HIGHER-DIMENSIONAL SHRINKING TARGET PROBLEM FOR BETA DYNAMICAL SYSTEMS

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Abstract

We consider the two-dimensional shrinking target problem in beta dynamical systems (for general $\beta > 1$) with general errors of approximation. Let f, g be two positive continuous functions. For any $x_0, y_0 \in [0, 1]$, define the shrinking target set

$$E(T_\beta, f, g) := \left\{ (x, y) \in [0, 1]^2 : \begin{array}{l} |T_\beta^n x - x_0| < e^{-S_n f(x)} \\ |T_\beta^n y - y_0| < e^{-S_n g(y)} \end{array} \text{ for infinitely many } n \in \mathbb{N} \right\},$$

where $S_n f(x) = \sum_{j=0}^{n-1} f(T_\beta^j x)$ is the Birkhoff sum. We calculate the Hausdorff dimension of this set and prove that it is the solution to some pressure function. This represents the first result of this kind for the higher-dimensional beta dynamical systems.

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1. Introduction

The study of the Diophantine properties of the distribution of orbits for a measure-preserving dynamical system has recently received much attention. Let $T : X \to X$ be a measure-preserving transformation of the system $(X, \mathcal{B}, \mu)$ with a consistent metric d. If the transformation T is ergodic with respect to the measure μ, Poincaré's recurrence theorem implies that, for almost every $x \in X$, the orbit $\{T^n x\}_{n=0}^\infty$ returns to an arbitrary but fixed neighbourhood of x infinitely often. That is, for any

$x_0 \in X$, for μ-almost all $x \in X$,

$$\liminf_{n \to \infty} d(T^n x, x_0) = 0.$$

Poincaré's recurrence theorem is qualitative in nature but it does motivate the study of the distribution of T-orbits of points in X quantitatively. In other words, a natural motivation is to investigate how fast the above limit infimum tends to zero. To this end, the spotlight is on the size of the set

$$D(T, \varphi) := \{x \in X : d(T^n x, x_0) < \varphi(n) \text{ for i.m. } n \in \mathbb{N}\},$$

where $\varphi : \mathbb{N} \to \mathbb{R}_{\geq 0}$ is a positive function such that $\varphi(n) \to 0$ as $n \to \infty$. Here and throughout, 'i.m.' is used for 'infinitely many'. The set $D(T, \varphi)$ can be viewed as the collection of points in X whose T-orbit hits a shrinking target infinitely many times. The set $D(T, \varphi)$ is the dynamical analogue of the classical inhomogeneous well-approximable set

$$W(\varphi) := \{x \in [0, 1) : |x - p/q - x_0| < \varphi(q) \text{ for i.m. } p/q \in \mathbb{Q}\}.$$

As one would expect, the 'size' of both of these sets depends upon the nature of the function φ, that is, how fast it approaches zero. The size of the set $W(\varphi)$ in terms of Lebesgue measure or Hausdorff measure and dimension has been established even in the higher-dimensional (linear form) settings; see [1, 8, 19] for further details. In contrast, not much is known for the higher-dimensional version of the set $D(T, \varphi)$ for general T.

Following the work of Hill and Velani [6], the Hausdorff dimension of the set $D(T, \varphi)$ has been determined for many dynamical systems, from the system of rational expanding maps on their Julia sets to conformal iterated function systems [15]. We refer the reader to [3] for a comprehensive discussion regarding the Hausdorff dimension of various dynamical systems. In this paper, we confine ourselves to the two-dimensional shrinking target problem in beta dynamical systems with general errors of approximation.

For a real number $\beta > 1$, define the transformation $T_\beta : [0, 1] \to [0, 1]$ by

$$T_\beta : x \mapsto \beta x \pmod{1}.$$

This map generates the β-dynamical system $([0, 1], T_\beta)$. It is well known that β-expansion is a typical example of an expanding nonfinite Markov system whose properties are reflected by the orbit of some critical point; in other words, it is not a subshift of finite type with mixing properties. This causes difficulties in studying metrical questions related to β-expansions. General β-expansions have been widely studied in the literature; see for instance [7, 9, 12–14] and references therein. In particular, the Hausdorff dimension, denoted throughout by $\dim_H$, of $D(T_\beta, \varphi)$ was

obtained in [13] and the Lebesgue measure and Hausdorff dimension of the set

$$D(T_\beta, \varphi_1, \varphi_2) := \left\{ (x, y) \in [0, 1]^2 : \begin{array}{l} |T_\beta^n x - x_0| < \varphi_1(n) \\ |T_\beta^n y - y_0| < \varphi_2(n) \end{array} \text{ for i.m. } n \in \mathbb{N} \right\}$$

was calculated in [9]. Here $x_0, y_0 \in [0, 1]$ are fixed and the approximating functions φ_1, φ_2 are positive functions of n.

In 2014, Bugeaud and Wang [2] calculated the Hausdorff dimension of the set with the error of approximation given by the ergodic sum, that is,

$$E(T_\beta, h) := \{x \in [0, 1] : |T_\beta^n x - x_0| < e^{-S_n h(x)} \text{ for i.m. } n \in \mathbb{N}\},$$

where h is a positive continuous function on $[0, 1]$ and $S_n h(x) = h(x) + \cdots + h(T_\beta^{n-1} x)$. Clearly the error of approximation is exponential depending upon the orbits $T_\beta x$. Note that it is still an open problem whether $e^{-S_n h(x)}$ implies the arbitrary function $\varphi(n)$ or not. However, $e^{-S_n f(x)}$ reduces to $\beta^{-n\tau}$ by considering $h(x) = \tau \log|T'(x)|$ for some $\tau > 0$. Thus, the result of [2] implies the Jarník–Besicovitch type result for the set under consideration.

In this paper, we extend Bugeaud and Wang's set $E(T_\beta, h)$ to the two-dimensional setting and calculate its Hausdorff dimension. Let f, g be two positive continuous function on $[0, 1]$ and let $x_0, y_0 \in [0, 1]$ be fixed. Define

$$E(T_\beta, f, g) := \left\{ (x, y) \in [0, 1]^2 : \begin{array}{l} |T_\beta^n x - x_0| < e^{-S_n f(x)} \\ |T_\beta^n y - y_0| < e^{-S_n g(y)} \end{array} \text{ for i.m. } n \in \mathbb{N} \right\}.$$

The set $E(T_\beta, f, g)$ is the set of all points (x, y) in the unit square such that the pair $\{(T^n x, T^n y)\}$ is in the shrinking rectangle $B(x_0, e^{-S_n f(x)}) \times B(y_0, e^{-S_n g(y)})$ for infinitely many n. The rectangle shrinks to zero at exponential rates given by $e^{-S_n f(x)}$ and $e^{-S_n g(y)}$. We shall prove the following result.

THEOREM 1.1. *Let f, g be two continuous functions on $[0, 1]$ with $f(x) \geq g(y)$ for all $x, y \in [0, 1]$. Then*

$$\dim_H E(T_\beta, f, g) = \min\{s_1, s_2\},$$

where

$$s_1 = \inf\{s \geq 0 : P(f - s(\log \beta + f)) + P(-g) \leq 0\},$$
$$s_2 = \inf\{s \geq 0 : P(-s(\log \beta + g)) + \log \beta \leq 0\}.$$

Here $P(\cdot)$ stands for the pressure function for the β-dynamical system associated to continuous potentials f and g. To keep the introductory section short, we formally give the definition of the pressure function in Section 2. The reason why the Hausdorff dimension is in terms of the pressure function is because of the dynamical nature of the set $E(T_\beta, f, g)$. For the detailed analysis of the properties of the pressure function and ergodic sums for general dynamical systems we refer the reader to [17, Ch. 9].

The proof of this theorem splits into two parts: establishing the upper bound and then the lower bound. The upper bound is relatively easier to prove by using the definition of Hausdorff dimension on the natural cover of the set. However, establishing the lower bound is challenging and the main substance of this paper. Actually, the main obstacle in determining the metrical properties of general β-expansions lies in the difficulty of estimating the length of a general cylinder and, since we are dealing with two-dimensional settings, the area of the cross-product of general cylinders. As far as the Hausdorff dimension is concerned, one does not need to take all points into consideration; instead, one may choose a subset of points with regular properties to approximate the set in question. This argument in turn requires some continuity of the dimensional number, when the system is approximated by its subsystem.

