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Hodge–Frobenius equations and the Hodge–Bäcklund transformation

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Linear and nonlinear Hodge-like systems for 1-forms are studied with an assumption equivalent to complete integrability substituted for the requirement of closure under exterior differentiation. The systems are placed in a variational context and properties of critical points are investigated. Certain standard choices of energy density are related by Bäcklund transformations which employ basic properties of the Hodge involution. These *Hodge–Bäcklund transformations* yield invariant forms of classical Bäcklund transformations that arise in diverse contexts. Some extensions to higher-degree forms are indicated.

1. Introduction

The study of vectors which are both divergence free and curl free can be traced back at least to Helmholtz's analysis of vortices and gradients [16]. The generalization to differential forms which are both closed and co-closed under exterior differentiation is the content of the Hodge equations (see, for example, [27, ch. 7]). The divergence-free condition is frequently relaxed in variational contexts, but generalizations of the curl-free condition remain rather rare.

Our goal is to study both linear and nonlinear variants of the Hodge equations for differential forms which are neither co-closed nor closed but which satisfy milder conditions having physical and geometric significance.

1.1. Organization of the paper

Sections 1–4 are mainly expository. We introduce the topic in § 1.2 with an example from fluid dynamics. Section 2 presents the equations in an invariant context. The linear case is studied in § 3, largely as motivation for the considerably more complex nonlinear case. Two geometric analogies are discussed in § 4. Technical results on the properties of solutions are presented in § 5. Section 6 shows that ideas introduced, in very different contexts, by Yang [45] and by Magnanini and

Talenti [21–23] can be given a unified interpretation in terms of equations studied in the preceding sections.

The proofs of theorems 5.4, 5.5 and 5.7, which are based on rather straightforward applications of nonlinear elliptic theory, are collected in the appendix. We note that these applications are only straightforward once solutions have been associated to a uniformly sub-elliptic operator; this is accomplished in § 5.2. The methods used in that section to derive uniform estimates are elementary and, in particular, do not require a delicate limiting argument that has become known as *Shiffman regularization* (see [36] and [37, appendix]).

1.2. A motivating example: steady, ideal flow

In models for the steady, adiabatic and isentropic flow of an ideal fluid, conservation of mass is represented by the *continuity equation*

$$\nabla \cdot (\rho(|\boldsymbol{v}|^2)\boldsymbol{v}) = 0, \tag{1.1}$$

where $\boldsymbol{v}$ denotes flow velocity and ρ denotes mass density. The dependence of ρ on $|\boldsymbol{v}|^2$ is a consequence of compressibility; in the incompressible limit, equation (1.1) merely says that the vector $\boldsymbol{v}$ has zero divergence. If the fluid is irrotational, then the velocity is curl free in the sense that

$$\nabla \times \boldsymbol{v} = 0. \tag{1.2}$$

Condition (1.2) implies, by the Poincaré lemma, that there exists locally a scalar flow potential $\varphi(\boldsymbol{x})$, where $\boldsymbol{x} \in \mathbb{R}^3$ denotes the position of a particle in the flow.

Perhaps the mildest weakening of the irrotationality condition results from replacing (1.2) by the integrability condition

$$\boldsymbol{v} \cdot \nabla \times \boldsymbol{v} = 0. \tag{1.3}$$

The replacement of the linear condition (1.2) by the nonlinear condition (1.3) as a side condition to equation (1.1) is likely to result in singular solutions, even in the subsonic regime. The usual arguments for reducing ρ to the conventional form, which depend on smoothness (see, for example, [5, ch. 1.2]), would not necessarily apply in such cases. This suggests that we consider whether a useful *a priori* bound can be placed on the size of the singular set for solutions of systems having the general form (1.1), (1.3). In § 5.2 we take the first step towards an answer to this difficult question, deriving sufficient conditions under which a solution remains bounded on an apparent singular set of given codimension. Despite the physical motivation (here and in various other examples scattered throughout the text), in this paper our main interest is in deriving hypotheses which are mathematically natural and apply to large classes of mass densities.

2. An invariant formulation

We now generalize the mathematical context of § 1.2. Let Ω be an open, finite domain of $\mathbb{R}^n$, $n \geqslant 2$, satisfying an interior sphere condition. Consider the following system (see [28, § 6], [29, § 4]):

$$\delta(\rho(Q)\omega) = 0, \tag{2.1}$$

$$\mathrm{d}\omega = \mathit{\Gamma} \wedge \omega, \tag{2.2}$$

for scalar-valued 1-forms ω and $\mathit{\Gamma}$, where $\mathit{\Gamma}$ is given and ω is unknown, d is the flat exterior derivative with formal adjoint δ, Q is a quadratic form in ω given by

$$Q(\omega) = *(\omega \wedge *\omega) \equiv \langle \omega, \omega \rangle, \tag{2.3}$$

where $*\colon \mathit{\Lambda}^k \to \mathit{\Lambda}^{n-k}$ is the Hodge involution and ρ is a positive, continuously differentiable function of Q (but a possibly singular function of $\boldsymbol{x}$).

Using (2.2), we find that

$$\omega \wedge \mathrm{d}\omega = -\mathit{\Gamma} \wedge \omega \wedge \omega = 0. \tag{2.4}$$

A 1-form ω that is the pointwise Riemannian inner product with a vector field $\boldsymbol{v}$ is said to be *dual* to $\boldsymbol{v}$. In this case, the left-hand side of (2.4) is equivalent to the left-hand side of (1.3), and any solution ω of equation (2.2) is dual to a solution $\boldsymbol{v}$ of equation (1.3).

The system (2.1), (2.2) is uniformly elliptic provided the differential inequality

$$0 < \kappa_1 \leqslant \frac{(\mathrm{d}/\mathrm{d}Q)[Q\rho^2(Q)]}{\rho(Q)} \leqslant \kappa_2 < \infty \tag{2.5}$$

is satisfied for constants κ_1, κ_2. In the context of fluid dynamics, one typically encounters the weaker condition

$$0 < \rho^2(Q) + 2Q\rho'(Q)\rho(Q). \tag{2.6}$$

Ideal flow governed by equation (1.1) is subsonic provided that (2.6) is satisfied. Moreover, there is typically a critical value Q_{crit} such that the right-hand side of (2.6) tends to zero in the limit as Q tends to Q_{crit}. In this case, equations (2.1) and (2.2) with ρ satisfying (2.6) are elliptic, but not uniformly so, and this condition has mathematical as well as physical interest.

If $\mathit{\Gamma} \equiv 0$, then the system (2.1), (2.2) degenerates to the *nonlinear Hodge equations* introduced in [38] on the basis of a conjecture in [4]. In that case, condition (2.2) generates a cohomology class, which is not true in the more general case studied here.

We show in § 2.1 that whenever ω is a 1-form, equation (2.2) possesses solutions of the form

$$\omega = \mathrm{e}^{\eta}\,\mathrm{d}u, \tag{2.7}$$

where u and η are 0-forms, and $\mathit{\Gamma}$ can be made *exact*: we can write $\mathit{\Gamma} = d\eta$. When ω is a k-form, a more general representation applies; that representation is discussed in § 2.2.

In § 5.1 we introduce a variant of equation (2.1) for differential k-forms ($k \geqslant 1$) satisfying equation (2.2) with $\mathit{\Gamma}$ exact ($\mathit{\Gamma} \equiv \mathrm{d}\eta$), namely

$$\delta[\rho(Q)\omega] = (-1)^{n(k+1)} * (\mathrm{d}\eta \wedge *\rho(Q)\omega), \tag{2.8}$$

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which arises as a variational equation of the *nonlinear Hodge energy*

$$E = \frac{1}{2}\int_{\Omega}\int_{0}^{Q}\rho(s)\,\mathrm{d}s\,\mathrm{d}\Omega. \tag{2.9}$$

2.1. The Frobenius theorem

Let Γ be fixed, and define

$$S \equiv \left\{\omega \in \Lambda(\Omega) \equiv \bigoplus_{k=1}^{n}\Lambda^k(\Omega) : \mathrm{d}\omega = \Gamma\wedge\omega\right\}.$$

Denote by $I \equiv I(S)$ the ideal generated by S. If $\Gamma \neq 0$, then clearly $\mathrm{d}I \neq \{0\}$, so the k-forms $\omega \in S$ do not generate cohomology classes.

Nevertheless, the ideal I is *closed*, i.e. $\mathrm{d}I \subset I$. In fact, a differential form $\alpha \in I$ is a linear combination of forms of type $\omega \wedge \beta$ with $\omega \in S$, $\beta \in \Lambda(\Omega)$. The latter satisfy

$$\mathrm{d}(\omega\wedge\beta) = \pm\omega\wedge(\Gamma\wedge\beta \pm \mathrm{d}\beta),$$

and thus satisfy $\mathrm{d}\alpha \in I$. This is an important fact, especially for exterior systems of 1-forms.

Following the approach used in [12, § 4.2], we define an exterior system $\{\omega^a\}$, $a = 1,\dots,r$, of r 1-forms in a space of dimension $n = r + s$ to be *completely integrable* if and only if there exist r independent functions g^a, $a = 1,\dots,r$, such that each of the 1-forms ω^a vanishes on the r-parameter family of s-dimensional hypersurfaces $\{g^a = k^a,\ a = 1,\dots,r\}$ generated by letting the constants k^a range over all r-tuples of real numbers.

Equivalently, we define $\{\omega^a\}_1^r$ to be *completely integrable* if and only if there exist a non-singular $r\times r$ matrix of functions ξ_b^a, and r independent functions g^b such that

$$\omega^a = \sum_{b=1}^{r}\xi_b^a\,\mathrm{d}g^b.$$

The *Frobenius theorem* asserts that an exterior system $\{\omega^a\}_1^r$ of 1-forms is completely integrable if and only if it generates a closed ideal of $\Lambda(\Omega)$.

Because a 1-form ω satisfying (2.2) generates a closed ideal, by the Frobenius theorem it can always be written in the form (2.7). (In this case, $r = 1$.) Thus

$$\mathrm{d}\omega = \mathrm{d}\eta\wedge\omega, \tag{2.10}$$

which shows that Γ can be chosen to be exact. Note the gauge invariance has the form $\Gamma \to \tilde{\Gamma} \equiv \Gamma + f(\boldsymbol{x})\omega$.

For this reason we call the system (2.1), (2.2) the *nonlinear Hodge–Frobenius equations for 1-forms.*

Unfortunately, the Frobenius theorem does not generalize to forms of arbitrary degree k, as the condition $\mathrm{d}I \subset I$ does not imply complete integrability if $k \neq 1$. However, this does not mean that there is nothing to be said about higher-degree forms. Relevant properties of such forms are described in the following section.

2.2. Recursive forms

An exterior differential form of degree k is said to be *recursive with coefficient* Γ if it satisfies (2.2). Let Ω be star shaped. It is known that one can define a *homotopy operator* $\mathcal{H} : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$, which satisfies

$$\omega = \mathrm{d}\mathcal{H}\omega + \mathcal{H}\,\mathrm{d}\omega. \tag{2.11}$$

This property can be used, among other things, to show that a closed form on a star-shaped domain is exact. We omit the formal definition of this operator (for this and further details, see [12, § 5.3]) and describe its main properties as follows:

(a) $\mathcal{H}$ is linear;

(b) $\mathcal{H}^2 = 0$;

(c) $\mathcal{H}\,\mathrm{d}\mathcal{H} = \mathcal{H}$, $\mathrm{d}\mathcal{H}\,\mathrm{d} = \mathrm{d}$;

(d) $(\mathrm{d}\mathcal{H})^2 = \mathrm{d}\mathcal{H}$, $(\mathcal{H}\,\mathrm{d})^2 = \mathcal{H}\,\mathrm{d}$.

