ONE-LEVEL DENSITY OF LOW-LYING ZEROS OF QUADRATIC HECKE *L*-FUNCTIONS OF IMAGINARY QUADRATIC NUMBER FIELDS

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Abstract

In this paper, we prove a one level density result for the low-lying zeros of quadratic Hecke *L*-functions of imaginary quadratic number fields of class number 1. As a corollary, we deduce, essentially, that at least $(19 - \cot(1/4))/16 = 94.27\ldots\%$ of the *L*-functions under consideration do not vanish at $1/2$.

1. Introduction

The theory of random matrices provides fruitful predictions on the behaviors of the zeros of *L*-functions by drawing analogues between the eigenvalues of ensembles of random matrices and the zeros of *L*-functions. In this connection, the density conjecture of Katz and Sarnak [23, 24] asserts that the statistics of the low-lying zeros (zeros near the central point) of various families of *L*-functions are the same as those of eigenvalues near 1 of a corresponding classical compact group. These statistics, the *n*-level densities, have been studied for various families of *L*-functions for suitably restricted test functions (see [2, 4, 5, 8, 9, 11, 12, 16–19, 22, 28, 29, 32, 34–36, 38]). The results are shown to agree with the density conjecture.

To study the *n*-level densities, one needs to form families of *L*-functions, and those of *L*-functions associated with quadratic characters are among the most natural ones to consider. For the family of quadratic Dirichlet *L*-functions, Özlük and Snyder [32] calculated the one-level density and all *n*-level densities were calculated later

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by Rubinstein [36]. The support of the Fourier transform of the test function in Rubinstein's result was enlarged by the first-named author in [12]. It is shown in [10, 27] that these results agree with the density conjecture and the family of quadratic Dirichlet *L*-functions is a symplectic family. Recently, the authors [15] studied another family of quadratic *L*-functions: quadratic Hecke *L*-functions of $\mathbb{Q}(i)$. Via its one level density computations, it is shown that it is a symplectic family.

Our goal in this paper is to extend the methods used in [15] to study the one-level density of the low-lying zeros of quadratic *L*-functions associated with the imaginary quadratic number fields $K = \mathbb{Q}(\sqrt{d})$ of class number 1. Here d is a negative, square-free integer and it is well known (see [21, (22.77)]) that the class number of K is 1 if and only if $d \in \{-1, -2, -3, -7, -11, -19, -43, -67, -163\}$. Since the case $d = -1$ has been studied in [15], we assume throughout the paper that

$$d \in \{-2, -3, -7, -11, -19, -43, -67, -163\}.$$

We note here that if K has class number 1, then all ideals in its ring of integers are principal. Thus, sums over ideals in this setting can be recast as sums of the relevant primary elements (see the discussion in Section 2). This reparametrization becomes more difficult if the class number is greater than 1, giving a stumbling block to further generalization.

The ring of integers O_K is a free $\mathbb{Z}$ module, $O_K = \mathbb{Z} + \omega_K \mathbb{Z}$, where

$$\omega_K = \begin{cases} \frac{1}{2}(1 + \sqrt{d}) & \text{if } d \equiv 1 \ (\text{mod } 4), \\ \sqrt{d} & \text{if } d \equiv 2, 3 \ (\text{mod } 4). \end{cases}$$

Let

$$C_K(X) = \{c \in O_K : (c, 2) = 1, \ c \text{ square-free}, \ X \le N(c) \le 2X\}.$$

We define $c_K = \sqrt{-2}$ when $d = -2$ and $c_K = 2$ otherwise. We shall define in Section 2.2 the primitive quadratic Kronecker symbol $\chi^{(-4c_K c)}$ for $c \in C_K(X)$. We denote the nontrivial zeros of the Hecke *L*-function $L(s, \chi^{(-4c_K c)})$ by $\frac{1}{2} + i\gamma_{\chi^{(-4c_K c)}, j}$. Without assuming the generalized Riemann hypothesis (GRH), we order them as

$$\cdots \le \operatorname{Re} \gamma_{\chi^{(-4c_K c)}, -2} \le \operatorname{Re} \gamma_{\chi^{(-4c_K c)}, -1} < 0 \le \operatorname{Re} \gamma_{\chi^{(-4c_K c)}, 1} \le \operatorname{Re} \gamma_{\chi^{(-4c_K c)}, 2} \le \cdots.$$

