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# Uniqueness of positive radial solutions for $n$-Laplacian Dirichlet problems

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We prove a theorem on the uniqueness of positive radial solutions to a Dirichlet problem of the $n$-Laplacian in a finite ball of $\mathbb{R}^n$. Our proofs use only elementary analysis based on an identity due to Erbe and Tang. The result can be applied to a large class of nonlinearities, including some polynomials and functions with exponential growth; in particular, the one recently studied by Adimurthi.

## 1. Introduction

We study the uniqueness of positive radial solutions for the Dirichlet problem of the quasilinear elliptic equation,

$$\left.\begin{aligned} \operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(u) = 0 \quad \text{in } B, \\ u > 0 \quad \text{in } B, \quad u = 0 \quad \text{on } \partial B, \end{aligned}\right\} \tag{1.1}$$

where $B$ is a finite ball in $\mathbb{R}^n$. When $n > m > 1$, there are in the literature a number of well-known uniqueness theorems for solutions of (1.1) and also of the associated problem,

$$\operatorname{div}(|\nabla u|^{m-2}\nabla u) + f(u) = 0, \quad u > 0 \quad \text{in } \mathbb{R}^n. \tag{1.1$'$}$$

Much less is known for the uniqueness of these problems in the case

$$1 < n \leqslant m,$$

especially when the nonlinear function $f(u)$ has exponential growth.

Very recently, Adimurthi [2] studied the uniqueness of (1.1); using the Emden–Fowler inversion technique of Atkinson and Peletier [4], he proved that if

$$n = m \geqslant 2, \qquad f(u) = u^{m-1}\mathrm{e}^u,$$

then (1.1) admits at most one radial solution. Uniqueness of radial solutions of (1.1$'$) with exponential nonlinearities was started by Pucci and Serrin [18], where more general elliptic operators were also considered. By using an identity developed in [8], together with some ingenious ideas of [17], they proved that if

$$2 \leqslant n \leqslant m, \quad f(u) = u^{p-1}[-\lambda u^p + (\exp(u^p) - 1 - u^p)], \quad \lambda > 0, \quad \tfrac{1}{2} < p \leqslant 1,$$

then (1.1$'$) has exactly one radial solution which approaches zero as $|x| \to \infty$.

In this paper, we shall follow the approach initiated in [8] and prove the following.



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THEOREM 1.1. *Let $n \leqslant m$. If $f$ satisfies*

(H1) $f \in C^1[0,\infty)$, $f(0)=0$, $f(u)>0$ *on* $(0,\infty)$,

*then (1.1′) admits no radial solution in* $\mathbb{R}^n$. *Moreover, if $f$ also satisfies*

(H2) $f(u)/u^{m-1}$ *is strictly increasing over* $(0,\infty)$,

(H3) $F(u)/f(u)$ *is increasing over* $(0,\infty)$,

*then the Dirichlet problem (1.1) has at most one radial solution.*

This result can be applied to a large class of nonlinearities, including some polynomials and functions which have exponential growth. As an example, we derive the following.

THEOREM 1.2. *Let $n \leqslant m$. If*

$$f(u) = u^{p_0}(u+a_1)^{p_1}\cdots(u+a_k)^{p_k}\exp(\lambda_1 u^{\beta_1}+\lambda_2 u^{\beta_2}+\cdots+\lambda_l u^{\beta_l}),$$

*where $p_0 \geqslant m-1$, $0 \leqslant \beta_i \leqslant 1$ and all other constants are non-negative, then (1.1) has at most one radial solution.*

Clearly, the function considered by Adimurthi is a special case of $f$ in theorem 1.2. More nonlinearities satisfying conditions of theorem 1.1 are given in §§ 4 and 5.

A radial solution $u$ of (1.1) is, in fact, a solution of the initial value problem

$$\left.\begin{aligned}\left((m-1)u''+\frac{n-1}{r}u'\right)|u'|^{m-2}+f(u)=0,\\ u(0)=\alpha>0,\quad u'(0)=0,\end{aligned}\right\} \tag{1.2}$$

where $r=|x| \geqslant 0$. When (H1) holds, then problem (1.2) has a unique solution $u=u(r,\alpha)\in C^2[0,b(a))$, where $[0,b(a))$ is the maximal interval on which $u>0$. Moreover, the solution depends continuously on $\alpha$, and obeys $u'<0$ for all $0<r<b(a)$ (see [14,15] and the appendix of [11]). Let

$$F(u)=\int_0^u f(s)\,\mathrm{d}s.$$

Let

$$\Phi(u)=\left(\frac{F(u)}{f(u)}\right)'-\frac{1}{m}+\frac{1}{n}, \tag{1.3}$$

and

$$P(r,u(r),u'(r)) = r^n\left[\frac{m-1}{m}|u'|^m+F(u)\right]+nr^{n-1}u'|u'|^{m-2}\frac{F(u)}{f(u)}. \tag{1.4}$$

