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On the relation between viscoelastic and magnetohydrodynamic flows and their instabilities

By GORDON I. OGILVIE[1,2]
AND MICHAEL R. E. PROCTOR[2]

[1]Institute of Astronomy, University of Cambridge, Madingley Road, Cambridge CB3 0HA, UK

[2]Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Centre for Mathematical Sciences, Wilberforce Road, Cambridge CB3 0WA, UK

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We demonstrate a close analogy between a viscoelastic medium and an electrically conducting fluid containing a magnetic field. Specifically, the dynamics of the Oldroyd-B fluid in the limit of large Deborah number corresponds to that of a magnetohydrodynamic (MHD) fluid in the limit of large magnetic Reynolds number. As a definite example of this analogy, we compare the stability properties of differentially rotating viscoelastic and MHD flows. We show that there is an instability of the Oldroyd-B fluid that is physically distinct from both the inertial and elastic instabilities described previously in the literature, but is directly equivalent to the magnetorotational instability in MHD. It occurs even when the specific angular momentum increases outwards, provided that the angular velocity decreases outwards; it derives from the kinetic energy of the shear flow and does not depend on the curvature of the streamlines. However, we argue that the elastic instability of viscoelastic Couette flow has no direct equivalent in MHD.

1. Introduction

1.1. *Viscoelastic and magnetohydrodynamic fluids*

In his investigation of the viscosity of gases, Clerk Maxwell (1867) proposed that the stress in a fluid obeys an equation of the form

$$(\text{stress}) + \tau \frac{\mathrm{d(stress)}}{\mathrm{d}t} = (\text{viscosity}) \times \frac{\mathrm{d(strain)}}{\mathrm{d}t}, \tag{1.1}$$

where τ is the relaxation time. If the time scale of the straining motion is long compared to τ, the second term on the left-hand side is negligible and the stress is proportional to the rate of strain. This Newtonian relation gives rise to viscous behaviour. However, if the time scale of the strain is short compared to τ, the first term on the left-hand side is negligible and the stress is proportional to the strain itself. This Hookean relation gives rise to elastic behaviour. The reason for the elastic response is that the rapid strain prevents the configuration of the molecules from relaxing towards an equilibrium distribution, and the stress is therefore 'frozen in' to the fluid.

Modern constitutive equations for viscoelastic fluids (Bird, Armstrong & Hassager 1987*a*) are usually expressed in a covariant tensorial form based on the principles set out by Oldroyd (1950). His liquid B is one of the most widely used nonlinear models of a viscoelastic fluid, and provides a fair representation of a dilute solution of a polymer of high molecular weight. It is based on Maxwell's equation (1.1), but cast in a form that satisfies the principle of material frame indifference. Moreover, it can also be derived from the kinetic theory of idealized extensible polymer molecules contained in a Newtonian solvent (Bird, Curtiss & Armstrong 1987*b*). The dimensionless number characterizing the ratio of the relaxation time to the time scale of the flow is the Deborah number (or Weissenberg number); when this is large, the polymeric stress is effectively 'frozen in' to the fluid.

In an electrically conducting fluid, the magnetic field $\boldsymbol{B}$ affects the dynamics through the bulk Lorentz force (e.g. Roberts 1967). This can be represented in terms of the Maxwell electromagnetic stress tensor,†

$$\mathbf{M} = \frac{\boldsymbol{B}\boldsymbol{B}}{\mu_0} - \frac{B^2}{2\mu_0}\mathbf{1}, \tag{1.2}$$

the two parts of which correspond to a tension in the field lines and an isotropic magnetic pressure. It is well known that, in a perfectly conducting fluid, the magnetic field is 'frozen in' to the fluid, in the sense that magnetic field lines can be identified with material lines (Alfvén 1950). Even in a fluid of finite conductivity, the magnetic field is effectively 'frozen in' for motions of sufficiently short time scale, or sufficiently large length scale, corresponding to a large magnetic Reynolds number. It follows that the Maxwell stress is also 'frozen in' to the fluid in a certain sense.

From a mathematical point of view, the 'freezing in' of a tensor field $\mathbf{X}(\boldsymbol{r}, t)$ in a flow with velocity field $\boldsymbol{u}(\boldsymbol{r}, t)$ can be expressed by the equation (e.g. Tur & Yanovsky 1993)

$$\frac{\partial \mathbf{X}}{\partial t} + \mathscr{L}_{\boldsymbol{u}}\mathbf{X} = \mathbf{0}, \tag{1.3}$$

where $\mathscr{L}$ is the Lie derivative. For a scalar field X, this gives the familiar expression $\partial X/\partial t + \boldsymbol{u}\cdot\nabla X = 0$, meaning that the numerical value of X is conserved by every fluid element. For a (contravariant) vector field $\boldsymbol{B}$ it gives the induction equation of ideal, incompressible magnetohydrodynamics (MHD), which implies that the magnetic field is advected and stretched in the same way as infinitesimal line elements. For a second-rank tensor field it results in the 'upper-convected derivative' that appears in the governing equation of the Oldroyd-B fluid (Bird *et al.* 1987*a*).

At a physical level, an analogy is to be seen between a polymer solution, containing extensible molecules that are advected and distorted by the flow and react on it through their tension, and an electrically conducting fluid, containing magnetic field lines that are also advected and distorted by the flow and react on it through their tension.

It follows that there is a physical and mathematical similarity between the dynamics of viscoelastic and MHD fluids. We will show that a formal analogy can be drawn between the Oldroyd-B fluid in the limit of large Deborah number and an MHD fluid in the limit of large magnetic Reynolds number. In other words, in this limit, the Maxwell stress in MHD obeys the equation of a Maxwell fluid.

1.2. *Instabilities of differentially rotating fluids*

Differentially rotating flows are common in astrophysics and geophysics, and have been studied extensively in the laboratory. The simplest form of differential rotation occurs when the angular velocity depends only on the cylindrical radius, $\Omega = \Omega(r)$,

† Here μ_0 is the permeability of free space. For non-relativistic flows, the electric field makes a negligible contribution to the stress.

and Couette flow between differentially rotating cylinders provides an excellent model system for investigating the dynamics of such flows. According to Rayleigh (1916), instability occurs in the absence of viscosity whenever the specific angular momentum $|r^2\Omega|$ decreases outwards. Most subsequent theoretical and experimental studies, starting with the classic work of Taylor (1923), have focused on the onset of Rayleigh's inertial instability in the presence of viscosity, and the interesting sequence of dynamical states that ensues.

Numerous variants of Couette flow have also been considered, and among these the Couette flow of viscoelastic fluids has received much attention. Early work in the 1960s (e.g. Thomas & Walters 1964; Giesekus 1966) examined the effect of viscoelasticity on the onset of Rayleigh's inertial instability. However, one of the most important recent results is the theoretical and experimental demonstration by Larson, Shaqfeh & Muller (1990) of a physically distinct instability in viscoelastic Couette flow. This is a purely elastic instability that occurs at sufficiently large Deborah number $\tau|\mathrm{d}\Omega/\mathrm{d}\ln r|$, even in the limit of negligible inertia, and irrespective of the sign of the angular momentum gradient.

The influence of a magnetic field on the stability of the Couette flow of an electrically conducting fluid has also been investigated theoretically and (to a lesser extent) experimentally. Again, early work (described by Chandrasekhar 1961) focused on the effect of the magnetic field on the onset of inertial instability. More importantly, Velikhov (1959) and Chandrasekhar (1960) uncovered a physically distinct instability in magnetized Couette flow. In the absence of viscosity and resistivity, and in the presence of a weak vertical magnetic field, this 'magnetorotational' instability occurs whenever the angular velocity $|\Omega|$ decreases outwards, irrespective of the sign of the angular momentum gradient.

The magnetorotational instability finds its most important applications in astrophysical fluid dynamics, where magnetic fields are prevalent and the astronomical length scales allow for large magnetic Reynolds numbers. The stability of differentially rotating flows is of considerable interest in astrophysics, especially in connection with accretion discs (e.g. Pringle 1981). These are usually thin discs of gas in circular orbital motion around a star or black hole. The angular velocity decreases outwards according to Kepler's third law, $\Omega \propto r^{-3/2}$, and the Reynolds numbers are extremely high (e.g. 10^{14}). Observations indicate that angular momentum is transported outwards through accretion discs at a much greater rate than allowed by viscosity, and understanding the origin of this 'anomalous viscosity' has been a major goal of accretion disc research.

The anomalous viscosity is usually attributed to turbulent transport. However, despite the very high Reynolds numbers, there is no convincing demonstration of any suitable hydrodynamic instability in circular Keplerian flow. Indeed, simple reasoning can be used to argue that hydrodynamic turbulence is unlikely to be self-sustaining in a flow that amply satisfies Rayleigh's stability criterion (Balbus & Hawley 1998). However, it has been demonstrated that MHD turbulence develops very readily in accretion discs, through the nonlinear development of the magnetorotational instability. Since the results of Velikhov (1959) and Chandrasekhar (1960) were rediscovered by Balbus & Hawley (1991) and their significance was appreciated, the magnetorotational instability has been analysed in considerable detail in the astrophysical literature.

