n\xi_1}{n\xi_1 + \xi_2}. \tag{8}$$

TABLE 2. Income distribution by age group

	Σ	$\widehat{\Omega}$	Ω
Austria	0.62	0.44	0.55
Belgium	0.61	0.42	0.49
Czech Republic	0.84	0.60	0.63
Denmark	0.56	0.38	0.51
Finland	0.57	0.40	0.49
France	0.53	0.26	0.33
Germany	0.57	0.36	0.42
Greece	0.78	0.48	0.52
Hungary	0.65	0.41	0.48
Italy	0.63	0.39	0.47
Netherlands	0.61	0.44	0.56
Norway	0.72	0.54	0.59
Poland	0.74	0.55	0.58
Portugal	0.64	0.38	0.47
Spain	0.66	0.42	0.52
Sweden	0.63	0.45	0.55
United Kingdom	0.66	0.44	0.47
United States	0.60	0.52	0.58

Note: Σ is the fraction of labor income obtained by the young individuals. $\widehat{\Omega}$ is the fraction of capital income obtained by the middle-aged individuals when pensions are considered part of the capital income of the old. Finally, Ω is the fraction of capital income obtained by middle-aged individuals when pensions are not considered as capital income of the old. The empirical strategy followed to obtain these fractions is explained in Appendix A.4.

Note that at a steady state with $a_t = a_{t-1}$, the fraction of productive capital simplifies to the following parameter:

$$\Omega = \frac{n\phi_1}{n\phi_1 + \phi_2}.$$

The fractions Σ and Ω measure the distribution of before-taxes labor and capital income by age group. In Appendix A.4, we use the distribution of total income by age group displayed in Table 1 and two plausible assumptions of the model, the old do not obtain labor income and the young do not obtain capital income, to obtain the values of Σ and Ω displayed in Table 2. This table shows huge differences across countries in the value of Σ and Ω. As an example, the largest value of Σ is 84%, whereas the minimum value is only 53% and the largest value of Ω almost doubles its minimum value. Note that these very large differences are the consequence of both differences in the relative size of the age groups and also differences in the mean income of each age group.

From the previous two market-clearing conditions, we also obtain that capital per efficiency unit of labor is

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$$k_t = \frac{N_{t-1}\phi_1 a_t + N_{t-2}\phi_2 a_{t-1}}{N_t \xi_1 + N_{t-1}\xi_2},$$

which can be rewritten as

$$k_t = \frac{\phi_1}{n\xi_1 + \xi_2} a_t + \frac{\phi_2}{n^2\xi_1 + n\xi_2} a_{t-1}. \quad \textbf{(9)}$$

We assume that the speculative asset is supplied in one unit at a price p_t in period t. New investments in this asset by young and middle-aged individuals are in quantities ϵ_t and $1-\epsilon_t$, respectively. Therefore, the values of this asset bought or sold by these agents are $B_{1,t} = b_{1,t}N_t = p_t\epsilon_t$ and $B_{2,t} = b_{2,t}N_{t-1} = p_t(1-\epsilon_t)$. Since this asset has no fundamental value, it is a bubble if $p_t = B_{1,t} + B_{2,t} > 0$, which happens when $nb_{1,t} + b_{2,t} > 0$. Finally, the market-clearing condition for the speculative asset at period $t+1$ is

$$N_{t+1}b_{1,t+1} + N_t b_{2,t+1} = R_{t+1}\left(N_t b_{1,t} + N_{t-1}b_{2,t}\right).$$

The left-hand side of the previous equation is the value of the speculative asset bought by young and middle-aged individuals, whereas the right-hand side is the value of the speculative asset sold by middle-aged and old individuals. The speculative asset sold in period $t+1$ is multiplied by the growth rate of the price, R_{t+1}, as it was purchased in period t. This equation can be rewritten as

$$nb_{1,t+1} + b_{2,t+1} = \frac{R_{t+1}}{n}\left(nb_{1,t} + b_{2,t}\right). \quad \textbf{(10)}$$

From the previous arguments, it follows that there is a bubble when $nb_{1,t} + b_{2t} > 0$, while a bubbleless equilibrium is given by $b_{1,t} = b_{2t} = 0$.

3. EQUILIBRIA WITHOUT BUBBLE

We start by analyzing the model when there is no bubble, that is $b_{1,t} = b_{2,t} = 0$. In this case, the household's budget constraint rewrites:

$$c_{1,t} = (1-\tau_w)\,\xi_1 w_t - a_{t+1}, \quad \textbf{(11)}$$

$$c_{2,t+1} = (1-\tau_w)\,\xi_2 w_{t+1} + (1-\tau_k)\,q_{t+1}\phi_1 a_{t+1}, \quad \textbf{(12)}$$

$$c_{3,t+2} = (1-\tau_k)\,q_{t+2}\phi_2 a_{t+1}. \quad \textbf{(13)}$$

Maximizing the utility under the budget constraints (11)–(13), we get

$$\frac{1}{(1-\tau_w)\,\xi_1 w_t - a_{t+1}} = \frac{(1-\tau_k)\,\beta q_{t+1}\phi_1}{(1-\tau_w)\,\xi_2 w_{t+1} + (1-\tau_k)\,q_{t+1}\phi_1 a_{t+1}} + \frac{\beta^2}{a_{t+1}}. \quad \textbf{(14)}$$

This equation equalizes the marginal cost, measured by the marginal utility of the young individual, of investing an additional unit of the illiquid asset when young with the marginal benefit, measured by the marginal utility of both middle-aged and old individuals times the returns from that investment obtained in the following two periods of life. From using (5) and (6), the previous equation can be rewritten as

$$k_{t+1} = \frac{\alpha\,(1-\tau_k)\,\phi_1 a_{t+1}}{(1-\alpha)\,(1-\tau_w)\,\xi_2} \times \left(\frac{\beta\,(1+\beta)\,(1-\tau_w)\,\xi_1\,(1-\alpha)\,Ak_t^{\alpha} - \left(1+\beta+\beta^2\right) a_{t+1}}{\left(1+\beta^2\right) a_{t+1} - \beta^2\,(1-\tau_w)\,\xi_1\,(1-\alpha)\,Ak_t^{\alpha}} \right). \quad \textbf{(15)}$$

Note that using (15), we can implicitly define a_{t+1} as a function of k_{t+1} and k_t. Substituting it into (9), we deduce that k_{t+1} implicitly depends on k_t and k_{t-1}. This explains that two initial conditions, $k_{-1} > 0$ and $k_0 > 0$, are required in the following definition of the equilibrium:

DEFINITION 1. *Given $k_{-1} \geqslant 0$ and $k_0 \geqslant 0$, an equilibrium without bubble is a path $\{k_t, a_t\}_{t=1}^{\infty}$ that solves the system of equations (9) and (15).*

In the following, we restrict our attention to steady states, because our main aim is to compare stationary equilibria with and without bubbles, and understand the role of the distribution of income by age group.

3.1. Steady State

We use (9) and (15) to show that there is a unique steady state, and using (7) and (8), it can be shown that the steady state values of productive investment, a^*, and capital, k^*, are

$$a^* = \frac{\Omega n \xi_1}{\phi_1 \Sigma} k^*, \quad \textbf{(16)}$$

$$k^* = \left(\frac{(1-\alpha)(1-\tau_w)(1-\Sigma)+\alpha(1-\tau_k)\Omega}{(1-\alpha)\beta^2(1-\tau_w)(1-\Sigma)+\left(\beta+\beta^2\right)\alpha(1-\tau_k)\Omega} + 1 \right)^{\frac{1}{\alpha-1}} \left(\frac{n\Omega}{A\phi_1(1-\alpha)(1-\tau_w)\Sigma} \right)^{\frac{1}{\alpha-1}}. \quad \textbf{(17)}$$

Note that the capital stock at the steady state increases with the fraction of labor income obtained by the young individuals, Σ, and it also increases with the fraction of capital income obtained by the middle-aged individuals, Ω. On the one hand, an increase in Σ rises the income obtained by the young individuals, who then increase investment in productive capital. On the other hand, an increase in Ω reduces the income obtained by old individuals. Young individuals then compensate this reduction by increasing the investment in the productive asset.

The previous arguments show that the willingness to postpone consumption is large when Σ and Ω are large, which suggests that in this case the equilibrium will be dynamically inefficient. This is analyzed in the following subsection. For the sake of simplicity, in the following subsection, we set taxes to zero. We analyze the effect of fiscal policy on the capital stock in Section 5.