Then $\Phi(u)$ and $P(r)$ satisfy an identity due to Erbe and Tang [8]; in a refined form provided by Pucci and Serrin [17], it reads

$$P(r)=\int_0^r n\tau^{n-1}|u'(\tau)|^m\Phi(u(\tau))\,\mathrm{d}\tau. \tag{1.5}$$

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This identity has been shown to be very useful in studying the uniqueness of radial solutions for the case $n > m$, which plays a crucial role in proving the uniqueness theorems for (1.1) in [8], and for (1.1′) in [17,19]. In the case $n = m$, the identity seems to be more powerful since the function $\Phi(u)$ in (1.3) reduces to the derivative of $F(u)/f(u)$.

To complete the proof of theorem 1.1, we assume for contradiction that (1.1) admits two distinct solutions $u_1(r)$ and $u_2(r)$, with the respective inverses $r_1(u)$ and $r_2(u)$. By an elementary argument, we shall show that under hypotheses (H1) and (H2) the two solutions must intersect in the interior of $B$, that is, $u_1(r)-u_2(r)$ vanishes at some point in $(0, b)$, where $b$ is the radius of $B$. Since $u_1 - u_2$ is also zero at $r = b$, $r_1(u) - r_2(u)$ has a critical point, at which we associate a quantity involving (1.4). By using (1.4) directly, we find the quantity negative, while using the identity (1.5) and condition (H3), we find it positive. This gives a contradiction.

We shall use theorem 1.1 to study the polynomial case in § 4 and the exponential case in § 5, where theorem 1.2 and some other results are proved. Though the verification of (H1)–(H3) is not difficult, it is, however, non-trivial.

## 2. Superlinearity and sublinearity

If a function $f(u)$ satisfies (H1) and (H2), then we call it a *superlinear function*; if $f(u)$ satisfies (H1) and $f(u)/u^{m-1}$ is a strictly decreasing function over $(0,\infty)$, then we call it a *sublinear function*.

PROPOSITION 2.1. *Let $f(u)$ be a superlinear function over $(0,\infty)$; that is, conditions (H1) and (H2) hold. If $u_1$ and $u_2$ are two solutions of (1.2) such that*

$$0 < u_1(r) < u_2(r) \quad \text{on } [0, r_0)$$

*for some $r_0 > 0$, then $u_1/u_2$ is strictly increasing on this interval.*

*Proof.* Let $\alpha_i = u_i(0)$, $i = 1, 2$, then $0 < \alpha_1 < \alpha_2$. Write $w_i(r) = |u_i'(r)|^{m-1}$. By (1.2), we find that

$$w_i' + \frac{n-1}{r} w_i - f(u_i) = 0.$$

Letting $r \downarrow 0$ in this identity and using L'Hospital's rule yields $w_i'(0) = f(\alpha_i)/n$. By L'Hospital's rule again, we have

$$\lim_{r\downarrow 0} \frac{w_1(r)}{w_2(r)} = \frac{w_1'(0)}{w_2'(0)} = \frac{f(\alpha_1)}{f(\alpha_2)} < \frac{\alpha_1^{m-1}}{\alpha_2^{m-1}},$$

where the last inequality holds because of (H2). Therefore,

$$\lim_{r\downarrow 0} \frac{u_2}{u_2'}\left(\frac{u_1}{u_2}\right)' = \lim_{r\downarrow 0}\left(\frac{u_1'}{u_2'} - \frac{u_1}{u_2}\right) = \left(\frac{f(\alpha_1)}{f(\alpha_2)}\right)^{1/(m-1)} - \frac{\alpha_1}{\alpha_2} < 0,$$

and so $(u_1/u_2)' > 0$ for $r$ sufficiently small.