Moreover, we normalize the γ's by setting

$$\widetilde{\gamma}_{\chi^{(-4c_K c)}, j} = \frac{\gamma_{\chi^{(-4c_K c)}, j}}{2\pi} \log X.$$

This is done so that the average spacing of the zeros is 1. We also define, for an even Schwartz class function ϕ,

$$S_K(\chi^{(-4c_K c)}, \phi) = \sum_j \phi(\widetilde{\gamma}_{\chi^{(-4c_K c)}, j}).$$

We further let $\Phi_X(t)$ be a nonnegative smooth function supported on $(1, 2)$ satisfying $\Phi_X(t) = 1$ for $t \in (1 + 1/U, 2 - 1/U)$ with $U = \log\log X$ and such that $\Phi_X^{(j)}(t) \ll_j U^j$ for all integers $j \geq 0$. We define the one-level density associated to the quadratic family of $\{L(s, \chi^{(-4c_K c)})\}$ with $c \in C_K(X)$ as

$$\frac{1}{\#C_K(X)} \sum_{(c,2)=1}^{*} S_K(\chi^{(-4c_K c)}, \phi).$$

This is a weighted count of the low-lying zeros in our family.

Our result is the following theorem.

THEOREM 1.1. *Suppose that the GRH is true. Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ has compact support in $(-2, 2)$; then*

$$\lim_{X \to +\infty} \frac{1}{\#C_K(X)} \sum_{(c,2)=1}^{*} S_K(\chi^{(-4c_K c)}, \phi) \Phi_X\left(\frac{N(c)}{X}\right) = \int_{\mathbb{R}} \phi(x) W_{\mathrm{USp}}(x) \, dx, \qquad (1\text{-}1)$$

where

$$W_{\mathrm{USp}}(x) = 1 - \frac{\sin(2\pi x)}{2\pi x}.$$

Here the '$*$' on the sum over c means that the sum is restricted to square-free elements c of O_K.

The kernel of the integral W_{USp} in (1-1) is the same function which occurs in the one-level density of the eigenvalues of unitary symplectic matrices. This shows that the family of quadratic Hecke L-functions of O_K is a symplectic family. We remark here that a kind of repulsion of zeros from the central point has been observed in symplectic families, that is, the probability of finding an L-function in a symplectic family with a low first zero above the central point is very small. See, for example, [26, 30] for detailed discussions of the phenomenon.

Our proof of Theorem 1.1 proceeds along the same lines of arguments as those in [15]. The allowable range of the support of the Fourier transform of the test function in (1-1) depends on certain character sums. To get a better estimation, we apply a two-dimensional Poisson summation over O_K to convert long character sums to shorter ones, which leads to a second main term in the computation of the one-level density. In this process, we need to evaluate explicitly certain Gauss sums at prime arguments, which we achieve by combining the quadratic reciprocity laws for imaginary quadratic number fields and the approaches used in [1, 3].

A conjecture that goes back to Chowla [6] asserts that $L(1/2, \chi) \neq 0$ for any primitive quadratic Dirichlet character χ. In general, it is believed that an L-function does not vanish at its central points unless there is a compelling reason (the sign of the functional equation is -1 or the L-function is associated with an elliptic curve of positive rank, a connection given by the Birch–Swinnerton-Dyer conjecture). Thus,

it stands to reason that the same statement should hold for Hecke characters as well. Motivated by this conjecture, we use Theorem 1.1 to deduce the following result.

COROLLARY 1.2. *Suppose that the GRH is true and that 1/2 is a zero of $L(s, \chi^{(-4c_Kc)})$ of order $m_c \geq 0$. As $X \to \infty$,*

$$\sum_{(c,2)=1}^{*} m_c \Phi_X\left(\frac{N(c)}{X}\right) \leq \left(\frac{\cot \frac{1}{4} - 3}{8} + o(1)\right)\#C_K(X). \tag{1-2}$$