Using (b), we observe that $\mathcal{H}\,\mathrm{d}\omega \in \ker \mathcal{H}$. This and (2.11) can be used to define the *exact part of* ω as $\omega_{\mathrm{e}} \equiv \mathrm{d}\mathcal{H}\omega$ and the *anti-exact part of* ω as $\omega_{\mathrm{a}} \equiv \omega - \omega_{\mathrm{e}} = \mathcal{H}\,\mathrm{d}\omega$.

Using (2.11), one can further show that $\mathcal{H}$ improves regularity. With no loss of generality, we prove this for forms having vanishing anti-exact part, i.e. forms ω such that

$$\omega = \omega_{\mathrm{e}}. \tag{2.12}$$

Note that, for any given form ω, no cancellations can occur between ω_{e} and ω_{a}, therefore ω_{e} is always as smooth as ω, and furthermore $\mathcal{H}\omega_{\mathrm{a}} = 0$. Thus, we can restrict our attention to forms satisfying (2.12). If ω is 1-form, then $\mathcal{H}\omega$ is a function and (2.12) implies that

$$\omega_i = \frac{\partial(\mathcal{H}\omega)}{\partial x^i} \quad \text{for all } i,$$

thus improving regularity. In the general case, $\mathcal{H}\omega \equiv (\mathcal{H}\omega)_I\,\mathrm{d}x^I$, where I is a multi-index satisfying $|I| = k - 1$. In order to improve regularity in this case one would need to control $\delta(\mathcal{H}\omega)$, i.e. all derivatives of type $\partial(\mathcal{H}\omega)_I/\partial x^i$ for $i \in I$. But $\delta(\mathcal{H}\omega) = 0$ from the Hodge decomposition theorem [27], as $\mathcal{H}\omega$ is anti-exact (using (b), above), so $\mathcal{H}$ is a smoothing operator on k-forms.

The important result for us is the following. Recursive k-forms with coefficient Γ on a star-shaped region can be represented as follows [12]:

$$\omega = \mathrm{e}^{\eta}[\mathrm{d}u + \mathcal{H}(\theta \wedge \mathrm{d}u)], \tag{2.13}$$

where $\eta = \mathcal{H}\Gamma$, $\theta = \mathcal{H}\,\mathrm{d}\Gamma$ and $u = \mathcal{H}(\mathrm{e}^{-\eta}\omega)$. Condition (2.2) implies that θ satisfies

$$\mathrm{d}\theta \wedge [\mathrm{d}u + \mathcal{H}(\theta \wedge \mathrm{d}u)] = 0.$$

For our purposes we can rewrite (2.13) as

$$\omega = \mathrm{e}^{\eta} g(\mathrm{d}u), \tag{2.14}$$

where g is a smooth linear operator, or, alternatively, as

$$\omega = \mathrm{e}^{\eta}\,\mathrm{d}u + \mathrm{e}^{\eta} h(\mathrm{d}u), \tag{2.15}$$

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where h is also a smooth linear operator, the coefficients of which depend on Γ. The latter variant yields better regularity. In fact, the form $h(\mathrm{d}u)$ is as smooth as u, provided Γ is smooth.

A particular case of the above occurs when the coefficient Γ is exact. In that case, the form ω satisfying (2.2) is said to be *gradient recursive*. For gradient recursive k-forms, (2.13) assumes the simpler form (2.7), in which g is the identity.

3. The linear case

Corresponding to the physical example of § 1.2, which illustrates the nonlinear form of equations (2.1) and (2.2), we can illustrate the linear case with an even simpler physical example.

Condition (1.3) arises when a rigid body rotates in the x,y-plane at constant angular velocity $\tilde{\omega}$. Taking the axis of rotation to lie at the origin of coordinates, we write the tangential velocity vector in the form

$$\boldsymbol{v} = v_1\hat{\imath} + v_2\hat{\jmath},$$

where

$$v_1 = -\tilde{\omega}y, \qquad v_2 = \tilde{\omega}x. \tag{3.1}$$

Then

$$\nabla \times \boldsymbol{v} = 2\tilde{\omega}\hat{k}, \tag{3.2}$$

so (1.3) is satisfied (see, for example, [26, exercise 4.4]). In what follows we take $\tilde{\omega} \equiv 1$ for simplicity.

Equations (3.1) imply that

$$\nabla \cdot \boldsymbol{v} = 0,$$

so we express the 1-form ω dual to $\boldsymbol{v}$ as a solution of the *linear* Hodge–Frobenius equations

$$\begin{aligned} \delta\omega &= 0, \\ \mathrm{d}\omega &= \Gamma \wedge \omega. \end{aligned} \tag{3.3}$$

Applying (2.7) and (2.10), we choose η to depend only on the distance r from the axis of rotation. Then

$$\mathrm{d}\eta \wedge \omega = \eta'(r) \cdot r\,\mathrm{d}x\,\mathrm{d}y. \tag{3.4}$$

In addition, equation (3.2) implies that

$$\mathrm{d}\eta \wedge \omega = \mathrm{d}\omega = 2\,\mathrm{d}x\,\mathrm{d}y. \tag{3.5}$$

Equating the right-hand sides of equations (3.4) and (3.5), we conclude that $\eta(r) = 2\log r$ and $\omega = r^2\,\mathrm{d}u$ for $u(x,y) = \arctan(y/x)$. Then $|\mathrm{d}u| = r^{-1}$, so the singular structure of u in the x,y-plane is analogous to that of the fundamental solution of Laplace's equation in $\mathbb{R}^3$. In particular, u is singular at the origin of the disc.

In this example, the Hodge–Frobenius equations themselves are only defined on the punctured disc, as

$$\Gamma \wedge \omega = (\Gamma_1\omega_2 - \Gamma_2\omega_1)\,\mathrm{d}x\,\mathrm{d}y = \mathrm{d}\omega = 2\,\mathrm{d}x\,\mathrm{d}y.$$

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Using (3.1), we can write this condition as an equation for the inner product

$$\begin{pmatrix} \Gamma_1 \\ \Gamma_2 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = 2,$$

which cannot be satisfied at the origin.

Thus, singular solutions arise naturally in both the linear and nonlinear Hodge–Frobenius equations.

Because equation (2.13) requires that the domain be star shaped, the conclusion that ω is representable as a product $f\,\mathrm{d}u$, where f is non-vanishing, does not follow, and is in fact violated in our example, in which $f = r^2$.

We have presented a particularly simple model, in which the role of condition (2.2) is especially transparent. For more sophisticated completely integrable models of rigid-body rotation see, for example, [6] and the references therein.

In the linear case we can accomplish easily what we cannot accomplish at all in the nonlinear case: an integrability condition sufficient to imply the smoothness of weak solutions.

PROPOSITION 3.1. *Let ω be a weak solution of the linear Hodge–Frobenius equations (3.3) on Ω. If $|\Gamma|$ is bounded and $\omega \in L^p(\Omega)$ for $p > n$, then ω is continuous.*

Proof. The Friedrichs mollification ω_h of ω is a classical solution of (3.3) (see, for example, [15, § 7.2]). Thus,

$$|\mathrm{d}\omega_h|^p + |\delta\omega_h|^p + |\omega_h|^p \leqslant (|\Gamma|^p + 1)|\omega_h|^p.$$

Integrate and apply the L^p Gaffney–Gärding inequality [18, lemma 4.7] to obtain

$$\|\nabla\omega_h\|_p \leqslant C(\Gamma)\|\omega_h\|_p.$$

Because $\omega \in L^p(\Omega)$ we can allow the mollification parameter h to tend to zero. The proof is completed by the Sobolev embedding theorem. □

4. Geometric analogies

4.1. Hypersurface-orthogonal vector fields

A unification of the two superficially different physical examples of §§ 1 and 3 can be found in their underlying geometry: in particular, in the relation of each vector field to hypersurfaces created by the level sets of an associated scalar function u. We can think of u informally as the potential for the conservative field that would result from taking Γ to be zero in equation (2.2). Whereas a conservative vector field is actually equal to ∇u, the vector fields in §§ 1 and 3 merely point in the same direction as ∇u.

A non-vanishing vector field $\boldsymbol{v}$ is said to be *hypersurface-orthogonal* whenever there exists a foliation of hypersurfaces orthogonal to $\boldsymbol{v}$. The foliated hypersurfaces can be represented as level sets of a scalar function u. That is, one can write

$$\boldsymbol{v} = \lambda(\boldsymbol{x})\nabla u$$

for a non-vanishing function λ. Conversely, a vector field which can be written in this way is clearly hypersurface-orthogonal. We conclude that a vector field $\boldsymbol{v}$ is

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hypersurface-orthogonal if and only if the 1-form ω dual to $\boldsymbol{v}$ satisfies

$$\omega = \lambda \, \mathrm{d}u,$$

with non-vanishing λ (i.e. if and only if ω is completely integrable). Other equivalent conditions now follow from the Frobenius theorem.

Hypersurface-orthogonal vector fields arise naturally in general relativity, particularly in connection with black-hole mechanics. Introducing a tensor field

$$\boldsymbol{B}_{\alpha\beta} \equiv \boldsymbol{v}_{\alpha;\beta},$$

where the semicolon denotes covariant differentiation with respect to the space-time metric connection, the condition that $\boldsymbol{v}$ be hypersurface-orthogonal implies that the antisymmetric part of $\boldsymbol{B}$, called the *rotation tensor*, vanishes [31, §§ 2.32 and 2.33]. For this reason, vector fields satisfying (2.2) are called *rotation free* in general relativity, which is somewhat confusing in the context of the examples in §§ 1 and 3.

4.2. Twisted Born–Infeld equations

A condition broadly analogous to (2.2) arises if ω is a Lie-algebra-valued 2-form satisfying the second Bianchi identity. In that case, replacing ω by F_A, where A is a Lie-algebra-valued 1-form, we have

$$\mathrm{d}F_A = -[A, F_A] \tag{4.1}$$

(where $[\cdot,\cdot]$ denotes the Lie bracket): an equation which resembles (2.2).

Precisely, let X be a vector bundle over a smooth, finite, oriented, n-dimensional Riemannian manifold M. Suppose that X has compact structure group $G \subset \mathrm{SO}(m)$. Let $A \in \Gamma(M, \mathrm{ad}\, X \otimes T^*M)$ be a connection 1-form on X having curvature 2-form

$$F_A = \mathrm{d}A + \tfrac{1}{2}[A, A] = \mathrm{d}A + A \wedge A,$$

where $[\cdot,\cdot]$ is the bracket of the Lie algebra $\mathfrak{I}$, the fibre of the adjoint bundle $\mathrm{ad}\, X$. Sections of the automorphism bundle $\mathrm{Aut}\, X$, called *gauge transformations*, act tensorially on F_A but affinely on A (see, for example, [24]).

Consider energy functionals having the form (2.9), where $Q = |F_A|^2 = \langle F_A, F_A \rangle$ is an inner product on the fibres of the bundle $\mathrm{ad}\, X \otimes \Lambda^2(T^*M)$. The inner product on $\mathrm{ad}\, X$ is induced by the normalized trace inner product on $\mathrm{SO}(m)$, and the inner product on $\Lambda^2(T^*M)$ is induced by the exterior product $*(F_A \wedge *F_A)$.

A non-abelian variational problem analogous to equations (2.2) and (2.8) is described briefly in [30, § 5.1]. One is led to consider smooth variations taken in the infinitesimal deformation space of the connection and having the explicit form

$$\begin{aligned} \mathrm{var}(E) &= \int_M \rho(Q)\, \mathrm{var}(Q)\, \mathrm{d}M \\ &= \int_M \rho(Q) \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} |F_{A+t\psi}|^2\, \mathrm{d}M \\ &= \int_M \rho(Q) \frac{\mathrm{d}}{\mathrm{d}t}\bigg|_{t=0} |F_A + tD_A\psi + t^2 \psi \wedge \psi|^2\, \mathrm{d}M, \end{aligned} \tag{4.2}$$

where $D_A = \mathrm{d} + [A, \cdot]$ is the exterior covariant derivative in the bundle. The Euler–Lagrange equations are

$$\delta(\rho(Q)F_A) = - * [A, *\rho(Q)F_A]. \tag{4.3}$$

In addition, we have the Bianchi identity (4.1).