Now, if the assertion of this proposition were not true, then there would be some $r_c \in (0, r_0)$ at which $(u_1/u_2)' = 0$ and $(u_1/u_2)'' \leqslant 0$; thus $u_1'u_2 = u_1u_2'$, together

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with (1.2), condition (H2) and the fact that $u_1(r_c) < u_2(r_c)$, yielding

$$\begin{aligned}\left(\frac{u_1}{u_2}\right)'' &= \frac{u_1''u_2 - u_1u_2''}{u_2^2} \\ &= \frac{1}{(m-1)u_2^2}(-f(u_1)u_2|u_1'|^{2-m} + f(u_2)u_1|u_2'|^{2-m}) \\ &= \frac{1}{(m-1)u_2} f(u_2)|u_1'|^{2-m}((u_1/u_2)^{m-1} - f(u_1)/f(u_2)) > 0.\end{aligned}$$

It gives an obvious contradiction, and so there must hold that $(u_1/u_2)' > 0$ on $(0, r_0)$. □

Using exactly the same proof we obtain the following.

PROPOSITION 2.1′. Let $f(u)$ be a sublinear function over $(0, \infty)$; that is, condition (H1) holds and $f(u)/u^{m-1}$ is a strictly decreasing function over $(0, \infty)$. If $u_1$ and $u_2$ are two solutions of (1.2) such that

$$0 < u_1(r) < u_2(r) \quad \text{on } [0, r_0)$$

for some $r_0 > 0$, then $u_1/u_2$ is strictly decreasing on this interval.

PROPOSITION 2.2. *Let $f(u)$ be a superlinear function over $(0, \infty)$. If $u_1$ and $u_2$ are two radial solutions of (1.1) with $u_1 \leqslant u_2$ over $B$, then $u_1 \equiv u_2$.*

*Proof.* Let $u_1$ and $u_2$ be two radial solutions of (1.1) with $u_1 \leqslant u_2$ over $B$, and $u_i(0) = \alpha_i > 0$, $i = 1, 2$. If $\alpha_1 = \alpha_2$, then $u_1 \equiv u_2$ and we are done. Hence we may assume that $\alpha_1 < \alpha_2$. Let $b$ be the radius of $B$, then

$$u_1(r) < u_2(r) \quad \text{for } 0 \leqslant r < b, \quad u_1(b) = u_2(b) = 0.$$

Applying proposition 2.1 and L'Hospital's rule, we obtain

$$\frac{u_1(r)}{u_2(r)} < \lim_{r \uparrow b} \frac{u_1(r)}{u_2(r)} = \frac{u_1'(b)}{u_2'(b)}, \quad 0 \leqslant r < b. \tag{2.1}$$

Now, by (1.2) and the fact that $u_i' < 0$ for $0 < r < b$, we have

$$(r^{n-1}|u_i'|^{m-1})' = r^{n-1}f(u_i); \quad \text{thus } b^{n-1}|u_i'(b)|^{m-1} = \int_0^b r^{n-1}f(u_i(r))\,\mathrm{d}r, \tag{2.2}$$

together with (2.1) and (H2), leading to

$$\begin{aligned} 0 &= \int_0^b r^{n-1}[f(u_1(r))|u_2'(b)|^{m-1} - f(u_2(r))|u_1'(b)|^{m-1}]\,\mathrm{d}r \\ &= \int_0^b r^{n-1}f(u_2(r))|u_2'(b)|^{m-1}\left(\frac{f(u_1(r))}{f(u_2(r))} - \left(\frac{u_1'(b)}{u_2'(b)}\right)^{m-1}\right)\mathrm{d}r \\ &< \int_0^b r^{n-1}f(u_2(r))|u_2'(b)|^{m-1}\left(\frac{f(u_1(r))}{f(u_2(r))} - \left(\frac{u_1(r)}{u_2(r)}\right)^{m-1}\right)\mathrm{d}r < 0.\end{aligned}$$

This is absurd and the proof is complete. □

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Using proposition 2.1′ and the proof of proposition 2.2, we can give a simple proof of the following well-known result.

PROPOSITION 2.2′. Let $f(u)$ be a sublinear function over $(0,\infty)$. Then (1.1) has at most one radial solution.

*Proof.* Let $u_1$ and $u_2$ be two solutions of (1.2) with $u_1(0)=\alpha_1$, $u_2(0)=\alpha_2$, $0<\alpha_1<\alpha_2$. Let $b(\alpha_i)$ (possibly infinity) be the unique point where $u_i$ vanishes. By proposition 2.1′, it is clear that $0<u_1<u_2$ on $(0,b(\alpha_1))$, since otherwise there would be some $\xi\in(0,b(\alpha_1))$ such that $u_1<u_2$ for $0\leqslant r<\xi$ and $u_1(\xi)=u_2(\xi)$; thus $u_1(\xi)/u_2(\xi)=1>u_1(0)/u_2(0)$, contradicting proposition 2.1′. Now, if $b(\alpha_1)<\infty$, then, by the argument of the proof of proposition 2.2, we have $u_2(b(\alpha_1))>0$ and so $b(\alpha_1)<b(\alpha_2)$. Hence, for any fixed ball $B$, there can be at most one $\alpha$ such that $b(\alpha)$ equals the radius of $B$. Thus (1.1) admits at most one radial solution. □