Moreover, as $X \to \infty$,

$$\#\left\{c \in C_K(X) : L\left(\frac{1}{2}, \chi^{(-4c_Kc)}\right) \neq 0\right\} \geq \left(\frac{19 - \cot \frac{1}{4}}{16} + o(1)\right)\#C_K(X) \tag{1-3}$$
$$\geq 0.9427 \cdot \#C_K(X).$$

PROOF. The proof is similar to that of [5, Corollary 2.1] and goes along the same lines as the optimization carried out in [22, Appendix A]. Suppose that ϕ satisfies the conditions of Theorem 1.1 and $\phi(0) = 1$. Discarding all terms in $S_K(\chi^{(-4c_Kc)}, \phi)$ with $\tilde{\gamma}_{\chi^{(-4c_Kc)}, j} \neq 0$,

$$\frac{1}{\#C_K(X)} \sum_{(c,2)=1}^{*} m_c \Phi_X\left(\frac{N(c)}{X}\right) \leq \int_{\mathbb{R}} \phi(x) W_{\text{USp}}(x) \, dx. \tag{1-4}$$

It thus remains to minimize the right-hand side of the above expression, by taking the infimum over all admissible ϕ.

Following the computations carried out in [5, 22],

$$\inf_{\phi \in \mathfrak{F}} \int_{\mathbb{R}} \phi(x) W_{\text{USp}}(x) \, dx = \frac{\cot(1/4) - 3}{8}. \tag{1-5}$$

Here $\mathfrak{F}$ consists of even Schwartz functions $\phi \in L^1(\mathbb{R})$ with $\phi(0) = 1$ and Fourier transform supported in $(-2, 2)$. The infimum is attained by the function ϕ whose Fourier transform $\hat{\phi}$ is

$$\hat{\phi}(y) = (h \star \check{h})(y).$$

Here $f \star g$ denotes the convolution of f and g, $h(y)$ is the even extension of

$$h(y) = \begin{cases} \dfrac{\sin(y/2 - (\pi + 1)/4)}{\sqrt{2}\sin(1/4) - \cos(\pi + 1)/4}, & 0 \leq y \leq 1, \\[2ex] 0, & y > 0, \end{cases}$$

and $\check{h}(y) = \overline{h(-y)}$.

The infimum of the right-hand side of (1-4) taken over all admissible ϕ's under the conditions of Theorem 1.1 does not exceed the quantity in (1-5). This proves (1-2).

Moreover, as we are dealing with a family of quadratic characters and hence the root numbers of these L-functions are 1, m_c must be even and thus at least 2 if positive. From this observation and (1-2), we arrive at (1-3). □

1.1. Notation. The following notation and conventions are used throughout the paper.

(i) We write $\Phi(t)$ for $\Phi_X(t)$.
(ii) $e(z) = \exp(2\pi i z) = e^{2\pi i z}$.
(iii) $f = O(g)$ or $f \ll g$ means that $|f| \leq cg$ for some unspecified positive constant c.
(iv) $f = o(g)$ means that $\lim_{x \to \infty} f(x)/g(x) = 0$.
(v) μ_K denotes the Möbius function on O_K.
(vi) $\chi_{[-1,1]}$ denotes the characteristic function of $[-1, 1]$.

2. Preliminaries

2.1. Imaginary quadratic number fields of class number 1. The following facts concerning $K = \mathbb{Q}(\sqrt{d})$ can be found in [21, Section 3.8]. The group of units $U_k = \{\pm 1\}$ based on our assumption on d. The discriminant D_K of K is

$$D_K = \begin{cases} d & \text{if } d \equiv 1 \pmod 4, \\ 4d & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

The different of K is the principal ideal $(\delta_K) = (\sqrt{D_K})$. The Kronecker symbol in $\mathbb{Z}$, $\chi_{D_K}(\cdot) = (D_k/\cdot)_{\mathbb{Z}}$, is a real primitive character of conductor $-D_K$ and the law of factorization of rational primes p into prime ideals in K asserts the following:

$$\chi_{D_K}(p) = 0, \text{then } p \text{ ramifies}, (p) = \mathfrak{p}^2 \quad \text{with } N(\mathfrak{p}) = p,$$
$$\chi_{D_K}(p) = 1, \text{then } p \text{ splits}, (p) = \mathfrak{p}\overline{\mathfrak{p}} \quad \text{with } \mathfrak{p} \neq \overline{\mathfrak{p}} \text{ and } N(\mathfrak{p}) = p,$$
$$\chi_{D_K}(p) = -1, \text{then } p \text{ is inert}, (p) = \mathfrak{p} \quad \text{with } N(\mathfrak{p}) = p^2.$$

In particular, the rational prime 2 splits when $d \equiv 1 \pmod 8$, is inert when $d \equiv 5 \pmod 8$ and ramifies when $d = -2$.