The paper is organised as follows. Section 2 is devoted to recalling some elementary properties of β-expansions. Short proofs are also given when we could not find any reference. Definitions and some properties of the pressure function are stated in this section as well. In Section 3 we prove the upper bound of Theorem 1.1. In Section 4 we prove the lower bound of Theorem 1.1, and since this takes up a large proportion of the paper we subdivide this section into several subsections.

2. Preliminaries

We begin with a brief account of some basic properties of β-expansions and we fix some notation. We then state and prove two propositions which give the covering and packing properties.

The β-expansion of real numbers was first introduced by Rényi [11], and is given by the following algorithm. For any $\beta > 1$, let

$$T_\beta(0) := 0, \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor, \ x \in [0, 1), \tag{2-1}$$

where $\lfloor \xi \rfloor$ is the integer part of $\xi \in \mathbb{R}$. By taking

$$\epsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1} x \rfloor \in \mathbb{N}$$

recursively for each $n \geq 1$, every $x \in [0, 1)$ can be uniquely expanded into a finite or an infinite sequence

$$x = \frac{\epsilon_1(x, \beta)}{\beta} + \frac{\epsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\epsilon_n(x, \beta)}{\beta^n} + \frac{T_\beta^n x}{\beta^n},$$

which is called the β-expansion of x, and the sequence $\{\epsilon_n(x, \beta)\}_{n \geq 1}$ is called the digit sequence of x. We also write the β-expansion of x as

$$\epsilon(x, \beta) = (\epsilon_1(x, \beta), \ldots, \epsilon_n(x, \beta), \ldots).$$

The system $([0, 1], T_\beta)$ is called a β-*dynamical system* or just a β-*system*.

DEFINITION 2.1. A finite or an infinite sequence $(w_1, w_2, \ldots)$ is said to be admissible (with respect to the base β), if there exists an $x \in [0, 1)$ such that the digit sequence of x equals $(w_1, w_2, \ldots)$.

Denote by Σ_β^n the collection of all admissible sequences of length n and by Σ_β that of all infinite admissible sequences.

Let us now turn to the infinite β-expansion of 1, which plays an important role in the study of β-expansions. Applying algorithm (2-1) to the number $x = 1$, then the number 1 can be expanded into a series, denoted by

$$1 = \frac{\epsilon_1(1,\beta)}{\beta} + \frac{\epsilon_2(1,\beta)}{\beta^2} + \cdots + \frac{\epsilon_n(1,\beta)}{\beta^n} + \cdots .$$

If the above series is finite, that is, there exists $m \geq 1$ such that $\epsilon_m(1,\beta) \neq 0$ but $\epsilon_n(1,\beta) = 0$ for $n > m$, then β is called a simple Parry number. In this case, we write

$$\epsilon^*(1,\beta) := (\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots) = (\epsilon_1(1,\beta), \ldots, \epsilon_{m-1}(1,\beta), \epsilon_m(1,\beta) - 1)^\infty,$$

where $(w)^\infty$ denotes the periodic sequence $(w, w, w, \ldots)$. If β is not a simple Parry number, we write

$$\epsilon^*(1,\beta) := (\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots) = (\epsilon_1(1,\beta), \epsilon_2(1,\beta), \ldots).$$

In both cases, the sequence $(\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots)$ is called the infinite β-expansion of 1 and we always have that

$$1 = \frac{\epsilon_1^*(\beta)}{\beta} + \frac{\epsilon_2^*(\beta)}{\beta^2} + \cdots + \frac{\epsilon_n^*(\beta)}{\beta^n} + \cdots .$$

The lexicographical order $\prec$ between infinite sequences is defined as

$$w = (w_1, w_2, \ldots, w_n, \ldots) \prec w' = (w_1', w_2', \ldots, w_n', \ldots)$$

if there exists $k \geq 1$ such that $w_j = w_j'$ for $1 \leq j < k$, while $w_k < w_k'$. The notation $w \preceq w'$ means that $w \prec w'$ or $w = w'$. This ordering can be extended to finite blocks by identifying the finite block $(w_1, w_2, \ldots, w_n)$ with the sequence $(w_1, w_2, \ldots, w_n, 0, 0, \ldots)$.

The following result due to Parry [10] is a criterion for the admissibility of a sequence.

LEMMA 2.2 (Parry [10]). *Let $\beta > 1$ be a real number. Then a nonnegative integer sequence $\epsilon = (\epsilon_1, \epsilon_2, \ldots)$ is admissible if and only if, for any $k \geq 1$,*

$$(\epsilon_k, \epsilon_{k+1}, \ldots) \prec (\epsilon_1^*(\beta), \epsilon_2^*(\beta), \ldots).$$

The following result of Rényi implies that the dynamical system $([0, 1], T_\beta)$ admits $\log \beta$ as its topological entropy.

LEMMA 2.3 (Rényi [11]). *Let $\beta > 1$. For any $n \geq 1$,*

$$\beta^n \leq \#\Sigma_\beta^n \leq \frac{\beta^{n+1}}{\beta - 1},$$

where $\#$ denotes the cardinality of a finite set.

It is clear from this lemma that

$$\lim_{n\to\infty} \frac{\log(\#\Sigma_\beta^n)}{n} = \log\beta.$$

For any $\epsilon_n := (\epsilon_1, \ldots, \epsilon_n) \in \Sigma_\beta^n$, call

$$I_n(\epsilon_n) := \{x \in [0, 1), \epsilon_j(x, \beta) = \epsilon_j, 1 \le j \le n\}$$

an nth-order cylinder (with respect to the base β). It is a left-closed and right-open interval with the left endpoint

$$\frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n}$$

and length

$$|I_n(\epsilon_n)| \le \frac{1}{\beta^n}.$$

Here and throughout the paper, we use $|\cdot|$ to denote the length of an interval. Note that the unit interval can be naturally partitioned into a disjoint union of cylinders; that is, for any $n \ge 1$,

$$[0, 1] = \bigcup_{\epsilon_n \in \Sigma_\beta^n} I_n(\epsilon_n).$$

One difficulty in studying the metric properties of β-expansions is that the length of a cylinder is not regular. It may happen that $|I_n(\epsilon_1, \ldots, \epsilon_n)| \ll \beta^{-n}$. Here $a \ll b$ is used to indicate that there exists a constant $c > 0$ such that $a \le cb$. We write $a \asymp b$ if $a \ll b \ll a$. The following notation plays an important role in bypassing this difficulty.

DEFINITION 2.4 (Full cylinder). A cylinder $I_n(\epsilon_n)$ is called full if it has maximal length, that is, if

$$|I_n(\epsilon_n)| = \frac{1}{\beta^n}.$$

Correspondingly, we also call the word $(\epsilon_1, \ldots, \epsilon_n)$, defining the full cylinder $I_n(\epsilon_n)$, a full word.

Next, we collect some properties about the distribution of full cylinders.

PROPOSITION 2.5 (Fan and Wang [5]). *An nth-order cylinder $I_n(\epsilon_n)$ is full if and only if, for any admissible sequence $\epsilon_m' := (\epsilon_1', \epsilon_2', \ldots, \epsilon_m') \in \Sigma_\beta^m$ with $m \ge 1$,*

$$(\epsilon_n, \epsilon_m') \in \Sigma_\beta^{n+m}.$$

Moreover.

$$|I_{n+m}(\epsilon_n, \epsilon_m')| = |I_n(\epsilon_n)| \cdot |I_m(\epsilon_m')|.$$

So, for any two full cylinders $I_n(\epsilon_n)$, $I_m(\epsilon_m')$, the cylinder $I_{n+m}(\epsilon_n, \epsilon_m')$ is also full.

LEMMA 2.6 (Bugeaud and Wang [2]). *For $n \geq 1$, among every $n + 1$ consecutive cylinders of order n, there exists at least one full cylinder.*

As a consequence, one has the following relationship between balls and cylinders.

PROPOSITION 2.7 (Covering property). *Let J be an interval of length β^{-l} with $l \geq 1$. Then it can be covered by at most $2(l + 1)$ cylinders of order l.*

PROOF. By Lemma 2.6, among any $2(l + 1)$ consecutive cylinders of order l, there are at least two full cylinders. So the total length of these intervals is larger than $2\beta^{-l}$. Thus J can be covered by at most $2(l + 1)$ cylinders of order l. □

The following result may be of independent interest.