REMARK 2.3. For the special case $m=2$, proposition 2.2 can be proved by a much simpler way. Let $u_1$ and $u_2$ be two solutions of (1.1) satisfying $u_1\leqslant u_2$ over a ball of radius $b$. Using (2.2), one gets

$$(r^{n-1}|u_1'(r)|)'u_2(r)-(r^{n-1}|u_2'(r)|)'u_1(r)=r^{n-1}[u_2f(u_1)-u_1f(u_2)],\quad 0<r<b.$$

Integrating the identity over $(0,b)$ and using integration by parts, we have

$$[r^{n-1}(|u_1'|u_2-|u_2'|u_1)]|_0^b=\int_0^b r^{n-1}(u_2f(u_1)-u_1f(u_2))\,\mathrm{d}r,$$

for which the left-hand side is clearly zero by the boundary conditions of $u_1$ and $u_2$, and the right-hand side is not zero by monotonicity of $f(u)/u$. This gives a contradiction.

The same method, however, does not work for the case $m>2$; though, by a similar argument, one may handle the case $1<m<2$.

REMARK 2.4. Another proof of proposition 2.2 was given in [8] using the first eigenvalue characterization of $m$-Laplace operator. The proof we give here is more interesting. It uses only an elementary analysis, and gives some important information on the behaviour of radial solutions.

## 3. Proof of theorem 1.1

Let $u(r,\alpha)$ be a solution of (1.2) defined on the maximal interval $[0,b(a))$ on which $u>0$. If $b(\alpha)<\infty$, then $u(b(\alpha))=0$ and $u'(b(\alpha))<0$, in which case we call $u$ a *crossing solution*; otherwise $u$ is positive on $(0,\infty)$ and we call it an *entire solution*. We first prove the following simple result.

PROPOSITION 3.1. *Let (H1) hold. If $n\leqslant m$, then $u(r,\alpha)$ is a crossing solution for each $\alpha>0$. Consequently, equation (1.1) admits no entire solution.*

*Proof.* Let $n\leqslant m$. Suppose for contradiction that $u$ is an entire solution. Then $u>0$ and $u'<0$ over $(0,\infty)$. By (1.2),

$$\begin{aligned}\frac{m-1}{r}(ru')'&=(m-1)u''+\frac{m-1}{r}u'\leqslant(m-1)u''+\frac{n-1}{r}u'\\&=-f(u)|u'|^{2-m}<0,\quad r>0,\end{aligned}$$

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showing that $ru'$ is decreasing on $(0, \infty)$ and so it approaches a limit as $r \to \infty$. Indeed, the limit must be zero since $u$ approaches a finite number as $r \to \infty$. Thus $ru' > 0$ for all $r > 0$, giving a contradiction. □

The assumption here is *nearly necessary*. When $n > m$ and the nonlinearity obeys $f(u) = u^p$, $p > ((m-1)n+m)/(n-m)$, the Sobolev critical exponent, *all* solutions of (1.2) are entire solutions. When $n < m$ and $f(u)$ is negative for small $u > 0$ and positive for large $u$, problem (1.2) *always* has entire solutions which approach zero as $r \to \infty$ (see [12]). For the case $n = m$, this proposition was first observed by Ni and Serrin [15].

Since $u'(r, \alpha) < 0$ as long as $u > 0$, the inverse of $u(r, \alpha)$, denoted by $r = r(u, \alpha)$, is well defined and is strictly decreasing on $(0, \alpha)$. We have

$$u_r = \frac{1}{r_u}, \qquad u_{rr} = -\frac{r_{uu}}{r_u^3},$$

hence $r = r(u, \alpha)$ satisfies the equation

$$(m-1)r'' = \frac{n-1}{r} r'^2 + f(u)|r'|^m r', \tag{3.1}$$

where the prime indicates differentiation with respect to $u$.

By a change of variable, we can write (1.4) as

$$P(u, r(u), r'(u)) = r^n(u)\left[\frac{m-1}{m|r'(u)|^m} + F(u)\right] + nQ(u)\frac{F(u)}{f(u)}, \tag{3.2}$$

where

$$Q(u) = \frac{r^{n-1}(u)}{r'(u)|r'(u)|^{m-2}}.$$

Now the identity (1.5) takes the form

$$P(u) = \int_\alpha^u n\Phi(s)Q(s)\,\mathrm{d}s. \tag{3.3}$$

For a pair of numbers $\alpha_1 > 0$, $\alpha_2 > 0$, we shall denote by $r_i(u)$ the inverse functions of $u(r, \alpha_i)$ and by $Q_i(u)$ and $P_i(u)$ the respective functions $Q(u)$ and $P(u)$ with $r(u)$ replaced by $r_i(u)$, $i = 1, 2$.

