MAL'TSEV CONDITIONS AND SPECTRA

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Let J be a cofinite set of positive integers which contains 1. In (1973) I proved that the following condition on a variety (equational class) $\mathscr{V}$ is Mal'tsev-definable: if $\mathfrak{A} \in \mathscr{V}$ and $\mathfrak{A}$ is finite, then $|\mathfrak{A}| \in J$. This article contains some subsidiary results, concerned mainly with a more detailed description of these Mal'tsev conditions. Many of our results arose upon considering a recent article of W. D. Neumann (1978).

To restate the above result in more detail, let us first recall that the *spectrum* of a variety $\mathscr{V}$ is the set of finite cardinalities of algebras in $\mathscr{V}$. Then that theorem asserts the existence, for each J, of a sequence $\varphi_1, \varphi_2, \varphi_3, \ldots$ of finite conjunctions of identities, such that Spec $\mathscr{V} \subseteq J$ if and only if some φ_i holds in $\mathscr{V}$ (after suitably interpreting the operation symbols appearing in φ_i as $\mathscr{V}$-terms). (These φ_i are loosely known as Mal'tsev 'conditions'. It is usually also required that φ_{i+1} should follow from φ_i after suitably interpreting the operation symbols of φ_{i+1} as φ_i-terms; in our case this is possible if and only if J is closed under multiplication, which we will assume to be true.) Here we are concerned with the questions: *Can the sequence $\varphi_1, \varphi_2, \ldots$ be taken finite? How many variables are needed for the identities appearing in this sequence? How easy is it to actually write this sequence?*

Our first theorem simply states that the sequence cannot be taken finite.

THEOREM 1. '*Spec $\mathscr{V} \subseteq J$' is not a strong Mal'tsev condition.*

This result is an immediate corollary of stronger results to be proved in Section 1.

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It contradicts a result 'proved' in Neumann (1978). In Section 1 we will also comment on some related false results of that article. In Section 2 we will address the second question, with this result:

THEOREM 2. (*p prime.*) *If* $J = \omega - \{p\}$, *then p variables are required for the Mal'tsev condition for* Spec $\mathscr{V} \subseteq J$.

Our third question has no precise answer, since 'easy' has no mathematical definition. But in Section 3 we will provide simple algorithms to show that writing $\varphi_1, \varphi_2, \ldots$ is just as easy as, and probably no easier than, writing a Horn sentence with spectrum J. This seems as close as one can come to answering Problem 4.2 of Neumann (1978), who asked for 'nice, explicit' conditions.

THEOREM 3. *There exist* $2^{\aleph_0}$ *varieties of varieties.* (See Section 4 for definitions.)

The results of this article were announced in W. Taylor (1978) and (1979).

1. Proof of Theorem 1

The reader should have a modest familiarity with general algebra and lattice theory. We will define all notions special to the theory of Mal'tsev conditions, but even so this article will be easier to read if one has some familiarity with one of the three background papers, Taylor (1973), Neumann (1974) or Baldwin and Berman (1977), or especially its immediate predecessor, Neumann (1978).

For varieties $\mathscr{V}_1, \mathscr{V}_2, \ldots$ (not necessarily of the same similarity type), we define their product $\prod \mathscr{V}_i$ to be the variety generated by all algebras of the form $(\prod A_i, F_t)_{t \in T}$, where each A_i is the underlying set of an algebra $\mathfrak{A}_i \in \mathscr{V}_i$, and the operations F_t are defined as follows. For every n and every sequence

$$p_i(x_1, \ldots, x_n) \ (i = 1, 2, \ldots)$$

with each p_i a derived operation of $\mathscr{V}_i$, we define $F: (\prod A_i)^n \rightarrow (\prod A_i)$ via

$$F(a_1, \ldots, a_n) = \langle p_1(a_1^1, \ldots, a_n^1), p_2(a_1^2, \ldots, a_n^2), \ldots \rangle;$$

these are all the operation of $\prod \mathscr{V}_i$. A basic result in the theory of Mal'tsev conditions (*op. cit.*) is that *a Mal'tsev condition is strong if and only if it is preserved under the formation of products of varieties.* And so Theorem 1 is an immediate corollary of the main result of this section:

PROPOSITION 1.1. *There exist varieties* $\mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3 \ldots$ *such that each* $\mathscr{V}_i$ *contains no non-trivial finite algebras, but* $\prod \mathscr{V}_i$ *has a subvariety equivalent to the variety of*

bounded distributive lattices. (In particular, $\prod \mathscr{V}_i$ contains algebras of every finite power.)