1.3. *Properties of the magnetorotational instability*

The properties of the magnetorotational instability have been reviewed by Balbus & Hawley (1998), and we recall some of the important features here. Its simplest manifestation is in an incompressible, inviscid, perfectly conducting fluid having angular velocity $\Omega(r)$ and containing a uniform magnetic field $\boldsymbol{B}$ parallel to the axis of rotation. An approximate local dispersion relation can be obtained for normal modes having growth rate s and wavevector $\boldsymbol{k}$ parallel to $\boldsymbol{B}$, and has the form

$$s^4 + s^2[4\Omega(\Omega - A) + 2\omega_{\mathrm{A}}^2] + \omega_{\mathrm{A}}^2(\omega_{\mathrm{A}}^2 - 4\Omega A) = 0, \tag{1.4}$$

where the quantity

$$A = -\frac{r}{2}\frac{\mathrm{d}\Omega}{\mathrm{d}r} \tag{1.5}$$

measures the differential rotation, and is known as Oort's first constant in the astrophysical literature. The Alfvén frequency is $\omega_{\mathrm{A}} = (\mu_0\rho)^{-1/2}\boldsymbol{k}\cdot\boldsymbol{B}$, and the combination

$$4\Omega(\Omega - A) = \frac{1}{r^3}\frac{\mathrm{d}}{\mathrm{d}r}(r^4\Omega^2) \tag{1.6}$$

is the Rayleigh discriminant, or the square of the epicyclic oscillation frequency. When this is positive, unstable normal modes with $s > 0$ can nevertheless be found provided that $4\Omega A > 0$, as is the case in astrophysical discs. The maximal growth rate, $s = |A|$, is achieved by a mode having $\omega_{\mathrm{A}}^2 = A(2\Omega - A)$.

More generally, and from a local perspective, rotation and shear in the correct relative orientation are required, and a weak magnetic field of any geometry is sufficient to initiate the instability, provided the fluid is sufficiently ionized. In the presence of significant dissipation, the ideal growth rate must compete with viscous and resistive damping, so that growth rates less than $|A|$ are achieved, or the instability may be suppressed altogether. An unstable mode must always bend the field lines, having a non-zero Alfvén frequency, and therefore the instability of a purely azimuthal (or toroidal) field is essentially non-axisymmetric.

A simple explanation of the instability can be given in terms of two fluid elements, connected by magnetic field lines, that are initially in circular orbit at the same radius. The fluid elements are then given angular momentum perturbations of opposite sign. The one receiving the positive perturbation moves to an orbit of larger radius and acquires a smaller angular velocity, lagging behind its partner. The tension of the magnetic field exerts a torque that pulls the lagging element forwards, enhancing the initial perturbation and leading to instability. A mechanical analogue, consisting of two orbiting particles connected by a weak spring, also exhibits instability.

1.4. *Plan of the paper*

The main purpose of this paper is to draw attention to the physical and mathematical similarity between viscoelasticity and MHD. As an example of the application of this idea, we explore in some detail the relation between the instabilities of differentially rotating viscoelastic and MHD flows. As described in § 1.2, previous investigations have uncovered instabilities of viscoelastic and MHD Couette flow that are physically distinct from Rayleigh's inertial instability. In the light of the analogy we describe, an obvious question is whether the elastic instability of Larson *et al.* (1990) is somehow related to the magnetorotational instability. We will argue that this is not the case, but will show that there is another instability of viscoelastic Couette flow that is the direct equivalent of the magnetorotational instability.

The remainder of this paper is organized as follows. In § 2 we set out the basic equations governing incompressible viscoelastic and MHD flows and present the analogy between them. We then discuss, in § 3, the possible sources of instability on the basic of energy considerations. In § 4 we define a model system consisting of plane Couette flow in a rotating channel, equivalent to cylindrical Couette flow in the narrow-gap limit, and formulate eigenvalue problems for the normal modes of the Oldroyd-B and MHD fluids. Some numerical solutions are presented in § 5 to illustrate the expected similarity between the two systems. In § 6 the existence of localized growing solutions in the two systems, satisfying the same magnetorotational dispersion relation, is demonstrated by an asymptotic analysis. Finally, the results are summarized and discussed in § 7.

2. Basic equations

2.1. *Viscoelastic fluid*

We first consider an incompressible viscoelastic fluid of uniform density ρ. The Oldroyd-B model is characterized by a solvent viscosity μ, a polymer viscosity μ_{p} and a relaxation time τ. The velocity field $\boldsymbol{U}$ obeys the solenoidal condition,

$$\nabla \cdot \boldsymbol{U} = 0, \tag{2.1}$$

and the equation of motion

$$\rho\left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}\right) = -\nabla \Psi + \nabla \cdot \boldsymbol{T} + \mu \nabla^2 \boldsymbol{U}. \tag{2.2}$$

Here $\Psi = p + \rho\Phi$ is the modified pressure (Φ being the gravitational potential) and $\boldsymbol{T}$ is the Oldroyd-B stress, which is a symmetric tensor field of second rank satisfying the constitutive equation (cf. equation (1.1))

$$\boldsymbol{T} + \tau\left[\frac{\partial \boldsymbol{T}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{T} - (\nabla \boldsymbol{U})^{\mathrm{T}} \cdot \boldsymbol{T} - \boldsymbol{T} \cdot \nabla \boldsymbol{U}\right] = \mu_{\mathrm{p}}[\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^{\mathrm{T}}], \tag{2.3}$$

where the superscript 'T' denotes the transpose of a second-rank tensor.

2.2. *MHD fluid*

We also consider an incompressible, electrically conducting fluid of uniform density ρ, viscosity μ and electrical conductivity σ. The velocity field $\boldsymbol{U}$ obeys the solenoidal condition,

$$\nabla \cdot \boldsymbol{U} = 0, \tag{2.4}$$

and the equation of motion

$$\rho\left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U}\right) = -\nabla \Psi + \nabla \cdot \boldsymbol{M} + \mu \nabla^2 \boldsymbol{U}. \tag{2.5}$$

Here $\Psi = p + \rho\Phi$ is again the modified pressure and $\boldsymbol{M}$ is the Maxwell stress given in equation (1.2). The magnetic field obeys the solenoidal condition,

$$\nabla \cdot \boldsymbol{B} = 0, \tag{2.6}$$

and the induction equation,

$$\frac{\partial \boldsymbol{B}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{B} = \boldsymbol{B} \cdot \nabla \boldsymbol{U} + \eta \nabla^2 \boldsymbol{B}, \tag{2.7}$$

which is derived from Maxwell's equations for the electromagnetic field and Ohm's law. Here $\eta = 1/(\mu_0\sigma)$ is the magnetic diffusivity.

2.3. *The formal analogy*

Instead of comparing the Oldroyd-B stress $\mathbf{T}$ and the Maxwell stress $\mathbf{M}$ directly, we take the polymeric and magnetic stress tensors to be

$$\mathbf{T}_{\mathrm{p}} = \mathbf{T} + \frac{\mu_{\mathrm{p}}}{\tau}\mathbf{1}, \quad \mathbf{T}_{\mathrm{m}} = \frac{\boldsymbol{B}\boldsymbol{B}}{\mu_0}, \tag{2.8}$$

which differ from $\mathbf{T}$ and $\mathbf{M}$ only by the addition of an isotropic part in each case. Specifically, $\mathbf{T}_{\mathrm{p}}$ does not include the equilibrium isotropic pressure nkT of the polymer molecules, and $\mathbf{T}_{\mathrm{m}}$ does not include the isotropic pressure $B^2/2\mu_0$ of the magnetic field. In an incompressible fluid, such terms can be taken care of by writing the two equations of motion in the form

$$\rho\left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{U}\right) = -\nabla\Psi_{\mathrm{p,m}} + \nabla\cdot\mathbf{T}_{\mathrm{p,m}} + \mu\nabla^2\boldsymbol{U}, \tag{2.9}$$

where

$$\Psi_{\mathrm{p}} = \Psi + \frac{\mu_{\mathrm{p}}}{\tau}, \quad \Psi_{\mathrm{m}} = \Psi + \frac{B^2}{2\mu_0} \tag{2.10}$$

are redefined modified pressures.