3.2. Dynamic Efficiency

The steady-state equilibrium is dynamically efficient when aggregate consumption increases with investment. This is a direct implication of the results obtained by Abel et al. (1989) and de la Croix and Michel (2002). As it is well known,

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this occurs when the return on investment is larger than population growth. In this model, this condition implies that $(\phi_1 + \phi_2/n)\, q > n$. Assuming that taxes are equal to zero, using (6) and (17), we obtain that the steady state is dynamically efficient when the following condition holds:

$$\left(\frac{(1-\alpha)(1-\Sigma)+\alpha\Omega}{(1-\alpha)\beta^2(1-\Sigma)+\alpha\Omega\left(\beta+\beta^2\right)}+1\right)\left(\frac{\alpha}{1-\alpha}\right)>\Sigma. \quad \textbf{(18)}$$

Using this condition, we get the following result:

PROPOSITION 1. *Assume that* $\tau_k = \tau_w = 0$. *The equilibrium is dynamically efficient if either (i)* $\Sigma < \Sigma_1$ *or (ii)* $\Sigma \in (\Sigma_1, \Sigma_2)$ *and* $\Omega < \overline{\Omega}$, *where* $\Sigma_1 = \frac{\alpha}{1-\alpha}\frac{1+\beta+\beta^2}{\beta+\beta^2}$, $\Sigma_2 = \frac{\alpha}{1-\alpha}\frac{1+\beta^2}{\beta^2}$ *and*

$$\overline{\Omega} = \left(\frac{\Sigma_2-\Sigma}{\Sigma-\Sigma_1}\right)\left(\frac{1+\beta^2}{\beta+\beta^2}\right)\left(\frac{1-\Sigma}{\Sigma_2}\right).$$

Proof. See Appendix A.1. ■

The result in Proposition 1 implies that the equilibrium is dynamically inefficient when either Σ or Ω is sufficiently large. This result is obtained because there is a positive relationship between the savings rate and the values of both Σ and Ω. In order to illustrate this mechanism that relates dynamic efficiency with the distribution of income by age group and that it is based on savings, we next show the relation between the savings rate and condition (18). We first use (5) and (6) to obtain $w/q = (1-\alpha)\, k/\alpha$. We use this equation, the expression of k^*, and (14) with zero taxes to obtain:

$$\frac{\xi_1 w}{a} = \frac{(1-\alpha)(1-\Sigma)+\alpha\Omega}{(1-\alpha)(1-\Sigma)\beta^2+\alpha\Omega\left(\beta+\beta^2\right)}+1, \quad \textbf{(19)}$$

where $a/\xi_1 w$ is the savings rate defined as the ratio between savings and the labor income of the young. Using (19), condition (18) can then be written as

$$\frac{1}{\Sigma}\left(\frac{\alpha}{1-\alpha}\right) > \frac{a}{\xi_1 w}.$$

Therefore, the steady-state equilibrium is dynamically efficient when the savings rate is smaller than $\alpha/(1-\alpha)$. This is exactly the same condition that the literature has obtained for dynamic efficiency. In fact, if $\Sigma = 1$, condition (18) simplifies to $\alpha/(1-\alpha) > \left(\beta+\beta^2\right)/\left(1+\beta+\beta^2\right)$, which is the condition obtained in Raurich and Seegmuller (2019). However, in this case, the savings rate and the condition for dynamic efficiency are independent from the distribution of income by age group. In contrast, as follows from (19), the savings rate increases with both Σ and Ω when $\Sigma < 1$. Note that this is a crucial difference that explains that dynamic efficiency depends on the income distribution by age group and it will also explain some of the main results in the following section.

4. EQUILIBRIA WITH A BUBBLE

We introduce in this section the portfolio decision of the consumer between a liquid speculative asset, $b_{1,t}$ and $b_{2,t+1}$, and an illiquid productive asset, a_{t+1}. Hence, the consumer decides a_{t+1}, $b_{1,t}$ and $b_{2,t+1}$ to maximize the utility (1) subject to the budget constraints (2)–(4). The solution to this maximization problem is characterized by the first-order conditions with respect to $b_{1,t}$, $b_{2,t+1}$, and a_{t+1}, which are, respectively,

$$\frac{1}{c_{1,t}} = \beta \frac{R_{t+1}}{c_{2,t+1}}, \tag{20}$$

$$\frac{1}{c_{2,t+1}} = \beta \frac{R_{t+2}}{c_{3,t+2}}, \tag{21}$$

$$\frac{1}{c_{1,t}} = \beta \frac{(1-\tau_k)\,\phi_1 q_{t+1}}{c_{2,t+1}} + \beta^2 \frac{(1-\tau_k)\,\phi_2 q_{t+2}}{c_{3,t+2}}. \tag{22}$$

From combining (20)–(22) and using (6), we obtain the following no-arbitrage condition between the returns from investing one unit in the speculative asset and the returns from investing the same unit in productive capital:

$$R_{t+1} = (1-\tau_k)\,\phi_1 \alpha A k_{t+1}^{\alpha-1} + \frac{(1-\tau_k)\,\phi_2 \alpha A k_{t+2}^{\alpha-1}}{R_{t+2}}. \tag{23}$$

This means that in the economy with bubbles, perfect consumption smoothing occurs. It is worth mentioning that, in the economy without bubbles, equation (23) does not hold. Because of incomplete asset markets, there is not such a perfect consumption smoothing.

In Appendix A.2, we combine (2)–(6), (20), (21), and (23) to obtain the following two equations:

$$b_{1,t} = \frac{\left(\beta+\beta^2\right)(1-\tau_w)\,\xi_1\,(1-\alpha)\,A k_t^{\alpha} - \frac{(1-\tau_w)\xi_2(1-\alpha)A k_{t+1}^{\alpha}}{R_{t+1}}}{1+\beta+\beta^2} - a_{t+1}, \tag{24}$$

$$b_{2,t+1} = \frac{\beta^2(1-\tau_w)\,\xi_2\,(1-\alpha)\,A k_{t+1}^{\alpha} + \beta^2(1-\tau_w)\,\xi_1\,(1-\alpha)\,A k_t^{\alpha} R_{t+1}}{1+\beta+\beta^2} + a_{t+1}\left[(1-\tau_k)\phi_1 \alpha A k_{t+1}^{\alpha-1} - R_{t+1}\right]. \tag{25}$$

DEFINITION 2. *Given $k_{-1} \geqslant 0$ and $k_0 \geqslant 0$, an equilibrium is a path of $\left\{a_t, k_t, b_{1,t}, b_{2,t}, R_t\right\}_{t=1}^{\infty}$ that solves the system of difference equations (23), (24), and (25) and the market-clearing conditions (9) and (10).*

We proceed to obtain the steady state, and then we characterize the distributions of income for which an equilibrium with bubbles exists and also the distributions for which these bubbles are productive, that is, are associated with a larger level of capital per unit of labor.

4.1. Steady State

We first use (10) and $nb_1 + b_2 > 0$ to obtain $R = n$. Next, from (23), we obtain that the steady-state value of capital in the equilibrium with bubbles, k, is

$$k = \left(\frac{(1-\tau_k)\,\phi_1 \alpha A}{\Omega n} \right)^{\frac{1}{1-\alpha}}. \tag{26}$$

We use (16) to deduce the steady-state value of productive investment, a. From (24), we obtain the steady-state value of the bubbles owned by the young individuals:

$$b_1 = \frac{(n\xi_1 + \xi_2)\,(1-\alpha)\,Ak^{\alpha}}{n}\,(1-\tau_w)\left(\Sigma - \Sigma_{b_1}\right), \tag{27}$$

where $\Sigma_{b_1} = \frac{1}{1+\beta+\beta^2} + \frac{\alpha(1-\tau_k)}{(1-\alpha)(1-\tau_w)}$. From (25), we obtain the steady-state value of the bubbles owned by the middle-aged individuals:

$$b_2 = (n\xi_1 + \xi_2)\,\alpha A k^{\alpha}\,(1-\tau_k)\left(\Omega - \Omega_{b_2}\right), \tag{28}$$

where $\Omega_{b_2} = 1 - \left(\frac{1-\alpha}{\alpha(\,1-\tau_k)}\right)\left(\frac{\beta^2}{1+\beta+\beta^2}\right)(1-\tau_w)$. Finally, as explained in Section 2, the price of the bubble is $N_{t-1}\,(nb_1 + b_2)$, where

$$nb_1 + b_2 = (n\xi_1 + \xi_2)\,(1-\alpha)\,Ak^{\alpha}\digamma,$$

and

$$\digamma = (1-\tau_w)\left(\Sigma - \Sigma_{b_1}\right) + \frac{\alpha\,(1-\tau_k)}{1-\alpha}\left(\Omega - \Omega_{b_2}\right).$$

Recall that b_1 is used to smooth consumption between young and middle-aged individuals, whereas b_2 is used to smooth consumption between middle-aged and old individuals. This explains that the sign of b_1 depends on Σ, whereas the sign of b_2 depends mainly on Ω. If $\Sigma > \Sigma_{b_1}$, then a large fraction of labor income is obtained by the young individuals. The bubble is then used to transfer consumption to the second period of life, that is, $b_1 > 0$. In contrast, if $\Sigma < \Sigma_{b_1}$, then a large part of labor income is obtained by middle-aged individuals. The bubble is then used to transfer consumption to the first period of life, $b_1 < 0$. Similarly, if $\Omega > \Omega_{b_2}$, then a large fraction of capital income is obtained by the middle-aged individuals. These individuals use the bubble to transfer consumption to the last period of life, that is, $b_2 > 0$. Obviously, the opposite occurs when $\Omega < \Omega_{b_2}$.

We next obtain conditions for which an equilibrium with bubbles exists.

PROPOSITION 2. *A steady state with a bubble exists if* $\Omega > \widetilde{\Omega}$ *where*

$$\widetilde{\Omega} = \left(\frac{1-\alpha}{\alpha\,(1-\tau_k)} \right)\left[\Sigma_3 - (1-\tau_w)\,\Sigma\right],$$

and $\Sigma_3 = \frac{\left(1-\beta^2\right)(1-\tau_w)}{1+\beta+\beta^2} + \frac{2\alpha(1-\tau_k)}{1-\alpha}$.