PROOF. Let L_i be the lattice
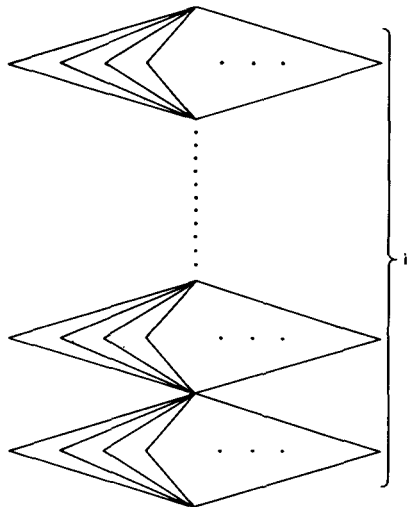
where each 'diamond' has a top and a bottom, $\aleph_0$ atoms, and nothing else: there are i diamonds mounted vertically. Define

$$\mathscr{V}_i = \text{HSP}\{(L_i, \wedge, \vee, \text{all constants})\}.$$

From the fact that each diamond is a simple lattice (well known and easy to prove), we see that any lattice in $\mathscr{V}_i$ with $0 \neq 1$ (that is, any non-trivial lattice in $\mathscr{V}_i$) must have infinitely many constants appearing in it as distinct elements. Hence $\mathscr{V}_i$ contains no non-trivial finite algebras. Now we check that $\prod \mathscr{V}_i$ has the desired properties.

Fix a non-principal ultrafilter $\mathscr{F}$ on ω; we will say that a property $p(i)$ of integers holds *almost everywhere* (a.e.) if and only if $\{i: p(i) \text{ is true}\} \in \mathscr{F}$. We will use $\mathscr{F}$ to define a congruence relation on $\prod L_i \in \prod \mathscr{V}_i$. First define

$$\varphi : \prod L_i \twoheadrightarrow \{0, 1\}$$

via

$$\varphi(a_1, a_2, \ldots) = \begin{cases} 0, & \text{if for some } n, \text{ height } (a_i) = n \text{ a.e.,} \\ 1, & \text{otherwise.} \end{cases}$$

We wish to see that $\ker \varphi$ is a congruence relation on L, that is, that $\varphi(a) = \varphi(b)$ implies $\varphi(\alpha(a)) = \varphi(\alpha(b))$ for every unary algebraic function $\alpha(x)$. Without loss of generality we may assume that $\varphi(a) = \varphi(b) = 0$ and $\varphi(\alpha(a)) = 1$; from which we will deduce $\varphi(\alpha(b)) = 1$.

By definition, $\alpha(a) \doteq \langle \alpha_i(a_i) \rangle$ for some unary algebraic functions α_i on L_i. Our assumptions tell us that a.e. $\alpha_i(a_i) > z > a_i$; in particular, a.e.

$$\alpha_i(a_i) \geqslant z > a_i \quad \text{with } z \vee\text{-prime}$$

This last condition tells us (by an easy induction on the complexity of $\alpha_i(x)$) that in fact, a.e., $\alpha_i(a_i)$ is the least possible value of $\alpha_i(x)$, hence, a.e.,

$$\alpha_i(b_i) \geqslant \alpha_i(a_i).$$

And so $\varphi(\alpha(b)) = 1$.

And so $\prod \mathscr{V}_i$-operations may be defined on $\{0, 1\}$ so that φ becomes a homomorphism. It is easy to see from the definitions that each of these operations will be monotonic, and hence must be a polynomial in $0, 1, \wedge, \vee$ (for example, by G. Birkhoff (1967, Theorem 5, p. 61)). Thus this homomorphic image generates a variety equivalent to that of bounded distributive lattices.