Lemma 3.2 (see [8, 17]). *Let (H1) hold. For $0 < u < \min\{\alpha_1, \alpha_2\}$, put*

$$S(u) = Q_1(u)/Q_2(u).$$

*Then $S'(u) > 0$ if and only if $r_1'(u) > r_2'(u)$.*

*Proof.* By (3.1), we obtain $Q'(u) = -r^{n-1}(u)r'(u)f(u)$. Using the quotient rule and noticing that $r_i'(u) < 0$ for $0 < u < \min\{\alpha_1, \alpha_2\}$, there holds

$$S'(u) = \left(\frac{r_1}{r_2}\right)^{n-1}\left(\frac{r_2'}{r_1'}\right)^{m-1} f(u)(|r_2'|^m - |r_1'|^m),$$

yielding the desired result since $f(u) > 0$ for $u > 0$. □

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The proof of theorem 1.1 we give below follows the approach of Erbe and Tang [8]. As in the original work of Peletier and Serrin [16], the intersection behaviour of a pair of solutions is investigated by working on their inverses $r_1(u)$ and $r_1(u)$, rather than the solutions themselves. The key ingredient is to evaluate

$$\Psi(u) = P_1(u)Q_2(u) - P_2(u)Q_1(u) \tag{3.4}$$

at any possible critical point of $r_1 - r_2$. The main purpose is to show that the evaluation leads to two different results for the same quantity by two different ways, one using (3.2) and the other using (3.3).

*Proof of theorem 1.1.* The non-existence of radial entire solutions are shown in proposition 3.1. We now prove the uniqueness part.

Suppose for contradiction that $u_1$ and $u_2$ are two distinct radial solutions of (1.1) on a ball $B$ of radius $b$. Then $u_1(b) = u_2(b) = 0$. By proposition 2.2, there also exists an $\xi \in (0, b)$ such that $u_1(\xi) = u_2(\xi)$. Let $u_i(0) = \alpha_i$ with $\alpha_1 < \alpha_2$. Let $r_i$ be the inverse of $u_i$, $i = 1, 2$. Then $r_1 - r_2$ is well defined on $[0, \alpha_1]$, and vanishes at $u = 0$ and also at some point in $(0, \alpha_1)$. Hence $r_1 - r_2$ must have a critical point in $(0, \alpha_1)$. Since $r_1'(u) \to -\infty$ as $u \uparrow \alpha_1$, we can fix a critical number $u_c \in (0, \alpha_1)$ such that

$$r_1'(u_c) = r_2'(u_c) < 0, \tag{3.5 a}$$

$$r_1'(u) < r_2'(u), \quad \text{for } u_c < u < \alpha_1. \tag{3.5 b}$$

Consequently, $r_1''(u) - r_2''(u) \leqslant 0$. By (3.1), we then find that

$$0 \geqslant (m-1)(r_1''(u_c) - r_2''(u_c)) = (n-1){r'}_1^2(u_c)\left(\frac{1}{r_1(u_c)} - \frac{1}{r_2(u_c)}\right)$$

and so $r_1(u_c) \geqslant r_2(u_c)$. Observe that $r_1(u_c) = r_2(u_c)$ is indeed impossible, since otherwise $r_1$ and $r_2$ would be identical by (3.5 *a*). Hence $r_1(u_c) > r_2(u_c)$. By (3.2) and (3.4), we then have

$$\begin{aligned} \Psi(u_c) &= (r_1^n(u_c)Q_2(u_c) - r_2^n(u_c)Q_1(u_c))\left[\frac{m-1}{m|r_1'(u_c)|^m} + F(u_c)\right] \\ &= -(r_1 - r_2)\cdot\frac{(r_1r_2)^{n-1}}{|r_1'|^{m-1}}\left[\frac{m-1}{m|r_1'|^m} + F(u_c)\right] < 0. \end{aligned} \tag{3.6}$$