Writing equation (2.2) in components as

$$\mathrm{d}\omega^a = \Gamma^a_b \wedge \omega^a,$$

we observe that if $-\Gamma$ is interpreted as a connection 1-form, then (2.2) can be interpreted as the vanishing of an exterior covariant derivative, which is the content of equation (4.1). Moreover, the well-known algebraic requirement that Γ must satisfy

$$(\mathrm{d}\Gamma^a_b - \Gamma^a_c \wedge \Gamma^c_b) \wedge \omega^b = 0$$

(see [12, equation (4.2.3)]) is the zero-curvature condition $[F_\Gamma, \omega] = 0$.

If G is abelian, then the Lie bracket vanishes and equations (4.3) reduce to the system

$$\delta\{\rho[Q(F_A)]F_A\} = \delta\{\rho[Q(\mathrm{d}A)]\,\mathrm{d}A\} = 0,$$

which is a nonlinear Hodge equation analogous to taking $\Gamma = 0$ in equations (2.1) and (2.2).

Equations (4.1) reduce, in the abelian case, to the equations for the equality of mixed partial derivatives,

$$\mathrm{d}^2 A = 0.$$

If $\rho \equiv 1$, then equations (4.3) are the Yang–Mills equations, describing quantum fields in the classical limit. These resemble a version of equation (2.8) for 2-forms with $\rho \equiv 1$, with the Bianchi identity (4.1) playing the role of equation (2.2).

Whereas the Yang–Mills equations do not have the nonlinear structure of (2.8) for non-constant ρ, those of the Born–Infeld model for electromagnetism are equivalent to (2.8) for differential forms of degree 2 with

$$\rho(Q) = (1 + |F_A|^2)^{-1/2}. \tag{4.4}$$

This model was introduced in [7] in order to produce a model of electromagnetism that does not diverge when the source is a point charge. Geometric aspects of the model are investigated in [13] and its analytic aspects are investigated in [45]. A mathematical generalization of the Born–Infeld model to non-abelian variational equations was proposed in [28] and further studied in [40] (see also [17] for a related problem). The equations assume the form of (4.1) and (4.3) for an appropriate choice of $\rho(Q)$.

Geometrically, the non-abelian model puts a twist in the principal bundle corresponding to the configuration space of solutions. Thus, equations (4.1) and (4.3) are called *twisted nonlinear Hodge equations* [30]. A different approach to generalizing nonlinear Hodge theory to bundle-valued connections, which is based on the formulation of a natural class of boundary-value problems, is introduced in [25], but interior estimates would be required in order to extend the theory of [25] to twisted forms of the equations considered in this paper. The derivation of such estimates is a goal of § 5.2.

5. Analysis

We do not expect rotational fields of any kind to be very smooth. In particular, assumption (2.2) may produce caustics (see, for example, the discussion in [31, § 2.4]). However, it is reasonable to seek conditions under which the field remains bounded at a singularity or under which the field equations remain uniformly elliptic.

Because these conditions will be derived for a large class of mass densities, the strength of the estimates obtained will depend on the integrability with respect to $\boldsymbol{x}$ of a given choice of mass density $\rho(Q(\boldsymbol{x}))$.

In what follows, we denote by C generic positive constants, the value of which may change from line to line. We follow an analogous convention for continually updated *small* positive constants ε. Repeated indices are to be summed from 1 to n.

5.1. Variational structure

The energy E of the field ω on $\varOmega$ is defined by equation (2.9), where $Q = Q(\omega)$ is defined as in (2.3) for $\omega \in \varLambda^1$ given by (2.7). Then, for all $\psi \in C_0^\infty(\varOmega)$, the variations of E are computed as

$$\begin{aligned}
\mathrm{var}(E) &= \frac{\mathrm{d}}{\mathrm{d}t}E(u+t\psi)\Big|_{t=0} \\
&= \frac{1}{2}\int_\varOmega \rho(Q)\frac{\mathrm{d}}{\mathrm{d}t}Q(u+t\psi)\Big|_{t=0}\,\mathrm{d}\varOmega \\
&= \frac{1}{2}\int_\varOmega \rho(Q)\frac{\mathrm{d}}{\mathrm{d}t}[\mathrm{e}^{2\eta}|\mathrm{d}(u+t\psi)|^2]\Big|_{t=0}\,\mathrm{d}\varOmega \\
&= \frac{1}{2}\int_\varOmega \rho(Q)\mathrm{e}^{2\eta}\frac{\mathrm{d}}{\mathrm{d}t}(|\mathrm{d}u|^2 + 2\langle \mathrm{d}u, t\,\mathrm{d}\psi\rangle + t^2|\mathrm{d}\psi|^2)\Big|_{t=0}\,\mathrm{d}\varOmega \\
&= \int_\varOmega \rho(Q)\mathrm{e}^{2\eta}\langle \mathrm{d}u, \mathrm{d}\psi\rangle\,\mathrm{d}\varOmega \\
&= \int_\varOmega \langle \rho(Q)\mathrm{e}^{2\eta}\,\mathrm{d}u, \mathrm{d}\psi\rangle\,\mathrm{d}\varOmega \\
&= \int_\varOmega \mathrm{d}\langle \rho(Q)\mathrm{e}^{2\eta}\,\mathrm{d}u, \psi\rangle\,\mathrm{d}\varOmega + \int_\varOmega \langle \delta[\rho(Q)\mathrm{e}^{\eta}\omega], \psi\rangle\,\mathrm{d}\varOmega \\
&= \int_\varOmega \langle \delta[\rho(Q)\mathrm{e}^{\eta}\omega], \psi\rangle\,\mathrm{d}\varOmega,
\end{aligned}$$

as ψ has compact support in $\varOmega$. At a critical point, $\mathrm{var}(E) = 0$, or

$$\delta[\rho(Q)\mathrm{e}^{\eta}\omega] = 0, \tag{5.1}$$

so the variational formulation yields a ‘weighted’ form of the continuity equation (2.1). The presence of this weight adds an inhomogeneous term to the unweighted variant. To see this, we write the local equation

$$-\partial_i[\rho(Q)\mathrm{e}^{\eta}\omega_i] = -(\partial_i\eta)\mathrm{e}^{\eta}\rho(Q)\omega_i - \mathrm{e}^{\eta}\partial_i[\rho(Q)\omega_i] = 0.$$

This local form corresponds to the invariant representation

$$e^{\eta}\delta[\rho(Q)\omega] = e^{\eta}\rho(Q)\langle\, d\eta, \omega\rangle.$$

Thus, we obtain equation (2.8) for 1-forms ω (i.e. for $k = 1$) or, equivalently, since $\delta\alpha = - * d * \alpha$ for all $\alpha \in \Lambda^1$, the equation

$$d * [\rho(Q)\omega] = -d\eta \wedge *[\rho(Q)\omega]. \tag{5.2}$$

5.1.1. *Variational equations for k-forms*

Recall that an exterior differential form ω of degree k which is recursive with coefficient Γ can be written in the form (2.15), where u is a $(k-1)$-form (which depends on ω), and where the function η and the linear operator h depend only on Γ (§ 2.2).

We now compute the variation of the energy functional (2.9) among all forms $\omega + t\alpha$ satisfying

$$d(\omega + t\alpha) = \Gamma \wedge (\omega + t\alpha).$$

Such forms satisfy (2.15) with fixed η and h. Therefore, the variations of E are computed as

$$\begin{aligned} \operatorname{var}(E) &= \frac{d}{dt}E(\omega + t\alpha)\bigg|_{t=0} \\ &= \frac{1}{2}\int_{\Omega}\rho(Q)\frac{d}{dt}Q(\omega + t\alpha)\bigg|_{t=0} d\Omega \\ &= \int_{\Omega}\langle\rho(Q)\omega, \alpha\rangle\, d\Omega, \end{aligned}$$

where $\alpha = e^{\eta}(dv + h(dv))$ for a $(k-1)$-form v that depends on α.

Thus,

$$\begin{aligned} \operatorname{var}(E) &= \frac{1}{2}\int_{\Omega}\langle e^{\eta}\rho(Q)\omega, dv + h(dv)\rangle\, d\Omega \\ &= \int_{\Omega}\langle e^{\eta}\rho(Q)\omega, G\, dv\rangle\, d\Omega \\ &= \int_{\Omega}\langle G^{T}e^{\eta}\rho(Q)\omega, dv\rangle\, d\Omega, \end{aligned}$$

where $G \equiv g_{ij}(\Gamma)$ is an $n \times n$ matrix and G^{T} is its transpose.

As the forms α (and thus v) are assumed to have compact support in Ω, setting $\operatorname{var}(E) = 0$ is equivalent to imposing the condition

$$\delta[G^{T}e^{\eta}\rho(Q)\omega] = 0.$$

Note that if ω is gradient recursive, then G is the identity matrix and we recover (5.1), or, equivalently, since, for all $\alpha \in \Lambda^k$,

$$\delta\alpha = (-1)^{nk+n+1} * d * \alpha, \tag{5.3}$$

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the equation

$$\mathrm{d} * [\rho(Q)\omega] = -\mathrm{d}\eta \wedge *[\rho(Q)\omega], \tag{5.4}$$

i.e. equation (2.8) for a gradient-recursive k-form ω with coefficient $\mathrm{d}\eta$.

The variations employed in this section, applied directly to a 2-form, are necessarily different from the variations of (4.2), which are applied instead to the Lie-algebra-valued connection 1-form A.

In the remainder of the paper we focus mainly on 1-forms. However, many of the results will extend easily to gradient-recursive k-forms. Moreover, the structure of expressions (2.14) and (2.15) suggests that, under appropriate technical hypotheses on the linear operators g or h, many of the results will also extend to recursive k-forms with coefficient Γ (not necessarily exact).

5.2. When are solutions bounded at a singularity?

Following [44], we find it convenient to introduce a function $H(Q)$, which is defined so that

$$H'(Q) = \tfrac{1}{2}\rho(Q) + Q\rho'(Q). \tag{5.5}$$

Then ellipticity is equivalent to the condition that H has positive derivative with respect to Q.

In [28, theorem 7 and corollary 8] and [29, theorem 6 and corollary 7], L^p conditions are derived which imply the boundedness of solutions to equations (2.1) and (2.2) on domains that include singular sets of given co-dimension. We call such theorems *partial removable singularities theorems* as they imply that although solutions may have jump discontinuities at the singularity they cannot blow up there. Those results require the mass density ρ to satisfy the inequality

$$C(K+Q)^q \leqslant H'(Q) \leqslant C^{-1}(Q+K)^q \tag{5.6}$$

for constants $q > 0$ and $K \geqslant 0$. (This hypothesis is imposed in [29], following [44, § 1]; a somewhat stronger hypothesis is imposed in [28].)

Conditions resembling (5.6) also arise in the theory of A-harmonic forms [10, condition (1.2)]. Like the ρ-harmonic forms of nonlinear Hodge theory, A-harmonic forms $\mathrm{d}u$ are closed (in fact, exact), but are not generally co-closed (cf. [10, equation (1.4)]). See also [1] and the references therein.

There is obvious interest in deriving estimates for densities which may not satisfy (5.6). The focus of this section is to obtain and exploit such estimates for a broad class of densities, using the variational form of the equations. However, we retain the condition that ρ is positive, which is natural for applications.

We impose an additional condition that arises from technical considerations. If $\rho'(Q) > 0$, we require that

$$H'(Q) \leqslant C\rho(Q). \tag{5.7}$$

(This inequality is satisfied automatically if $\rho'(Q) \leqslant 0$.) If $\rho'(Q) < 0$, we require instead that

$$H'(Q) \geqslant C\rho(Q). \tag{5.8}$$

(This inequality is satisfied automatically if $\rho'(Q) \geqslant 0$.) Note that (5.8) implies (2.6) under our assumption $\rho(Q) > 0$. In the case where $\rho'(Q) < 0$, equation (2.8)

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is uniformly elliptic (i.e. (2.5) is satisfied) whenever (5.8) is satisfied and ρ is *non-cavitating*, that is, bounded below away from zero. In the case when $\rho'(Q) > 0$, condition (2.5) is satisfied whenever $\rho(Q)$ is bounded above.