This proposition provides a counterexample to many of the assertions appearing in Neumann (1978), most notably 4.1, which claims that 'Spec $\mathscr{V} \subseteq J$' is a strong Mal'tsev condition. It obviously also falsifies 3.1, which claims that Spec $(\prod \mathscr{V}_i)$ is the multiplicative closure of $\bigcup$ Spec $\mathscr{V}_i$ (a result which *is* true for finite products). The root of these troubles is Lemma 2.3, in whose proof that author has now found a mistake. The interested reader may consult the article and check that 6.1 and 7.1 are also false *via* this counterexample. (For 6.1, take $S = \{2\}$, $\mathscr{V}$ the variety of 'sets' (no operations) and $\mathscr{V}_i \subseteq \mathscr{V}$ the trivial variety defined by $x = y$.)

The *finite* product of varieties has a structure theory so strong that we can consider a finite product just as 'known' as its factors are (see, for example, Taylor (1973), p. 358 or (1975, p. 266). But clearly this is far from true for infinite products of varieties, even for their finite members. Calculations of infinite products (or merely some of their subvarieties, as we have done here) are rare in the literature; for a couple of examples see, for example, Taylor (1974), p. 348 and Bulman-Fleming and Taylor (1976), p. 191. Of course every proof that this or that Mal'tsev condition is not strong implicitly contains such a calculation; for references see Taylor (1973), p. 375. And so it would be interesting to see some more infinite products worked out in detail. We mention in this regard just one problem among many that occur naturally.

PROBLEM 1.2. Do there exist varieties $\mathscr{V}_i$ such that Spec $\mathscr{V}_i = \{1\}$ for each i, but $\prod \mathscr{V}_i$ contains a subvariety equivalent to 'sets' or 'pointed sets'? (That is, this subvariety has only constant operations.)

2. Proof of Theorem 2

Let A be a $(p+1)$-element set, and define $F: A^p \to A$ and $T: A^3 \to A$ as follows:

$$F(a_1, \ldots, a_p) = \begin{cases} \text{the remaining member of } A \text{ if } a_1, \ldots, a_p \text{ are all distinct} \\ a_1, \quad \text{otherwise.} \end{cases}$$

$$T(a, b, c) = \begin{cases} a & \text{if } a \neq b, \\ c & \text{if } a = b. \end{cases}$$

(That is, T is the *ternary discriminator*.) The *quasiprimal algebra* theory of Foster and Pixley (see Quackenbush (1979) for an exposition with references) tells us that the finite algebras in $\mathscr{V} = \mathbf{HSP}(A; F, T)$ are the products of subalgebras of $(A; F, T)$. Since p is prime, we easily check that $\mathscr{V}$ contains no p-element algebras. The proof will be complete when we show that there exists a p-element algebra in $\mathscr{V}^{p-1}$. (In general, $\mathscr{V}^n$ denotes the variety $\supseteq \mathscr{V}$ which is defined by all n-variable identities of $\mathscr{V}$—see Cohn (1963), p. 173 or Jónsson, McNulty and Quackenbush (1975).) For our algebra, we take $\mathfrak{B} = (B; F, T)$ where B is a p-element set, $F(b_1, \ldots, b_n) = b_1$ and T is again the ternary discriminator. To see that $\mathfrak{B} \in \mathscr{V}^{p-1}$, it is enough by Cohn (1963, loc. cit.) to show that every $(p-1)$-generated subalgebra of $\mathfrak{B}$ is in $\mathscr{V}$; and this is obvious.

The example given in this proof can doubtless be greatly refined. We will extend it only to show the existence of a single Mal'tsev condition requiring $\aleph_0$ variables. A similar example appears at the end of Taylor (1972), but that example is really a *generalized* Mal'tsev condition (in the sense of Taylor (1973), p. 387), namely, it requires the use of a distinguished operation. Recall that $\mathscr{V}^n$ is defined in the proof just above.

THEOREM 2.1. *There exists a Mal'tsev definable class K such that for each n there exists a variety $\mathscr{V} \in K$ with $\mathscr{V}^n \notin K$.*

PROOF. For each prime p, define $\mathfrak{A}_p$ to be the $(p+1)$-element algebra defined at the beginning of the above proof of Theorem 2, and define $\mathscr{V}_p$ to be $\mathbf{HSP}\,\mathfrak{A}_p$. And, finally, define $\mathscr{W}_p$ to be the product variety $\mathscr{V}_2 \times \mathscr{V}_3 \times \ldots \times \mathscr{V}_p$. (Defined in Section 1 above. In some of our publications (1973), (1974), (1975) we used '$\otimes$' for this kind of product, but we now feel it is reasonable to follow the example of Freyd (1966) and Neumann (1978) and reserve $\otimes$ for a different varietal product, which is closely related to tensor product of rings.)