However, by (3.5 *b*) and lemma 3.2,

$$S(u) < S(u_c) \quad \text{for } u_c < u < \alpha_1.$$

Combining this with (3.3), (3.4), we obtain

$$\begin{aligned} \Psi(u_c) &= \int_{\alpha_1}^{u_c} n\Phi(u)[Q_1(u)Q_2(u_c) - Q_2(u)Q_1(u_c)]\,\mathrm{d}u + \int_{\alpha_1}^{\alpha_2} n\Phi(u)Q_2(u)Q_1(u_c)\,\mathrm{d}u \\ &\geqslant \int_{\alpha_1}^{u_c} n\Phi(u)[Q_1(u)Q_2(u_c) - Q_2(u)Q_1(u_c)]\,\mathrm{d}u \\ &= \int_{\alpha_1}^{u_c} n\Phi(u)Q_2(u)Q_2(u_c)[S(u) - S(u_c)]\,\mathrm{d}u \geqslant 0, \end{aligned}$$

where the first inequality holds since $Q_i(s) < 0$ on $(0, \alpha_i)$, and $\Phi(u) \geqslant 0$ follows from (1.3), (H3) and the fundamental assumption $n \leqslant m$. The second inequality follows from (3.5) and lemma 3.2. This gives a contradiction to (3.6). The proof is completed. □

## 4. Polynomial case

Clearly, condition (H3) is valid if

$$f^2(u) > F(u)f'(u) \quad \text{for } u > 0. \tag{4.1}$$

Usually, it is not easy to verify (4.1) directly due to the technical difficulty in evaluating $F(u)$. To avoid this difficulty, we may use a stronger condition,

$$f'^2(u) > f(u)f''(u) \quad \text{for } u > 0. \tag{4.2}$$

LEMMA 4.1. *Let (H1) and (H2) hold. Then (4.2) implies (H3).*

*Proof.* By (H1) and (H2), there holds $uf'(u) \geqslant (m-1)f(u) > 0$. Write

$$h(u) = f^2(u)/f'(u) - F(u).$$

If (4.2) holds, then

$$\liminf_{u \downarrow 0} h(u) \geqslant 0 \quad \text{and} \quad h'(u) = f(f'^2 - ff'')/f'^2 > 0,$$

implying that $h(u) > 0$ for $u > 0$. Thus (4.1), and hence (H3), is valid. □

It is an elementary fact that a polynomial

$$a_0 + a_1 u + a_2 u^2 + \cdots + a_l u^l, \quad l > 0, \quad a_l \neq 0,$$

can be factored into linear factors and irreducible quadratic factors. A quadratic function is irreducible if it cannot be factored over the real numbers. Motivated by this observation, we shall consider a function that can be expressed in a form

$$f(u) = u^{p_0}(u + a_1)^{p_1} \cdots (u + a_k)^{p_k}(u^2 + b_1 u + c_1)^{q_1} \cdots (u^2 + b_s u + c_s)^{q_s}, \tag{4.3}$$

where each quadratic form is irreducible, and

$$p_0 \geqslant m - 1, \quad a_i, p_i > 0, \quad i = 1, 2, \dots, k, \quad b_j \geqslant 0, \quad c_j, q_j > 0, \quad j = 1, 2, \dots, s. \tag{4.4}$$

Note that here $p_i$ and $q_j$ are not necessarily integers.

THEOREM 4.2. *Let $n \leqslant m$. If $f(u)$ is given by (4.3), (4.4) and*

$$b_j^2 \geqslant 2c_j, \quad j = 1, 2, \dots, s, \tag{4.5}$$

*then (1.1) has at most one radial solution.*

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*Proof.* Clearly, condition (H1) is satisfied. Using the product rule we find that

$$f'(u) = f(u)\bigg(\frac{p_0}{u} + \frac{p_1}{u+a_1} + \cdots + \frac{p_k}{u+a_k} + \frac{q_1(2u+b_1)}{u^2+b_1u+c_1} + \cdots + \frac{q_s(2u+b_s)}{u^2+b_su+c_s}\bigg).$$

Hence

$$uf'(u) \geqslant p_0 f(u) \geqslant (m-1)uf(u),$$

where the first strict inequality holds if $k > 0$ or $s > 0$; the second strict one holds if $p_0 > m-1$. Thus (H2) is valid except when

$$k = s = 0, \qquad p_0 = m-1,$$

for which $f(u) \equiv u^{m-1}$ and so (1.1) reduces to a 'linear' problem and the uniqueness holds trivially. Finally, by (4.5), we find that

$$\frac{f'^2 - ff''}{f^2} = -(\frac{f'}{f})' = \frac{p_0}{u^2} + \frac{p_1}{(u+a_1)^2} + \cdots + \frac{q_s(2u^2+2b_su+b_s^2-2c_s)}{(u^2+b_su+c_s)^2} > 0.$$

Thus (4.2), and hence (H3), holds. The uniqueness now follows from theorem 1.1. □

If $s = 0$, then $f(u)$ has no irreducible quadratic factors and (4.5) holds vacuously.