According to the Oldroyd-B constitutive equation (2.3) and the induction equation (2.7), $\mathbf{T}_{\mathrm{p}}$ and $\mathbf{T}_{\mathrm{m}}$ satisfy the equations

$$\frac{\partial \mathbf{T}_{\mathrm{p}}}{\partial t} + \boldsymbol{U}\cdot\nabla\mathbf{T}_{\mathrm{p}} - (\nabla\boldsymbol{U})^{\mathrm{T}}\cdot\mathbf{T}_{\mathrm{p}} - \mathbf{T}_{\mathrm{p}}\cdot\nabla\boldsymbol{U} = -\frac{1}{\tau}\left(\mathbf{T}_{\mathrm{p}} - \frac{\mu_{\mathrm{p}}}{\tau}\mathbf{1}\right), \tag{2.11}$$

$$\frac{\partial \mathbf{T}_{\mathrm{m}}}{\partial t} + \boldsymbol{U}\cdot\nabla\mathbf{T}_{\mathrm{m}} - (\nabla\boldsymbol{U})^{\mathrm{T}}\cdot\mathbf{T}_{\mathrm{m}} - \mathbf{T}_{\mathrm{m}}\cdot\nabla\boldsymbol{U} = \frac{\eta}{\mu_0}[\boldsymbol{B}\nabla^2\boldsymbol{B} + (\nabla^2\boldsymbol{B})\boldsymbol{B}]. \tag{2.12}$$

In the limits $\tau\to\infty$ and $\eta\to 0$, corresponding to large Deborah number and large magnetic Reynolds number respectively, the right-hand sides of these equations are negligible. The polymeric and magnetic stresses then satisfy identical equations, involving the same upper-convected derivative, and they appear identically in the equation of motion of the fluid. Therefore the formal analogy can be expressed symbolically as

$$\lim_{\tau\to\infty}(\text{Oldroyd-B fluid}) = \lim_{\eta\to 0}(\text{MHD fluid}). \tag{2.13}$$

We note that $\mathbf{T}_{\mathrm{m}}$ is a positive semi-definite tensor having one non-negative eigenvalue and two zero eigenvalues. Joseph (1990) has shown that $\mathbf{T}_{\mathrm{p}}$ also retains a positive definite character when it evolves according to equation (2.11). This is required on physical grounds, because in the derivation of the Oldroyd-B constitutive equation from kinetic theory, $\mathbf{T}_{\mathrm{p}} \propto \langle \boldsymbol{d}\boldsymbol{d}\rangle$, where $\boldsymbol{d}$ is the separation of the ends of a polymer molecule, and the angle brackets denote an average (Bird *et al.* 1987*b*). Equation (2.11) shows that $\mathbf{T}_{\mathrm{p}}$ attempts to return to isotropy on the relaxation time, but in a shear flow at large Deborah number, this tendency is overcome and one eigenvalue of $\mathbf{T}_{\mathrm{p}}$ does indeed dominate, as required by the MHD analogy.

As we have shown, the induction equation of ideal MHD provides an equation for the magnetic stress tensor, which is comparable to the constitutive equation of the Oldroyd-B fluid. One might ask whether the constitutive equation can be reduced to something resembling an induction equation. This is indeed so: at any instant we may express the positive definite tensor $\boldsymbol{T}_\mathrm{p}$ in terms of *three* vector fields $\boldsymbol{B}_i$,

$$\boldsymbol{T}_\mathrm{p} = \frac{1}{\mu_0}\sum_{i=1}^{3} \boldsymbol{B}_i\boldsymbol{B}_i. \tag{2.14}$$

Equation (2.11) is recovered if the fields evolve according to the induction-like equations

$$\frac{\partial \boldsymbol{B}_i}{\partial t} + \boldsymbol{U}\cdot\nabla\boldsymbol{B}_i = \boldsymbol{B}_i\cdot\nabla\boldsymbol{U} - \frac{1}{2\tau}\left(\boldsymbol{B}_i - \frac{\mu_0\mu_\mathrm{p}}{\tau}\boldsymbol{Q}_i\right), \tag{2.15}$$

provided that the fields $\boldsymbol{Q}_i$ satisfy

$$\frac{1}{2}\sum_{i=1}^{3}(\boldsymbol{B}_i\boldsymbol{Q}_i + \boldsymbol{Q}_i\boldsymbol{B}_i) = \boldsymbol{1}. \tag{2.16}$$

To find the fields $\boldsymbol{Q}_i$, let $\boldsymbol{B}$ be the matrix whose columns are $(\boldsymbol{B}_1, \boldsymbol{B}_2, \boldsymbol{B}_3)$, and similarly for $\boldsymbol{Q}$. In matrix notation, equation (2.16) reads

$$\tfrac{1}{2}(\boldsymbol{B}\boldsymbol{Q}^\mathrm{T} + \boldsymbol{Q}\boldsymbol{B}^\mathrm{T}) = \boldsymbol{1} \tag{2.17}$$

and is satisfied when

$$\boldsymbol{Q} = (\boldsymbol{1} + \boldsymbol{A})\boldsymbol{C}, \tag{2.18}$$

where $\boldsymbol{A}$ is an arbitrary antisymmetric matrix and $\boldsymbol{C}$ is the inverse of $\boldsymbol{B}^\mathrm{T}$. This means that

$$\boldsymbol{Q}_i = \boldsymbol{C}_i + \boldsymbol{\Omega}\times\boldsymbol{C}_i, \tag{2.19}$$

where $\boldsymbol{\Omega}$ is an arbitrary vector field. The vector fields $\boldsymbol{C}_i$ are just the reciprocal vectors to $\{\boldsymbol{B}_i\}$, e.g.

$$\boldsymbol{C}_1 = \frac{\boldsymbol{B}_2\times\boldsymbol{B}_3}{\boldsymbol{B}_1\cdot(\boldsymbol{B}_2\times\boldsymbol{B}_3)}. \tag{2.20}$$

The non-uniqueness of the fields $\boldsymbol{Q}_i$ reflects the fact that the representation (2.14) is partially redundant: we are expressing a tensor field with six independent components in terms of three vector fields each having three independent components. It is therefore permissible to impose three constraints on the fields $\boldsymbol{B}_i$, and then $\boldsymbol{\Omega}$ will no longer be arbitrary. For example, it may be convenient to require that the fields $\boldsymbol{B}_i$ be solenoidal. Now equation (2.15) implies that

$$\left(\frac{\partial}{\partial t} + \boldsymbol{U}\cdot\nabla\right)(\nabla\cdot\boldsymbol{B}_i) = -\frac{1}{2\tau}\left(\nabla\cdot\boldsymbol{B}_i - \frac{\mu_0\mu_\mathrm{p}}{\tau}\nabla\cdot\boldsymbol{Q}_i\right), \tag{2.21}$$

and so the solenoidal property is preserved if $\boldsymbol{\Omega}$ is chosen such that

$$0 = \nabla\cdot\boldsymbol{Q}_i = \nabla\cdot(\boldsymbol{C}_i + \boldsymbol{\Omega}\times\boldsymbol{C}_i), \quad i = 1, 2, 3. \tag{2.22}$$

Choosing the fields $\boldsymbol{B}_i$ to be solenoidal also ensures that

$$\nabla\cdot\boldsymbol{T}_\mathrm{p} = \frac{1}{\mu_0}\sum_{i=1}^{3}\boldsymbol{B}_i\cdot\nabla\boldsymbol{B}_i, \tag{2.23}$$

for direct comparability with the Lorentz force.

3. Energetics and instability

3.1. *MHD fluid*

Some insight into the possible instabilities of viscoelastic and MHD flows can be obtained on the basis of energy considerations. Instabilities typically release energy stored in the basic state and use this to allow a perturbation to grow in time. This restricts the class of flows that can exhibit instability, and limits the growth rates that can be achieved.

We start by considering the case of the MHD fluid, which is more straightforward. Starting from the equations of § 2.2 it is possible to derive an energy equation of the form

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{F} = -D, \tag{3.1}$$

where

$$E = \tfrac{1}{2}\rho U^2 + \frac{B^2}{2\mu_0} \tag{3.2}$$

is the energy density,

$$\boldsymbol{F} = (E + \Psi)\boldsymbol{U} - \frac{1}{\mu_0}(\boldsymbol{B} \cdot \boldsymbol{U})\boldsymbol{B} - \mu \boldsymbol{U} \times (\nabla \times \boldsymbol{U}) - \frac{\eta}{\mu_0}\boldsymbol{B} \times (\nabla \times \boldsymbol{B}) \tag{3.3}$$

is the energy flux, and

$$D = \mu|\nabla \times \boldsymbol{U}|^2 + \frac{\eta}{\mu_0}|\nabla \times \boldsymbol{B}|^2 \tag{3.4}$$

is the dissipation rate.