PROPOSITION 2.8 (Packing property). *Fix $0 < \epsilon < 1$. Let n_0 be an integer such that $2n^2\beta < \beta^{(n-1)\epsilon}$ for all $n \geq n_0$. Let $J \subset [0, 1]$ be an interval of length r with $0 < r < 2n_0\beta^{-n_0}$. Then inside J there exists a full cylinder I_n satisfying*

$$r \geq |I_n| > r^{1+\epsilon}.$$

PROOF. Let $n > n_0$ be the integer such that

$$2n\beta^{-n} \leq r < 2(n - 1)\beta^{-n+1}.$$

Since every cylinder of order n is of length at most β^{-n}, the interval J contains at least $2n - 2 \geq n + 1$ consecutive cylinders of order n. Thus, by Lemma 2.6, it contains a full cylinder of order n and we denote such a cylinder by I_n. By the choice of n_0, we have

$$r \geq |I_n| = \beta^{-n} > (2(n - 1)(\beta^{-n+1}))^{1+\epsilon} > r^{1+\epsilon}.$$

This completes the proof. □

We now define a sequence of numbers β_N approximating β from below. For any N with $\epsilon_N^*(\beta) \geq 1$, define β_N to be the unique real solution to the algebraic equation

$$1 = \frac{\epsilon_1^*(\beta)}{\beta_N} + \frac{\epsilon_2^*(\beta)}{\beta_N^2} + \cdots + \frac{\epsilon_N^*(\beta)}{\beta_N^N}.$$

Then β_N approximates β from below and the β_N-expansion of unity is

$$(\epsilon_1^*(\beta), \ldots, \epsilon_{N-1}^*(\beta), \epsilon_N^*(\beta) - 1)^\infty.$$

More importantly, by the admissible sequence criterion, we have, for any $\epsilon_n \in \Sigma_{\beta_N}^n$ and $\epsilon_m' \in \Sigma_{\beta_N}^m$, that

$$(\epsilon_n, 0^N, \epsilon_m') \in \Sigma_{\beta_N}^{n+N+m}, \tag{2-2}$$

where 0^N means a zero word of length N.

From assertion (2-2), we get the following proposition.

PROPOSITION 2.9. *For any $\epsilon_n \in \Sigma_{\beta_N}^n$, $I_{n+N}(\epsilon_n, 0^N)$ is a full cylinder. So,*

$$\frac{1}{\beta^{n+N}} \leq |I_n(\epsilon_n)| \leq \frac{1}{\beta^n}.$$

We end this section with a definition of the pressure function for a β-dynamical system associated to some continuous potential g:

$$P(g, T_\beta) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\epsilon_n \in \Sigma_\beta^n} \sup_{y \in I_n(\epsilon_n)} e^{S_n g(y)}, \tag{2-3}$$

where $S_n g(y)$ denotes the ergodic sum $\sum_{j=0}^{n-1} g(T_\beta^j y)$. Since g is continuous, the limit does not depend upon the choice of y. The existence of the limit (2-3) follows from subadditivity:

$$\log \sum_{(\epsilon_n, \epsilon_m') \in \Sigma_\beta^{n+m}} e^{S_{n+m} g(y)} \leq \log \sum_{\epsilon_n \in \Sigma_\beta^n} e^{S_n g(y)} + \log \sum_{\epsilon_m' \in \Sigma_\beta^m} e^{S_m g(y)}.$$

The reader is referred to [16] for more details.

3. Proof of Theorem 1.1: the upper bound

As is typical in determining the Hausdorff dimension of a set, we split the proof of Theorem 1.1 into two parts: the upper bound and the lower bound.

For any $\epsilon_n = (\epsilon_1, \ldots, \epsilon_n) \in \Sigma_\beta^n$ and $\omega_n = (\omega_1, \ldots, \omega_n) \in \Sigma_\beta^n$, we always take

$$x^* = \frac{\epsilon_1}{\beta} + \frac{\epsilon_2}{\beta^2} + \cdots + \frac{\epsilon_n}{\beta^n}$$

to be the left endpoint of $I_n(\epsilon_n)$ and

$$y^* = \frac{\omega_1}{\beta} + \frac{\omega_2}{\beta^2} + \cdots + \frac{\omega_n}{\beta^n}$$

to be the left endpoint of $I_n(\omega_n)$.

Instead of directly considering the set $E(T_\beta, f, g)$, we consider a closely related limit supremum set

$$\overline{E}(T_\beta, f, g) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{\epsilon_n, \omega_n \in \Sigma_\beta^n} J_n(\epsilon_n) \times J_n(\omega_n),$$

where

$$J_n(\epsilon_n) = \{x \in [0, 1] : |T_\beta^n x - x_0| < e^{-S_n f(x^*)}\},$$
$$J_n(\omega_n) = \{y \in [0, 1] : |T_\beta^n y - y_0| < e^{-S_n g(y^*)}\}.$$

In the sequel it is clear that the set $\overline{E}(T_\beta, f, g)$ is easier to handle. Since f and g are continuous functions, for any $\delta > 0$ and n large enough, we have

$$|S_n f(x) - S_n f(x^*)| < n\delta, \quad |S_n g(y) - S_n g(y^*)| < n\delta.$$

Thus we have

$$\overline{E}(T_\beta, f + \delta, g + \delta) \subset E(T_\beta, f, g) \subset \overline{E}(T_\beta, f - \delta, g - \delta).$$

Therefore, to calculate the Hausdorff dimension of the set $E(T_\beta, f, g)$, it is sufficient to determine the Hausdorff dimension of $\overline{E}(T_\beta, f, g)$.

The length of $J_n(\epsilon_n)$ satisfies

$$|J_n(\epsilon_n)| \leq 2\beta^{-n} e^{-S_n f(x^*)},$$

since, for every $x \in J_n(\epsilon_n)$, we have

$$\left| x - \left(\frac{\epsilon_1}{\beta} + \cdots + \frac{\epsilon_n + x_0}{\beta^n} \right) \right| = \frac{|T_\beta^n x - x_0|}{\beta^n} < \beta^{-n} e^{-S_n f(x^*)}.$$

Similarly,

$$|J_n(\omega_n)| \leq 2\beta^{-n} e^{-S_n f(y^*)}.$$

So, $\overline{E}(T_\beta, f, g)$ is a limit supremum set defined by a collection of rectangles. There are two ways to cover a single rectangle $J_n(\epsilon_n) \times J_n(\omega_n)$, as follows.

3.1. Covering by shorter side length. Recall that $f(x) \geq g(y)$ for all $x, y \in [0, 1]$. This implies that $J_n(\epsilon_n)$ is shorter in length than $J_n(\omega_n)$. Then the rectangle $J_n(\epsilon_n) \times J_n(\omega_n)$ can be covered by

$$\frac{\beta^{-n} e^{-S_n g(y^*)}}{\beta^{-n} e^{-S_n f(x^*)}} = \frac{e^{S_n f(x^*)}}{e^{S_n g(y^*)}}$$

balls of side length $\beta^{-n} e^{-S_n f(x^*)}$.

Since, for each N,

$$\overline{E}(T_\beta, f, g) \subseteq \bigcup_{n=N}^{\infty} \bigcup_{\epsilon_n, \omega_n \in \Sigma_\beta^n} J_n(\epsilon_n) \times J_n(\omega_n),$$

the s-dimensional Hausdorff measure $\mathcal{H}^s$ of $\overline{E}(T_\beta, f, g)$ can be estimated as

$$\mathcal{H}^s(\overline{E}(T_\beta, f, g)) \leq \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{\epsilon_n, \omega_n \in \Sigma_\beta^n} \frac{e^{S_n f(x^*)}}{e^{S_n g(y^*)}} \left(\frac{1}{\beta^n e^{S_n f(x^*)}} \right)^s.$$

Define

$$s_1 = \inf\{s \geq 0 : P(f - s(\log \beta + f)) + P(-g) \leq 0\}.$$

Then from the definition of the pressure function (2-3), it is clear that

$$P(f - s(\log \beta + f)) + P(-g) \le 0 \iff \sum_{n=1}^{\infty} \sum_{\epsilon_n, \omega_n \in \Sigma_{\beta}^n} \frac{e^{S_n f(x^*)}}{e^{S_n g(y^*)}} \left(\frac{1}{\beta^n e^{S_n f(x^*)}} \right)^s < \infty.$$

Hence, for any $s > s_1$,

$$\mathcal{H}^s(\overline{E}(T_{\beta}, f, g)) = 0.$$

Hence, it follows that $\dim_{\mathrm{H}}(\overline{E}(T_{\beta}, f, g)) \le s_1$.