2.2. Quadratic characters and Kronecker symbols. The symbol $(\cdot/n)$ is the quadratic residue symbol in the ring of integers O_K. For a prime $\varpi \in O_K, (\varpi, 2) = 1$, the quadratic character is defined for $a \in O_K, (a, \varpi) = 1$ by $(a/\varpi) \equiv a^{(N(\varpi)-1)/2} \pmod{\varpi}$ with $(a/\varpi) \in \{\pm 1\}$. When $\varpi | a$, we define $(a/\varpi) = 0$. Then the quadratic character can be extended to any composite n with $(N(n), 2) = 1$ multiplicatively. We further define $(\cdot/n) = 1$ when n is a unit in O_K.

For any $n \in O_K, (n, 2) = 1$, let $n = a + b\omega_K$ and $n\omega_K = c + d\omega_K$ with $a, b, c, d \in \mathbb{Z}$. Then $\chi(n) = i^w$, where $w = (b^2 - a + 2)c + (a^2 - b + 2)d + ad$, defines a map defined modulo $4O_K$ and we have the following quadratic reciprocity law for quadratic

imaginary number fields [25, Theorem 8.15] that for $(mn, 2) = 1$,

$$\left(\frac{n}{m}\right)\left(\frac{m}{n}\right) = (-1)^{(N(n)-1)/2 \ (N(m)-1)/2} \chi(m)^{N(m) \ (N(n)-1)/2} \chi(n)^{N(n) \ (N(m)-1)/2}.$$

It follows from the above quadratic reciprocity law that when $(mn, 2) = 1$ and $m \equiv 1 \pmod 4$, then (see [25, Proposition 8.17])

$$\left(\frac{n}{m}\right)\left(\frac{m}{n}\right) = 1. \tag{2-1}$$

One also has the following supplementary laws (see [25, Proposition 4.2(ii), (iii)]):

$$\left(\frac{-1}{n}\right) = (-1)^{(N(n)-1)/2}, \quad \left(\frac{2}{n}\right) = \left(\frac{2}{N(n)}\right)_{\mathbb{Z}}, \tag{2-2}$$

where $(\cdot/\cdot)_{\mathbb{Z}}$ denotes the Jacobi symbol in $\mathbb{Z}$. Moreover, it follows from [25, Proposition 4.2(iii)] that for $a \in \mathbb{Z}, m \in O_K, (a, 2) = 1$,

$$\left(\frac{m}{a}\right) = \left(\frac{N(m)}{a}\right)_{\mathbb{Z}}. \tag{2-3}$$

We deduce from (2-2) that when $d \neq -2$, for $n \equiv 1 \pmod{4c_Kc}$,

$$\left(\frac{4c_K}{n}\right) = 1. \tag{2-4}$$

Now assume that $d = -2$ and $n \equiv 1 \pmod{4c_Kc}$. In this case, we have $c_K = \omega_K$. We write $n = a + b\omega_K$ with $a, b \in \mathbb{Z}$ to see that $a \equiv 1 \pmod 8$ and we can write $b = 2^l b'$ with $(b', 2) = 1, b', l \in \mathbb{Z}, l \geq 0$. We consider

$$\left(\frac{a}{a + b\omega_K}\right) = \left(\frac{-b\omega_K}{a + b\omega_K}\right) = \left(\frac{-b}{a + b\omega_K}\right)(c_K a + b\omega_K). \tag{2-5}$$

On the other hand, as $a \equiv 1 \pmod 8$, it follows from (2-3) that when $a, b \in \mathbb{Z}, a \equiv 1 \pmod 8$, then

$$\left(\frac{b}{a}\right) = 1, \quad \left(\frac{\omega_K}{a}\right) = \left(\frac{N(\omega_K)}{a}\right)_{\mathbb{Z}} = \left(\frac{2}{a}\right)_{\mathbb{Z}} = 1.$$