Densities which satisfy (5.6) satisfy hypotheses (5.7) and (5.8). However, there are many densities which satisfy (5.7) and (5.8) but do not satisfy (5.6). Among the latter are densities for which the value of the exponent q on the left-hand side of inequality (5.6) differs from its value on the right-hand side, and certain densities for which the value of q in (5.6) is negative.

As a simple illustration, consider the class of densities

$$\rho(Q) = (K + Q)^q, \quad -\tfrac{1}{2} < q < 0,\ K > 0. \tag{5.9}$$

Such densities do not satisfy condition (5.6) and cavitate as Q tends to infinity. They arise, for example, in connection with models of pseudo-plastic non-Newtonian fluids [3]. In this section we will obtain sufficient conditions for a bound on $Q\rho(Q)$ which is valid as Q tends to infinity. Under additional conditions, this bound extends to possibly singular solutions of the system (2.2), (2.8) with density given by (5.9). Note that, for such densities, a bound on the product $Q\rho(Q)$ implies an asymptotic bound on the norm Q of the solution itself.

Because conditions (5.7) and (5.8) are used in a crucial way to establish both the subelliptic estimates and the ellipticity of the second-order operator, they appear to provide a mathematically natural generalization of condition (5.6).

Lemma 5.1. *Let condition (5.8) be satisfied. Then H can be chosen so that*

$$Q\rho(Q) \leqslant CH(Q). \tag{5.10}$$

Proof. If $\rho'(Q)$ is non-negative, then (5.10) is always satisfied with $C = 2$ if we choose $H(0) \geqslant 0$. (Note that assumption (5.8) is not needed in this case.) In order to see this, let

$$\Phi(Q) \equiv 2H(Q) - Q\rho(Q).$$

Then $\Phi(0) = H(0) \geqslant 0$ and

$$\Phi'(Q) = 2H'(Q) - \rho(Q) - Q\rho'(Q) = Q\rho'(Q) \geqslant 0.$$

Thus, $\Phi(Q)$ remains non-negative on the entire range of Q.

If $\rho'(Q) \leqslant 0$, we assume (5.8). Then, in particular,

$$2H'(Q) \geqslant \varepsilon\rho(Q), \tag{5.11}$$

where we take ε to be so small that it lies in the interval $(0, 1)$. Inequality (5.11) can be written in the form

$$(1 - \varepsilon)\rho(Q) + 2Q\rho'(Q) \geqslant 0. \tag{5.12}$$

Define a constant c by the formula

$$c = \frac{1 + \varepsilon}{2\varepsilon}.$$

In terms of c, (5.12) can be written in the form

$$(c - 1)\rho(Q) + 2(c - \tfrac{1}{2})Q\rho'(Q) \geqslant 0.$$

We can convert this expression into the differential inequality

$$2cH'(s) \geqslant \frac{\mathrm{d}}{\mathrm{d}s}[s\rho(s)], \quad s \in [0, Q].$$

Integrate the inequality over s, using $H(0) = 0$. We obtain (5.10). □

LEMMA 5.2. *Let the 1-forms Γ and ω smoothly satisfy (2.2) and (2.8). Let $\rho > 0$ satisfy conditions (5.7) and (5.8). Then*

$$\Delta H + (-1)^{3n}\nabla \cdot \{*[\omega \wedge *(\rho'(Q)\,\mathrm{d}Q \wedge \omega)]\} + C(|\Gamma|^2 + |\nabla\Gamma|)H \geqslant 0. \tag{5.13}$$

Proof. We have [44, § 1]

$$\begin{aligned}\langle\omega, \Delta[\rho(Q)\omega]\rangle &= \partial_i\langle\omega, \partial_i(\rho(Q)\omega)\rangle - \langle\partial_i\omega, \partial_i(\rho(Q)\omega)\rangle \\ &= \Delta H(Q) - [\rho(Q)\langle\partial_i\omega, \partial_i\omega\rangle + \rho'(Q)\langle\partial_i\omega, \omega\rangle\partial_i Q],\end{aligned} \tag{5.14}$$

where

$$\Delta H(Q) = \partial_i(\partial_i H(Q)) = \partial_i(H'(Q)\partial_i Q).$$

Setting $\partial_i Q = 2\langle\partial_i\omega, \omega\rangle$, we rewrite (5.14) in the form

$$\langle\omega, \Delta[\rho(Q)\omega]\rangle = \Delta H(Q) - \rho(Q)|\nabla\omega|^2 - 2Q\rho'(Q)|\mathrm{d}|\omega||^2. \tag{5.15}$$

Applying equation (5.15) to the operator identity $\Delta = -(\delta\,\mathrm{d} + \mathrm{d}\delta)$ and using (2.8), we write

$$\begin{aligned}0 &= \langle\omega, \Delta[\rho(Q)\omega]\rangle + \langle\omega, \delta\mathrm{d}(\rho(Q)\omega)\rangle + \langle\omega, \mathrm{d}\delta(\rho(Q)\omega)\rangle \\ &= \Delta H(Q) - \gamma + \langle\omega, \delta(\mathrm{d}\rho \wedge \omega)\rangle + \tau_1 + \tau_2,\end{aligned} \tag{5.16}$$

where

$$\begin{aligned}\gamma &= \rho(Q)|\nabla\omega|^2 + 2Q\rho'(Q)|\mathrm{d}|\omega||^2, \\ \tau_1 &= \langle\omega, \delta(\rho(Q)\,\mathrm{d}\omega)\rangle\end{aligned}$$

and

$$\tau_2 = \langle\omega, \mathrm{d}[\rho(Q)\langle\Gamma, \omega\rangle]\rangle.$$

Define

$$L_\omega(H) \equiv \Delta H + \langle\omega, \delta(\mathrm{d}\rho(Q) \wedge \omega)\rangle.$$

Then (5.16) can be written in the compact form

$$L_\omega H + \tau_1 + \tau_2 = \gamma. \tag{5.17}$$

We have

$$\begin{aligned}\tau_1 &= \langle\omega, \rho(Q)\delta(\Gamma \wedge \omega) - \langle\mathrm{d}\rho(Q), \mathrm{d}\omega\rangle\rangle \\ &\leqslant |\omega|[\rho(Q)|\delta(\Gamma \wedge \omega)| + |\mathrm{d}\rho(Q)||\mathrm{d}\omega|] \\ &\leqslant |\omega|[\rho(Q)|\nabla\Gamma||\omega| + \rho(Q)|\Gamma||\nabla\omega| + |\rho'(Q)\mathrm{d}Q|\,|\mathrm{d}\omega|].\end{aligned}$$

Applying (2.2) to the last term on the right-hand side and using

$$|\mathrm{d}Q| = 2|\omega||\mathrm{d}|\omega||,$$

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we obtain

$$\begin{aligned}\tau_1 &\leqslant Q\rho(Q)|\nabla\Gamma| + \rho(Q)|\Gamma||\omega||\nabla\omega| + 2Q|\rho'(Q)||\Gamma||\omega|\,|\mathrm{d}|\omega|| \\ &\leqslant Q\rho(Q)|\nabla\Gamma| + \rho(Q)\left[\frac{Q|\Gamma|^2}{2\varepsilon} + \tfrac{1}{2}\varepsilon|\nabla\omega|^2\right] + 2Q|\rho'(Q)|\left(\tfrac{1}{2}\varepsilon|\mathrm{d}|\omega||^2 + \frac{|\Gamma|^2 Q}{2\varepsilon}\right).\end{aligned} \tag{5.18}$$

Estimating τ_2 yields the same terms:

$$\begin{aligned}\tau_2 &\leqslant Q\rho(Q)|\nabla\Gamma| + |\omega||\Gamma|[\rho(Q)|\nabla\omega| + |\omega|\nabla\rho(Q)|] \\ &= Q\rho(Q)|\nabla\Gamma| + |\omega||\Gamma|\rho(Q)|\nabla\omega| + 2Q|\Gamma||\rho'(Q)||\omega||\mathrm{d}|\omega||,\end{aligned} \tag{5.19}$$

which is bounded by the right-hand side of (5.18). That is,

$$\begin{aligned}\tau_1 + \tau_2 \leqslant 2Q\rho(Q)|\nabla\Gamma| &+ \frac{Q}{\varepsilon}[\rho(Q) + 2Q|\rho'(Q)|]|\Gamma|^2 \\ &+ \varepsilon\rho(Q)|\nabla\omega|^2 + 2Q\varepsilon|\rho'(Q)||\mathrm{d}|\omega||^2.\end{aligned} \tag{5.20}$$

Applying inequality (5.20) to equation (5.17) we obtain, in the case where $\rho'(Q) > 0$, the estimate

$$\begin{aligned}L_\omega(H) + 2\left[Q\rho(Q)|\nabla\Gamma| + \frac{QH'(Q)}{\varepsilon}|\Gamma|^2\right] &\geqslant (1-\varepsilon)[\rho(Q)|\nabla\omega|^2 + 2Q\rho'(Q)|\mathrm{d}|\omega||^2] \\ &\geqslant (1-\varepsilon)H'(Q)|\mathrm{d}|\omega||^2,\end{aligned} \tag{5.21}$$

the inequality on the right following from Kato's inequality. We apply lemma 5.1 and (5.7) to terms on the extreme left-hand side of inequality (5.21):

$$\begin{aligned}2\left[Q\rho(Q)|\nabla\Gamma| + \frac{QH'(Q)}{\varepsilon}|\Gamma|^2\right] &\leqslant C[H(Q)|\nabla\Gamma| + Q\rho(Q)|\Gamma|^2] \\ &\leqslant C(|\nabla\Gamma| + |\Gamma|^2)H(Q).\end{aligned}$$

We now have, for the case where $\rho'(Q) > 0$, the estimate

$$L_\omega(H) + C(|\nabla\Gamma| + |\Gamma|^2)H \geqslant (1-\varepsilon)H'(Q)|\mathrm{d}|\omega||^2 \geqslant 0. \tag{5.22}$$

In the case where $\rho'(Q) < 0$, we also apply (5.20) to (5.17), but here we obtain

$$\begin{aligned}&L_\omega(H) + 2\left[Q\rho(Q)|\nabla\Gamma| + \frac{QH'(Q)}{\varepsilon}|\Gamma|^2\right] \\ &\quad\geqslant (1-\varepsilon)[\rho(Q)|\nabla\omega|^2 + 2Q\rho'(Q)|\mathrm{d}|\omega||^2] + 4\varepsilon Q\rho'(Q)|\mathrm{d}|\omega||^2.\end{aligned} \tag{5.23}$$

In the case where $\rho'(Q) < 0$, condition (5.8) yields

$$-Q\rho'(Q) < \tfrac{1}{2}\rho(Q) \leqslant CH'(Q),$$

and thus,

$$Q|\rho'(Q)| \leqslant CH'(Q). \tag{5.24}$$

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(Note that (5.24) is automatic for $\rho'(Q) \geqslant 0$.) Applying Kato's inequality and (5.24) to (5.23) yields

$$L_\omega(H) + 2\left[Q\rho(Q)|\nabla\Gamma| + \frac{QH'(Q)}{\varepsilon}|\Gamma|^2\right] \geqslant (1-\varepsilon-4\varepsilon C)H'(Q)|\mathrm{d}|\omega\|^2 \\ \geqslant 0.$$

Applying (5.7) and (5.10) to the left-hand side of this inequality, we obtain (5.22) (for updated ε) for the case where $\rho'(Q) < 0$ as well.