Consider a perturbative solution of the equations of § 2.2 in which upper-case symbols $(\boldsymbol{U}, \boldsymbol{B}, \Psi)$ denote the basic state (not necessarily steady) and lower-case symbols $(\boldsymbol{u}, \boldsymbol{b}, \psi)$ denote the Eulerian perturbations. Using the linearized equations, it is then possible to derive the energy-like equation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \frac{b^2}{2\mu_0}\right) + \nabla \cdot \boldsymbol{F}' = \left(-\rho \boldsymbol{u}\boldsymbol{u} + \frac{\boldsymbol{b}\boldsymbol{b}}{\mu_0}\right) : \nabla \boldsymbol{U} + (\boldsymbol{b} \times \boldsymbol{u}) \cdot \boldsymbol{J} - \mu|\nabla \times \boldsymbol{u}|^2 - \frac{\eta}{\mu_0}|\nabla \times \boldsymbol{b}|^2 \tag{3.5}$$

governing the perturbations, where $\boldsymbol{F}'$ is a certain flux and $\boldsymbol{J} = \mu_0^{-1}\nabla \times \boldsymbol{B}$ is the current density in the basic state. The quantity differentiated with respect to time is the part of the energy density at second order in the perturbation amplitude that must grow in any instability. Provided that the instability is local, so that it does not depend on a particular choice of boundary conditions, the term $\nabla \cdot \boldsymbol{F}'$ cannot play an essential role in this equation, because it will vanish on integration over the volume of the fluid in the case of periodic boundary conditions or, in many cases, physical boundary conditions. Therefore any local instability must derive its energy either from the kinetic energy of the basic flow, through the term involving $\nabla \boldsymbol{U}$, or from the magnetic energy, through the term involving $\boldsymbol{J}$. For kinetic energy to be released, there must be a velocity *gradient*, because a uniform flow can be eliminated by a Galilean transformation and therefore cannot be a source of instability. A potential magnetic field $(\boldsymbol{J} = \boldsymbol{0})$ also cannot be a source of instability, as it minimizes the magnetic energy in a region subject to the magnetic flux through its boundary being prescribed (e.g. Priest 1982).

In the case of a potential magnetic field it is possible to place an upper bound on the growth rate of any local instability. Let

$$\mathbf{S} = \tfrac{1}{2}[\nabla \boldsymbol{U} + (\nabla \boldsymbol{U})^{\mathrm{T}}] \tag{3.6}$$

be the rate-of-strain tensor of the basic flow, and $(\lambda_1, \lambda_2, \lambda_3)$ its eigenvalues. Its quadratic form satisfies the inequalities

$$\min(\lambda_1, \lambda_2, \lambda_3) \leqslant \frac{\mathbf{S} : \boldsymbol{xx}}{x^2} \leqslant \max(\lambda_1, \lambda_2, \lambda_3) \tag{3.7}$$

and therefore

$$\frac{|\mathbf{S} : \boldsymbol{xx}|}{x^2} \leqslant \max(|\lambda_1|, |\lambda_2|, |\lambda_3|). \tag{3.8}$$

It follows from equation (3.5) that the largest possible growth rate of any local instability is the largest eigenvalue, in absolute value, of the rate-of-strain tensor of the basic flow. When the magnetic field is not potential the growth rate can be increased by at most $(\mu_0/\rho)^{1/2}|\boldsymbol{J}|/2$.

3.2. *Viscoelastic fluid*

By working with the representation (2.14) of the polymeric stress in terms of three vector fields $\boldsymbol{B}_i$ it is possible to derive a similar energy-like equation

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}\rho u^2 + \sum_{i=1}^{3} \frac{b_i^2}{2\mu_0}\right) + \nabla \cdot \boldsymbol{F}'' = \left(-\rho \boldsymbol{uu} + \sum_{i=1}^{3} \frac{\boldsymbol{b}_i \boldsymbol{b}_i}{\mu_0}\right) : \nabla \boldsymbol{U} + \sum_{i=1}^{3} (\boldsymbol{b}_i \times \boldsymbol{u}) \cdot \boldsymbol{J}_i$$

$$+ \frac{1}{\mu_0} \sum_{i=1}^{3} (\nabla \cdot \boldsymbol{b}_i) \boldsymbol{u} \cdot \boldsymbol{B}_i - \mu |\nabla \times \boldsymbol{u}|^2 - \frac{1}{\tau} \sum_{i=1}^{3} \frac{b_i^2}{2\mu_0} + \frac{\mu_{\mathrm{p}}}{2\tau^2} \sum_{i=1}^{3} \boldsymbol{b}_i \cdot \boldsymbol{q}_i \tag{3.9}$$

governing linear perturbations from any basic state, where $\boldsymbol{F}''$ is a certain flux and $\boldsymbol{J}_i = \mu_0^{-1} \nabla \times \boldsymbol{B}_i$ by analogy with MHD. If we constrain the representation such that the fields are solenoidal, then the term involving $\nabla \cdot \boldsymbol{b}_i$ vanishes. The argument proceeds almost as before, with gradients in the basic flow or the basic stress providing potential sources of energy for the disturbance. The final term, involving $\boldsymbol{b}_i \cdot \boldsymbol{q}_i$, is a third possible source of energy, but the τ^{-2} dependence suggests that the effect of this term may be expected to be small in the limit of large Deborah number.

4. Plane Couette flow in a rotating channel

As a minimal model of a differentially rotating flow, we consider a linear shear flow (plane Couette flow) in a rotating channel. This is equivalent to cylindrical Couette flow in the limit of a narrow gap, if the angular velocities of the two cylinders are not widely disparate. All effects of curvature are then neglected.

4.1. *Basic state and boundary conditions*

We adopt Cartesian coordinates (x, y, z) in a frame of reference rotating with uniform angular velocity $\Omega\, \boldsymbol{e}_z$. The only change required to the equations in the rotating frame is the inclusion of the Coriolis force. The centrifugal force, which is derivable from a potential, can be absorbed into the modified pressure. The rotation of the frame does not affect any of the other equations. The equation of motion therefore becomes

$$\rho\left(\frac{\partial \boldsymbol{U}}{\partial t} + \boldsymbol{U} \cdot \nabla \boldsymbol{U} + 2\Omega\, \boldsymbol{e}_z \times \boldsymbol{U}\right) = \cdots. \tag{4.1}$$

We consider flow in the channel $0 < x < d$ between a stationary plane boundary $x = 0$ and a moving plane boundary $x = d$ with velocity $-2Ad\,\boldsymbol{e}_y$. The non-slip and impermeable boundary conditions

$$\boldsymbol{U} = \boldsymbol{0} \quad \text{at} \quad x = 0, \qquad \boldsymbol{U} = -2Ad\,\boldsymbol{e}_y \quad \text{at} \quad x = d \tag{4.2}$$

apply. The basic flow is the plane Couette flow,

$$\boldsymbol{U} = -2Ax\,\boldsymbol{e}_y. \tag{4.3}$$

A modified pressure quadratic in x is required to balance the Coriolis force.

When this model is taken as a local representation of a differentially rotating flow with angular velocity $\Omega(r)$, the shear parameter A is to be interpreted as Oort's first constant. When the Rayleigh discriminant $4\Omega(\Omega - A)$ is positive, the flow of an inviscid, unmagnetized flow is linearly stable to axisymmetric perturbations. For a Keplerian flow in which $\Omega \propto r^{-3/2}$, we have $A/\Omega = 3/4$. (In the rheological literature, the shear rate $|2A|$ would usually be called $\dot{\gamma}$.)

In the case of the viscoelastic fluid, the non-zero stress components associated with the basic flow are

$$T_{xy} = T_{yx} = -2A\mu_{\mathrm{p}}, \quad T_{yy} = 8A^2\tau\mu_{\mathrm{p}}, \tag{4.4}$$

which provide the steady solution of equation (2.3). The polymeric stress defined in equation (2.8) is

$$\boldsymbol{T}_{\mathrm{p}} = \frac{\mu_{\mathrm{p}}}{\tau}\begin{bmatrix} 1 & -De & 0 \\ -De & 2De^2 + 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{4.5}$$

where $De = 2A\tau$ is the Deborah number, and this can be represented in the form (2.14) using the three solenoidal fields

$$\boldsymbol{B}_{1,2} = \left(\frac{\mu_0\mu_{\mathrm{p}}}{2\tau}\right)^{1/2}\begin{bmatrix} -1 \\ De \pm (De^2 + 1)^{1/2} \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_3 = \left(\frac{\mu_0\mu_{\mathrm{p}}}{\tau}\right)^{1/2}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4.6}$$

Note that, for large De, the field $\boldsymbol{B}_1$ is much greater than the other two and corresponds to a uniform magnetic field almost exactly in the y-direction.

In the case of the MHD fluid, we suppose that a uniform magnetic field $\boldsymbol{B} = B_y\,\boldsymbol{e}_y$ is imposed. We also suppose the boundaries to be perfectly conducting, so that the additional boundary conditions

$$B_x = \frac{\partial B_y}{\partial x} = \frac{\partial B_z}{\partial x} = 0 \tag{4.7}$$

apply at $x = 0$ and $x = d$.

We note that the magnetic stress tensor in the MHD fluid resembles the polymeric stress tensor in the viscoelastic fluid if De is large and we identify

$$\frac{B_y^2}{\mu_0} \leftrightarrow 8A^2\tau\mu_{\mathrm{p}}. \tag{4.8}$$

The energy considerations of §3 are not affected by the rotation of the frame of reference, because the Coriolis force does no work on the fluid. The eigenvalues of the rate-of-strain tensor are $(A, -A, 0)$. As the magnetic field is uniform, the maximal growth rate of any local instability, at least in the MHD case, is $|A|$. Incidentally, this proves the conjecture of Balbus & Hawley (1992) that the magnetorotational instability, with a suitably chosen wavevector and in the absence of dissipation, achieves the largest possible growth rate of any local shear instability. (In an inviscid, unmagnetized fluid, the largest possible growth rate of Rayleigh's inertial instability is $\sqrt{4\Omega(A-\Omega)}$. This is always less than or equal to $|A|$, with equality in the case $A = 2\Omega$.)