$\mathscr{W}_p$ is defined by a single identity φ_p—see Baker (1977), McKenzie (1975), Padmanabhan and Quackenbush (1973) and Taylor (1979), 9.14. The sequence $(\varphi_2, \varphi_3, \varphi_5, \ldots)$ defines a Mal'tsev condition as described in our introduction; we

take K to be the class of varieties obeying this Mal'tsev condition; that is $\mathcal{V} \in K$ if and only if some φ_p holds in $\mathcal{V}$ (with the φ_p-operations interpreted as $\mathcal{V}$-terms).

Since the sequence of primes p is infinite, it will be enough to establish that $\mathcal{V}_p \in K$ but $\mathcal{V}_p^{p-1} \notin K$. The former fact is immediate from basic facts about product varieties. To see the latter, we assume that $\mathcal{V}_p^{p-1} \in K$ and derive a contradiction. First note that for some q, $\mathcal{V}_p^{p-1}$ satisfies φ_q; since φ_q entails $\varphi_{q'}$ for $q < q'$ (this is why we took finite products), we may assume that $q > p$. (As in the earlier proof of this section) the theory of quasiprimal algebras (op. cit.) tells us that $\mathcal{V}_2, \mathcal{V}_3, ..., \mathcal{V}_q$ have no q-element algebras, and the theory of finite products of varieties (Taylor (1973), p. 358, (1975), p. 266) tells us that $\mathcal{W}_q$ has no q-element model. It will be enough to show that $\mathcal{V}_p^{p-1}$ does have a q-element model, for this will contradict the fact that $\mathcal{V}_p^{p-1}$ satisfies φ_q, the defining identity for $\mathcal{W}_q$. But obviously $(\{1, ..., q\}, F, T) \in \mathcal{V}_p^{p-1}$ for $F(a_1, ..., a_p) = a_1$ and T the ternary discriminator.

3. Spectral Mal'tsev conditions and Horn formulas

Recently, Michael Morley has called to our attention both difficulties and successes associated with the problem of writing a Horn sentence φ_J in pure equality theory which says, of an n-element set, 'either n is infinite or $n \in J$'. As above, J is a cofinite set of integers containing 1 and closed under multiplication. It is easy to see the existence of an algorithm which proceeds from the finite complement of J to yield φ_J, and Morley has provided a simpler algorithm than was known before.

In this section we will address Neumann's request for 'nice, explicit' conditions by providing an algorithm (3.3 below) which easily converts φ_J into a Mal'tsev condition $\varphi_1, \varphi_2, ...$ for J (described in the introduction). At the same time, we will give a new algorithm for constructing φ_J, which differs conceptually from previous ones in that it proceeds very simply from equational spectrum considerations. This algorithm for φ_J will show that in some sense constructing $\varphi_1, \varphi_2, ...$ is just as hard as constructing φ_J, and so no improvement can really be made on our algorithm $\varphi_J \to \varphi_1, \varphi_2, ...$.

We give our main points in synopsis form, with detailed algorithms for 3.2 and 3.3 to follow.

3.1. *There exists an algorithm to convert (the finite complement of) J into a finite conjunction of identities e_J whose spectrum is J.* Such an algorithm may be found in McKenzie (1975); the number of steps required, and the length of e_J, are no more than kn^2, where $n = $ maximum $m \notin J$.

3.2. *We will give an algorithm to convert any identities e_J with spectrum J to a Horn sentence φ_J as above. The length of φ_J is linear, and the number of steps required is quadratic, in the length of e_J.*

3.3. *We will give an algorithm to convert any φ_J into a Mal'tsev condition $\varphi_1, \varphi_2, \ldots$ for J, in which each φ_i has spectrum J.* More precisely, in time linear in $m = $ length (φ_J), our algorithm produces a simple algorithm A_J for the sequence $\{\varphi_i\}$. (To produce an individual φ_i takes Km^i steps—but this means very little, since the individual φ_i have no invariant meaning in this problem; and since individual φ_i may be repeated or deleted at will, even their growth rate has no real significance.)