It remains only to show that the operator $L_\omega(H)$ can be put into divergence form, at the cost of absorbing another lower-order term. The second term of $L_\omega(H)$ can be written in the form

$$\begin{aligned}\langle\omega, \delta(\mathrm{d}\rho(Q)\wedge\omega)\rangle &= *[\omega\wedge *\delta(\mathrm{d}\rho(Q)\wedge\omega)] \\ &= *\mathrm{d}[\omega\wedge *(\mathrm{d}\rho(Q)\wedge\omega)] - *[\mathrm{d}\omega\wedge *(\mathrm{d}\rho(Q)\wedge\omega)],\end{aligned}$$

where

$$\begin{aligned}-*[\mathrm{d}\omega\wedge *(\mathrm{d}\rho(Q)\wedge\omega)] &= -*[\Gamma\wedge\omega\wedge *(\mathrm{d}\rho(Q)\wedge\omega)] \\ &\geqslant -2Q|\Gamma\|\rho'(Q)\|\omega\|\mathrm{d}|\omega\|,\end{aligned}$$

which is estimated in the same way as the last term in the sum on the far right-hand side of (5.19).

Taking into account (5.3) we obtain, for any k-form α,

$$\begin{aligned}*\mathrm{d}\alpha &= (-1)^{k(n-k)} * \mathrm{d}(**)\alpha \\ &= (-1)^{k(n-k)}(*\mathrm{d}*) * \alpha \\ &= (-1)^{2kn+n+1-k^2}\delta * \alpha.\end{aligned} \tag{5.25}$$

Taking $k = 1$ and

$$\alpha = *[\omega\wedge *(\rho'(Q)\,\mathrm{d}Q\wedge\omega)]$$

in equation (5.25), we can express the operator δ in that equation as a divergence. This allows us to write (5.22) (again updating C and ε) in the form

$$\Delta H + (-1)^{3n}\nabla\cdot\{*[\omega\wedge *(\rho'(Q)\mathrm{d}Q\wedge\omega)]\} \\ + C(|\Gamma|^2 + |\nabla\Gamma|)H \geqslant (1-\varepsilon)H'(Q)|\mathrm{d}|\omega\|^2 \geqslant 0.$$

This completes the proof of lemma 5.2. □

LEMMA 5.3. *Under the hypotheses of lemma 5.2, the operator*

$$\mathcal{L}_\omega(H) \equiv \Delta H + (-1)^{3n}\nabla\cdot\{*[\omega\wedge *(\rho'(Q)\,\mathrm{d}Q\wedge\omega)]\}$$

is a uniformly elliptic operator on H.

Proof. We argue as in [44, § 1] and [29, § 4], but without using hypothesis (5.6).

Define a map $\beta_\omega : \Lambda^0 \to \Lambda^{k+1}$ by the explicit formula

$$\beta_\omega : \mu \to \mathrm{d}\mu\wedge\omega \tag{5.26}$$

for $\mu \in \Lambda^0$ and $\omega \in \Lambda^k$. Then we can write the variational form of the Hodge–Frobenius equations (2.2), (2.7) and (2.8) in the alternate form

$$\mathrm{d}\sigma_i = -\beta_{\sigma_i}(\eta), \quad \sigma = 1, 2, \tag{5.27}$$

where $\sigma_1 = *\rho(Q)\omega$ and $\sigma_2 = -\omega$. The 'irrotational' case $\mathrm{d}\omega = 0$ can be recovered as the special case of (5.27) in which

$$\beta_\omega(\eta) = \mathrm{d}(\eta\omega),$$

in which case equation (5.26) implies that

$$\mathrm{d}\eta \wedge \omega = \mathrm{d}\eta \wedge \omega + \eta\, \mathrm{d}\omega$$

(cf. [29, § 4] and [44, § 1]).

Moreover, writing

$$\beta_\omega(g) = dg \wedge \omega$$

for some 0-form g we compute, for arbitrary compactly supported $\mu \in \Lambda^{k+1}$,

$$\begin{aligned}\langle \mu, \mathrm{d}g \wedge \omega \rangle &= *(\mathrm{d}g \wedge (**)(\omega \wedge *\mu)) \\ &= \langle \mathrm{d}g, *(\omega \wedge *\mu) \rangle \\ &= \langle g, \delta * (\omega \wedge *\mu) \rangle.\end{aligned}$$

So the map $\beta_\omega^* : \Lambda^{k+1} \to \Lambda^0$ defined by the explicit formula

$$\beta_\omega^*(\mu) = \delta * (\omega \wedge *\mu)$$

is the formal adjoint of β_ω.

In terms of the maps β_ω and β_ω^*, we can write

$$\begin{aligned}(-1)^{n+1}\nabla \cdot \{*[\omega \wedge *(\rho'(Q)\mathrm{d}Q \wedge \omega)]\} &= \beta_\omega^* \beta_\omega[\rho] \\ &= \beta_\omega^*[\mu_\omega(H)]\end{aligned}$$

for μ_ω satisfying

$$\mu_\omega(H) = \frac{\rho'(Q)}{H'(Q)}\, \mathrm{d}H \wedge \omega.$$

Using β_ω^*, we write the inequality of lemma 5.2 in the form

$$\mathcal{L}_\omega(H) + \text{ lower-order terms } \geqslant 0,$$

where

$$\mathcal{L}_\omega(H) = \Delta H - \beta_\omega^*[\mu_\omega(H)].$$

Writing

$$\mathcal{L}_\omega(H) = \partial_k(\alpha^{jk}\partial_j)H,$$

we find that if $\rho'(Q) < 0$, then (5.24) implies that

$$1 \leqslant \alpha_{kj} + \frac{Q|\rho'(Q)|}{H'(Q)} \leqslant \alpha_{kj} + C.$$

If $\rho'(Q) \geqslant 0$, then we write

$$\nabla \cdot \left\{ \left[1 - \frac{Q\rho'(Q)}{H'(Q)} \right] \nabla H \right\} = \nabla \cdot \left\{ \left[\frac{\rho(Q)}{2H'(Q)} \right] \nabla H \right\}.$$

Condition (5.7) implies that there is a positive constant c such that

$$\tfrac{1}{2}c \leqslant \frac{\rho(Q)}{2H'(Q)} \leqslant 1,$$

where c is the reciprocal of the constant C in (5.7). This completes the proof of lemma 5.3. □

These three lemmas easily yield the following.

THEOREM 5.4. *Under the hypotheses of lemma 5.2, the product $Q\rho(Q)$ is locally bounded above by the L^2-norm of H.*

The proof of theorem 5.4 is given in § A.1.

Of course, the integrability of H depends on ρ. But for any given ρ in $C^1(Q)$, H can be computed explicitly by integrating (5.5). Even if ρ cavitates, theorem 5.4 yields asymptotic information about the fastest rate at which Q can blow up. Nevertheless, theorem 5.4 is ultimately not very useful, due to the hypothesis that the solutions are smooth. It would become more useful if it could be applied to singular solutions. We will find that the partial removable singularities theorems proven in [29] for singular sets of prescribed codimension extend to our conditions on ρ under slightly different hypotheses.

Initially, we treat the special case of an isolated point singularity, for which the proof is somewhat simpler than the proof for higher-order singularities and the range of applicable dimensions somewhat larger.

THEOREM 5.5. *Let the hypotheses of lemma 5.2 be satisfied on $\Omega \setminus \{p\}$, where p is a point of $\mathbb{R}^n$ and $n > 2$. If $H \in L^{2n/(n-2)}(\Omega)$ and if the function*

$$f \equiv |\nabla \Gamma| + |\Gamma|^2 \tag{5.28}$$

is sufficiently small in $L^{n/2}(\Omega)$, then H is an $H^{1,2}$-weak solution in a neighbourhood of the singularity.

The proof of theorem 5.5 is given in § A.2.

COROLLARY 5.6. *Let the hypotheses of theorem 5.5 be satisfied and, in addition, let the function f given by (5.28) satisfy the growth condition*

$$\int_{B_r(\boldsymbol{x}_0)\cap\Omega} |f|^{n/2} \,\mathrm{d}\Omega \leqslant Cr^{\kappa} \tag{5.29}$$

for some $\kappa > 0$, where $B_r(\boldsymbol{x}_0)$ is an n-disc of radius r, centred at $\boldsymbol{x}_0$. Then the conclusion of theorem 5.4 remains valid.

Proof. Apply [27, theorem 5.3.1] to the conclusion of theorem 5.5, following the proof of theorem 5.4. □

Note that the singularity in corollary 5.6 is in the solution, rather than in the underlying metric (cf. [41], in which metric point singularities are considered in the case when $\Gamma \equiv 0$). Corollary 5.6 extends to higher-order singularities in spaces of sufficiently high dimension, but the proof requires more delicate test functions.

THEOREM 5.7. *Let the pair ω and Γ smoothly satisfy equations (2.2) and (2.8), with ρ satisfying the hypotheses of lemma 5.2, on $\Omega \setminus \Sigma$. Here Σ is a compact singular set of dimension $0 \leqslant m < n-4$, completely contained in a sufficiently small n-disc D which is itself completely contained in the interior of Ω. Let $H(Q)$ lie in $L^{2\beta\gamma_1}(D) \cap L^{2\gamma_2}(D)$, where $\beta = (n-m-\varepsilon)/(n-m-2-\varepsilon)$ for $\frac{1}{2} < \gamma_1 < \gamma_2$. If the function f given by (5.28) satisfies the growth condition (5.29), then the conclusion of theorem 5.4 remains valid.*

The proof of theorem 5.7 is given in § A.3.

The conditions imposed on ρ in this section also lead to extensions of known results for the conventional case $\Gamma \equiv 0$. In particular, we consider equations which, expressed in components, have the weak form

$$\int_\Omega [\rho(Q) u_{x_k}]_{x_i} \varphi_{x_k} \, \mathrm{d}\Omega = 0, \tag{5.30}$$

where $\varphi \in C_0^\infty(\Omega)$ is arbitrary and $Q = |\mathrm{d}u|^2$. Such equations have been intensively studied in cases for which $\rho(|\mathrm{d}u|^2)$ grows as a power of $|\mathrm{d}u|$ [9, equation (3.10)]. Equation (5.30) can be interpreted as a weak derivative with respect to x_i of the system (2.1), (2.2) with $\Gamma \equiv 0$.

Define

$$\mathfrak{H}(Q) \equiv Q\rho^2(Q).$$

The system (2.1), (2.2) is elliptic precisely when $\mathfrak{H}'(Q) > 0$. The following result extends [29, theorem 1], which requires the derivative of ρ with respect to Q to be non-positive. If ρ is specified to be non-increasing in Q, then the integrability of $\mathfrak{H}(Q)$ can be shown to follow from finite energy. In the general case, we impose this integrability as an independent hypothesis on the weak solution.

THEOREM 5.8. *Let the scalar function $u(\boldsymbol{x})$ satisfy equation (5.30) with ρ bounded, positive and non-cavitating, and with $\mathfrak{H}(Q)$ integrable. Assume conditions (2.6) and (5.7). Then, for every n-disc D_R of radius R completely contained in Ω, there is a positive number $\delta > 0$ such that*

$$\sup_{Q \in D_{(1-\delta)R}} \mathfrak{H}(Q) \leqslant CR^{-n} \int_{D_R} \mathfrak{H}(Q) * 1,$$

where C depends on ρ and δ but not on Q or R.

Proof. Define Q_* to be the (possibly infinite) value of Q for which

$$\sup_{Q \in \Omega} \rho(Q) = \rho(Q_*).$$

Then

$$\frac{\mathfrak{H}'(Q)}{\rho^2(Q_*)} = \frac{2\rho(Q)H'(Q)}{\rho^2(Q_*)} \leqslant C\left(\frac{\rho(Q)}{\rho(Q_*)}\right)^2 \leqslant C, \tag{5.31}$$

where C is the constant in (5.7).

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As in [29], we initially estimate smooth solutions, and subsequently extend the result to weak solutions, recovering the derivatives as limits of difference quotients.