3.2. Covering by longer side length. From the previous subsection (Section 3.1), it is clear that only one ball of side length $\beta^{-n} e^{-S_n g(y^*)}$ is needed to cover the rectangle $J_n(\epsilon_n) \times J_n(\omega_n)$. Hence, in this case, the s-dimensional Hausdorff measure $\mathcal{H}^s$ of $\overline{E}(T_{\beta}, f, g)$ can be estimated as

$$\mathcal{H}^s(\overline{E}(T_{\beta}, f, g)) \le \liminf_{N \to \infty} \sum_{n=N}^{\infty} \sum_{\epsilon_n, \omega_n \in \Sigma_{\beta}^n} \left(\frac{1}{\beta^n e^{S_n g(y^*)}} \right)^s.$$

Define

$$s_2 = \inf\{s \ge 0 : P(-s(\log \beta + g)) + \log \beta \le 0\}.$$

Then, from the definitions of the pressure function and Hausdorff measure, it follows that, for any $s > s_2$, $\mathcal{H}^s(\overline{E}(T_{\beta}, f, g)) = 0$. Hence,

$$\dim_{\mathrm{H}}(\overline{E}(T_{\beta}, f, g)) \le s_2.$$

4. Theorem 1.1: the lower bound

It should be clear from the previous section that proving the upper bound requires only a suitable covering of the set $\overline{E}(T_{\beta}, f, g)$. In contrast, proving the lower bound is a challenging task, requiring all possible coverings to be considered; it therefore represents the main problem in metric Diophantine approximation (in various settings). The following principle, commonly known as the mass distribution principle [4], is used frequently for this purpose.

PROPOSITION 4.1 (Falconer [4]). *Let E be a Borel measurable set in $\mathbb{R}^d$ and μ be a Borel measure with $\mu(E) > 0$. Assume that there exist two positive constants c, δ such that, for any set U with diameter $|U|$ less than δ, $\mu(U) \le c|U|^s$. Then $\dim_{\mathrm{H}} E \ge s$.*

Specifically, the mass distribution principle replaces the consideration of all coverings by the construction of a particular measure μ and it is typically deployed in two steps:

- construct a suitable Cantor subset $\mathcal{F}_{\infty}$ of $\overline{E}(T_{\beta}, f, g)$ and a probability measure μ supported on $\mathcal{F}_{\infty}$;
- show that, for any fixed $c > 0$, μ satisfies the condition that, for any measurable set U of sufficiently small diameter, $\mu(U) \le c|U|^s$.

If this can be done, then by the mass distribution principle, it follows that

$$\dim_H(\overline{E}(T_\beta, f, g)) \geq \dim_H(\mathcal{F}_\infty) \geq s.$$

The substantive, intricate part of this entire process is the construction of a suitable Cantor type subset $\mathcal{F}_\infty$ which supports a probability measure μ. In the remainder of this paper, we construct a suitable Cantor type subset of the set $\overline{E}(T_\beta, f, g)$ and demonstrate that it satisfies the mass distribution principle.

4.1. Construction of the Cantor subset. We construct the Cantor subset $\mathcal{F}_\infty$ iteratively. Start by fixing an $\epsilon > 0$ and assume that $f(x) \geq (1 + \epsilon)g(y) \geq g(y)$ for all $x, y \in [0, 1]$. We construct a Cantor subset level by level and note that each level depends on its predecessor. Choose a rapidly increasing subsequence $\{m_k\}_{k \geq 1}$ of positive integers with m_1 large enough.

4.1.1. Level 1 of the Cantor set. Let $n_1 = m_1$. For any $U_1, W_1 \in \Sigma_{\beta_N}^{n_1}$ ending with the zero word of order N (that is, 0^N), let $x_1^* \in I_{n_1}(U_1)$, $y_1^* \in I_{n_1}(W_1)$. From Proposition 2.8, it follows that there are two full cylinders $I_{k_1}(K_1), I_{l_1}(L_1)$ such that

$$I_{k_1}(K_1) \subset B(x_0, e^{-S_{n_1} f(x_1^*)}),$$
$$I_{l_1}(L_1) \subset B(y_0, e^{-S_{n_1} g(y_1^*)}),$$

and

$$e^{-S_{n_1} f(x_1^*)} > \beta^{-k_1} > (e^{-S_{n_1} f(x_1^*)})^{1+\epsilon},$$

$$e^{-S_{n_1} g(y_1^*)} > \beta^{-l_1} > (e^{-S_{n_1} g(y_1^*)})^{1+\epsilon} = e^{-S_{n_1}(1+\epsilon)g(y_1^*)}.$$

So, we get a subset $I_{n_1+k_1}(U_1, K_1) \times I_{n_1+l_1}(W_1, L_1)$ of $J_{n_1}(U_1) \times J_{n_1}(W_1)$. Since $f(x) \geq (1 + \epsilon)g(y)$ for all $x, y \in [0, 1]$, it follows that $k_1 \geq l_1$. It should be noted that K_1 and L_1 depend on U_1 and W_1, respectively. Consequently, for different U_1 and W_1, the choice of K_1 and L_1 may be different.

The first level of the Cantor set is defined as

$$\mathcal{F}_1 = \{I_{n_1+k_1}(U_1, K_1) \times I_{n_1+l_1}(W_1, L_1) : U_1, W_1 \in \Sigma_{\beta_N}^{n_1} \text{ ending with } 0^N\},$$

which is composed of a collection of rectangles. Next, we cut each rectangle into balls with the radius as the shorter side length of the rectangle:

$$I_{n_1+k_1}(U_1, K_1) \times I_{n_1+l_1}(W_1, L_1)$$
$$\rightarrow \{I_{n_1+k_1}(U_1, K_1) \times I_{n_1+k_1}(W_1, L_1, H_1) : H_1 \in \Sigma_{\beta_N}^{k_1-l_1}\}.$$

Then we get a collection of balls

$$\mathcal{G}_1 = \{I_{n_1+k_1}(U_1, K_1) \times I_{n_1+k_1}(W_1, L_1, H_1) : U_1, W_1 \in \Sigma_{\beta_N}^{n_1} \text{ ending with } 0^N, H_1 \in \Sigma_{\beta_N}^{k_1-l_1}\}.$$

4.1.2. Level 2 of the Cantor set. Fix a $J_1 = I_{n_1+k_1}(\Gamma_1) \times I_{n_1+k_1}(\Upsilon_1)$ in $\mathcal{G}_1$. We define the local sublevel $\mathcal{F}_2(J_1)$ as follows.

Choose a large integer m_2 such that

$$\frac{\epsilon}{1+\epsilon} \cdot m_2 \log \beta \geq (n_1 + \sup\{k_1 : I_{n_1+k_1}(\Gamma_1)\}) \|f\|,$$

where $\|f\| = \sup\{|f(x)| : x \in [0, 1]\}$.

Write $n_2 = n_1 + k_1 + m_2$. Just like the first level of the Cantor set, for any $U_2, W_2 \in \Sigma_{\beta_N}^{m_2}$ ending with 0^N, applying Proposition 2.8 to $J_{n_2}(\Gamma_1, U_2) \times J_{n_2}(\Upsilon_1, W_2)$, we can get two full cylinders $I_{k_2}(K_2), I_{l_2}(L_2)$ such that

$$I_{k_2}(K_2) \subset B(x_0, e^{-S_{n_2}f(x_2^*)}), \quad I_{l_2}(L_2) \subset B(y_0, e^{-S_{n_2}g(y_2^*)})$$

and

$$e^{-S_{n_2}f(x_2^*)} > \beta^{-k_2} > (e^{-S_{n_2}f(x_2^*)})^{1+\epsilon},$$

$$e^{-S_{n_2}g(y_2^*)} > \beta^{-l_2} > (e^{-S_{n_2}g(y_2^*)})^{1+\epsilon} = e^{-S_{n_2}(1+\epsilon)g(y_2^*)},$$

where $x_2^* \in I_{n_2}(\Gamma_1, U_2)$, $y_2^* \in I_{n_2}(\Upsilon_1, W_2)$.