3.4. *If $\varphi_1, \varphi_2, \ldots$ is any Mal'tsev condition for J, then, from some N onward, each φ_n has spectrum J.* This is an easy consequence of 3.1; moreover such an N is recursive in (a Turing machine for) the sequence $\varphi_1, \varphi_2, \ldots$. Our contention that knowing $\varphi_1, \varphi_2 \ldots$ is probably no easier than knowing φ_J rests on the observation that, as soon as we know this N, we can recover φ_J from φ_N, using 3.2. While one can easily imagine silly ways to make N very large and difficult to compute, reasonable procedures—which get right down to business—generally have N small and apparent. In fact $N = 1$ both in the general case given in 3.3 above and in the special case $(J = \omega - \{2\})$ given on p. 383 of Taylor (1973).

For the definition and basic theory of *Horn formulas*, see, for example, Chang and Keisler (1973), Sections 6.2, 6.3, where there is a proof of the existence of φ_J. For our first algorithm, the reader does not need the full definition, only our assurance that the constructed φ_J is Horn.

ALGORITHM 3.2 *from e_J to φ_J.* We know that e_J has the form

$$\forall x_1 \ldots x_s \bigwedge_{k=1}^{t} \sigma_k = \tau_k,$$

where σ_k and τ_k are terms in $x_1, \ldots, x_s$. Take one new variable y_α for each subterm α of each σ_i and τ_i. For each $i = 1, \ldots, s$, we take the formula

(3.5) $x_i = y_{\sigma_i}.$

If $\alpha = F(\alpha_1, \ldots, \alpha_p)$, $\beta = F(\beta_1, \ldots, \beta_p)$, then we take the 'basic Horn formula':

(3.6) $\left(\bigwedge_{i=1}^{p} y_{\alpha_i} = y_{\beta_i} \right) \rightarrow y_\alpha = y_\beta.$

And, finally, for $k = 1, \ldots, t$ we take

(3.7) $y_{\sigma_k} = y_{\tau_k}$

(with σ_k, τ_k as given in φ_n). The required φ_J is then

$$\forall x_1 \ldots x_s \exists \ldots y_\sigma \ldots \bigwedge_{q=1}^{N} H_q,$$

where $H_1, \ldots, H_N$ are the formulas given in (3.5), (3.6) and (3.7). This completes our description of the algorithm; to see that the constructed φ_J is as desired, we will need this lemma:

LEMMA. *For φ_J as defined in the algorithm, and any finite set X,*

$$|X| \in J \text{ if and only if } X \Vdash \varphi_J.$$

PROOF. If $|X| \in J$, then X is the universe of an algebra obeying e_J. We simply use the operations defined on this algebra to find all the y_σ so that (3.5), (3.6) and (3.7) all become true.

Conversely, if $X \Vdash \varphi_J$, then (3.5), (3.6) and (3.7) obviously provide a way of unambiguously defining the operations of e_J on X so that e_J becomes true.

ALGORITHM 3.3 *from φ_J to $\varphi_1, \varphi_2, \ldots$.* We shall assume that φ_J is given in the form

$$\varphi_J = \forall x_1 \ldots x_s \exists y_1 \ldots y_t \bigwedge_{i=1}^{n} \left[\left(\bigwedge_{j=1}^{m} \alpha_{ij} = \beta_{ij} \right) \rightarrow \gamma_i = \delta_i \right]^!$$

where each α, β, γ and δ is one of the x's or one of the y's. (The existence of this form follows of course from the lemma just above, or from traditional theorems in model theory—for example, Exercise 6.3.4 on p. 367 of Chang and Keisler (1973); in any case an arbitrary Horn sentence in pure equality theory is easily converted to this form.) We next claim that it is easy to convert φ_J so that each γ and each δ is one of the x's. This is an easy inductive argument based on the observation that φ_J is obviously equivalent to

$$\forall x_0 x_1 \ldots x_s \exists y_1 \ldots y_t \bigwedge_{i=2}^{n} \left[\left(\bigwedge_{j=1}^{m} \alpha_{ij} = \beta_{ij} \right) \rightarrow \gamma_i = \delta_i \right]$$

$$\wedge \left[(x_0 = \gamma_1) \wedge \left(\bigwedge_{j=1}^{m} \alpha_{ij} = \beta_{ij} \right) \rightarrow x_0 = \delta_1 \right].$$