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Proof. A bubble exists when its price is positive, which occurs when $nb_1 + b_2 > 0$. Using (27) and (28), the previous inequality implies that $\Omega > \widetilde{\Omega}$. ■

From Proposition 2, it follows that a bubble may only exist when either Σ or Ω are sufficiently large. A bubble may only exist if either the young individuals buy the speculative asset ($b_1 > 0$), or the middle-aged individuals buy this asset ($b_2 > 0$). As already explained, the young individuals buy the speculative asset if they obtain a sufficiently large income, which requires large Σ. Similarly, middle-aged individuals buy this asset when they obtain a sufficiently large amount of income, which requires a sufficiently large value of Ω.

Fiscal policy modifies the distribution of income among individuals, and hence, it directly affects the existence of a bubble. The following proposition summarizes the effect of fiscal policy on the existence of bubbles:

PROPOSITION 3. *The following fiscal policies facilitate the existence of an equilibrium with bubbles: (i) a reduction in the labor income taxes when they are mainly paid by the young individuals* ($\Sigma > \frac{1-\beta^2}{1+\beta+\beta^2}$)*; (ii) an increase in the labor income taxes when they are mainly paid by the middle-aged individuals* ($\Sigma < \frac{1-\beta^2}{1+\beta+\beta^2}$)*; (iii) an increase in the capital income taxes.*

Proof. The results follow directly from a simple comparative static analysis on the function $\digamma$. ■

The effect of an increase in the labor income tax on the existence of bubbles depends on the value of Σ. If Σ is large, the labor income tax is mainly a tax on the income of young individuals, whereas if Σ is small, this tax is mainly paid by middle-aged agents. When labor income taxes are mainly paid by young individuals, these taxes limit young individuals' capacity to postpone consumption using bubbles, whereas when labor income taxes are mainly paid at the middle age, they facilitate that individuals use bubbles to postpone consumption toward middle age. Thus, when Σ is high, an increase in the labor tax hinders the possibility of bubbles, whereas the opposite occurs when Σ is small.

Finally, capital income taxes reduce the after-tax income of both middle-aged and old individuals. Since capital has a lower return, traders have more incentive to invest in the speculative asset. Therefore, an increase in these taxes facilitates the existence of bubbles that will be used to postpone consumption.

4.2. Productive Bubbles

Bubbles are a financial instrument that facilitates consumption smoothing, and hence, individuals do not need to use productive capital to smooth consumption. As a consequence, the introduction of bubbles modifies the stock of productive capital, which may either increase or decrease. More specifically, bubbles are productive when $k > k^*$. From the comparisons between these two stocks of capital, it is easy to show that the bubble is unproductive if and only if the equilibrium

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without bubbles is dynamically inefficient. In this case, as in Tirole (1985), the bubble is used to postpone consumption and, as a consequence, productive investment declines. The bubbly steady state corresponds to the golden rule.

We have shown that a bubble may exist when the young generation obtains a large fraction of the labor income and when the middle-aged generation obtains a large fraction of the capital income. We have also shown that if these two fractions are not too large then the steady state without bubbles is dynamically efficient and, hence, the bubble is productive. The following proposition summarizes these findings and provides a complete characterization of the conditions implying the existence of productive bubbles. For the sake of simplicity, we assume that taxes are equal to zero in the rest of the section.

PROPOSITION 4. *Assume that* $\tau_k = \tau_w = 0$. *The steady-state equilibrium satisfies the following properties.*

1. *If* $\Sigma < \Sigma_1$, *then (i) the bubble exists and is productive when* $\Omega > \widetilde{\Omega}$ *and (ii) the bubble does not exist when* $\Omega < \widetilde{\Omega}$.
2. *If* $\Sigma > \Sigma_1$, *then (i) the bubble exists and is not productive when* $\Omega > \max\left\{\widetilde{\Omega}, \overline{\Omega}\right\}$, *(ii) the bubble exists and is productive when* $\Omega \in \left(\widetilde{\Omega}, \overline{\Omega}\right)$, *and (iii) the bubble does not exist when* $\Omega < \widetilde{\Omega}$.

Proof. From Proposition 2, it is immediate to show that the bubble exists if $\Omega > \widetilde{\Omega}$. From Proposition 1, it is easy to show that the equilibrium without bubbles is dynamically efficient and the bubble is productive if either $\Sigma < \Sigma_1$ or $\Sigma > \Sigma_1$ and $\Omega < \overline{\Omega}$, where the expressions of $\overline{\Omega}$ and Σ_1 are defined in Proposition 1. ■

Proposition 4 provides the main result of the paper. It shows that the distribution of income by age group crucially determines the existence of productive bubbles. It extends the analysis provided in Raurich and Seegmuller (2019), where it is already shown that bubbles can increase the stock of productive capital when productive investment is an illiquid investment. However, that paper restricts its attention to the case where $\Sigma = 1$ and, hence, productive bubbles only arise if $\Sigma_1 > 1$. Therefore, the existence of productive bubbles does not depend on the distribution of income by age group. Here, this distribution plays a crucial role not only on the existence of productive bubbles but also on their features, that is, whether they are characterized by $b_i < 0$ or $b_i > 0$. This is studied in the following proposition. Let

$$\kappa_1 = \frac{\left(1-\beta^2\right)\left(\beta+2\beta^2\right)}{\left(1+\beta+\beta^2\right)(2+\beta)}, \kappa_2 = \frac{\beta^2}{1+\beta+\beta^2}, \kappa_3 = \frac{\beta/2+\beta^2}{1+\beta+\beta^2}, \kappa_4 = \frac{\beta+2\beta^2}{1+\beta+\beta^2}.$$

PROPOSITION 5. *Assume that* $\tau_k = \tau_w = 0$. *We distinguish among the following cases that correspond to different parametric regions:*

1. *If* $\frac{\alpha}{1-\alpha} \in (\kappa_1, \kappa_2) \cup (\kappa_3, \kappa_4)$, *then productive bubbles satisfy* $b_1 < 0$ *and* $b_2 > 0$. *It requires* $\Sigma < \Sigma_{b_1}$.

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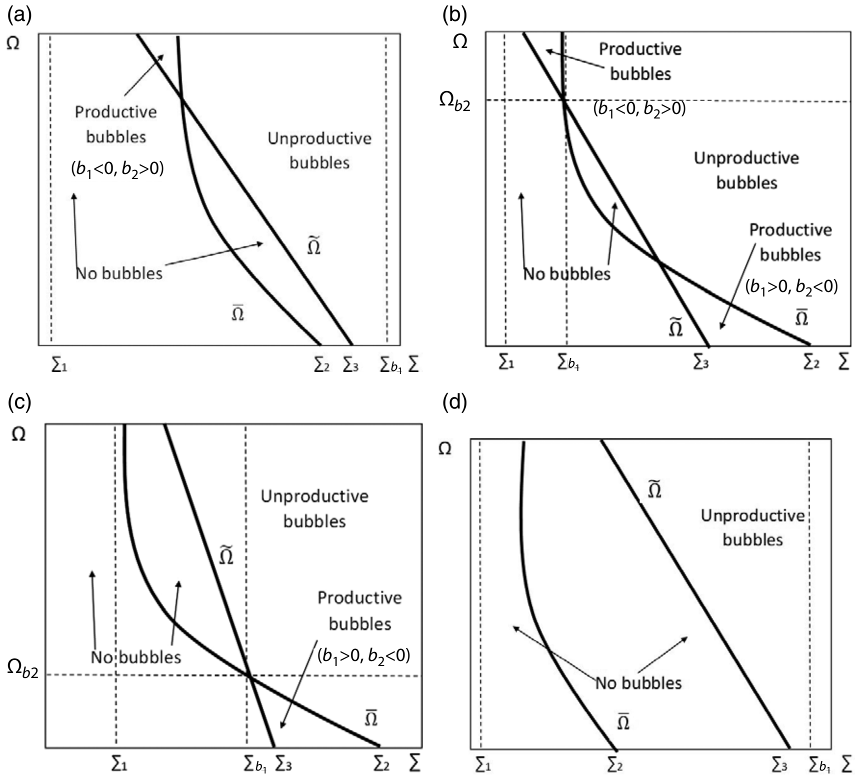

FIGURE 1. Bubbles and the distribution of income.