We choose test functions φ^i having the form

$$\varphi^i(\boldsymbol{x}) = u_{x_i}\tilde{\mathfrak{H}}^{\alpha/2}\zeta^2,$$

where $\alpha > 0$, $\zeta(\boldsymbol{x}) \in C_0^\infty(D_R)$, and

$$\tilde{\mathfrak{H}} \equiv \mathfrak{H}(Q) + \varepsilon \tag{5.32}$$

for a small positive parameter ε (cf. [9, § 3]). Then

$$\begin{aligned}[\rho(Q)u_{x_k}]_{x_i}\varphi^i_{x_k} &= 2H'(Q)(u_{x_i x_k})^2\tilde{\mathfrak{H}}^{\alpha/2}\zeta^2 \\ &\quad + \alpha\rho(Q)[H'(Q)]^2|\nabla Q|^2\tilde{\mathfrak{H}}^{(\alpha-2)/2}\zeta^2 \\ &\quad + 2H'(Q)Q_{x_k}\tilde{\mathfrak{H}}^{\alpha/2}\zeta\zeta_{x_k} \\ &\equiv i_1 + i_2 + i_3.\end{aligned}$$

Here

$$\begin{aligned}i_1 &\geqslant 2H'(Q)u_{x_i x_k}u_{x_i x_k}Q\rho^2(Q)\tilde{\mathfrak{H}}^{(\alpha-2)/2}\zeta^2 \\ &= \tfrac{1}{2}\rho^2(Q)H'(Q)|\nabla Q|^2\tilde{\mathfrak{H}}^{(\alpha-2)/2}\zeta^2 \\ &\geqslant C|\nabla(\tilde{\mathfrak{H}}^{(\alpha+2)/4})|^2\zeta^2,\end{aligned}$$

where the last inequality follows from (5.31) and the constant C depends on α, $\rho^{-1}(Q_*)$, the constant of (5.31) and the lower bound of $\rho(Q)$.

Similarly,

$$i_2 \geqslant \frac{\rho(Q)}{\rho(Q_*)}i_2 \geqslant C|\nabla(\tilde{\mathfrak{H}}^{(\alpha+2)/4})|^2\zeta^2$$

and

$$i_3 = 2H'(Q)Q_{x_j}\tilde{\mathfrak{H}}^{\alpha/2}\zeta(\boldsymbol{x})\zeta_{x_j}.$$

The latter quantity can be estimated by applying, as in the proof of lemma 5.2, the elementary algebraic inequality *(Young's inequality)*:

$$ab \geqslant -\left(\frac{\tilde{\varepsilon}}{2}a^2 + \frac{1}{2\tilde{\varepsilon}}b^2\right).$$

Now, take

$$a = 2H'(Q)\tilde{\mathfrak{H}}^{(\alpha-2)/4}\zeta Q_{x_j}, \qquad b = \tilde{\mathfrak{H}}^{(\alpha+2)/4}\zeta_{x_j}$$

and $\tilde{\varepsilon} = \tilde{\delta}\rho^2(Q)$ for small $\tilde{\delta} > 0$.

Putting these estimates together, we obtain

$$\int_\Omega |\nabla(\tilde{\mathfrak{H}}^{(\alpha+2)/4})|^2\zeta^2\,\mathrm{d}\Omega \leqslant C\int_\Omega \tilde{\mathfrak{H}}^{(\alpha+2)/2}|\nabla\zeta|^2\,\mathrm{d}\Omega.$$

The proof for the smooth case is completed by applying the Moser iteration as in [20, (9.5.8)–(9.5.12)] and subsequently letting the parameter ε in (5.32) tend to zero. The proof is extended to the general case by applying the difference-quotient method as in [29, equation (13)] (see also [44, lemma 2]). □

6. Hodge–Bäcklund transformations

Different choices of ρ may sometimes be related by a special kind of Bäcklund transformation which is based on properties of the Hodge involution. We call these transformations *Hodge–Bäcklund*. Although this term does not seem to have been used up to now, such generalized Bäcklund transformations have a long history in diverse fields of mathematical physics. Our aim in this section is to unify these various transformations, to place them in an invariant context and extend them to the completely integrable case.

Historically, the term *Bäcklund transformation* has been defined in many ways (see, for example, [33] for the classical theory). In what follows we will use it in the general sense of a function that maps a solution a of a differential equation A into a solution b of a differential equation B and vice versa, where B may equal A but b will not equal a.

6.1. Transformation of the Chaplygin mass density

The mass density for the adiabatic and isentropic subsonic flow of an ideal fluid has the form

$$\rho(Q) = \left(1 - \frac{\gamma - 1}{2} Q\right)^{1/(\gamma-1)} \tag{6.1}$$

for $Q \in [0, 2/(\gamma+1))$, where γ is the *adiabatic constant*: the ratio of specific heats for the gas. The adiabatic constant for air is 1.4. Choosing $\gamma = 2$, we obtain, by an independent physical argument originally introduced for one spatial dimension in [32], the mass density for shallow hydrodynamic flow in the tranquil regime [42, equation (10.12.5)]. If we choose γ to be -1 (a physically impossible choice), we obtain the density of the minimal surface equation [19, 39]

$$\rho(Q) = \frac{1}{\sqrt{1+Q}}. \tag{6.2}$$

Flow governed by this density is called *Chaplygin flow*. Despite the fact that the numbers -1 and 1.4 are not particularly close, this choice of mass density has many attractive properties as an approximation for (6.1) (see, for example, [11] and [5, ch. 5]). These properties are, in general, retained in the case of completely integrable flow described in § 1.2.

If $\Gamma \equiv 0$, equations (2.1) and (2.2) with $\rho(Q)$ given by (6.2) describe, for $k = 1$, non-parametric minimal surfaces embedded in Euclidean space. If $k = 2$, they describe electromagnetic fields in the Born–Infeld model, as in (4.4).

More generally, we have the following result, which extends an argument introduced for the case $\Gamma \equiv 0$ by Yang [45] (see also [2] and [40, theorem 2.1]).

Theorem 6.1. *Let the 1-form ω satisfy equations (2.2) and (2.8), with ρ satisfying (6.2). Then there exists an $(n-1)$-form ξ with $|\xi| < 1$, satisfying equations analogous to (2.2) and (2.8), but with $\Gamma \equiv \mathrm{d}\eta$ replaced by $\hat{\Gamma} \equiv \mathrm{d}\hat{\eta} = -\mathrm{d}\eta$ and $\rho(Q)$ replaced by*

$$\hat{\rho}(|\xi|^2) \equiv \frac{1}{\sqrt{1-|\xi|^2}}. \tag{6.3}$$

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Proof. Equation (2.8) can be interpreted as the assertion that the $(n-1)$-form

$$\xi = *[\rho(Q)\omega] = *\left[\frac{\omega}{\sqrt{1+|\omega|^2}}\right] \tag{6.4}$$

satisfies

$$\mathrm{d}\xi = \mathrm{d}\hat{\eta} \wedge \xi, \tag{6.5}$$

that is, equation (2.2) with η replaced by $\hat{\eta} \equiv -\eta$. As a consequence, we conclude that this $(n-1)$-form is also gradient recursive and, on domains with trivial de Rham cohomology, there exists an $(n-2)$-form σ such that $\xi = \mathrm{e}^{\hat{\eta}}\,\mathrm{d}\sigma$ (cf. § 2.2).

Because the Hodge involution is an isometry,

$$|\xi|^2 = \frac{|\omega|^2}{1+|\omega|^2}$$

or, equivalently,

$$1 - |\xi|^2 = \frac{1}{1+|\omega|^2}. \tag{6.6}$$

Note that equation (6.6) implies $|\xi|^2 < 1$, as well as

$$\rho(|\omega|^2)\hat{\rho}(|\xi|^2) = \frac{1}{\sqrt{1+|\omega|^2}}\frac{1}{\sqrt{1-|\xi|^2}} = 1.$$

This, together with (6.4), directly yields

$$\begin{aligned} *\hat{\rho}(|\xi|^2)\xi &= *^2\rho(|\omega|^2)\hat{\rho}(|\xi|^2)\omega \\ &= (-1)^{n-1}\omega. \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{d} * (\hat{\rho}(|\xi|^2)\xi) &= (-1)^{n-1}\,\mathrm{d}\omega \\ &= (-1)^{n-1}\,\mathrm{d}\eta \wedge \omega \\ &= \mathrm{d}\eta \wedge *(\hat{\rho}(|\xi|^2)\xi) \\ &= -\mathrm{d}\hat{\eta} \wedge *(\hat{\rho}(|\xi|^2)\xi), \end{aligned}$$

which is equivalent to equation (2.8) for the gradient recursive $(n-1)$-form ξ with coefficient $d\hat{\eta}$, where ρ has been replaced by $\hat{\rho}$ (see also (5.4)). This completes the proof of theorem 6.1. □

The above argument carries over to any pairing of functions $\rho(|\omega|^2)$, $\hat{\rho}(|\xi|^2)$, as long as their product is 1. Moreover, it carries over essentially unchanged to gradient-recursive k-forms ω, as these would automatically yield gradient-recursive $(n-k)$-forms ξ.

The same argument, with small modifications, extends theorem 6.1 to general (non-gradient-recursive) k-forms, in which we write equations (2.2) and (2.8) in terms of Γ and $\hat{\Gamma}$ rather than in terms of η and $\hat{\eta}$.

In addition to the original, 'irrotational' version of theorem 6.1 introduced in [45], other aspects of the duality of mass densities for nonlinear Hodge equations are presented in [38] and [30, § 4.2]. (The irrotational case of the above argument is reviewed in [30, § 4.1]; note the recurring misprint in the two paragraphs following equation (35) of that reference: dω should be du.) Densities of the form (6.3) arise in the study of maximal space-like hypersurfaces [8] and, in a completely different way, harmonic diffeomorphisms [43].

6.2. Transformation of the complex eikonal equation (after Magnanini and Talenti)

As an example of the diverse fields in which these transformations arise, and of the simplifying and unifying role of the Hodge involution, we describe an example from complex optics. The description of the local Bäcklund transformations follows the analysis of Magnanini and Talenti [21], in which these transformations were introduced (see also [22, 23]). We then reproduce the argument of [21] in a simpler, invariant context using the Hodge operator.

The *eikonal equation* in $\mathbb{R}^2$ can be written in the form

$$\psi_x^2 + \psi_y^2 + \nu^2 = 0, \tag{6.7}$$

where $\nu(x, y)$ is a given real-valued function. If we write the solution $\psi(x, y)$ as a complex function having the form

$$\psi(x, y) = u(x, y) + \mathrm{i}v(x, y)$$

for real-valued functions u and v, then (6.7) is equivalent to the first-order system

$$u_x^2 + u_y^2 - v_x^2 - v_y^2 + \nu^2 = 0, \tag{6.8}$$

$$u_x v_x + u_y v_y = 0. \tag{6.9}$$

The function ν corresponds physically to the refractive index of the medium through which the wavefront represented by the function ψ propagates. If

$$u_x^2 + u_y^2 > 0, \tag{6.10}$$

then, by equation (6.8), v_x and v_y cannot both vanish. In that case, equation (6.9) can be expressed, in the language of proportions, by the assertion that either

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} : \sqrt{v_x^2 + v_y^2} = \pm \begin{bmatrix} -u_y \\ u_x \end{bmatrix} : \sqrt{u_x^2 + u_y^2}, \tag{6.11}$$

or

$$u_x = u_y = 0. \tag{6.12}$$

Treating these relations as a coupled system of scalar equations, we have, under the same hypothesis,

$$\frac{v_x}{\sqrt{v_x^2 + v_y^2}} = \pm \frac{-u_y}{\sqrt{u_x^2 + u_y^2}}$$

and, using equation (6.8),

$$v_x = \pm(-u_y)\sqrt{\frac{\nu^2}{u_x^2 + u_y^2} + 1}. \tag{6.13}$$

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Similarly, the scalar equation

$$\frac{v_y}{\sqrt{v_x^2+v_y^2}} = \pm\frac{u_x}{\sqrt{u_x^2+u_y^2}},$$

which also follows from (6.11), implies, by analogous operations, the equation

$$v_y = \pm u_x\sqrt{\frac{\nu^2}{u_x^2+u_y^2}+1}. \tag{6.14}$$

We can write the coupled system (6.13), (6.14) as a vector equation of the form

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \pm\sqrt{1+\frac{\nu^2}{u_x^2+u_y^2}}\begin{bmatrix} -u_y \\ u_x \end{bmatrix}, \tag{6.15}$$

or as a single (exact) equation for 1-forms,

$$\mathrm{d}v = \pm\sqrt{1+\frac{\nu^2}{u_x^2+u_y^2}}(-u_y\,\mathrm{d}x + u_x\,\mathrm{d}y). \tag{6.16}$$

This implies the local existence of a solution to the divergence-form equation

$$\frac{\partial}{\partial x}\left(\sqrt{1+\frac{\nu^2}{u_x^2+u_y^2}}u_x\right) + \frac{\partial}{\partial y}\left(\sqrt{1+\frac{\nu^2}{u_x^2+u_y^2}}u_y\right) = 0, \tag{6.17}$$

whenever condition (6.10) is satisfied.