Obviously, we get a subset $I_{n_2+k_2}(\Gamma_1, U_2, K_2) \times I_{n_2+l_2}(\Upsilon_1, W_2, L_2)$ of $J_{n_2}(\Gamma_1, U_2) \times J_{n_2}(\Upsilon_1, W_2)$ and $k_2 \geq l_2$. Then, the second level of the Cantor set is defined as

$$\mathcal{F}_2(J_1) = \{I_{n_2+k_2}(\Gamma_1, U_2, K_2) \times I_{n_2+l_2}(\Upsilon_1, W_2, L_2) : U_2, W_2 \in \Sigma_{\beta_N}^{m_2} \text{ ending with } 0^N\},$$

which is composed of a collection of rectangles.

Next, we cut each rectangle into balls with the radius as the shorter side length of the rectangle:

$$I_{n_2+k_2}(\Gamma_1, U_2, K_2) \times I_{n_2+l_2}(\Upsilon_1, W_2, L_2) \rightarrow \{I_{n_2+k_2}(\Gamma_1, U_2, K_2)$$
$$\times I_{n_2+k_2}(\Upsilon_1, W_2, L_2, H_2) : H_2 \in \Sigma_{\beta_N}^{k_2-l_2}\} := \mathcal{G}_2(J_1).$$

Therefore, the second level is defined as

$$\mathcal{F}_2 = \bigcup_{J \in \mathcal{G}_1} \mathcal{F}_2(J), \quad \mathcal{G}_2 = \bigcup_{J \in \mathcal{G}_1} \mathcal{G}_2(J).$$

4.1.3. From level $i-1$ to level i. Assume that the $(i-1)$th level of the Cantor set $\mathcal{G}_{i-1}$ has been defined. Let $J_{i-1} = I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})$ be a generic element in $\mathcal{G}_{i-1}$. We define the local sublevel $\mathcal{F}_i(J_{i-1})$ as follows.

Choose a large integer m_i such that

$$\frac{\epsilon}{1+\epsilon} \cdot m_i \log \beta \geq (n_{i-1} + \sup\{k_{i-1} : I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1})\}) \|f\|. \tag{4-1}$$

Write $n_i = n_{i-1} + k_{i-1} + m_i$. For each $U_i, W_i \in \Sigma_{\beta_N}^{m_i}$ ending with 0^N, apply Proposition 2.8 to

$$J_{n_i}(\Gamma_{n_{i-1}+k_{i-1}}, U_i) \times J_{n_i}(\Upsilon_{n_{i-1}+k_{i-1}}, W_i).$$

We can get two full cylinders $I_{k_i}(K_i), I_{l_i}(L_i)$ such that

$$I_{k_i}(K_i) \subset B(x_0, e^{-S_{n_i}f(x_i^*)}), I_{l_i}(L_i) \subset B(y_0, e^{-S_{n_i}g(y_i^*)})$$

and

$$e^{-S_{n_i}f(x_i^*)} > \beta^{-k_i} > (e^{-S_{n_i}f(x_i^*)})^{1+\epsilon},$$

$$e^{-S_{n_i}g(y_i^*)} > \beta^{-l_i} > (e^{-S_{n_i}g(y_i^*)})^{1+\epsilon} = e^{-S_{n_i}(1+\epsilon)g(y_i^*)},$$

where $x_i^* \in I_{n_i}(\Gamma_{i-1}, U_i)$, $y_i^* \in I_{n_i}(\Upsilon_{i-1}, W_i)$.

Obviously, we get a subset $I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i)$ of $J_{n_i}(\Gamma_{i-1}, U_i) \times J_{n_i}(\Upsilon_{i-1}, W_i)$ and $k_i \geq l_i$. Then, the ith level of the Cantor set is defined as

$$\mathcal{F}_i(J_{i-1}) = \{I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i) : U_i, W_i \in \Sigma_{\beta_N}^{m_i} \text{ ending with } 0^N\},$$

which is composed of a collection of rectangles. As before, we cut each rectangle into balls with the radius as the shorter side length of the rectangle:

$$I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i) \rightarrow \{I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i)$$
$$\times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i) : H_i \in \Sigma_{\beta_N}^{k_i-l_i}\} := \mathcal{G}_i(J_{i-1}).$$

Therefore, the ith level is defined as

$$\mathcal{F}_i = \bigcup_{J \in \mathcal{G}_{i-1}} \mathcal{F}_i(J), \quad \mathcal{G}_i = \bigcup_{J \in \mathcal{G}_{i-1}} \mathcal{G}_i(J).$$

Finally, the Cantor set is defined as

$$\mathcal{F}_\infty = \bigcap_{i=1}^{\infty} \bigcup_{J \in \mathcal{F}_i} J = \bigcap_{i=1}^{\infty} \bigcup_{I \in \mathcal{G}_i} I.$$

It is straightforward to see that $\mathcal{F}_\infty \subset \overline{E}(T_\beta, f, g)$.

REMARK 4.2. It should be noted that the integer k_i depends upon Γ_{i-1} and U_i. Assume that f is strictly positive, otherwise replace f by $f + \epsilon$. Since m_i can be chosen as $m_i \gg n_{i-1}$ for all n_{i-1}, we have

$$\beta^{-k_i} \asymp e^{-S_{n_i}f(x_i^*)} = (e^{-S_{m_i}f(T_\beta^{n_{i-1}+k_{i-1}}x_i^*)})^{1+\epsilon},$$

where $x_i^* \in I_{n_{i-1}+k_{i-1}+m_i}(\Gamma_{i-1}, U_i)$. In other words, k_i is almost dependent only on U_i and

$$\beta^{-k_i} \asymp e^{-S_{m_i}f(x_i')}, x_i' \in I_{m_i}(U_i). \tag{4-2}$$

The same is true for l_i:

$$\beta^{-l_i} \asymp e^{-S_{m_i}f(y_i')}, y_i' \in I_{m_i}(W_i). \tag{4-3}$$

4.2. Supporting measure. We now construct a probability measure μ supported on $\mathcal{F}_\infty$, which is defined by distributing masses among the cylinders with nonempty intersection with $\mathcal{F}_\infty$. The process splits into two cases: when $s_0 > 1$ and $0 \leq s_0 \leq 1$.

4.2.1. Case I: $s_0 > 1$. In this case, for any $1 < s < s_0$, notice that

$$\frac{e^{S_n f(x')}}{e^{S_n g(y')}} \left(\frac{1}{\beta^n e^{S_n f(x')}} \right)^s \le \left(\frac{1}{\beta^n e^{S_n g(y')}} \right)^s.$$

This means that the covering of the rectangle $J_n(U) \times J_n(W)$ by balls of shorter side length is preferable and therefore, it reasonable to define the probability measure on smaller balls. To this end, let s_i be the solution to the equation

$$\sum_{U,W \in \Sigma_{\beta_N}^{m_i}} \frac{e^{S_{m_i} f(x'_i)}}{e^{S_{m_i} g(y'_i)}} \left(\frac{1}{\beta^{m_i} e^{S_{m_i} f(x'_i)}} \right)^s = 1,$$

where $x'_i \in I_{m_i}(U_i)$, $y'_i \in I_{m_i}(W_i)$.

By the continuity of the pressure function $P(T_\beta, f)$ with respect to β [14, Theorem 4.1], it can be shown that $s_i \to s_0$ when $m_i \to \infty$. Thus, without loss of generality, we choose all m_i large enough such that $s_i > 1$ for all i and $|s_i - s_0| = o(1)$.

We systematically define the measure μ on the Cantor set by defining it on the basic cylinders first. Recall that for level 1 of the Cantor set construction, we assumed that $n_1 = m_1$. For sublevels of the Cantor set, say $\mathcal{F}_k$, the role of m_k is to denote the number of positions where the digits can be chosen (almost) freely, while n_k denotes the length of a word in $\mathcal{F}_k$ before shrinking.