2. *If* $\frac{\alpha}{1-\alpha} \in (\max\{\kappa_1, \kappa_2\}, \kappa_3)$*, then productive bubbles satisfy* $b_1 < 0$ *and* $b_2 > 0$ *when* $\Sigma < \Sigma_{b_1}$ *and* $b_1 > 0$ *and* $b_2 < 0$ *otherwise.*
3. *If* $\frac{\alpha}{1-\alpha} \in (\kappa_2, \kappa_1)$*, then productive bubbles satisfy* $b_1 > 0$ *and* $b_2 < 0$*. It requires* $\Sigma > \Sigma_{b_1}$*.*
4. *If* $\frac{\alpha}{1-\alpha} < \min\{\kappa_2, \kappa_1\}$ *or* $\frac{\alpha}{1-\alpha} > \kappa_4$*, then the equilibrium does not exhibit productive bubbles.*

Proof. See Appendix A.3. ■

This proposition implies that, depending on the values of α and β, we can distinguish among four possible cases. In the first case, bubbles are productive only when $b_1 < 0$ and $b_2 > 0$. Panel a of Figure 1 shows this case by displaying the relationship between Ω and Σ implied by the functions $\widetilde{\Omega}$ and $\overline{\Omega}$ when $b_1 < 0$ and $b_2 > 0$ is the only possible productive bubble.[6] Observe from Panel a that productive bubbles emerge when Σ is small and Ω is large. This implies that productive bubbles arise when $b_1 < 0$ and $b_2 > 0$ if the middle-aged individuals obtain a sufficiently large fraction of total income. In this case, individuals use the bubble to transfer consumption from the middle age to the other two periods of life. On

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the one hand, middle-aged individuals postpone consumption, which implies that $b_2 > 0$. On the other hand, middle-aged individuals transfer consumption to the young individuals, which implies that $b_1 < 0$.

In the second case, bubbles can be productive when either $b_1 < 0$ and $b_2 > 0$ or when $b_1 > 0$ and $b_2 < 0$. This case is displayed in Panel b of Figure 1. This figure shows that, as in the previous case, bubbles are productive when $b_1 < 0$ and $b_2 > 0$ if the middle-aged obtains a sufficiently large fraction of income (Σ small and Ω large). The figure also shows that bubbles are productive when $b_1 > 0$ and $b_2 < 0$ if the middle-aged individuals obtain a small fraction of income (Σ large and Ω small). In this case, consumption smoothing implies that consumption is transferred from the young and old individuals to the middle-aged individuals. In the third case of the previous proposition, bubbles can be productive only when $b_1 > 0$ and $b_2 < 0$. This case is displayed in Panel c of Figure 1. As in the second case, this productive bubble arises when the middle-aged individuals obtain a small fraction of total income. Finally, the last case of the proposition is displayed in Panel d of Figure 1. In this case, productive bubbles do not exist for any income distribution.

From inspection of Figure 1, we obtain clear insights about the effects of non-marginal increases in Σ and Ω that change the characteristics of the equilibrium. On the one hand, an increase in Σ facilitates the existence of an equilibrium with bubbles. These bubbles can be productive or unproductive, depending on the value of Ω. A large value of Σ implies that the fraction of income obtained by young individuals is large and, hence, young individuals are willing to hold the bubble to postpone consumption. On the other hand, an increase in Ω also facilitates the existence of an equilibrium with bubbles. A larger value of Ω increases the income obtained by middle-aged individuals. These individuals are then willing to hold the bubble to postpone consumption.

Proposition 5 shows that bubbles can be productive in two very different situations: (i) when $b_1 < 0$ and $b_2 > 0$ and (ii) when $b_1 > 0$ and $b_2 < 0$. To obtain an intuition on the existence of these two different cases of productive bubbles, it is worth to consider equation (14). This equation governs the investment decision in the absence of bubbles by equating the marginal utility cost of productive investment with the marginal utility benefit. It follows that a bubble is productive when either reduces the utility cost of investment or increases the utility benefit of this investment. These two different effects of bubbles explain the two situations in which bubbles are productive. In the first situation, the bubble is used to transfer consumption to the young ($b_1 < 0$). This transfer reduces the marginal utility cost of investment of the young, who then increase productive investment.

In the second situation, the bubble transfers consumption to middle-aged individuals. As we have explained, the second situation occurs when middle-aged individuals obtain a relatively small fraction of total income because the return of capital is obtained mainly by the old and most of labor income is obtained by the young. As the return on the illiquid asset is mostly obtained by the old, investment

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in this asset is not an effective instrument to transfer consumption to the middle-aged. The bubble introduces an asset that provides the liquidities necessary to transfer consumption from the old to the middle-aged individuals. This transfer decreases the marginal utility of the middle-aged individuals and increases the marginal utility of the old individuals. Given that most of the return of the illiquid asset is obtained when old, this transfer increases the marginal utility benefit of investment. Hence, the bubble increases the benefit from the investment in the illiquid asset, which explains that the bubble is productive.

To summarize, bubbles can be productive either because they reduce the cost of investment or because they increase the benefit from this investment. To the best of our knowledge, this second mechanism is new and it implies a productive bubble that transfers income from the young and old to the middle-aged individuals.

This second mechanism requires that the savings of the young are larger in the economy with bubbles than in the economy without bubbles. If the savings of the young are high enough in the economy with bubbles, productive investment increases even though part of the savings are used to transfer consumption to the middle-aged individuals ($b_1 > 0$). In order to show more explicitly this argument, we compare the savings rate in the economy with bubbles with the savings rate in the economy without bubbles. The savings rate is defined as the ratio between assets accumulated when young and the income of the young individuals. We first use (5) and (24) to obtain the savings rate in the economy with bubbles when tax rates are equal to zero,

$$\frac{a+b_1}{\xi_1 w}=\frac{\beta+\beta^2-\frac{1-\Sigma}{\Sigma}}{1+\beta+\beta^2}.$$

Note that in the economy with bubbles young individuals accumulate both productive assets and speculative assets. Using (19), we obtain the savings rate in the economy without bubbles, where the young individuals only accumulate productive assets, that is,

$$\frac{a}{\xi_1 w}=\frac{(1-\alpha)(1-\Sigma)\beta^2+\alpha\Omega\left(\beta+\beta^2\right)}{(1-\alpha)(1-\Sigma)\left(1+\beta^2\right)+\alpha\Omega\left(1+\beta+\beta^2\right)}.$$

Note that both expressions of the savings rate are different when $\Sigma < 1$, whereas they coincide when $\Sigma = 1$. As a consequence, when $\Sigma = 1$, productive capital is larger with bubbles if and only if $b_1 < 0$. It follows that the second case of productive bubbles is not possible. On the contrary, when $\Sigma < 1$, capital can be larger with bubbles even if $b_1 > 0$, since the savings rates can be larger in the economy with bubbles. From the comparison between the two savings rates, it follows that the savings rate of the economy with bubbles is larger when the following condition on the distribution of income by age group holds:

$$\Omega<\left(\Sigma-\frac{1+\beta^2}{1+\beta+\beta^2}\right)\frac{1-\alpha}{\alpha}.$$

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This condition implies that the savings rate is larger in the economy with bubbles when either Σ is sufficiently large or when Ω is sufficiently small. Therefore, these two conditions show that the savings rate is larger when the middle-aged individuals are poor, which is precisely the condition that makes bubbles be productive when $b_1 > 0$ and $b_2 < 0$.

5. FISCAL POLICY

We proceed to study the effect of fiscal policies on production both in the economy without bubbles and in the economy with bubbles. This will allow us to characterize those fiscal policies that promote productive bubbles. At this point, it is important to clarify that the effect of fiscal policy on production follows directly from the effect that fiscal policy has on the stock of productive capital.

Using equation (17), it can be shown that the steady-state stock of productive capital of the economy without bubbles, k^*, decreases when (i) the tax rate on the labor income increases if this tax is mainly paid by young individuals (Σ close to 1); (ii) the tax rate on the labor income decreases if this tax is mainly paid by middle-aged individuals (Σ close to 0); and (iii) the tax rate on capital income increases. The effects of labor income taxes are explained because, in the absence of bubbles, productive capital is used to smooth consumption. Therefore, an increase in the labor income tax paid by the young individuals reduces their income net of taxes, which causes a reduction in productive investment. An increase in the labor income tax paid by the middle-aged individuals reduces their after-tax income. Young individuals then increase investment in productive capital to postpone consumption. Finally, taxes on capital income reduce the return from productive capital, which implies a raise of the discounted income. Therefore, young households consume more, which causes the reduction in productive investment.

Using (26), we can easily see that the steady-state stock of productive capital of the economy with bubbles, k, decreases following an increase of the tax on capital income. This result follows from the fact that this tax reduces the return from productive investment and there is a no-arbitrage condition between holding capital and the bubble. As a direct implication, this stock of productive capital does not depend on the tax on labor income.

The previous results imply that the effect on the stock of capital of taxes on labor income depends on the existence of bubbles. As a consequence, fiscal policy may make bubbles productive or unproductive. To study the effect of fiscal policy, we compare the stocks of capital k and k^*, and we show that $k^* < k$ when $\Psi > 0$, where

$$\Psi = 1 + \frac{(1-\alpha)(1-\tau_w)(1-\Sigma) + \alpha(1-\tau_k)\Omega}{(1+\beta)\beta\alpha(1-\tau_k)\Omega + \beta^2(1-\alpha)(1-\tau_w)(1-\Sigma)} - \frac{(1-\alpha)(1-\tau_w)\Sigma}{\alpha(1-\tau_k)}.$$

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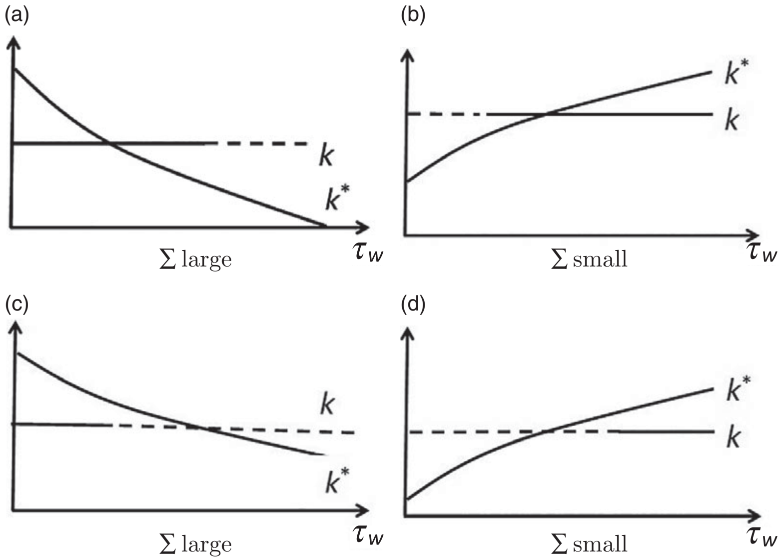

FIGURE 2. The effect of fiscal policies on capital.