Equations (6.8), (6.9) and (6.11) can also be solved for u_x and u_y, in addition to being solvable for v_x and v_y as in (6.13) and (6.14). Under the hypothesis that either

$$v_x^2 + v_y^2 = \nu^2 \tag{6.18}$$

or

$$v_x^2 + v_y^2 > \nu^2, \tag{6.19}$$

one obtains, by completely analogous arguments to those applied to v_x and v_y, the vector equation

$$\begin{bmatrix} u_x \\ u_y \end{bmatrix} = \mp\sqrt{1-\frac{\nu^2}{v_x^2+v_y^2}}\begin{bmatrix} -v_y \\ v_x \end{bmatrix} \tag{6.20}$$

and the equation

$$\mathrm{d}u = \mp\sqrt{1-\frac{\nu^2}{v_x^2+v_y^2}}(-v_y\,\mathrm{d}x + v_x\,\mathrm{d}y) \tag{6.21}$$

for 1-forms. Note that equation (6.21) is exact either if (6.18) holds, or if (6.19) holds and the divergence-form equation

$$\frac{\partial}{\partial x}\left(\sqrt{1-\frac{\nu^2}{v_x^2+v_y^2}}v_x\right) + \frac{\partial}{\partial y}\left(\sqrt{1-\frac{\nu^2}{v_x^2+v_y^2}}v_y\right) = 0 \tag{6.22}$$

is satisfied.

6.2.1. *Hodge–Bäcklund interpretation*

Equations (6.17) and (6.22) define a Bäcklund transformation $u \to v$ and its inverse $v \to u$ [23]. These equations can be written in the form of nonlinear Hodge equations:

$$\mathrm{d} * (\rho(Q)\omega) = 0, \tag{6.23}$$

$$\mathrm{d}\omega = 0 \tag{6.24}$$

(that is, as equations (2.1) and (2.2) for $\Gamma \equiv 0$). Either

$$\rho(|\omega|^2) = \sqrt{1 + \frac{\nu^2}{|\omega|^2}}, \tag{6.25}$$

corresponding to equation (6.17), or

$$\hat{\rho}(|\xi|^2) = \sqrt{1 - \frac{\nu^2}{|\xi|^2}}, \tag{6.26}$$

corresponding to equation (6.22). In either case we assume that Q does not vanish, by analogy with equations (6.10), (6.12), (6.18) and (6.19).

Take ρ as in (6.25). If the domain is simply connected, then equation (6.24) implies, analogously to § 5.1, that there is a 0-form u such that

$$\omega = \mathrm{d}u,$$

and a 0-form v such that

$$\mathrm{d}v = \pm * \left(\sqrt{1 + \frac{\nu^2}{|\omega|^2}}\omega\right) = \pm * \left(\sqrt{|\mathrm{d}u|^2 + \nu^2}\frac{\omega}{|\omega|}\right). \tag{6.27}$$

Because the Hodge operator is an isometry, which is illustrated locally by (6.11), we have

$$|\mathrm{d}v|^2 = |\mathrm{d}u|^2 + \nu^2.$$

Thus, the Hodge–Bäcklund transformation (6.27) yields an invariant form of (6.8). Unlike classical Bäcklund transformations of the eikonal equation, in this case the Cauchy–Riemann equations are not satisfied. Rather,

$$u_x = \mp\rho(Q)v_y \quad \text{and} \quad u_y = \pm\rho(Q)v_x,$$

which is sufficient for the orthogonality condition (6.9).

Now take $\hat{\rho}$ as in (6.26). Arguing as before, we conclude that there is a 0-form $\tilde{u}$ such that $\omega = \mathrm{d}\tilde{u}$ and a 0-form $\tilde{v}$ such that

$$\mathrm{d}\tilde{v} = \pm * \left(\sqrt{Q - \nu^2}\frac{\omega}{|\omega|}\right).$$

We obtain

$$|\mathrm{d}\tilde{v}|^2 = |\mathrm{d}\tilde{u}|^2 - \nu^2.$$

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Letting $\tilde{u} = \pm\mathrm{i}u$ and $\tilde{v} = \pm\mathrm{i}v$, we obtain a mapping taking solutions to equations (2.2) and (2.8) with $\Gamma \equiv 0$ and $\hat{\rho}$ satisfying (6.26) into solutions of that system with $\Gamma \equiv 0$ and ρ satisfying (6.25).

These arguments extend immediately to the Hodge–Frobenius case: for example, by replacing (6.27) with the expansion

$$\mathrm{e}^{\hat{\eta}}\,\mathrm{d}v = \pm *\left(\sqrt{\mathrm{e}^{2\eta}|\mathrm{d}u|^2 + \nu^2}\,\frac{\omega}{|\omega|}\right)$$

and squaring both sides. They also extend in a straightforward way to gradient-recursive k-forms and, with some modifications, to general k-forms (see the remarks following theorem 6.1).

Motivated by these examples, we offer a general definition of the *Hodge–Bäcklund transformation*. It is a map taking a solution a of a nonlinear Hodge–Frobenius equation having mass density ρ_A into a solution b of a nonlinear Hodge–Frobenius equation having mass density ρ_B and vice versa, where B may equal A but b will not equal a.

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Appendix A. Methods from elliptic theory

In this appendix we collect the proofs of theorems 5.4, 5.5 and 5.7, which follow directly from the association of solutions to a uniformly subelliptic operator via lemmas 5.1–5.3.

A.1. Proof of theorem 5.4

We require the following well-known extension of de Giorgi–Nash–Moser theory.

Theorem A.1 (Morrey [27, theorem 5.3.1]). *Let $n > 2$. Let $U \in H^{1,2}(D)$ for each n-disc $D \subset\subset \Omega$, where $U(\boldsymbol{x}) \geqslant 1$ and define an L^2-function $W = U^{\lambda}$ for some $\lambda \in [1, 2)$. Suppose that*

$$\int_{\Omega} (a^{\alpha\beta}\partial_\beta W \partial_\alpha \zeta + fW\zeta)\,\mathrm{d}\Omega \leqslant 0 \tag{A 1}$$

for all $\zeta \in C_0^\infty(\Omega)$ with $\zeta(\boldsymbol{x}) \geqslant 0$, where the coefficients a and f are measurable with $f \in L^{n/2}(\Omega)$. Let the matrix a satisfy the ellipticity condition

$$C|\xi|^2 \leqslant a^{\alpha\beta}(\boldsymbol{x})\xi_\alpha\xi_\beta$$

for $|a(\boldsymbol{x})| \leqslant M$ at almost every $\boldsymbol{x} \in \Omega$ and for all ξ. Moreover, let the growth condition (5.29) be satisfied. If $U \in L^2(\Omega)$, then U is bounded on each n-disc $D \subset\subset \Omega$ and satisfies

$$|U(\boldsymbol{x})|^2 \leqslant C a_0^{-n} \int_{D_{(R+a_0)}(\boldsymbol{x}_0)} |U(\boldsymbol{y})|^2\,\mathrm{d}\boldsymbol{y},$$

where $\boldsymbol{x}$ is a point of $D_R(\boldsymbol{x}_0) \subset \Omega$.

Slightly modified conditions will extend Morrey's result to $n = 2$ [27, § 5.4].

We apply theorem A.1, taking $U(\boldsymbol{x}) = H(Q(\boldsymbol{x}))+1$ and $\lambda = 1$. Let a be given by the matrix α of lemma 5.3 and let f be given by (5.28). Then f satisfies the local growth condition (5.29) by smoothness. The inequality of lemma 5.2 completes the proof of theorem 5.4.

A.2. Proof of theorem 5.5

Without loss of generality, take p to be the origin of coordinates. Write the operator $\mathcal{L}_\omega$ in the form

$$\mathcal{L}_\omega(H) = \Delta H + \nabla \cdot [T(H)]$$

for

$$T = (-1)^{3n} * \{\omega \wedge *[\rho'(Q)\,\mathrm{d}Q \wedge \omega]\}.$$

Then

$$\nabla[T(H)] = \partial_k(a^{jk}\partial_j H),$$

by arguments analogous to those of lemma 5.3. Moreover, lemma 5.3 implies that

$$|\nabla H|^2 \leqslant (1 + a^{jk})\partial_j H \cdot \partial_k H \leqslant C|\nabla H|^2. \tag{A 2}$$

If f is given by (5.28), then

$$\int_\Omega \{[\nabla H + T(H)] \cdot \nabla\zeta - CfH\zeta\}\,\mathrm{d}\Omega \leqslant 0 \tag{A 3}$$

for all non-negative test functions $\zeta \in C_0^\infty(\Omega)$ which vanish in a neighbourhood of the origin. Define

$$\zeta = (\eta\bar{\eta}_\nu)^2 H,$$

where $\eta \in C_0^\infty(D)$ for an n-disc D of radius R, containing the origin, and itself completely contained in the interior of Ω. Recalling that we have taken $H(0) = 0$ and that (2.6) implies that $H'(Q) > 0$, we conclude that H is a non-negative function. Let $\bar{\eta}_\nu$ be given by the sequence [14]

$$\bar{\eta}_\nu(\boldsymbol{x}) = \begin{cases} 0, & |\boldsymbol{x}| \leqslant \nu^{-2}, \\ \dfrac{\log(\nu^2|\boldsymbol{x}|)}{\log(\nu^2 R)}, & \nu^{-2} < |\boldsymbol{x}| < R, \\ 1, & R \leqslant |\boldsymbol{x}|. \end{cases}$$

Note that $\bar{\eta}_\nu$ vanishes on a neighbourhood of the origin for any finite parameter ν, but as ν tends to ∞, $\bar{\eta}_\nu$ converges pointwise to 1, whereas $\nabla\bar{\eta}_\nu$ converges to zero in $L^n(D)$.