Corollary 4.3. *Let $n \leqslant m$. If*

$$f(u) = u^{p_0}(u+a_1)^{p_1} \cdots (u+a_k)^{p_k}, \quad p_0 \geqslant m-1, \quad a_i, p_i > 0, \quad i = 1,2,\dots,k,$$

*then (1.1) has at most one radial solution.*

In fact, if $f(u) > 0$ for all $u > 0$ and is factored into the form (4.3), then, necessarily, $a_i \geqslant 0$, $i = 1,2,\dots,k$, and $c_j \geqslant 0$, $j = 1,2,\dots,s$. However, it is possible that $b_j < 0$. When this happens, the verification of (4.2) may be very complicated. Consider, for example,

$$f_1(u) = u + u^5, \qquad f_2(u) = u^2 + u^6, \qquad f_3(u) = u^3 + u^7.$$

These can be factored as

$$f_i(u) = u^i(u^2 + \sqrt{2}u + 1)(u^2 - \sqrt{2}u + 1), \quad i = 1,2,3,$$

where the second quadratic factor has one negative coefficient. By a simple calculation we find that $f_1$ satisfies neither (4.1) nor (4.2); $f_2$ satisfies the weaker condition (4.1) but not (4.2); and $f_3$ satisfies both (4.1) and (4.2).

More generally, consider

$$f(u) = u^p + u^q, \quad m-1 \leqslant p < q. \tag{4.6}$$

Of course, if $p$ and $q$ are not integers, then this function may not have a factored expression like (4.3). Indeed, conditions (H1) and (H2) are fulfilled and the verification of (H3) can be done directly. Applying theorem 1.1, we have the following.

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Theorem 4.4. *Let $n \leqslant m$. Let $f(u)$ be given by (4.6). If either*

$$p+q+2 \geqslant (p-q)^2 \quad or \quad [p+q+2-(p-q)^2]^2 \leqslant 4(p+1)(q+1), \tag{4.7}$$

*then (1.1) has at most one radial solution.*

*Proof.* Since

$$f^2 - Ff' = \frac{1}{p+1}u^{2p} + \frac{p+q+2-(p-q)^2}{(p+1)(q+1)}u^{p+q} + \frac{1}{q+1}u^2,$$

condition (H3) is valid when either condition of (4.7) holds. The uniqueness now follows from theorem 1.1. □

Our method provides some partial results to the uniqueness of (1.1) for polynomials with positive coefficients and the lowest-order term having a degree not less than $m-1$. The assumption on $b_j$ in theorem 4.2 is very restrictive. Even for the simple model (4.6), our results are far from complete. Certainly, the hypothesis (4.7) can be weakened. As shown by Adimurthi and Yadava [3], the assertion of theorem 4.4 remains valid when

$$f(u) = u^{m-1} + u^q, \quad m-1 < q < \infty, \quad n = m,$$

a case not completely included in our result. It remains an open problem if the assumption (4.7) can be completely removed in theorem 4.4.

For the case $n > m$, some similar problems have been studied extensively; here we mention only briefly the work on a simple model

$$\left.\begin{array}{c} \Delta u + \lambda u^p + u^q = 0 \quad \text{in } B_1, \quad 1 < p < q, \\ u > 0 \quad \text{in } B_1, \quad u = 0 \quad \text{on } \partial B_1, \end{array}\right\} \tag{4.8}$$

where $B_1$ is the unit ball in $\mathbb{R}^n$, $n \geqslant 3$. As observed by Brezis and Nirenberg [6], and later proved by Atkinson and Peletier [5], when

$$n = 3, \quad 1 < p < 3, \quad q = 5,$$

problem (4.8) has *at least two* solutions for some $\lambda > 0$. Erbe and Tang [8] showed that if the dimension $n \geqslant 6$ and

$$1 \leqslant p < q \leqslant (n+2)/(n-2), \tag{4.9}$$

then (4.8) admits *exactly* one solution for *any* $\lambda > 0$. Hence, for a subcritical nonlinearity, uniqueness for the Dirichlet problem (4.8) holds in higher dimensions $n \geqslant 6$ but not in the lower dimension $n = 3$. For the cases $n = 4, 5$, it is unknown if there can be some pair of numbers $(p, q)$ in the range of (4.9) such that (4.8) admits more than one solution.