Straightforward comparative statics on the function Ψ show that bubbles may become productive as a consequence of the following fiscal policies: (i) an increase in the labor income taxes when they are mainly paid by the young individuals (Σ close to 1) and (ii) a reduction in the labor income taxes when they are mainly paid by the middle-aged individuals (Σ close to 0). As explained before, an increase in the labor income taxes paid by the young makes individuals use bubbles to transfer consumption to the first period of life. As a consequence, bubbles either disappear or become productive. Obviously, the effect is the opposite when the fiscal policy consists of increasing the taxes paid by the middle-aged individuals, either existence of bubbles is facilitated or bubbles become unproductive. Finally, an increase in the taxes on capital income has an ambiguous effect on the existence of productive bubbles. This is explained by the fact that these taxes reduce the stock of capital both when the equilibrium exhibits bubbles and when it does not exhibit bubbles.

These results on the effect of fiscal policy on the stock of capital are summarized in Figure 2 that shows how both stocks of productive capital depend on the taxes on labor income. Panel a shows the effect of the labor income tax when Σ is close to 1. It shows that if the tax rate on the labor income is sufficiently small, then the bubble will be used to postpone consumption and, hence, it will be unproductive. To see this, note that $k < k^*$ for low values of this tax rate. As the tax rate increases, the bubble becomes productive and, eventually, the bubble disappears. Panel b shows the effects of the tax rate on labor income when Σ is close to 0. These effects are the opposite from the ones displayed in Panel a. When this tax rate is sufficiently small, the bubble may not exist. When the tax rate increases, a productive bubble exists. Finally, for sufficiently large values of the tax rate, $k^* > k$ and, hence, the bubble becomes unproductive.

Figure 2 introduces an important implication for fiscal policy. It shows that marginal increases in the labor income taxes that do not affect the existence of bubbles have no effect on the stock of productive capital in the economy with bubbles. However, when Σ is close to 1, a non-marginal increase in the tax rate on the labor income that makes the bubble disappear will cause a dramatic reduction in the stock of capital since the only long-run equilibrium is the steady state without bubble. When Σ is close to 0, a large decline in the stock of productive capital would also occur if we instead consider a non-marginal reduction in the tax rate on the labor income, since this tax reduction eliminates bubbles in this case. These results point out an important discontinuity in the effects that fiscal policy has on production. They also highlight the crucial role played by the distribution of income to design the fiscal policy and to evaluate which generation will benefit or suffer from the tax variation.

The effects illustrated in the first two panels of Figure 2 are obtained when fiscal policy makes a productive bubble disappear. However, for a different distribution of income by age group, the same fiscal policy may cause the disappearance of an unproductive bubble. This possibility is illustrated in Panels c and d of Figure 2, that display, respectively, the effects of taxes on the labor income when Σ is close to 1 and Σ is close to 0. These two panels show that in this case a non-marginal change in the tax rates that eliminates the bubble will cause an increase, instead of a decrease in the stock of capital once the economy reaches the bubbleless steady state.

We conclude from the previous discussions that the effect of fiscal policies crucially depends on the distribution of income by age group. In what follows, we illustrate this conclusion by performing a simulation of the model based on a plausible parametrization. We fix the value of the parameters as follows. First, without loss of generality A, ξ_1 and ϕ_1 are normalized to one. Second, $\alpha = 0.3$, which implies a labor income share equal to 70%.[7] Third, $\beta = 0.93$, which implies a savings rate of 7%.[8] Apart from these parameters that are assumed to be common across countries, we consider two sets of country-specific parameters. First, the values of ξ_2 and of ϕ_2 are set so that Σ and Ω coincide with the values that are displayed in Table 2.[9] Second, tax rates and the population growth rate are obtained from the Organisation for Economic Co-operation and Development (OECD) data set and they are displayed in Table 3.[10]

For the economy described in the parametrization, we obtain $\Sigma_1 = 0.667$, $\kappa_1 = 0.044$, $\kappa_2 = 0.309$, $\kappa_3 = 0.475$, $\kappa_4 = 0.951$, and $\Sigma_{b_1} = 0.786$. Given these values, we can conclude that the US and many European economies satisfy $\Sigma < \Sigma_1$, which implies that these economies are described by Case 1 in Proposition 4. As a consequence, in the absence of taxes, bubbles are productive. Furthermore, given the values of α and of the κ_s, this parametrization corresponds to Case 2 in Proposition 5. Since the US and most European economies satisfy $\Sigma < \Sigma_{b_1}$, bubbles are productive because they transfer consumption from the middle-aged

TABLE 3. Taxes and population growth

	τ_w	τ_k	n
Austria	0.50	0.25	0.96
Belgium	0.55	0.34	0.98
Czech Republic	0.43	0.19	1.14
Denmark	0.36	0.23	0.94
Finland	0.44	0.20	0.89
France	0.48	0.38	0.97
Germany	0.49	0.30	0.84
Greece	0.39	0.26	1.13
Hungary	0.49	0.19	1.07
Italy	0.48	0.31	0.99
Netherlands	0.36	0.25	0.91
Norway	0.37	0.27	1.08
Poland	0.35	0.19	1.10
Portugal	0.42	0.29	1.02
Spain	0.40	0.28	1.16
Sweden	0.43	0.22	1.01
United Kingdom	0.31	0.20	1.05
United States	0.32	0.39	1.00

Source: OECD Database. The population growth rate is obtained from the ratio between the population in the interval 25–44 years and the population in the interval 45–64. The population growth rate is obtained for all countries in the year 2013, except for Belgium, France, Greece, Netherlands, and Poland that it is obtained in the year 2012.

period to the other two periods of life, that is, $b_1 < 0$ and $b_2 > 0$. We next use this calibration to discuss the effects of fiscal policy.

The numerical exercise consists of three parts. The purpose of the first part is to show that under plausible parameter values, obtained from cross-country comparisons, we can observe very different situations regarding the possibility of bubbles and their characteristics. The results from this numerical analysis are displayed in Table 4.[11] This table shows the value of the capital stock in both economies (bubble and no bubble) and the value of $\digamma$. The sign of $\digamma$ determines the existence of the bubble, with a negative sign implying that the economy does not exhibit a bubble. As it is clear from this table, according to the model, only the US economy may exhibit a bubble. This bubble is productive, as follows from the comparison between the two capital stocks. In contrast, none of the European economies may exhibit a bubble according to the model. From the comparison between the fundamentals of the European economies and those of the US economy, displayed in Tables 2 and 3, it follows that the main difference is fiscal policy. In fact, there are no relevant differences between US and European economies in the population growth rate or in the distribution of income by age group. The only clear difference with respect to European economies is the larger taxes on capital income and the smaller taxes on labor income. This different fiscal policy implies that the

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TABLE 4. Results from the simulation

	k^*	k	F
Austria	0.12	–	−0.55
Belgium	0.12	–	−0.63
Czech Republic	0.16	–	−0.07
Denmark	0.17	–	−0.50
Finland	0.16	–	−0.65
France	0.19	–	−0.74
Germany	0.19	–	−0.70
Greece	0.20	–	−0.09
Hungary	0.14	–	−0.63
Italy	0.16	–	−0.50
Netherlands	0.18	–	−0.32
Norway	0.16	–	−0.05
Poland	0.18	–	−0.13
Portugal	0.18	–	−0.39
Spain	0.15	–	−0.27
Sweden	0.14	–	−0.46
United Kingdom	0.23	–	−0.30
United States	0.15	0.19	0.01

tax burden in European economies is more concentrated on the young individuals, which limits investment in productive capital and prevents the existence of an equilibrium with bubbles.

In the second part of the numerical exercise, we show that the effect of fiscal policy on productive capital depends on the distribution of income. To this end, we simulate the model when we set the value of the tax rates in the European economies at the level of the USA. The results are shown in Table 5, where we distinguish between three groups of European economies. The first group, formed by six countries (Belgium, Denmark, Finland, France, Germany, and Italy), does not exhibit bubbles with this new fiscal policy. These economies are characterized by very low values of Ω, which, as follows from the analysis of the previous section, hinders the existence of bubbles. The second group, formed by four economies (Czech Republic, Greece, Norway, and Poland), may exhibit unproductive bubbles. These are economies characterized by large values of both Σ and Ω, and hence, individuals in these economies could use the bubble to postpone consumption. Finally, the last group of countries, formed by seven countries (Austria, Hungary, Netherlands, Portugal, Spain, Sweden, and United Kingdom), may exhibit productive bubbles. These seven economies are characterized by intermediate values of both Σ and Ω. This distribution of income together with the fiscal policy facilitates that individuals use the bubble to smooth consumption by placing resources from the middle-aged toward the young and the old. The increase in the disposable income of the young makes the bubble productive.