Inequality (A 3) now assumes the form

$$\begin{aligned} 0 &\leqslant \int_D [(\mathcal{L} + f)H] \cdot (\eta\bar{\eta}_\nu)^2 H * 1 \\ &= \int_D \{[\partial_k(1 + a^{jk})\partial_j + f]H\}(\eta\bar{\eta}_\nu)^2 H * 1 \\ &= -\int (1 + a^{jk})\partial_j H \cdot \partial_k[(\eta\bar{\eta}_\nu)^2 H] * 1 + \int_D fH^2(\eta\bar{\eta}_\nu)^2 * 1, \end{aligned}$$

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where

$$
\begin{aligned}
-\int_D (1+a^{jk})\partial_j H\cdot\partial_k[(\eta\overline{\eta}_\nu)^2 H]*1 &= -2\int_D (1+a^{jk})\overline{\eta}_\nu^2\eta(\partial_k\eta)H\cdot\partial_j H*1\\
&\quad -2\int_D (1+a^{jk})\eta^2\overline{\eta}_\nu(\partial_k\overline{\eta}_\nu)H\cdot\partial_j H*1\\
&\quad -\int_D (1+a^{jk})(\eta\overline{\eta}_\nu)^2(\partial_j H\cdot\partial_k H)*1\\
&\equiv -(2i_1+2i_2+i_3).
\end{aligned}
$$

That is,

$$
\begin{aligned}
i_3 &= \int_D (1+a^{jk})(\eta\overline{\eta}_\nu)^2(\partial_j H\cdot\partial_k H)*1\\
&\leqslant 2(|i_1|+|i_2|)+\int_D (\eta\overline{\eta}_\nu)^2 f H^2*1.
\end{aligned}
$$

Estimating the integrals on the right individually, we have

$$
\begin{aligned}
2i_1 &= 2\int_D (1+a^{jk})\overline{\eta}_\nu^2\eta(\partial_k\eta)H\cdot\partial_j H*1\\
&\leqslant \varepsilon\int_D (\overline{\eta}_\nu\eta)^2|\nabla H|^2*1+\frac{1}{\varepsilon}\int_D \overline{\eta}_\nu^2|\nabla\eta|^2H^2*1,\\
2i_2 &= 2\int_D (1+a^{jk})\eta^2\overline{\eta}_\nu(\partial_k\overline{\eta}_\nu)H\cdot\partial_j H*1\\
&\leqslant \varepsilon\int_D (\overline{\eta}_\nu\eta)^2|\nabla H|^2*1+\frac{1}{\varepsilon}\int_D \eta^2|\nabla\overline{\eta}_\nu|^2H^2*1\\
&\equiv \varepsilon i_{21}+\frac{1}{\varepsilon}i_{22},
\end{aligned}
$$

where the small constants ε depend on the constant C in the upper inequality of (A 2), and

$$
i_{22}=\int_D \eta^2|\nabla\overline{\eta}_\nu|^2H^2*1\leqslant C\|\nabla\overline{\eta}_\nu\|_n^2\|\eta H\|_{2n/(n-2)}^2. \tag{A 4}
$$

The right-hand side of (A 4) tends to zero as ν tends to infinity. Absorbing small constants on the left, now using the lower inequality of (A 2), we conclude that

$$
\begin{aligned}
&(1-2\varepsilon)\int_D (\overline{\eta}_\nu\eta)^2|\nabla H|^2*1\\
&\quad\leqslant C(\varepsilon)\biggl(\int_D \overline{\eta}_\nu^2|\nabla\eta|^2H^2*1+\|\nabla\overline{\eta}_\nu\|_n^2\|\eta H\|_{2n/(n-2)}^2+\|f\|_{n/2}\|\eta\overline{\eta}_\nu H\|_{2n/(n-2)}^2\biggr),
\end{aligned} \tag{A 5}
$$

where

$$\|\eta\overline{\eta}_\nu H\|^2_{2n/(n-2)} \leqslant C\int_D |\nabla(\eta\overline{\eta}_\nu H)|^2 * 1$$
$$\leqslant C\biggl(\int_D |\nabla(\eta\overline{\eta}_\nu)|^2 H^2 * 1 + \int_D (\eta\overline{\eta}_\nu)^2|\nabla H|^2 * 1\biggr). \qquad \text{(A 6)}$$

The first integral on the far right-hand side of inequality (A 6) has essentially already been estimated in (A 4), and the second can be subtracted from the left-hand side of (A 5) provided that its coefficient in (A 5), the $L^{n/2}$-norm of f over Ω, is sufficiently small. Letting ν tend to infinity, in the limit we have

$$\int_D \eta^2|\nabla H|^2 * 1 \leqslant C\int_D |\nabla\eta|^2 H^2 * 1.$$

We conclude that H is a weak solution in a neighbourhood of the singularity. This completes the proof of theorem 5.5.

A.3. Proof of theorem 5.7

6.3.1. *Outline*

The result follows from the application of lemmas 5.2 and 5.3 and theorem 5.4, to the proofs of [29, theorem 6 and corollary 7]. The absence of condition (5.6) results in a small change of L^p conditions on the solution.

6.3.2. *Details*

Initially proceed as in § A.2, but choose the test of functions $\overline{\eta}_\nu$ to be a sequence of functions possessing the following properties:

(a) $\overline{\eta}_\nu \in [0,1]$ for all ν;

(b) $\overline{\eta}_\nu = 0$ in a neighbourhood of Σ for all ν;

(c) $\lim_{\nu\to\infty}\overline{\eta}_\nu = 1$ almost everywhere;

(d) $\lim_{\nu\to\infty}\|\nabla\overline{\eta}_\nu\|_{L^{n-m-\varepsilon}} = 0$

(cf. [35, lemma 2 and p. 73]). The function $\mathcal{F}$ is given by [34]:

$$\mathcal{F}(H) = \begin{cases} H^{\gamma_2}, & 0 \leqslant H \leqslant \ell, \\ (1/\gamma_1)[\gamma_2\ell^{\gamma_2-\gamma_1}H^{\gamma_1} - (\gamma_1-\gamma_2)\ell^{\gamma_2}], & H \geqslant \ell. \end{cases}$$

The functions $\mathcal{F}(H)$ and

$$G(H) \equiv \mathcal{F}(H)F'(H) - \gamma_2$$

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satisfy [34, p. 280] (see also [14, § 3])

$$\begin{aligned}\mathcal{F}(H) &\leqslant \frac{\gamma_2}{\gamma_1}\ell^{\gamma_2-\gamma_1}H^{\gamma_1},\\ H\mathcal{F}'(H) &\leqslant \gamma_2\mathcal{F},\\ |G(H)| &\leqslant \mathcal{F}(H)\mathcal{F}'(H),\\ G'(H) &\geqslant C\mathcal{F}'(H)^2.\end{aligned}$$

Replace the test function in § A.2 by the test function

$$\zeta = (\eta\overline{\eta}_\nu)^2 G(H),$$

where η is defined as in § A.2, and substitute this value into (A 3). We obtain

$$\begin{aligned}0 &\leqslant \int_D \{[\partial_k(1+a^{jk})\partial_j + f]H\}(\eta\overline{\eta}_\nu)^2 G(H) * 1\\ &= -\int_D (1+a^{jk})\partial_j H\cdot\partial_k[(\eta\overline{\eta}_\nu)^2 G(H)] * 1 + \int_D f\eta^2\overline{\eta}_\nu^2 H\cdot G(H) * 1\end{aligned}$$

or

$$\begin{aligned}2\int_D &(1+a^{jk})(\eta\overline{\eta}_\nu)\partial_k(\eta\overline{\eta}_\nu)(\partial_j H)G(H) * 1\\ &+\int_D (1+a^{jk})(\eta\overline{\eta}_\nu)^2 G'(H)\partial_j H\partial_k H * 1 \leqslant \int_D f(\eta\overline{\eta}_\nu)^2 H\cdot G(H) * 1.\end{aligned}$$

Writing this inequality in short-hand form,

$$I_1 + I_2 \leqslant I_3, \tag{A 7}$$

we proceed analogously to inequalities (28)–(35) of [28], making the following changes from the notation of [28] to our notation: $\psi \to \overline{\eta}$, $Q \to H$, $H \to \mathcal{F}$, $\varXi \to G$, $\varPhi \to f$. Explicitly,

$$\begin{aligned}I_1 &= 2\int_D (1+a^{jk})(\eta\overline{\eta}_\nu)\partial_k(\eta\overline{\eta}_\nu)\cdot G(H)\partial_j H * 1\\ &\geqslant -C\int_D (\eta\overline{\eta}_\nu)|\nabla(\eta\overline{\eta}_\nu)|\,|\mathcal{F}(H)\nabla\mathcal{F}(H)| * 1\\ &\geqslant -\varepsilon\int_D (\eta\overline{\eta}_\nu)^2|\nabla\mathcal{F}(H)|^2 * 1 - C(\varepsilon)\int_D |\nabla(\eta\overline{\eta})|^2\mathcal{F}(H)^2 * 1,\\ I_2 &\geqslant C\int_D (\eta\overline{\eta}_\nu)^2|\mathcal{F}'(H)|^2|\nabla H|^2 * 1\\ &\geqslant C\int_D (\eta\overline{\eta}_\nu)^2|\nabla\mathcal{F}(H)|^2 * 1,\end{aligned}$$

$$\begin{aligned} I_3 &= \int_D f(\eta\overline{\eta}_\nu)^2 H \cdot G(H) * 1 \\ &\leqslant \int_D |f|(\eta\overline{\eta}_\nu)^2 |H \cdot \mathcal{F}'(H)||\mathcal{F}(H)| * 1 \\ &\leqslant \gamma_2 \int_D |f|(\eta\overline{\eta}_\nu)^2 |\mathcal{F}(H)|^2 * 1 \\ &\leqslant \|f\|_{n/2} \biggl(\int_D |\eta\overline{\eta}_\nu \mathcal{F}(H)|^{2n/(n-2)} * 1 \biggr)^{(n-2)/n}. \end{aligned} \tag{A 8}$$

As in (A 6), we apply the Sobolev inequality to the right-hand side of this expression, followed by the Minkowski and Schwartz inequalities:

$$\begin{aligned} &\biggl(\int_D |\eta\overline{\eta}_\nu \mathcal{F}(H)|^{2n/(n-2)} * 1 \biggr)^{(n-2)/n} \\ &\quad \leqslant C \int_D |\nabla[\eta\overline{\eta}_\nu \mathcal{F}(H)]|^2 * 1 \\ &\quad \leqslant C \biggl[\int_D |(\nabla\eta)\overline{\eta}_\nu \mathcal{F}(H)|^2 * 1 + \int_D |(\nabla\overline{\eta}_\nu)\eta\mathcal{F}(H)|^2 * 1 + \int_D (\eta\overline{\eta}_\nu)^2 |\nabla\mathcal{F}(H)|^2 * 1 \biggr] \\ &\quad \equiv I_{31} + I_{32} + I_{33}. \end{aligned}$$

The term I_{33} can be subtracted from the left-hand-side of inequality (A 7), as its coefficient in (A 8), the $L^{n/2}$ norm of f, is small on small discs as a consequence of condition (5.29). Moreover,

$$\begin{aligned} I_{32} &= \int_D |(\nabla\overline{\eta}_\nu)\eta\mathcal{F}(H)|^2 * 1 \\ &\leqslant C(\gamma_1, \gamma_2, \ell) \int_D (\nabla\overline{\eta}_\nu)^2 \eta^2 H^{2\gamma_1} * 1 \\ &\leqslant C \|\nabla\overline{\eta}_\nu\|^2_{n-m-\varepsilon} \|H^{2\gamma_1}\|_\beta. \end{aligned}$$

Letting ν tend to infinity, the term on the right-hand side is zero for every value of ℓ. Now letting ℓ tend to infinity and using Fatou's inequality, we conclude that

$$\int_D \eta^2 |\nabla(H^{\gamma_2})|^2 * 1 \leqslant C \int_D |\nabla\eta|^2 H^{2\gamma_2} * 1.$$

Apply theorem A.1, taking $U = H^{\gamma_2}$. Then $U \in H^{1,2}(D)$ and $W = U^\lambda$ satisfies inequality (A 1) for $\lambda = 1/\gamma_2$. Because $\gamma_2 > \frac{1}{2}$, we can conclude that $\lambda < 2$, as is required by theorem A.1. We now want to check that we can choose $\gamma_2 \leqslant 1$, in order to obtain $\lambda \geqslant 1$, which is also required by theorem A.1. Because H lies in the space $L^{2\beta\gamma_1}(D) \cap L^{2\gamma_2}(D)$, we let $\gamma_1\beta = \gamma_2$. Substituting the definition of β, we find that we can choose $\gamma_2 \leqslant 1$ for $\gamma_1 > \frac{1}{2}$, provided $m + 4 < n$, which is satisfied by hypothesis. This completes the proof of theorem 5.7.

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