We deduce from the above and (2-1) that

$$\left(\frac{a}{a + b\omega_K}\right) = \left(\frac{a + b\omega_K}{a}\right) = \left(\frac{b\omega_K}{a}\right) = \left(\frac{b}{a}\right)\left(\frac{\omega_K}{a}\right) = 1, \tag{2-6}$$

$$\left(\frac{-b}{a + b\omega_K}\right) = \left(\frac{-2^l b'}{a + b\omega_K}\right) = \left(\frac{2^l}{a + b\omega_K}\right)\left(\frac{a + b\omega_K}{-b'}\right) = \left(\frac{a}{-b'}\right) = 1.$$

Combining (2-5) and (2-6), we see that (2-4) still holds for $n \equiv 1 \pmod{4c_Kc}$ when $d = -2$. The above discussions allow us to define a quadratic Dirichlet character $\chi^{(-4c_Kc)} \pmod{4c_Kc}$ for any element $c \in O_K, (c, 2) = 1$ such that for any

$n \in (O_K/(4c_KcO_K))^*$,

$$\chi^{(-4c_Kc)}(n) = \left(\frac{-4c_Kc}{n}\right).$$

One deduces from (2-4) and the quadratic reciprocity that $\chi^{(-4c_Kc)}(n) = 1$ when $n \equiv 1 \pmod{4c_Kc}$. It follows from this that $\chi^{(-4c_Kc)}(n)$ is well defined. As $\chi^{(-4c_Kc)}(n)$ is clearly multiplicative and of order 2 and is trivial on units, it can be regarded as a quadratic Hecke character (mod $4c_Kc$) of trivial infinite type. We denote $\chi^{(-4c_Kc)}$ for this Hecke character as well and we call it the Kronecker symbol. Furthermore, when c is square-free, $\chi^{(-4c_Kc)}$ is nonprincipal and primitive. To see this, we write $c = u_c \cdot \varpi_1 \cdots \varpi_k$ with u_c a unit and ϖ_j being primes. Suppose that $\chi^{(-4c_Kc)}$ is induced by some χ modulo c' with $\varpi_j \nmid c'$; then, by the Chinese remainder theorem, there exists an n such that $n \equiv 1 \pmod{4c_Kc/\varpi_j}$ and $(\varpi_j/n) \neq 1$. It follows from this and (2-1) that $\chi(n) = 1$ but $\chi^{(-4c_Kc)}(n) \neq 1$, which is a contradiction. Thus, $\chi^{(-4c_Kc)}$ can only be possibly induced by some χ modulo $4a_Kc$, where $a_K = \omega_K$ or $\overline{\omega_K}$ when $d \equiv 1 \pmod 8$ (note that in this case $d = -7$ and $2 = \omega_K\overline{\omega_K}$) and $a_K = 1$ otherwise. Suppose that it is induced by some χ modulo $4a_Kc$; then, by the Chinese remainder theorem, there exists an n such that $n \equiv 1 \pmod c$ and $n \equiv 1 + 4b_K \pmod 8$, where $b_K = \omega_K$ when $d \equiv 5 \pmod 8$ and $b_K = a_K$ otherwise. When $d = -2$, we can further take this n to satisfy $n \in \mathbb{Z}$ (for example, we can take any $n \in \mathbb{Z}$ satisfying $n \equiv 1 \pmod{N(c)}$ and $n \equiv 5 \pmod 8$). As this $n \equiv 1 \pmod{4a_K}$, it follows that $n \equiv 1 \pmod{4a_Kc}$ and hence $\chi(n) = 1$ but $\chi^{(-4c_Kc)}(n) = (c_K/n) = -1$ (note that $(u/n) = 1$ when u is a unit in O_K), where the last equality follows from (2-2) when $d \neq -2$ and (2-3) when $d = -2$. This implies that $\chi^{(-4c_Kc)}$ is primitive. This also shows that $\chi^{(-4c_Kc)}$ is nonprincipal.