Let $I_{n_1+k_1}(U_1, K_1) \times I_{n_1+k_1}(W_1, L_1, H_1)$ be a generic cylinder in $\mathcal{G}_1$. Then define

$$\mu(I_{n_1+k_1}(U_1, K_1) \times I_{n_1+k_1}(W_1, L_1, H_1)) = \left(\frac{1}{\beta^{m_1} e^{S_{m_1} f(x'_1)}} \right)^{s_1},$$

where $x'_1 \in I_{m_1}(U_1)$. Assume that the measure on the cylinders of order $i - 1$ has been well defined. To define a measure on the ith cylinder, let $I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i)$ be a generic ith cylinder in $\mathcal{G}_i$. Define the probability measure μ as

$$\mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i))$$
$$= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} f(x'_i)}} \right)^{s_i},$$

where $x'_i \in I_{m_i}(U_i)$. The measure of a rectangle in $\mathcal{F}_i$ is then given as

$$\mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i))$$
$$= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \#\Sigma_\beta^{k_i-l_i} \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} f(x'_i)}} \right)^{s_i}$$
$$\asymp \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \frac{e^{S_{m_i} f(x'_i)}}{e^{S_{m_i} g(y'_i)}} \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} f(x'_i)}} \right)^{s_i},$$

where the last inequality follows from estimates (4-2) and (4-3).

4.2.2. Estimation of the μ-measure of cylinders. For any $i \geq 1$ consider the generic cylinder,

$$I := I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i).$$

We would like to show by induction that, for any $1 < s < s_0$,

$$\mu(I) \leq |I|^{s/(1+\epsilon)}.$$

When $i = 1$, the length of I is given as

$$|I| = \beta^{-m_1-k_1} \geq \beta^{-m_1} \cdot (e^{-S_{n_1}f(x_1^*)})^{1+\epsilon} = \beta^{-m_1} \cdot (e^{-S_{m_1}f(x_1^*)})^{1+\epsilon}.$$

But, by the definition of the measure μ, it is clear that

$$\mu(I) \leq |I|^{s_1} \leq |I|^{s/(1+\epsilon)}.$$

Now we consider the inductive process. Assume that

$$\mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \leq |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})|^{s/(1+\epsilon)}.$$

Let

$$I = I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i)$$

be a generic cylinder in $\mathcal{G}_i$. One one hand, its length satisfies

$$
\begin{aligned}
|I| = \beta^{-n_i-k_i} &= |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})| \times \beta^{-m_i} \times \beta^{-k_i} \\
&\geq |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})| \times \beta^{-m_i}(e^{-S_{n_i}f(x_i^*)})^{1+\epsilon},
\end{aligned}
$$

where $x_i^* \in I_{n_i}(\Gamma_i, U_i)$.

We compare $S_{n_i}f(x_i^*)$ and $S_{m_i}f(x_i')$. By (4-1) we have

$$
\begin{aligned}
|S_{n_i}f(x_i^*) - S_{m_i}f(x_i')| &= |S_{n_{i-1}+k_{i-1}}f(x_i^*)| \\
&\leq (n_{i-1}+k_{i-1})\|f\| \\
&\leq \frac{\epsilon}{1+\epsilon} m_i \log\beta,
\end{aligned}
$$

where $x_i' \in I_{m_i}(U_i)$. So we get

$$|I| \geq |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})| \times (\beta^{-m_i}e^{-S_{m_i}f(x_i')})^{1+\epsilon}. \tag{4-4}$$

On the other hand, by the definition of the measure μ and induction, we have that

$$
\begin{aligned}
\mu(I) = \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) &\times (\beta^{-m_i}e^{-S_{m_i}f(x_i')})^{s_i} \\
\leq |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})|^{s/(1+\epsilon)} &((\beta^{-m_i}e^{-S_{m_i}f(x_i')})^{1+\epsilon})^{s/(1+\epsilon)} \\
\leq |I|^{s/(1+\epsilon)}. &
\end{aligned}
$$

In the following steps, for any $(x, y) \in \mathcal{F}_\infty$, we estimate the measure of $I_n(x) \times I_n(y)$ compared with its length β^{-n}. By the construction of $\mathcal{F}_\infty$, there exists $\{k_i, l_i\}_{i\geq 1}$ such

that for all $i \geq 1$,

$$(x, y) \in I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i).$$

For any $n \geq 1$, let $i \geq 1$ be the integer such that

$$n_{i-1} + k_{i-1} < n \leq n_i + k_i = n_{i-1} + k_{i-1} + m_i + k_i.$$

Step 1: $n_{i-1} + k_{i-1} + m_i + l_i \leq n \leq n_i + k_i = n_{i-1} + k_{i-1} + m_i + k_i$.

The cylinder $I_n(x) \times I_n(y)$ contains $\beta^{n_i+k_i-n}$ cylinders in $\mathcal{G}_i$ with order $n_i + k_i$. Note that by the definition of $\{k_j, l_j\}_{1 \leq j \leq i}$, the first i pairs $\{k_j, l_j\}_{1 \leq j \leq i}$ depend only on the first n_i digits of (x, y). So the measures of the subcylinders of order $n_i + k_i$ are the same. So, the measure of $I_n(x) \times I_n(y)$ can be estimated as

$$\mu(I_n(x) \times I_n(y)) = \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1}))$$
$$\times (\beta^{-m_i} e^{-S_{m_i} f(x_i')})^{s_i} \times \beta^{n_i+k_i-n}.$$

Thus by the measure estimation of cylinders of order $n_{i-1} + k_{i-1}$ and the choice of k_i, one has that

$$\mu(I_n(x) \times I_n(y)) \leq (\beta^{-n_{i-1}-k_{i-1}})^{s/(1+\epsilon)} (\beta^{-m_i-k_i})^{s/(1+\epsilon)} \times \beta^{n_i+k_i-n}$$
$$= (\beta^{-n_i-k_i})^{s/(1+\epsilon)} \times \beta^{n_i+k_i-n}$$
$$\leq (\beta^{-n})^{s/(1+\epsilon)},$$

by noting that $n \leq n_i + k_i$ and $s/(1 + \epsilon) > 1$.

Step 2: $n_{i-1} + k_{i-1} + m_i \leq n \leq n_i + l_i = n_{i-1} + k_{i-1} + m_i + l_i$.

Recalling the definition of $n_i + k_i$, the first i pairs $\{k_j, l_j\}_{1 \leq j \leq i}$ depend only on the first n_i digits of (x, y). So the measures of the subcylinders in $\mathcal{G}_i$ with order $n_i + k_i$ are the same. It is clear that the cylinder $I_n(x) \times I_n(y)$ contains $\beta^{k_i-l_i}$ cylinders of order $n_i + k_i$. So, the measure of $I_n(x) \times I_n(y)$ can be estimated as

$$\mu(I_n(x) \times I_n(y)) = \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1}))$$
$$\times (\beta^{-m_i} e^{-S_{m_i} f(x_i'^*)})^{s_i} \times \beta^{k_i-l_i}.$$

Thus by the measure estimation of cylinders of order $n_{i-1} + k_{i-1}$ and the choice of k_i, one has that

$$\mu(I_n(x) \times I_n(y)) \leq (\beta^{-n_{i-1}-k_{i-1}})^{s/(1+\epsilon)} (\beta^{-m_i-k_i})^{s/(1+\epsilon)} \times \beta^{k_i-l_i}$$
$$= (\beta^{-n_i-k_i})^{s/(1+\epsilon)} \times \beta^{k_i-l_i}$$
$$\leq (\beta^{-n_i-l_i})^{s/(1+\epsilon)}$$
$$\leq (\beta^{-n})^{s/(1+\epsilon)},$$

by noting that $n \leq n_i + l_i$ and $s/(1 + \epsilon) > 1$.

Step 3: $n_{i-1} + k_{i-1} \leq n \leq n_{i-1} + k_{i-1} + m_i$.