A standard compactness argument now allows one to prove that for a variety $\mathcal{V}$, Spec $\mathcal{V} \subseteq J$ if and only if for some terms $F_k^\lambda (x_1, \ldots, x_s)$ $(1 \leqslant \lambda \leqslant N, 1 \leqslant k \leqslant t)$, $\mathcal{V}$ satisfies

$$\forall x_1 \ldots x_s \bigvee_{\lambda=1}^{N} \bigwedge_{i=1}^{n} \left[\left(\bigwedge_{j=1}^{m} \alpha_{ij}^\lambda = \beta_{ij}^\lambda \right) \rightarrow \gamma_i = \delta_i \right],$$

where α_{ij}^λ is obtained from β_{ij} by leaving each x alone, and converting each y_k to $F_k^\lambda (x_1, \ldots, x_s)$. (We omit a detailed proof.) An application of the distributive law

for $\wedge$ and $\vee$, together with the fact that $\mathscr{V}$ is closed under formation of products, tells us that $\operatorname{Spec}\mathscr{V}\subseteq J$ if and only if for some terms F_k^λ as above and some $\theta\colon n^N\to n$, $\mathscr{V}$ satisfies

$$\forall x_1\ldots x_s \bigwedge_{\gamma\in n^N}\left[\left(\bigwedge_{\lambda=1}^{N}\bigwedge_{j=1}^{m}\alpha_{\gamma(\lambda),\,j}^{\lambda}=\beta_{\gamma(\lambda),\,j}^{\lambda}\right)\to\gamma_{\theta(\gamma)}=\delta_{\theta(\gamma)}\right]$$

And this final condition is easily converted into a Mal'tsev condition (see, for example, p. 397 of Taylor (1973)).

4. Varieties of varieties

On p. 115 of Neumann (1978), Neumann defined a class K of varieties to be a *variety of varieties* if and only if K is closed under formation of product varieties, subvarieties and images of pure forgetful functors. He then asks whether the lattice of varieties of varieties is 'quite sparse'. Here we will prove the easy Theorem 3 of the introduction, which states that there are $2^{\aleph_0}$ varieties of varieties.

PROOF OF THEOREM 3. Isbell (1970) and Vaughan-Lee (1970) established that there are $2^{\aleph_0}$ varieties of *monoids* (semigroups with a unit element in the similarity type). (Also see Biryukov (1965) and Evans (1968).) It will thus be enough to give, for each variety $\mathscr{M}$ of monoids, a variety $K_\mathscr{M}$ of varieties such that distinct $\mathscr{M}$'s yield distinct $K_\mathscr{M}$'s.

If $\mathscr{V}$ is any variety (of any similarity type), then the unary operations of $\mathscr{V}$ form a monoid $M(\mathscr{V})$, in which the product FG is given by $FG(x)=F(G(x))$. (In fact $M(\mathscr{V})$ is simply the degree-1 part of the *clone*—see, for example, Taylor (1973), pp. 361–364—or the endomorphism monoid of the object $\mathbf{1}$ in the 'algebraic theory' of $\mathscr{V}$—see, for example, (op. cit. p. 391). We now define

$$K_\mathscr{M}=\{\mathscr{V}\colon M(\mathscr{V})\in\mathscr{M}\}.$$

It is routine to check that $K_\mathscr{M}$ is a variety of varieties. Now if $\mathscr{M}\ne\mathscr{N}$, we may without loss of generality find a monoid $M\in\mathscr{M}-\mathscr{N}$. Let $\mathscr{V}_M$ be the variety of all '*M-sets*', that is $\mathscr{V}_M$ has one unary operation $\bar{f}$ for each $f\in M$, subject to the laws

$$\bar{f}(\bar{g}(x))=\bar{f}\,\bar{g}(x),$$

$$\bar{1}(x)=x.$$

One easily checks that $M(\mathscr{V}_M)=M$, and hence $\mathscr{V}_M\in K_\mathscr{M}-K_\mathscr{N}$.

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