4.2. *Dimensionless groups*

We introduce the kinematic viscosities $\nu = \mu/\rho$ and $\nu_{\mathrm{p}} = \mu_{\mathrm{p}}/\rho$. The four dimensionless parameters of the viscoelastic system are the Rossby number, the Reynolds number, the Deborah number and the viscosity ratio, defined by

$$Ro = \frac{A}{\Omega}, \quad Re = \frac{2Ad^2}{\nu}, \quad De = 2A\tau, \quad S = \frac{\nu}{\nu_{\mathrm{p}}}, \tag{4.9}$$

respectively. For reference, the Taylor number is $Ta = Re^2 Ro^{-1}(1 - Ro^{-1})$.

The four dimensionless parameters of the MHD system are the Rossby number, the Reynolds number, the magnetic Reynolds number and the Chandrasekhar number, defined by

$$Ro = \frac{A}{\Omega}, \quad Re = \frac{2Ad^2}{\nu}, \quad Rm = \frac{2Ad^2}{\eta}, \quad Q = \frac{B_y^2 d^2}{\mu_0 \rho \nu \eta}, \tag{4.10}$$

respectively. The identification (4.8) corresponds to

$$Q \leftrightarrow \frac{2Rm\, De}{S}. \tag{4.11}$$

We can therefore quote the magnetic field strength in terms of an effective Deborah number for the MHD system, to make a direct comparison easier.

We are interested in comparing the behaviour of the viscoelastic and MHD systems in the limit of large De and large Rm. This limit could be approached in many different ways, but we choose to do this as one might in an ideal experiment, by keeping ρ, ν, ν_{p}, τ, η and d fixed while increasing Ω, A and B_y together. This means that Ro and S are fixed while $Re \propto Rm \propto De$ and $Q \propto De^2$. The elasticity, De/Re, is fixed in this process.

4.3. *Linear perturbations*

We now consider small deviations from the above state, such that the Eulerian perturbation of velocity, say, is

$$\mathrm{Re}[\boldsymbol{u}(x)\exp(st + \mathrm{i}k_y y + \mathrm{i}k_z z)], \tag{4.12}$$

where s is the growth rate (in general complex) and k_y and k_z are real wavenumber components. Differentiation of the perturbations with respect to x will be denoted by a prime, and we define $k^2 = k_y^2 + k_z^2$.

4.4. *Viscoelastic fluid*

The perturbations of the viscoelastic fluid satisfy the equations

$$u_x' + \mathrm{i}k_y u_y + \mathrm{i}k_z u_z = 0, \tag{4.13}$$

$$\rho[(s - 2\mathrm{i}Axk_y)u_x - 2\Omega u_y] = -\psi' + t_{xx}' + \mathrm{i}k_y t_{xy} + \mathrm{i}k_z t_{xz} + \mu(u_x'' - k^2 u_x), \tag{4.14}$$

$$\rho[(s - 2\mathrm{i}Axk_y)u_y + 2(\Omega - A)u_x] = -\mathrm{i}k_y\psi + t_{xy}' + \mathrm{i}k_y t_{yy} + \mathrm{i}k_z t_{yz} + \mu(u_y'' - k^2 u_y), \tag{4.15}$$

$$\rho(s-2\mathrm{i}Axk_y)u_z = -\mathrm{i}k_z\psi + t'_{xz} + \mathrm{i}k_y t_{yz} + \mathrm{i}k_z t_{zz} + \mu(u''_z - k^2 u_z), \tag{4.16}$$

$$t_{xx} + \tau[(s-2\mathrm{i}Axk_y)t_{xx} - 2\mathrm{i}T_{xy}k_y u_x] = 2\mu_\mathrm{p} u'_x, \tag{4.17}$$

$$t_{xy} + \tau[(s-2\mathrm{i}Axk_y)t_{xy} + 2At_{xx} - T_{xy}(u'_x + \mathrm{i}k_y u_y) - \mathrm{i}T_{yy}k_y u_x] = \mu_\mathrm{p}(u'_y + \mathrm{i}k_y u_x), \tag{4.18}$$

$$t_{xz} + \tau[(s-2\mathrm{i}Axk_y)t_{xz} - \mathrm{i}T_{xy}k_y u_z] = \mu_\mathrm{p}(u'_z + \mathrm{i}k_z u_x), \tag{4.19}$$

$$t_{yy} + \tau[(s-2\mathrm{i}Axk_y)t_{yy} + 4At_{xy} - 2T_{xy}u'_y - 2\mathrm{i}T_{yy}k_y u_y] = 2\mathrm{i}\mu_\mathrm{p}k_y u_y, \tag{4.20}$$

$$t_{yz} + \tau[(s-2\mathrm{i}Axk_y)t_{yz} + 2At_{xz} - T_{xy}u'_z - \mathrm{i}T_{yy}k_y u_z] = \mathrm{i}\mu_\mathrm{p}(k_y u_z + k_z u_y), \tag{4.21}$$

$$t_{zz} + \tau[(s-2\mathrm{i}Axk_y)t_{zz}] = 2\mathrm{i}\mu_\mathrm{p}k_z u_z. \tag{4.22}$$

These constitute a sixth-order system of linear ODEs to be solved for the eigenvalue s. The dependent variables may be taken as $(\psi, u_x, u_y, u'_y, u_z, u'_z)$. The boundary conditions $u_x = u_y = u_z = 0$ apply at $x = 0, d$.

These equations must be solved numerically in general. However, it is instructive to analyse further the case of unsheared or 'axisymmetric' modes ($k_y = 0$), which correspond to axisymmetric modes in cylindrical geometry. In this case, the equations have constant coefficients and can be combined into a single equation for u_x,

$$[qs + (q\nu + \nu_\mathrm{p})\,\mathrm{D}]^2\,\mathrm{D}u_x + 4\Omega(\Omega - A)q^2k_z^2u_x - 4\Omega A\tau\nu_\mathrm{p}k_z^2\,\mathrm{D}u_x = 0, \tag{4.23}$$

where $q = 1 + \tau s$ and D is the operator

$$\mathrm{D} = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + k_z^2. \tag{4.24}$$

This equation may be investigated analytically in an approximate way by considering solutions of a simple trigonometric form $u_x \propto \sin(k_x x)$, although these cannot satisfy all six physical boundary conditions. (In § 5 below we compute the global solutions of this equation numerically.) The local dispersion relation corresponding to these solutions,

$$[qs + (q\nu + \nu_\mathrm{p})(k_x^2 + k_z^2)]^2(k_x^2 + k_z^2) + 4\Omega(\Omega - A)q^2k_z^2 - 4\Omega A\tau\nu_\mathrm{p}k_z^2(k_x^2 + k_z^2) = 0, \tag{4.25}$$

is a quartic equation for s with real coefficients. It can be shown that the principle of the exchange of stabilities holds: instability first sets in at a stationary bifurcation ($s = 0$), which occurs when the constant term passes through zero, i.e. when

$$(\nu + \nu_\mathrm{p})^2(k_x^2 + k_z^2)^3 + 4\Omega(\Omega - A)k_z^2 - 4\Omega A\tau\nu_\mathrm{p}k_z^2(k_x^2 + k_z^2) = 0. \tag{4.26}$$

Suppose that Rayleigh's criterion for stability, $4\Omega(\Omega - A) > 0$, is satisfied, and that $4\Omega A > 0$. When τ is increased from zero to a sufficiently large value, a bifurcation occurs and axisymmetric instability ensues. To understand this we note that, when $\nu = 0$, and in the limit $\tau \gg |s|^{-1}$ with $k_z^2 \gg k_x^2$, the dispersion relation (4.25) becomes identical to the ideal magnetorotational dispersion relation (1.4) for a vertical magnetic field and vertical wavevector, provided that we identify $B_z^2 \leftrightarrow \mu_0\mu_\mathrm{p}/\tau$. This is precisely what is suggested by the field $\boldsymbol{B}_3$ of equation (4.6). Although the principal analogy is with a uniform magnetic field in the y-direction, such a field provides no restoring force to axisymmetric perturbations and we see instead the effect of the much weaker field $\boldsymbol{B}_3$. Therefore the axisymmetric viscoelastic instability, which we verify numerically in § 5 below, can be understood as being analogous to a magnetorotational instability deriving from the weak vertical field.