Assume that $U_i = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{m_i})$, $W_i = (\omega_1, \omega_2, \ldots, \omega_{m_i})$. We start off by denoting $l = n - (n_{i-1} + k_{i-1})$ and $h = m_i - l$. Then

$$
\begin{aligned}
&\mu(I_n(x) \times I_n(y)) \\
&= \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} \mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i)) \times \beta^{k_i - l_i} \\
&= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} (\beta^{-m_i} e^{-S_{m_i} f(x_i')})^{s_i} \times \beta^{k_i - l_i} \\
&= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} \frac{e^{S_{m_i} f(x_i')}}{e^{S_{m_i} g(y_i')}} (\beta^{-m_i} e^{-S_{m_i} f(x_i')})^{s_i}.
\end{aligned}
$$

Then by the estimation of the measure of cylinders of order $n_{i-1} + k_{i-1}$ and letting $(\widetilde{x_i'}, \widetilde{y_i'}) = (T_\beta^l x_i', T_\beta^l y_i')$, we get

$$
\begin{aligned}
\mu(I_n(x) \times I_n(y)) &\leq (\beta^{-n_{i-1}-k_{i-1}})^{s/(1+\epsilon)} \cdot \frac{e^{S_l f(x_i')}}{e^{S_l g(y_i')}} \cdot (\beta^{-l} e^{-S_l f(x_i')})^{s_i} \\
&\quad \times \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} \frac{e^{S_h f(\widetilde{x_i'})}}{e^{S_h g(\widetilde{y_i'})}} \cdot (\beta^{-h} e^{-S_h f(\widetilde{x_i'})})^{s_i}.
\end{aligned}
$$

The first part can be estimated as

$$
\begin{aligned}
(\beta^{-n_{i-1}-k_{i-1}})^{s/(1+\epsilon)} \cdot \frac{e^{S_l f(x_i')}}{e^{S_l g(y_i')}} \cdot (\beta^{-l} e^{-S_l f(x_i')})^{s_i} &\leq (\beta^{-(n_{i-1}+k_{i-1}+l)})^{s/(1+\epsilon)} \\
&= (\beta^{-n})^{s/(1+\epsilon)},
\end{aligned}
$$

since

$$
\frac{e^{S_l f(x_i')}}{e^{S_l g(y_i')}} \cdot (e^{-S_l f(x_i')})^{s_i} \leq 1, \quad \text{for } s_i \geq 1.
$$

To estimate the second part, we first recall that we defined s_i to be the solution of the equation

$$
\sum_{U,W \in \Sigma_{\beta_N}^{m_i}} \frac{e^{S_n f(x_i')}}{e^{S_n g(y_i')}} \left(\frac{1}{\beta^n e^{S_n f(x_i')}} \right)^s = 1.
$$

Therefore,

$$1 = \sum_{U_1, W_1 \in \Sigma^l_{\beta_N}} \frac{e^{S_l f(x'_i)}}{e^{S_l g(y'_i)}} \left(\frac{1}{\beta^l e^{S_l f(x'_i)}} \right)^{s_i} \times \sum_{U_2, W_2 \in \Sigma^h_{\beta_N}} \frac{e^{S_h f(\widetilde{x'_i})}}{e^{S_h g(\widetilde{y'_i})}} \left(\frac{1}{\beta^l e^{S_h f(\widetilde{x'_i})}} \right)^{s_i}.$$

So, by arguments similar to [14, pages 2095–2097] and [18, pages 1331–1332], we derive that

$$\sum_{U_2, W_2 \in \Sigma^h_{\beta_N}} \frac{e^{S_h f(\widetilde{x'_i})}}{e^{S_h g(\widetilde{y'_i})}} \left(\frac{1}{\beta^l e^{S_h f(\widetilde{x'_i})}} \right)^{s_i} \leq \beta^{l\epsilon}.$$

Therefore,

$$\mu(I_n(x) \times I_n(y)) \leq \beta^{-n \cdot s/(1+\epsilon)} \cdot \beta^{l\epsilon} \leq (\beta^{-n})^{s/(1+\epsilon)-\epsilon}.$$

As far as the measure of a general ball $B(x, r)$ with $\beta^{-n-1} \leq r < \beta^{-n}$ is concerned, we notice that it can intersect at most three cylinders of order n. Thus,

$$\mu(B(x, r)) \leq 3(\beta^{-n})^{s/(1+\epsilon)-\epsilon} \leq 3\beta^s r^{s/(1+\epsilon)-\epsilon} \leq 3\beta^2 r^{s/(1+\epsilon)-\epsilon}.$$

So, finally, an application of the mass distribution principle (Proposition 4.1) yields that

$$\dim_H \overline{E}(T_\beta, f, g) \geq s_0.$$

4.2.3. Case II: $0 \leq s_0 \leq 1$. The arguments are similar to Case I but the calculations are different. In this case, for any $s < s_0 \leq 1$, it is trivial that

$$\frac{e^{S_n f(x')}}{e^{S_n g(y')}} \left(\frac{1}{\beta^n e^{S_n f(x')}} \right)^s \geq \left(\frac{1}{\beta^n e^{S_n g(y')}} \right)^s.$$

This means that the covering of the rectangle $J_n(U) \times J_n(W)$ by balls of larger side length is more preferable and therefore, it reasonable to define the probability measure of the rectangle to be the same measure as for the cylinder of order $n_i + l_i$.

Just as in Case I, let s_i be the solution to the equation

$$\sum_{U, W \in \Sigma^{m_i}_{\beta_N} \text{ ending with } 0^N} \left(\frac{1}{\beta^{m_i} e^{S_{m_i} g(y'_i)}} \right)^s = 1,$$

where $y'_i \in I_{m_i}(W_i)$. By the continuity of the pressure function $P(T_\beta, f)$ with respect to β we can assume that for all m_i large enough we have that $s_i < 1$ for all i and $|s_i - s_0| = o(1)$.

We first define the measure μ on the basic cylinders. Let $I_{n_1+k_1}(U_1, K_1) \times I_{n_1+l_1}(W_1, L_1)$ be a generic cylinder in $\mathcal{F}_1$. Then define

$$\mu(I_{n_1+k_1}(U_1, K_1) \times I_{n_1+l_1}(W_1, L_1)) = \left(\frac{1}{\beta^{m_1} e^{S_{m_1} g(y'_1)}} \right)^{s_1},$$

where $y_1' \in I_{m_1}(W_1)$. Then the measure is evenly distributed on its subcylinders in $\mathcal{G}_1$. So, for a generic cylinder $I_{n_1+k_1}(U_1, K_1) \times I_{n_1+k_1}(W_1, L_1, H_1)$ in $\mathcal{G}_1$, define

$$
\begin{aligned}
\mu(I_{n_1+k_1}(U_1, K_1) \times I_{n_1+k_1}(W_1, L_1, H_1)) &= \frac{1}{\#\Sigma_\beta^{k_1-l_1}} \left(\frac{1}{\beta^{m_1} e^{S_{m_1} g(y_1')}} \right)^{s_1} \\
&\asymp \frac{1}{\beta^{k_1-l_1}} \left(\frac{1}{\beta^{m_1} e^{S_{m_1} g(y_1')}} \right)^{s_1}.
\end{aligned}
$$

Assume that the measure on the cylinders of order $(i-1)$ has been well defined. Then to define the measure on the ith cylinder we proceed as follows. Let $I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i)$ be a generic cylinder in $\mathcal{F}_i$. Then define

$$
\begin{aligned}
&\mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i)) \\
&= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} g(y_i')}} \right)^{s_i},
\end{aligned}
$$

where $y_i' \in I_{m_i}(W_i)$. By the definition of k_i, l_i, the measure of a cylinder in $\mathcal{G}_i$ is then given as

$$
\begin{aligned}
&\mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i)) \\
&= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \frac{1}{\#\Sigma_\beta^{k_i-l_i}} \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} g(y_i')}} \right)^{s_i} \\
&\asymp \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \frac{e^{S_{m_i} g(y_i')}}{e^{S_{m_i} f(x_i')}} \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} g(y_i')}} \right)^{s_i}.
\end{aligned}
$$

4.2.4. Estimation of the μ-measure of cylinders. We first show by induction that, for any $i \geq 1$ and a generic cylinder

$$
I := I_{n_{i-1}+k_{i-1}+m_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_{i-1}+k_{i-1}+m_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i),
$$

we have

$$
\mu(I) \leq |I|^{s/(1+\epsilon)}.
$$

When $i = 1$, on the one hand, the length of I is given as

$$
|I| = \beta^{-m_1-k_1} \geq \beta^{-m_1} \cdot (e^{-S_{n_1} f(x_1')})^{1+\epsilon} = \beta^{-m_1} \cdot (e^{-S_{m_1} f(x_1')})^{1+\epsilon}.
$$

But on the other hand, by the definition of the measure μ, it is clear that

$$
\begin{aligned}
\mu(I) &\leq \frac{e^{S_{m_1} g(y_1')}}{e^{S_{m_1} f(x_1')}} \cdot \left(\frac{1}{\beta^{m_1} e^{S_{m_1} g(y_1')}} \right)^{s_1} \\
&\leq (\beta^{-m_1} e^{-S_{m_1} f(x_1')})^{s_1} \\
&\leq |I|^{s/(1+\epsilon)},
\end{aligned}
$$

by noting that $s_1 < 1$.