If the function involves both supercritical and subcritical growth, then the situation becomes more complicated. Budd and Norbury [7] showed that (4.8) can admit *infinitely many* solutions at some critical value $\lambda > 0$ if $p = 1$, $q > (n+2)/(n-2)$ and $3 \leqslant n \leqslant 9$.

## 5. Exponential case

Our theorem can also be applied to

$$f(u) = g(u)\exp(\lambda u^{\beta}), \quad \lambda \geqslant 0, \quad \beta \geqslant 0. \tag{5.1}$$

THEOREM 5.1. *Let $n \leqslant m$. Let $f(u)$ be given by (5.1). Then (1.1) has at most one radial solution if the following hold.*

(i) $\beta \leqslant 1$.

(ii) *$g(u)$ satisfies the assumptions (H1), (H2), (4.2).*

*Proof.* Clearly, if $g$ satisfies (H1), then $f$ does. Since

$$f'(u) = g'(u)\exp(\lambda u^{\beta}) + \lambda\beta g(u)u^{\beta-1}\exp(\lambda u^{\beta}),$$

$\lambda \geqslant 0$, $\beta \geqslant 0$ and $g$ satisfies (H1) and (H2), we have

$$uf'(u) \geqslant ug'(u)\exp(\lambda u^{\beta}) > (m-1)g(u)\exp(\lambda u^{\beta}) = (m-1)f(u),$$

so $f$ satisfies (H2). Moreover, if $g(u)$ satisfies (4.2) and $\beta \leqslant 1$, then

$$\begin{aligned}(f'^2 - ff'')/f^2 &= -(f'/f)' \\ &= -(g'/g + \lambda\beta u^{\beta-1})' \\ &= -(g'/g)' - \lambda\beta(\beta-1)u^{\beta-2} \\ &\geqslant -(g'/g)' \\ &= ({g'}^2 - gg'')/g^2 \geqslant 0.\end{aligned}$$

Thus $f$ satisfies (4.2) and also (H3). Now the result follows from theorem 1.1. □

This theorem simply says that if $f$ is given by (5.1) and $\beta \leqslant 1$, then the verification of conditions of theorem 1.1 for $f(u)$ reduces to that for $g(u)$. By induction, we have the following.

COROLLARY 5.2. *Let $n \leqslant m$. Let*

$$f(u) = g(u)\exp(\lambda_1 u^{\beta_1} + \lambda_2 u^{\beta_2} + \cdots + \lambda_l u^{\beta_l}),$$
$$\lambda_i \geqslant 0, \quad 0 \leqslant \beta_i \leqslant 1, \quad i = 1, 2, \dots, l.$$

*If $g(u)$ satisfies (H1), (H2) and (4.2), then (1.1) has at most one radial solution.*

Using this corollary, together with results of § 4, we then obtain the following result.

THEOREM 5.3. *Let $n \leqslant m$. If $f(u)$ is equal to one of the following functions*

$$u^{p_0}(u+a_1)^{p_1}\cdots(u+a_k)^{p_k}(u^2+b_1u+c_1)^{q_1}$$
$$\cdots(u^2+b_su+c_s)^{q_s}\exp(\lambda_1 u^{\beta_1} + \cdots \lambda_l u^{\beta_l}),$$
$$u^{p_0}(u+a_1)^{p_1}\cdots(u+a_k)^{p_k}\exp(\lambda_1 u^{\beta_1} + \lambda_2 u^{\beta_2} + \cdots + \lambda_l u^{\beta_l}),$$
$$(u^p + u^q)\exp(\lambda_1 u^{\beta_1} + \lambda_2 u^{\beta_2} + \cdots + \lambda_l u^{\beta_l}), \quad m-1 \leqslant p < q,$$

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*where* $\lambda_i \geqslant 0$, $0 \leqslant \beta_i \leqslant 1$ *and all other constants satisfy conditions (4.4), (4.5) and (4.7), then (1.1) has at most one radial solution.*

We conjecture that the assertion of this theorem continues to hold under a weaker condition on $\beta_i$,

$$0 \leqslant \beta_i < n/(n-1), \quad i = 1, 2, \dots, l.$$

For the case $n = m = 2$, an interesting non-uniqueness result of (1.1) with a function $f(u)$ which behaves like $u^{-p}\exp(u^2)$, $p > 0$, for large $u$ was given by De Figueiredo and Ruf [9]. Some other important results on the existence and non-existence of radial solutions of (1.1) in the critical case ($\beta = 2$ in (5.1)) or supercritical case ($\beta > 2$ in (5.1)) can be found in [1,4,10,13].

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