4.5. *MHD fluid*

The perturbations of the MHD fluid satisfy the equations

$$u_x' + \mathrm{i}k_y u_y + \mathrm{i}k_z u_z = 0, \tag{4.27}$$

$$\rho[(s - 2\mathrm{i}Axk_y)u_x - 2\Omega u_y] = -\psi_\mathrm{m}' + \mathrm{i}\mu_0^{-1}k_y B_y b_x + \mu(u_x'' - k^2 u_x), \tag{4.28}$$

$$\rho[(s - 2\mathrm{i}Axk_y)u_y + 2(\Omega - A)u_x] = -\mathrm{i}k_y\psi_\mathrm{m} + \mathrm{i}\mu_0^{-1}k_y B_y b_y + \mu(u_y'' - k^2 u_y), \tag{4.29}$$

$$\rho(s - 2\mathrm{i}Axk_y)u_z = -\mathrm{i}k_z\psi_\mathrm{m} + \mathrm{i}\mu_0^{-1}k_y B_y b_z + \mu(u_z'' - k^2 u_z), \tag{4.30}$$

$$b_x' + \mathrm{i}k_y b_y + \mathrm{i}k_z b_z = 0, \tag{4.31}$$

$$(s - 2\mathrm{i}Axk_y)b_x = \mathrm{i}k_y B_y u_x + \eta(b_x'' - k^2 b_x), \tag{4.32}$$

$$(s - 2\mathrm{i}Axk_y)b_y + 2Ab_x = \mathrm{i}k_y B_y u_y + \eta(b_y'' - k^2 b_y), \tag{4.33}$$

$$(s - 2\mathrm{i}Axk_y)b_z = \mathrm{i}k_y B_y u_z + \eta(b_z'' - k^2 b_z). \tag{4.34}$$

These constitute a tenth-order system of linear ODEs to be solved for the eigenvalue s. The dependent variables may be taken as $(\psi_\mathrm{m}, u_x, u_y, u_y', u_z, u_z', b_y, b_y', b_z, b_z')$. The boundary conditions $u_x = u_y = u_z = b_y' = b_z' = 0$ apply at $x = 0, d$. To eliminate b_x from the problem, differentiate equation (4.31) to find b_x'', then substitute into equation (4.32) to find

$$(s - 2\mathrm{i}Axk_y + \eta k^2)b_x = \mathrm{i}k_y B_y u_x - \eta(\mathrm{i}k_y b_y' + \mathrm{i}k_z b_z'). \tag{4.35}$$

Therefore b_x is determined algebraically in terms of the dependent variables, and can be substituted where needed. It automatically satisfies the boundary condition $b_x = 0$ at $x = 0, d$. A difficulty would arise if the quantity $s - 2\mathrm{i}Axk_y + \eta k^2$ were to vanish at any point. As this is a complex function, it is 'unlikely' that both real and imaginary parts would vanish simultaneously. In any case, it could vanish only for a decaying mode, and such modes are of no interest here.

Although the linearized equations (4.27)–(4.34) for the MHD fluid appear quite different from those of the viscoelastic fluid, equations (4.13)–(4.22), they can be seen to correspond in the limits $\tau \to \infty$, $\eta \to 0$ if we identify $T_{yy} \leftrightarrow B_y^2/\mu_0$, $t_{xy} \leftrightarrow B_y b_x/\mu_0$, $t_{yy} \leftrightarrow 2B_y b_y/\mu_0$ and $t_{yz} \leftrightarrow B_y b_z/\mu_0$, while $T_{xy}, t_{xx}, t_{xz}, t_{zz} \leftrightarrow 0$.

In the special case of axisymmetric modes ($k_y = 0$) the velocity and magnetic perturbations are decoupled. The magnetic perturbation always decays if $\eta > 0$. The remaining equations can be combined into a single equation for u_x,

$$(s + \nu D)^2 D u_x + 4\Omega(\Omega - A)k^2 u_x = 0. \tag{4.36}$$

Stability is assured if Rayleigh's criterion, $4\Omega(\Omega - A) > 0$, is satisfied.

5. Numerical investigation

We solve the eigenvalue problems defined in §§4.4 and 4.5 for non-axisymmetric modes numerically by the shooting method. The arbitrary normalization $\psi(0) = 1$ is adopted, and the equations are integrated from $x = 0$ to $x = 1$. For the viscoelastic system, the boundary conditions at $x = 1$ impose three conditions on the three unknown quantities s, $u_y'(0)$ and $u_z'(0)$. Newton–Raphson iteration is applied to converge on a solution. For the MHD system, shooting in $\mathbf{C}^5$ is required.

In the absence of viscosity and resistivity, the MHD problem becomes identical to the 'Cartesian model' studied by Ogilvie & Pringle (1996) in their investigation

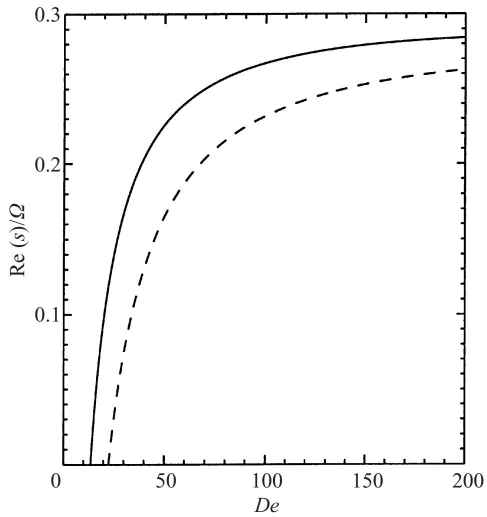

FIGURE 1. Variation of the growth rate of the first unstable non-axisymmetric mode (at $k_y = 1$) with the Deborah number, for the viscoelastic fluid (solid line) and the MHD fluid (dashed line). Note that the Reynolds number is $Re = (24/5)De$.

of the magnetorotational instability in the presence of an azimuthal (or toroidal) magnetic field. We recall some results of that analysis: (i) the instability requires a non-zero azimuthal wavenumber k_y so that the magnetic field lines are bent by the perturbation; (ii) as k_z is increased, unstable modes emerge from the continuous spectrum of Alfvén waves and the eigenvalues approach limit points; (iii) the largest growth rates are attained in the limit $k_z \to \infty$, when the normal modes are localized near a boundary (although, as we show below, solutions also exist that grow rapidly but transiently in the interior of the fluid); (iv) the maximal growth rate, A, is attained for an Alfvén frequency $\omega_A = (15/16)^{1/2}\Omega$ in the Keplerian case $A/\Omega = 3/4$.

For numerical purposes it is convenient to adopt d and Ω^{-1} as units of length and time. We adopt $Ro = 3/4$, which is stable according to Rayleigh's criterion and is suggested by astrophysical applications, and take $\nu = \nu_p = \eta$ for simplicity.

In the presence of dissipation, all modes decay in the limit $k_z \to \infty$. Therefore we restrict attention to a moderate value, $k_z d = \pi$, at which the growth rates are appreciable (but not optimal, and always less than A). We select the optimal Alfvén frequency, as described in § 1.3, by choosing $k_y d = 1$ and $De/Re = 5/24$. The most unstable mode is one with the fewest nodes in its eigenfunction. The variation of its growth rate with De is shown in figure 1. As De increases, the eigenvalues of the mode in the viscoelastic and MHD systems converge. The eigenfunctions are also in close agreement at $De = 150$, as shown in figure 2.

We have also solved numerically for axisymmetric unstable modes in the viscoelastic problem, as anticipated in § 4.4. The growth rates of the most unstable modes are shown in figure 3. They are smaller than for the non-axisymmetric modes, consistent with the idea that the axisymmetric instability is analogous to a magnetorotational instability deriving from the weak vertical field $\boldsymbol{B}_3$.

6. Asymptotic analysis

We now present an asymptotic analysis demonstrating the existence of localized, non-axisymmetric growing solutions of the perturbation equations for both systems,

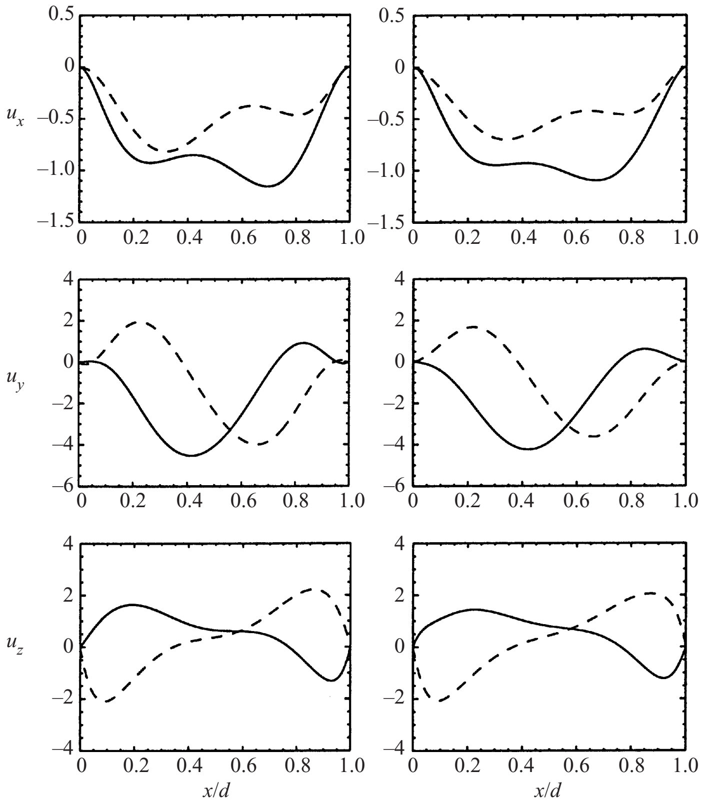

FIGURE 2. Eigenfunctions of a non-axisymmetric unstable mode at $De = 150$, for the viscoelastic fluid (left) and the MHD fluid (right). Real and imaginary parts are shown with solid and dashed lines, respectively.