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TABLE 5. Results from the simulation

	k^*	k	F
Austria	0.18	0.22	0.05
Belgium	0.20	–	−0.05
Czech Republic	0.20	0.14	0.69
Denmark	0.18	–	−0.16
Finland	0.21	–	−0.14
France	0.28	–	−0.42
Germany	0.29	–	−0.21
Greece	0.23	0.19	0.42
Hungary	0.20	0.23	0.06
Italy	0.22	–	−0.01
Netherlands	0.19	0.23	0.04
Norway	0.18	0.17	0.34
Poland	0.19	0.17	0.38
Portugal	0.22	0.25	0.02
Spain	0.17	0.18	0.13
Sweden	0.18	0.21	0.07
United Kingdom	0.22	0.24	0.06
United States	0.15	0.19	0.01

We assume that $\tau_k = 0.39$ and $\tau_w = 0.32$

We conclude from this cross-country analysis that under plausible distributions of income, observed in European economies, the effects on production of the same fiscal policy can vary substantially.

An interesting remark is obtained from the comparison of the stocks of capital in Tables 4 and 5. From this comparison, it follows that the proposed change in fiscal policy could cause a substantial increase in the stock of capital of the European economies.[12] The results in Table 5 show that under our calibration the average increase in the stock of capital of these European economies would be 25% if the economy remains in an equilibrium without bubbles. At this point, it is important to introduce some words of caution on the large effects of fiscal policy obtained in the previous analysis. First, the changes in the stock of capital are obtained by comparing two different steady states. Thus, these effects of fiscal policy may only occur in the long run. Second, the effects of fiscal policy crucially depend on the value of Σ and Ω. To obtain these values, we have introduced assumptions on the distribution of labor and capital income by age group that may introduce biases on the actual values of both Ω and Σ. Third, our economy is a simplified model that does not consider many other effects of fiscal policies. Therefore, the results in Table 5 should only be considered as illustrative of the large effects that fiscal policies may have when they modify the distribution of income by age group.

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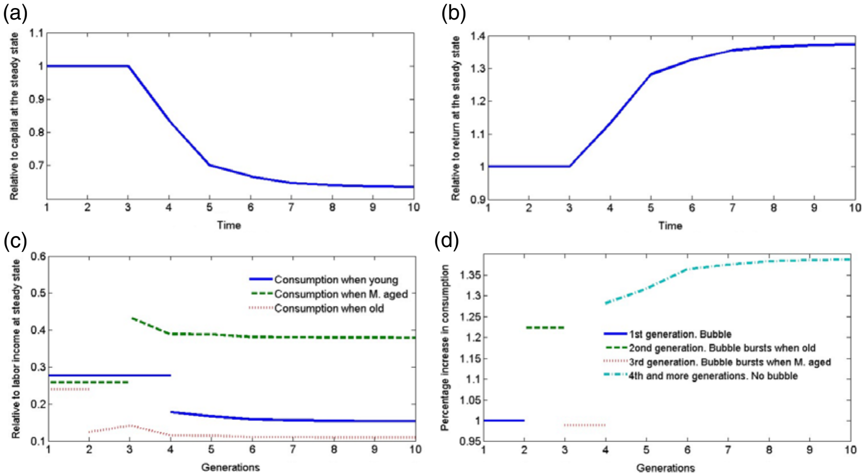

FIGURE 3. Bubble bursts due to a tax reform. (a) Capital. (b) Return of capital. (c) Consumption. (d) Utility cost.

In the last part of the numerical exercise, we analyze the dynamic effects of a fiscal policy that causes the transition from the bubbly steady state to the bubbleless steady state. To this extent, we study the effects for the US economy of a fiscal policy that raises taxes on labor income from the current level, 32%, to the average level in the European economies, 42.65%. This tax reform concentrates the tax burden on the young and middle-aged individuals. As a consequence, the willingness to buy the bubble of young and middle-aged individuals decreases and the bubble bursts. Figure 3 shows the dynamic consequences of this fiscal policy, assuming that it is introduced in period 3 when the US economy is in the bubbly steady state. Due to the fiscal policy, the bubble bursts, causing a dynamic transition from the saddle path stable bubbly steady state to the stable bubbleless steady state.[13] Panel a displays the transition of the capital stock relative to the capital stock in the steady state of the bubbly economy. It shows that the capital stock experiences a substantial decrease of 35%, which is a consequence of both the bursting of the productive bubble and the increase in taxes paid by young individuals. Panel b shows that the return on capital relative to its value in the bubbly steady state increases substantially, which is a direct consequence of the reduction in the capital stock.

Panels c and d show, respectively, the levels of consumption in each period of life and of utility for the different generations. Throughout the life of the first generation, the bubble persists. Therefore, the consumption and utility levels of this generation correspond to those achieved in the steady state. The tax reform and the resulting bubble bursts occur when individuals of the second generation are old and those of the third generation are middle-aged. Therefore, the fourth and the rest of generations live in an economy without bubbles. We measure consumption in each period of life as a percentage of the present value of life time labor

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income at the bubbly steady state, and we measure the utility cost of not being at the bubbly steady state by the percentage increase in consumption in each period of life necessary to reach the same level of utility than in the bubbly steady state.[14]

The bursting of the bubble eliminates a financial instrument used for consumption smoothing. As a result, consumption smoothing among different periods of life declines after the bubble bursts. This is shown in Panel c. During the bubble, which is characterized by $b_1 < 0$ and $b_2 > 0$, consumptions in each period of life reach quite similar values. The bursting of the bubble reduces the consumption of the young and old individuals and substantially increases consumption of middle-aged individuals. The reduction in consumption smoothing and the decrease of production (capital) explain the large increase in the utility cost of these generations living in an economy without bubbles (4th and more generations). Finally, the two generations that are alive when the bubble bursts experience opposite effects. The second generation, whose members are old when the bubble bursts, experiences a substantial utility loss because its members loose the value of the speculative asset purchased when they were middle-aged individuals. In contrast, the generation whose members are middle-aged when the bubble bursts experience a slight improvement in its utility, explained by the fact that the members of this generation do not need to compensate the previous generation for the bubbles that they short sell when they were young individuals.

6. CONCLUDING REMARKS

We are interested in the interplay between the distribution of income by age group and productive bubbles. We have studied an OLG model with agents who live three periods, in which productive investment done in the first period of life is an illiquid investment whose return occurs in the following two periods. The bubble is a liquid speculative investment that facilitates intertemporal consumption smoothing. Our main result shows that the distribution of labor and capital income by age group determines both the existence of bubbles and their effect on production. We first show that if a large part of the labor income is obtained by middle-aged individuals and a large part of the capital income is obtained by old individuals then the equilibrium does not exhibit bubbles. We also show that if the fraction of labor income obtained by the young individuals is large and the fraction of capital income obtained by the middle-aged individuals is also large then an equilibrium with unproductive bubbles exists. These bubbles are used to postpone consumption. Finally, we show that the equilibrium exhibits productive bubbles in two different situations: when the middle-aged individuals obtain a large fraction of total income and when these individuals obtain a small fraction of total income. In the first case, bubbles are productive because they are used to transfer consumption to the young individuals, who then increase investment in the productive asset. In the second case, bubbles are productive because the savings rate is larger in the equilibrium with bubbles.

Fiscal policies cause large changes in the distribution of income by age group, and as a consequence, they modify the effect that bubbles have on production and they can either facilitate or hinder the existence of bubbles. In particular, we show that large capital income taxes facilitate the existence of an equilibrium with bubbles. We also show that the effect of an increase in the labor income taxes depends on the age group of the tax payers. We conclude that the same fiscal policy may have very different effects on production depending on the distribution of income by age group. This conclusion is illustrated numerically by showing the effect that a fiscal policy has on several European economies.

Our analysis can be used to study the effects of shocks that modify the distribution of income by age group. An interesting example is population aging that will increase the size of the oldest age group. As a consequence, it will reduce the value of Ω in the following years, which will reduce the stock of productive capital. Our results suggest that population aging can be particularly harmful in those economies where productive bubbles finance a large stock of productive capital, as these bubbles, due to the reduction in Ω, may not be sustainable.

In this model, the bubble is the only asset that provides liquidities. Thus, an extension of this paper would be to include in the analysis other assets that also provide liquidity, as for example credit. As we have shown in a related but simplest version of this model (Raurich and Seegmuller (2019)), the introduction of a loan market does not alter the main conclusions. Therefore, we conjecture that the results in this paper would hold provided we introduce credit constraints. When the constraint binds, the bubble is still necessary to provide liquidity, which implies that the results obtained in this model should follow.

NOTES

1. The data sources used in Table 1 are the US census and the Eurostat. US government census provides mean income for the following age groups: young (age 25–44), middle-aged (age 45–64), and old (65 and over). Eurostat provides the same data in 2015 for the different European economies shown in Table 1 and for the following age groups: young (age 25–49), middle-aged (age 50–64), and old (65 and over). As explained in Appendix A.4, we normalize the age groups by the number of years individuals belong to each age group in order to make age groups homogeneous and comparable across countries.