Just as in Case I, we consider the inductive process. Assume that

$$\mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \leq |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})|^{s/(1+\epsilon)}.$$

Let

$$I = I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i, H_i)$$

be a generic cylinder in $\mathcal{G}_i$. By (4-4) we get

$$|I| \geq |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})| \times (\beta^{-m_i} e^{-S_{m_i} f(x_i')})^{1+\epsilon}.$$

From the definition of the measure μ, induction and the fact that $s_i < 1$, it follows that

$$\begin{aligned}
\mu(I) &= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \frac{e^{S_{m_i} g(y_i')}}{e^{S_{m_i} f(x_i')}} \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} g(y_i')}}\right)^{s_i} \\
&\leq |I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})|^{s/(1+\epsilon)} ((\beta^{-m_i} e^{-S_{m_i} f(x_i')})^{1+\epsilon})^{s/(1+\epsilon)} \\
&\leq |I|^{s/(1+\epsilon)} \\
&= (\beta^{-n_i-k_i})^{s/(1+\epsilon)} \\
&\asymp (\beta^{-m_i-k_i})^{s/(1+\epsilon)}.
\end{aligned}$$

So, for a rectangle

$$J = I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i)$$

in $\mathcal{F}_i$, we have that

$$\begin{aligned}
\mu(J) &= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \left(\frac{1}{\beta^{m_i} e^{S_{m_i} g(y_i')}}\right)^{s_i} \\
&\leq (\beta^{-n_{i-1}-k_{i-1}})^{s/(1+\epsilon)} (\beta^{-m_i} \beta^{-l_i})^{s_i} \\
&\leq (\beta^{-n_i-l_i})^{s/(1+\epsilon)}.
\end{aligned}$$

For any $(x, y) \in \mathcal{F}_\infty$, we estimate the measure of $I_n(x) \times I_n(y)$ compared with its length β^{-n}. By the construction of $\mathcal{F}_\infty$, there exists $\{k_i, l_i\}_{i \geq 1}$ such that for all $i \geq 1$,

$$(x, y) \in I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i,).$$

For any $n \geq 1$, let $i \geq 1$ be the integer such that

$$n_{i-1} + k_{i-1} < n \leq n_i + k_i = n_{i-1} + k_{i-1} + m_i + k_i.$$

Step I: $n_{i-1} + k_{i-1} + m_i + l_i \leq n \leq n_i + k_i = n_{i-1} + k_{i-1} + m_i + k_i.$

In this case, the cylinder can intersect only one rectangle in $\mathcal{F}_i$, so

$$\begin{aligned}
\mu(I_n(x) \times I_n(y)) &= \mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i)) \\
&\leq (\beta^{-n_i-l_i})^{s/(1+\epsilon)} \\
&\leq (\beta^{-n})^{s/(1+\epsilon)}.
\end{aligned}$$

Step II: $n_{i-1} + k_{i-1} + m_i \leq n \leq n_i + l_i = n_{i-1} + k_{i-1} + m_i + l_i$.

Now the cylinder $I_n(x) \times I_n(y)$ contains $\beta^{n_i+l_i-n}$ cylinders in $\mathcal{F}_i$ with order $n_i + l_i$. Note that by the definition of $\{k_j, l_j\}_{1 \leq j \leq i}$, the first i pairs $\{k_j, l_j\}_{1 \leq j \leq i}$ depend only on the first n_i digits of (x, y). So the measures of the subcylinders of order $n_i + k_i$ are the same. So, the measure of $I_n(x) \times I_n(y)$ can be estimated as

$$\mu(I_n(x) \times I_n(y)) = \mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+k_i}(\Upsilon_{i-1}, W_i, L_i)) \times \frac{1}{\beta^{n-n_i-l_i}}$$

$$\leq (\beta^{-n_i-l_i})^{s/(1+\epsilon)} \times \frac{1}{\beta^{n-n_i-l_i}}$$

$$\leq (\beta^{-n})^{s/(1+\epsilon)}.$$

Step III: $n_{i-1} + k_{i-1} \leq n \leq n_{i-1} + k_{i-1} + m_i$.

Assume $U_i = (\epsilon_1, \epsilon_2, \ldots, \epsilon_{m_i})$, $W_i = (\omega_1, \omega_2, \ldots, \omega_{m_i})$. Write $l = n - (n_{i-1} + k_{i-1})$ and $h = m_i - l$. Then

$$\mu(I_n(x) \times I_n(y))$$

$$= \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} \mu(I_{n_i+k_i}(\Gamma_{i-1}, U_i, K_i) \times I_{n_i+l_i}(\Upsilon_{i-1}, W_i, L_i))$$

$$= \mu(I_{n_{i-1}+k_{i-1}}(\Gamma_{i-1}) \times I_{n_{i-1}+k_{i-1}}(\Upsilon_{i-1})) \times \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} (\beta^{-m_i} e^{-S_{m_i} g(y_i')})^{s_i}.$$

Then by the estimate of the measure of cylinders of order $n_{i-1} + k_{i-1}$ and letting $\widetilde{y_i'} = T_\beta^l y_i'$, we get

$$\mu(I_n(x) \times I_n(y))$$

$$\leq (\beta^{-n_{i-1}-k_{i-1}})^{s/(1+\epsilon)} \cdot (\beta^{-l} e^{-S_l g(y_i')})^{s_i} \times \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} (\beta^{-h} e^{-S_h g(\widetilde{y_i'})})^{s_i}$$

$$\leq (\beta^{-n})^{s/(1+\epsilon)} \cdot \sum_{\substack{(\epsilon_{l+1},\ldots,\epsilon_{m_i}) \in \Sigma_\beta^l \\ (\omega_{l+1},\ldots,\omega_{m_i}) \in \Sigma_\beta^h}} (\beta^{-h} e^{-S_h g(\widetilde{y_i'})})^{s_i}.$$

Recall the definition of s_i:

$$\sum_{U,W \in \Sigma_{\beta_N}^{m_i}} \left(\frac{1}{\beta^{m_i} e^{S_{m_i} g(y_i')}} \right)^s = 1.$$

Then

$$1 = \sum_{U_1,W_1 \in \Sigma_{\beta_N}^l} \left(\frac{1}{\beta^l e^{S_l g(y_1')}} \right)^{s_i} \cdot \sum_{U_2,W_2 \in \Sigma_{\beta_N}^h} \left(\frac{1}{\beta^l e^{S_h g(\widetilde{y_1'})}} \right)^{s_i},$$

where $y_1'^* \in I_l(U_1), \widetilde{y_1'} \in I_h(W_2)$.

So, with a similar argument to the previous section, we have that

$$\sum_{U_2, W_2 \in \Sigma_{\beta_N}^l} \left(\frac{1}{\beta^h e^{S_h g(\widetilde{y_h'})}} \right)^{s_i} \leq \beta^{l\epsilon}.$$

Therefore,

$$\mu(I_n(x) \times I_n(y)) \leq (\beta^{-n})^{s/(1+\epsilon)} \cdot \beta^{l\epsilon} \leq (\beta^{-n})^{s/(1+\epsilon)-\epsilon}.$$

Notice that a general ball $B(x,r)$ with $\beta^{-n-1} \leq r < \beta^{-n}$ can intersect at most three cylinders of order n. Therefore, the measure of the general ball can be estimated as

$$\mu(B(x,r)) \leq 3(\beta^{-n})^{s/(1+\epsilon)-\epsilon} \leq 3\beta^s r^{s/(1+\epsilon)-\epsilon} \leq 3\beta^2 r^{s/(1+\epsilon)-\epsilon}.$$

So, finally, by using the mass distribution principle we have the lower bound of the Hausdorff dimension of this case,

$$\dim_{\mathrm{H}} \overline{E}(T_\beta, f, g) \geq s_0.$$

Hence, combining both cases, we have the desired conclusion.

Acknowledgments

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