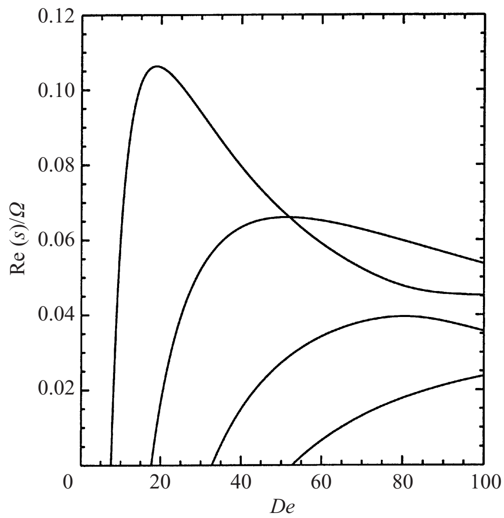

FIGURE 3. Variation of the growth rates of axisymmetric unstable modes ($k_y = 0$) with the Deborah number, for the viscoelastic fluid. The eigenfunctions of modes appearing successively as De is increased have increasing numbers of nodes.

consistent at leading order with the magnetorotational dispersion relation (1.4). We are interested again in the limit of large Deborah number and large magnetic Reynolds number.

6.1. *MHD fluid*

We consider a solution of the perturbation equations for the MHD fluid, localized in a layer near an arbitrary point $x = x_0$. Let

$$x = x_0 + \epsilon X, \tag{6.1}$$

where $\epsilon \ll 1$ is an ordering parameter and $X = O(1)$ within the layer of interest. We introduce the scalings

$$k_z = \epsilon^{-3/2}\tilde{k}_z, \quad \nu = \epsilon^4\tilde{\nu}, \quad \eta = \epsilon^4\tilde{\eta}, \tag{6.2}$$

implying that the vertical wavelength is even shorter than the width of the layer, and that dissipation has only a weak effect on the solution. The solution will have an exponential time-dependence at leading order, but we relax the assumption of a normal mode and allow the solution to evolve freely on a long time scale captured by the slow time coordinate $T = \epsilon t$. This is achieved through the replacement

$$s \mapsto s_0 + \epsilon\frac{\partial}{\partial T} + O(\epsilon^2) \tag{6.3}$$

in the perturbation equations, and we also replace

$$\frac{\mathrm{d}}{\mathrm{d}x} \mapsto \epsilon^{-1}\frac{\partial}{\partial X} \tag{6.4}$$

within the layer. A consistent expansion scheme for the perturbations is of the form

$$\left.\begin{aligned} u_x &= u_{x0}(X,T) + \epsilon u_{x1}(X,T) + O(\epsilon^2), \\ u_y &= u_{y0}(X,T) + \epsilon u_{y1}(X,T) + O(\epsilon^2), \\ u_z &= \epsilon^{1/2}[u_{z0}(X,T) + O(\epsilon)], \\ \psi_{\mathrm{m}} &= \epsilon^2[\psi_0(X,T) + O(\epsilon)], \\ b_x &= b_{x0}(X,T) + \epsilon b_{x1}(X,T) + O(\epsilon^2), \\ b_y &= b_{y0}(X,T) + \epsilon b_{y1}(X,T) + O(\epsilon^2), \\ b_z &= \epsilon^{1/2}[b_{z0}(X,T) + O(\epsilon)]. \end{aligned}\right\} \tag{6.5}$$

From equations (4.28), (4.29), (4.32) and (4.33) at leading order we obtain the algebraic system

$$\left.\begin{aligned} \rho(\hat{s}u_{x0} - 2\Omega u_{y0}) &= \mathrm{i}\mu_0^{-1}k_y B_y b_{x0}, \\ \rho[\hat{s}u_{y0} + 2(\Omega - A)u_{x0}] &= \mathrm{i}\mu_0^{-1}k_y B_y b_{y0}, \\ \hat{s}b_{x0} &= \mathrm{i}k_y B_y u_{x0}, \\ \hat{s}b_{y0} + 2Ab_{x0} &= \mathrm{i}k_y B_y u_{y0}, \end{aligned}\right\} \tag{6.6}$$

where $\hat{s} = s_0 - 2\mathrm{i}Ak_y x_0$. These may be combined into the single equation

$$\{\hat{s}^4 + \hat{s}^2[4\Omega(\Omega - A) + 2\omega_{\mathrm{A}}^2] + \omega_{\mathrm{A}}^2(\omega_{\mathrm{A}}^2 - 4\Omega A)\}u_{x0} = 0, \tag{6.7}$$

where $\omega_{\mathrm{A}}^2 = k_y^2 B_y^2/\mu_0\rho$. This has a non-trivial solution,

$$u_{x0} = F(X, T), \tag{6.8}$$

if and only if $\hat{s}$ satisfies the magnetorotational dispersion relation (1.4). We then deduce u_{y0}, b_{x0} and b_{y0} in terms of F, and also u_{z0}, ψ_0 and b_{z0} from equations (4.27), (4.30) and either (4.31) or (4.34) at leading order.

From equations (4.28), (4.29), (4.32) and (4.33) at order ϵ we similarly obtain

$$\{\hat{s}^4 + \hat{s}^2[4\Omega(\Omega - A) + 2\omega_{\mathrm{A}}^2] + \omega_{\mathrm{A}}^2(\omega_{\mathrm{A}}^2 - 4\Omega A)\}u_{x1} = R, \tag{6.9}$$

where the right-hand side R depends on F and its derivatives. Given that $\hat{s}$ has been chosen to satisfy the dispersion relation, the solvability condition for this equation is $R = 0$, which results in an evolutionary equation for F,

$$\frac{\partial F}{\partial T} = a\frac{\partial^2 F}{\partial X^2} + (\mathrm{i}bX - c)F. \tag{6.10}$$

This is a modified diffusion equation containing constant coefficients

$$a = \frac{(\hat{s} + \omega_{\mathrm{A}}^2)^3}{2\tilde{k}_z^2\hat{s}[\hat{s}^4 + 2\omega_{\mathrm{A}}^2\hat{s}^2 + \omega_{\mathrm{A}}^2(\omega_{\mathrm{A}}^2 + 4\Omega^2)]}, \tag{6.11}$$

$$b = 2Ak_y, \tag{6.12}$$

$$c = \frac{\tilde{k}_z^2[\tilde{\nu}(\hat{s} + \omega_{\mathrm{A}}^2)^2 + 4\tilde{\eta}\Omega^2\omega_{\mathrm{A}}^2]}{\hat{s}^4 + 2\omega_{\mathrm{A}}^2\hat{s}^2 + \omega_{\mathrm{A}}^2(\omega_{\mathrm{A}}^2 + 4\Omega^2)}. \tag{6.13}$$

When the conditions for instability are met, $\hat{s}$ is real and positive and therefore a, b and c are real and a and c are positive. A particular solution of equation (6.10), corresponding to an initial condition $F(X, 0) = \delta(X)$, and valid for $T > 0$ in the absence of boundaries, is the Green function

$$F = (4\pi aT)^{-1/2}\exp\left[-\left(\frac{a^2b^2T^4 + 12acT^2 - 6\mathrm{i}abXT^2 + 3X^2}{12aT}\right)\right], \tag{6.14}$$

as can be obtained by Fourier-transform methods. The Green function decays as $T \to \infty$ for any fixed X, or as $|X| \to \infty$ for any fixed T.

It follows that localized solutions exist that grow exponentially at leading order, following the magnetorotational dispersion relation. The envelope of the solution evolves more slowly in time but ultimately decays superexponentially, so that the instability grows for many e-folding times before the development of very short length scales leads to decay. If we insisted on having a normal-mode solution, equation (6.10) would become an Airy equation in X. It can be shown (Ogilvie 1997) that localized solutions of this type do exist, but only near the boundaries of the fluid.

The reason for considering disturbances that are localized in x is that it provides a convenient method of demonstrating the existence of growing solutions without resorting to numerical analysis. Provided that the localization scale δx is long compared to the vertical wavelength and to the characteristic dissipative scales, the growth rate is insensitive to δx. Terquem & Papaloizou (1996) have shown that, in the limit of ideal MHD, a continuous spectrum of infinitely localized growing disturbances exists.