2. See Abel et al. (1989) for an analysis of dynamic efficiency in OLG models.

3. The existence of bubbles has been studied in OLG models by Samuelson (1958), Tirole (1985), and Weil (1987), and more recently, by Bosi and Seegmuller (2010), Caballero et al. (2006), Fahri and Tirole (2012), Hillebrand et al. (2004), or Martin and Ventura (2012), Martin and Ventura (2016). There is a large literature that also studies the possibility of bubbles in models with infinitely lived agents. Some relevant references of this literature are Hirano and Yanagawa (2017), Kamihigashi (2008), Kocherlakota (1992), Kocherlakota (2009), Kunieda (2017), and Miao and Wang (2018). Finally, there are other theories of bubbles, as for example the greater fool bubble models. The survey by Barlevy (2015) summarizes the main findings in this literature.

4. Note that middle-aged individuals obtain a large (small) fraction of total income when the fraction of labor income obtained by the young is small (large) and when the fraction of capital income obtained by the middle-aged is large (small). Thus, the two situations in which bubbles can be productive correspond to polar cases of the distribution of income by age group.

5. Taxes on bubble returns could have been introduced. If they were introduced, the after-tax return from the bubbles would be $\widetilde{R} = 1 + (R - 1)(1 - \tau_b)$, where τ_b is the tax rate on bubble returns. As $R = n$ will hold at a bubbly steady state, then $\widetilde{R} = 1 + (n - 1)(1 - \tau_b) = n - (n - 1)\tau_b$. However, these taxes will reduce the increase of the price of the bubble if $n > 1$ and, hence, they would be a subsidy when the household is a short seller of the bubble. To avoid this problem, we do not introduce these taxes.

6. The productive bubbles obtained in Raurich and Seegmuller (2019) are a particular example of the bubbles obtained in Case 1.

7. There is not a consensus in the literature on the value of the labor income share. In a recent paper, Koh, Santaeulalia-Llopis and Zheng (2016) show that in the USA the labor income share is stable and close to 70% if intellectual property capital is not considered as a form of capital income. We choose this stable value of the labor income share for our steady-state analysis.

8. Using the OECD savings rate, we obtain that the average savings rate in the period 1970–2015 in the countries displayed in Table 1 is equal to 7%. The average savings rate for these countries obtained from our simulation is also 7% when $\beta = 0.93$.

9. Appendix A.4 provides a detailed explanation of the procedure followed to obtain the values of Σ and of Ω in Table 2.

10. The population growth rate n is obtained from the OECD data set as the ratio between the size of the young population and the size of the middle-aged population.

11. The results in Table 4 cannot be used for cross-country comparisons in the level of the GDP per capita, as countries may differ in both the efficiency units of labor and in the technology.

12. United Kingdom is an exception. Taxes are substantially lower in this country and, hence, the change in fiscal policy will increase taxes and reduce productive capital.

13. The characteristic polynomial associated with the system of equations that characterizes the bubbly equilibrium is of order five. As a consequence, the analysis of stability is beyond the scope of this paper. Therefore, the results on the stability of the two steady states are obtained numerically for the calibrated economy. We conducted several robustness analyses and conclude that these results are a robust finding. In particular, we obtain that the bubbleless steady state is stable when we consider the equilibrium for which $b_{1,t} = b_{2,t} = 0$ for all t. We also obtain that the bubbly steady state is saddle path stable because the characteristic polynomial only has two roots with a modulus larger than one and the equilibrium has two non-predetermined variables: R_{t+1} and $b_{2,t+1}$. The rest of variables, k_t, a_t, and $b_{1,t}$, are predetermined. Note that $b_{1,t}$ is predetermined because its value affects adults decisions about the value of $b_{2,t+1}$.

14. We follow Lucas (2003) and define the utility cost as the permanent increase in consumption necessary to reach the level of utility at the bubbly steady state. This increase is equal to $\exp\left[\left(u_{first} - u_{other}\right) / \left(1 + \beta + \beta^2\right)\right]$, where u_{first} is the utility of the first generation, which reaches the level of utility corresponding to the bubbly steady state, and u_{other} is the level of utility of any other generation.

15. We consider that 20 is the number of years individuals are old. This is approximately the value of the life expectancy at 65 in the economies considered.

16. The labor income share in 2014 is obtained from the Penn World Table.

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APPENDIX A

A.1. PROOF OF PROPOSITION 1

We rewrite condition (18) as

$$(1-\alpha)(1-\Sigma)\beta^2(\Sigma_2-\Sigma) > \alpha\Omega\left(\beta+\beta^2\right)(\Sigma-\Sigma_1),$$

where Σ_2 and Σ_1 are defined in the main text.

As $\left(1+\beta^2\right)/\beta^2 > \left(1+\beta+\beta^2\right)/\left(\beta+\beta^2\right)$, there are only three possibilities:

(i) $\Sigma > \Sigma_2$ and condition (18) is not satisfied;
(ii) $\Sigma_1 < \Sigma < \Sigma_2$ and condition (18) is satisfied when $\Omega < \overline{\Omega}$, where $\overline{\Omega}$ is obtained from the above equation; and
(iii) $\Sigma < \Sigma_1$ and condition (18) is always satisfied.

A.2. EQUILIBRIUM WITH BUBBLES

We first use (2), (3), and (4), to rewrite equations (20) and (21) as

$$b_{1,t} = \frac{\left(\beta+\beta^2\right)\left(\left(1-\tau_w\right)\xi_1 w_t - a_{t+1}\right) - \frac{(1-\tau_w)\xi_2 w_{t+1}}{R_{t+1}} - (1-\tau_k)\left[q_{t+1}\phi_1 + \frac{q_{t+2}}{R_{t+2}}\phi_2\right]\frac{a_{t+1}}{R_{t+1}}}{1+\beta+\beta^2}, \tag{A1}$$

$$b_{2,t+1} = \frac{\beta^2(1-\tau_w)\xi_2 w_{t+1} + \beta^2(1-\tau_k)q_{t+1}\phi_1 a_{t+1} + \beta^2\left((1-\tau_w)\xi_1 w_t - a_{t+1}\right)R_{t+1} - (1+\beta)\frac{(1-\tau_k)q_{t+2}}{R_{t+2}}\phi_2 a_{t+1}}{1+\beta+\beta^2}. \tag{A2}$$

From using (5) and (6), equations (A1) and (A2) can be rewritten as

$$b_{1,t} = \frac{\left(\beta+\beta^2\right)\left((1-\tau_w)\xi_1(1-\alpha)Ak_t^\alpha - a_{t+1}\right) - \frac{(1-\tau_w)\xi_2(1-\alpha)Ak_{t+1}^\alpha}{R_{t+1}} - (1-\tau_k)\left[\alpha Ak_{t+1}^{\alpha-1}\phi_1 + \frac{\alpha Ak_{t+2}^{\alpha-1}}{R_{t+2}}\phi_2\right]\frac{a_{t+1}}{R_{t+1}}}{1+\beta+\beta^2}, \tag{A3}$$

$$b_{2,t+1} = \frac{\beta^2\left(1-\tau_w\right)\xi_2\left(1-\alpha\right)Ak_{t+1}^\alpha + \beta^2\left(1-\tau_k\right)\alpha Ak_{t+1}^{\alpha-1}\phi_1 a_{t+1}}{1+\beta+\beta^2} + \tag{A4}$$

$$+\frac{\beta^2\left[\left(1-\tau_w\right)\xi_1\left(1-\alpha\right)Ak_t^\alpha - a_{t+1}\right]R_{t+1} - \frac{(1+\beta)(1-\tau_k)\alpha Ak_{t+2}^{\alpha-1}\phi_2 a_{t+1}}{R_{t+2}}}{1+\beta+\beta^2}.$$

We use (23) to rewrite (A3) and (A4) as (24) and (25) in the main text.

A.3. PROOF OF PROPOSITION 5

We recall that we assume $\tau_k = \tau_w = 0$. Then, a bubble exists iff:

$$\Omega > \widetilde{\Omega}(\Sigma) = \left(\frac{1-\alpha}{\alpha}\right)(\Sigma_3 - \Sigma).$$

Note that $\widetilde{\Omega}(\Sigma)$ is a strictly decreasing line ($\widetilde{\Omega}'(\Sigma) < 0$), $\widetilde{\Omega}(\Sigma_3) = 0$ and $\widetilde{\Omega}(\Sigma') = 1$, with $\Sigma' = \frac{1-\beta^2}{1+\beta+\beta^2} + \frac{\alpha}{1-\alpha}$.

We also recall that a bubble is productive iff $\Sigma < \Sigma_1$ or $\Sigma \in (\Sigma_1, \Sigma_2)$ and

$$\Omega < \overline{\Omega}(\Sigma) = \left(\frac{\Sigma_2 - \Sigma}{\Sigma - \Sigma_1}\right)\left(\frac{1+\beta^2}{\beta+\beta^2}\right)\left(\frac{1-\Sigma}{\Sigma_2}\right).$$

It can be shown that $\Sigma_1 < \Sigma_2$, $\overline{\Omega}(\Sigma_1) = +\infty$, $\overline{\Omega}(\Sigma_2) = 0$ and $\overline{\Omega}(1) = 0$. Moreover,

$$\overline{\Omega}'(\Sigma) = \frac{1}{\Sigma_2(\Sigma-\Sigma_1)^2}\frac{1+\beta^2}{\beta+\beta^2}\left[(\Sigma_1-\Sigma_2)(1-\Sigma) - (\Sigma_2-\Sigma)(\Sigma-\Sigma_1)\right] < 0,$$

for all $\Sigma \in (\Sigma_1, \min\{\Sigma_2; 1\})$. In addition,

$$\overline{\Omega}''(\Sigma) = \frac{1+\beta^2}{(\beta+\beta^2)\Sigma_2}\left[\frac{2(\Sigma_2-\Sigma_1)(1-\Sigma)}{(\Sigma-\Sigma_1)^3} + \frac{2(\Sigma_2-\Sigma_1)}{(\Sigma-\Sigma_1)^2}\right] > 0,$$

for all $\Sigma \in (\Sigma_1, 1)$. Hence, $\overline{\Omega}(\Sigma)$ is a convex function, decreasing for all Σ such that $\overline{\Omega}(\Sigma) \geqslant 0$.

We further note that $\widetilde{\Omega}\left(\Sigma_{b_1}\right) = \Omega_{b_2}$ and $\overline{\Omega}\left(\Sigma_{b_1}\right) = \Omega_{b_2}$. This means that $\widetilde{\Omega}(\Sigma)$ and $\overline{\Omega}(\Sigma)$ cross once at the point $(\Sigma, \Omega) = (\Sigma_{b_1}, \Omega_{b_2})$. Since $\widetilde{\Omega}(\Sigma)$ is a line and $\overline{\Omega}(\Sigma)$ is convex, they cross at most twice.

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We know that, on the one hand, $b_1 > 0$ if $\Sigma > \Sigma_{b_1}$ and $b_2 > 0$ if $\Omega > \Omega_{b_2}$ and, on the other hand, a bubble is productive if $\Omega < \overline{\Omega}(\Sigma)$. Since $\Omega = \overline{\Omega}(\Sigma)$ goes through $(\Sigma_{b_1}, \Omega_{b_2})$ and is a convex function, decreasing for all Σ such that $\overline{\Omega}(\Sigma) \geqslant 0$, a bubble cannot be productive if $b_1 > 0$ and $b_2 > 0$, whatever the values of Σ_{b_1} and Ω_{b_2}. Hence, a bubble is productive if either $b_1 < 0$ and $b_2 > 0$ or $b_1 > 0$ and $b_2 < 0$.

The existence of productive bubbles with $b_1 < 0$ and $b_2 > 0$ requires either $\Sigma_1 \geqslant \Sigma'$, which is equivalent to:

$$\frac{\alpha}{1-\alpha} \geqslant \frac{\left(1-\beta^2\right)\left(\beta+\beta^2\right)}{1+\beta+\beta^2},$$

or $\Sigma_1 < \Sigma'$ and $\overline{\Omega}\left(\Sigma'\right) > 1$, that is,

$$\frac{\left(1-\beta^2\right)\left(\beta+\beta^2\right)}{1+\beta+\beta^2} > \frac{\alpha}{1-\alpha} > \frac{\left(1-\beta^2\right)\left(\beta+2\beta^2\right)}{\left(1+\beta+\beta^2\right)(2+\beta)}.$$

Moreover, $\Sigma' < 1$ if and only if:

$$\frac{\alpha}{1-\alpha} < \frac{\beta+2\beta^2}{1+\beta+\beta^2}.$$

When these inequalities are satisfied, there is a non-empty set of Σ such that there exists a productive bubble for $\Omega = 1$. By continuity, this result holds for $\Omega < 1$ but sufficiently close to 1. We deduce the existence of productive bubbles with $b_1 < 0$ and $b_2 > 0$ for $\frac{\alpha}{1-\alpha} \in \left(\frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta+2\beta^2}{1+\beta+\beta^2}\right)$. This occurs if $\Sigma < \Sigma_{b_1}$.

To show the existence of productive bubbles with $b_1 > 0$ and $b_2 < 0$, we first prove that $\Sigma_3 < 1$ is equivalent to:

$$\frac{\alpha}{1-\alpha} < \frac{\beta/2+\beta^2}{1+\beta+\beta^2},$$

and $\Sigma_3 < \Sigma_2$ iff:

$$\frac{\alpha}{1-\alpha} > \frac{\beta^2}{1+\beta+\beta^2}.$$

If these two inequalities are satisfied, there is a non-empty interval for Σ such that there exists a productive bubble for $\Omega = 0$. By continuity, this result holds for $\Omega > 0$ but sufficiently close to 0. We deduce the existence of productive bubbles with $b_1 > 0$ and $b_2 < 0$ for $\frac{\alpha}{1-\alpha} \in \left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$. This occurs if $\Sigma > \Sigma_{b_1}$.

We deduce the different cases of Proposition 5 comparing the two intervals $\left(\frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta+2\beta^2}{1+\beta+\beta^2}\right)$ and $\left(\frac{\beta^2}{1+\beta+\beta^2}, \frac{\beta/2+\beta^2}{1+\beta+\beta^2}\right)$ and taking into account that $\max\left\{\frac{(1-\beta^2)(\beta+2\beta^2)}{(1+\beta+\beta^2)(2+\beta)}, \frac{\beta^2}{1+\beta+\beta^2}\right\} < \frac{\beta/2+\beta^2}{1+\beta+\beta^2} < \frac{\beta+2\beta^2}{1+\beta+\beta^2}$.

A.4. EMPIRICAL STRATEGY TO OBTAIN Σ AND Ω

In this appendix, we describe how the data in Table 2 on the distribution of gross labor and capital income by age group have been obtained. The data sources used are the US census and the Eurostat. US government census provides mean income and total population in 2015 for the following age groups: young (age 25–44), middle-aged (age 45–64), and old (65 and over). Eurostat provides the same data in 2015 for the different European economies shown in Table 1 and for the following age groups: young (age 25–49), middle-aged (age 50–64), and old (65 and over). As the number of years people belong to each age group is different with the Eurostat data, we divide total income of each age group by the number of years individuals belong to each age group.[15] This normalization makes the different age groups comparable. From using these data, we obtain the total income of each age group, and the total income of the economy is obtained as the sum of the income of each age group. The fraction of total income obtained by each age group is displayed in Table 1.

We use the labor income share and total income to obtain for each country the labor income and the capital income.[16] Consistent with the assumptions in the model, we assume that (i) the young individuals do not obtain capital income and (ii) the old individuals do not obtain labor income. Based on these assumptions, we obtain Σ as the ratio between the income of the young and the total labor income in the economy, and we obtain $\widehat{\Omega}$ as the difference between one and the ratio between the income of the old and the total capital income of the economy. The values of Σ and $\widehat{\Omega}$ are displayed in Table 2.

The value of Σ and $\widehat{\Omega}$ are obviously biased because of the two aforementioned assumptions. To measure how problematic are these two assumptions, we use the US census data to obtain that the fraction of labor income obtained by the old individuals is only 4% and the net worth owned by the young is only 9.4%. These small numbers imply that the two assumptions are not too strong and, hence, the bias in the measures of Σ and $\widehat{\Omega}$ should be small.

A more serious problem with the data is that the income of the old also includes the pensions they receive, which should not be considered as capital income. Using the notation introduced in Section 2 and the definition of $\widehat{\Omega}$, we obtain

$$\widehat{\Omega}_t = 1 - \frac{q_{t+2}\phi_2 a_{t+1} + p_{t+2}}{q_{t+1}\phi_1 a_{t+1} n + q_{t+2}\phi_2 a_{t+1} + p_{t+2}},$$

where p_{t+2} are the pensions received by individuals when old. Note that $\widehat{\Omega}_t$ is the difference between one and the ratio between the income of the old and the total capital income. As follows from the data, pensions are included in the income of the old and also in the total income. $\widehat{\Omega}_t$ at the steady state simplifies as

$$\widehat{\Omega} = \frac{n\phi_1}{n\phi_1 + \phi_2 + \frac{p}{qa}},$$

where p is the steady-state value of the pension. Let σ be the replacement rate of pensions and, hence, $p = \sigma\xi_2 w$. Using the replacement rate, (5), (6), and (9), we obtain

$$\Omega = \widehat{\Omega}\left[1 + (1-\Sigma)\,\frac{\sigma\,(1-\alpha)}{\alpha n}\right],$$

where $\Omega = n\phi_1/\,(n\phi_1 + \phi_2)$ is the fraction of capital income obtained by the middle-aged individuals and that we have used in the main text of this paper. The previous equation clearly shows that $\widehat{\Omega}$ is a biased measure of the distribution of capital income by age group when pensions are introduced. In the last step of our empirical strategy, we use this equation to obtain the value of Ω. To this end, we must obtain the values of σ, α, and n. The value of σ is obtained from OECD data set 2014, where the replacement rate is defined as the gross pension divided by the gross pre-retirement wage and, hence, it corresponds to our definition of σ. The value of α is obtained from the labor income share in the Penn World Table 2014, and the value of n is obtained from OECD data as the ratio between total population age 45–64 divided by total population age 65 and over. The value of Ω obtained from this analysis is displayed in the last column of Table 2.

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