6.2. *Viscoelastic fluid*

A very similar analysis can be carried out for the viscoelastic fluid. The additional requirements are the scaling $\tau = \epsilon^{-4}\tilde{\tau}$ and the expansions

$$\left.\begin{aligned} t_{xx} &= O(\epsilon^4), \\ t_{xy} &= t_{xy0}(X,T) + \epsilon t_{xy1}(X,T) + O(\epsilon^2), \\ t_{xz} &= O(\epsilon^{9/2}), \\ t_{yy} &= t_{yy0}(X,T) + \epsilon t_{yy1}(X,T) + O(\epsilon^2), \\ t_{yz} &= \epsilon^{1/2}[t_{yz0}(X,T) + O(\epsilon)], \\ t_{zz} &= O(\epsilon^7). \end{aligned}\right\} \tag{6.15}$$

Otherwise the analysis is so similar that we do not repeat it in detail. Equations (6.7) and (6.9) are obtained exactly as before, provided that we identify $\omega_{\mathrm{A}}^2 = k_y^2 T_{yy}/\rho$ in the dispersion relation. Exactly the same evolutionary equation (6.10) is also obtained with the sole exception that the term involving $\tilde{\eta}$ does not appear in the coefficient c.

Therefore this method also establishes the correspondence between the viscoelastic and MHD fluids in the limit of large *De* and large *Rm*, and demonstrates the existence of the magnetorotational instability in the Oldroyd-B fluid.

7. Discussion

We have demonstrated a close analogy between a viscoelastic medium and an electrically conducting fluid containing a magnetic field. Both an Oldroyd-B fluid, in the limit of large Deborah number, and a magnetohydrodynamic fluid, in the limit of large magnetic Reynolds number, feature a stress tensor that is nearly ‘frozen in’ to the fluid in a precise mathematical sense. As a definite example of this analogy, we have examined a local model of a differentially rotating fluid, consisting of plane Couette flow in a rotating channel. The stress tensor in the case of a viscoelastic fluid resembles the Maxwell stress corresponding to a magnetic field aligned with the flow.

Our analysis demonstrates that there is a detailed correspondence between instabilities in the two systems. We have identified a direct equivalent of the magnetorotational instability in the viscoelastic fluid. It exists when the angular velocity and relative vorticity are antiparallel (or when the angular velocity decreases outwards) and the maximal growth rate is equal to the shear parameter, or Oort constant, A. It is distinguished most clearly from Rayleigh’s inertial instability by the fact that it occurs even when the specific angular momentum increases outwards. It is also distinct from the elastic instability described by Larson *et al.* (1990), which depends on the curved geometry of Couette flow and exists in the elastic limit, $De/Re \to \infty$.

We have also found an axisymmetric viscoelastic instability that can be understood as being analogous to the magnetorotational instability of a vertical magnetic field. This reflects the fact that the polymeric stress tensor $\boldsymbol{T}_{\mathrm{p}}$ can be decomposed into three effective magnetic fields, one of which is a uniform field almost aligned with the flow and another of which is a uniform vertical field. Although the vertical field is much weaker in the limit of large *De*, it provides the dominant restoring force for axisymmetric disturbances.

The instability discussed by Larson *et al.* (1990) does not appear in our analysis because we have neglected the curvature of the streamlines. Although Larson *et al.* (1990) considered the limit of a narrow gap, the characteristic growth rates they obtained are smaller than the growth rates we have discussed, by a factor of order $\epsilon^{1/2}$, where ϵ is the ratio of the gap width to the radius. Being purely elastic in nature, the instability of Larson *et al.* (1990) must derive its energy from the elastic energy stored in the flow, rather than the shear energy which is the source for inertial and magnetorotational instabilities.

It is natural to enquire whether there is an MHD equivalent of the instability of Larson *et al.* (1990), in which energy is derived from the magnetic field. In cylindrical Couette flow at large *De* the dominant stress component has the form $T_{\phi\phi} \propto r^{-4}$, which can be identified with an azimuthal (or toroidal) magnetic field $B_\phi \propto r^{-2}$. Typically, toroidal pinch configurations are unstable to modes that rely on the curvature of the magnetic field lines and derive energy from the magnetic configuration. The most dangerous are the $m = 1$ 'kink' modes and $m = 0$ 'sausage' modes. However, in the absence of fluid motion the profile $B_\phi \propto r^{-2}$ is sufficiently steep to be stable to all perturbations (Tayler 1973). If the profile of $T_{\phi\phi}$ happened to be less steep, for example $T_{\phi\phi} \propto r^{-1}$, it is likely that there would be a viscoelastic equivalent of the kink instability. We therefore conclude that the instability of Larson *et al.* (1990) is not directly related to an MHD instability, but relies on inherently viscoelastic effects not captured by our analogy. This conclusion is supported by an examination of the physical explanation that Larson *et al.* give for their instability.

Since the work of Larson *et al.* (1990) there have been a number of related theoretical and experimental studies of viscoelastic Couette flow, some of which have examined the effects of inertia and non-axisymmetry (e.g. Avgousti & Beris 1993; Steinberg & Groisman 1998; Baumert & Muller 1999). Some of these might have been expected to reveal the analogue of the magnetorotational instability. However, it appears that the cases usually investigated are those in which either the outer or inner cylinder is stationary, or the inner cylinder rotates at twice the angular velocity of the outer cylinder with only a narrow gap between the two. This means that, whenever the analogue of the magnetorotational instability might have occurred, the system is unstable to Rayleigh's inertial instability. In order to separate the two effects, it would be valuable to examine cases in which the angular velocity decreases outwards but the specific angular momentum increases. Interestingly, the magnetorotational instability has never been demonstrated in laboratory experiments. Although there is currently much effort towards this goal (e.g. Goodman & Ji 2002), the technical requirements are considerable and the system is constrained by the very small magnetic Prandtl numbers of liquid metals. A viscoelastic magnetorotational experiment might prove to be less demanding and easier to visualize.

We conclude with some further perspectives on the analogy between viscoelastic and MHD flows.

The analogy is asymptotic in nature and therefore not perfect. Viscoelastic and MHD flows deviate from simple stress freezing in different ways: the viscoelastic stress relaxes, while the magnetic field diffuses. The classes of exactly steady solutions of the two systems are therefore different, because after a sufficiently long time either relaxation or diffusion will have its effect. For example, while there is only one solution for viscoelastic Couette flow, magnetized Couette flow can be set up with either vertical or azimuthal current-free magnetic fields. In this sense, the analogy is more applicable to dynamical, time-dependent situations than to steady flows.

Renardy (1997) has analysed the large-*De* limit of steady, two-dimensional flows of the upper-convected Maxwell fluid (obtained by setting the solvent viscosity μ of the Oldroyd-B fluid to zero). Noting that the stress $\boldsymbol{T}$ has one dominant eigenvalue in this limit, he writes $\boldsymbol{T} = \rho \boldsymbol{u}\boldsymbol{u}$, where ρ and $\boldsymbol{u}$ are a fictitious density and a fictitious velocity field. When inertial forces are negligible, $\nabla \cdot \boldsymbol{T}$ must balance the pressure gradient and ρ and $\boldsymbol{u}$ are then found to satisfy the steady Euler equations. Through our analogy, a connection can be seen here with the work of Moffatt (1985), which makes use of the analogy between steady Euler flows and magnetostatic equilibria in which the Lorentz force balances the pressure gradient.

There is a close connection between the polymeric stress $\boldsymbol{T}_\mathrm{p}$ at *moderate De* and the mean stress tensor $\langle \boldsymbol{BB} \rangle/\mu_0$ of a *disordered* magnetic field, as occurs in MHD turbulence. Under these conditions, $\boldsymbol{T}_\mathrm{p}$ and $\langle \boldsymbol{BB} \rangle/\mu_0$ may have three positive eigenvalues of comparable magnitude. Indeed, Ogilvie (2001) has suggested the use of Maxwellian viscoelastic models, with Deborah numbers of order unity, for MHD turbulence in accretion discs.

Besides Couette flow, another problem that has received much attention is the stability of a planar jet or shear layer with respect to two-dimensional disturbances. Azaiez & Homsy (1994) and Rallison & Hinch (1995) examined the equivalent problem for a viscoelastic fluid, noting the potentially stabilizing influence of a polymer additive. Taking a limit in which *Re* and *De* tend to infinity while maintaining a finite ratio, they derived an elastic equivalent of Rayleigh's stability equation. An analogy exists between this problem and that of the stability of a similar flow of an ideal, electrically conducting fluid with a magnetic field parallel to the flow, a problem studied since the 1950s. In a recent study, Hughes & Tobias (2001) derived a magnetic Rayleigh equation (their equation (3.5)) that is exactly equivalent to the elastic Rayleigh equation given by Rallison & Hinch (1995, p. 314).

In this paper we have drawn attention to a useful analogy between viscoelastic and MHD flows, and have discussed the relation between instabilities of differential rotation in the two systems. We anticipate that much further use can be made of this analogy.

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