2.3. The Gauss sums.   For any $n \in O_K$, $(n, 2) = 1$, the quadratic Gauss sum $g_K(n)$ is defined by

$$g_K(n) = \sum_{x \bmod n} \left(\frac{x}{n}\right)\widetilde{e}_K\left(\frac{x}{n}\right),$$

where

$$\widetilde{e}_K(z) = \exp\left(2\pi i\left(\frac{z}{\sqrt{D_K}} - \frac{\bar{z}}{\sqrt{D_K}}\right)\right).$$

The following well-known relation (see [33, page 195]) holds for all n:

$$|g_K(n)| = \begin{cases} \sqrt{N(n)} & \text{if } n \text{ is square-free,} \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of $g_K(n)$, one easily derives the relation

$$g_K(n_1n_2) = \left(\frac{n_2}{n_1}\right)\left(\frac{n_1}{n_2}\right)g_K(n_1)g_K(n_2), \quad (n_1, n_2) = 1.$$

For primes in O_K that are coprime to $2D_K$ and $(1-d)/4$ when $d \equiv 1 \pmod 4$, we now introduce the notion of primary. When a prime is also a rational prime p, then we say that it is primary when $p > 0$. When a prime ϖ satisfies $N(\varpi) = p$ with p a rational prime, then we write $\varpi = a + b\omega_K$ to see that

$$p = \begin{cases} a^2 + ab + b^2 \dfrac{1-d}{4} & \text{if } d \equiv 1 \pmod 4, \\ a^2 - db^2 & \text{if } d \equiv 2, 3 \pmod 4. \end{cases} \tag{2-7}$$

Note that p is also coprime to $2D_k$ and $(1-d)/4$ when $d \equiv 1 \pmod 4$. One deduces easily from this and the above expression for p that $(ab, p) = 1$.

We first consider the case when $d \equiv 1 \pmod 4$. In this case,

$$\left(\frac{-b(a + b(1-d)/4)}{p} \right)_{\mathbb{Z}} = \left(\frac{a^2}{p} \right)_{\mathbb{Z}} = 1.$$

It follows that

$$\left(\frac{b}{p} \right)_{\mathbb{Z}} = \left(\frac{-(a + b(1-d)/4)}{p} \right)_{\mathbb{Z}}.$$

We now define the prime $\varpi = a + b\omega_K$ to be primary if $b \equiv 1 \pmod 4$ or else $a + b(1-d)/4 \equiv -1 \pmod 4$ when $(b, 2) \neq 1$. Note that when $(b, 2) \neq 1$, we must have $(a, 2) = 1$ for otherwise we will have $2 | p$, contradicting the fact that p is coprime to 2. Since $U_K = \{\pm 1\}$, it is easy to see that each prime ideal in K has a unique primary generator ϖ. When ϖ is primary, we deduce via quadratic reciprocity and (2-7) that

$$\left(\frac{b}{p} \right)_{\mathbb{Z}} = \left(\frac{-(a + b(1-d)/4)}{p} \right)_{\mathbb{Z}} = 1. \tag{2-8}$$

Now suppose that $d = -2$. In this case, we define the prime $\varpi = a + b\omega_K$ to be primary if $b = 2^k b'$ with $k, b' \in \mathbb{Z}, k \geq 0, b' \equiv 1 \pmod 4$. Note that when $k \geq 1$, then again $(a, 2) = 1$ and it follows from (2-7) that $p \equiv 1 \pmod 8$; hence, $(2/p)_{\mathbb{Z}} = 1$. Again each prime ideal in K has a unique primary generator ϖ and, when ϖ is primary, we deduce via quadratic reciprocity and (2-7) that

$$\left(\frac{b}{p} \right)_{\mathbb{Z}} = 1. \tag{2-9}$$

We now evaluate the Gauss sum at each primary prime ϖ. We have the following result.

LEMMA 2.1. *Let ϖ be a primary prime in O_K. Then*

$$g_K(\varpi) = \begin{cases} N(\varpi)^{1/2} & \text{if } N(\varpi) \equiv 1 \pmod 4, \\ -iN(\varpi)^{1/2} & \text{if } N(\varpi) \equiv -1 \pmod 4. \end{cases}$$

PROOF. First, assume that $p = N(\varpi)$ is a rational prime so that $p = N(\varpi) = \varpi\overline{\varpi}$ with $(\varpi, \overline{\varpi}) = 1$ (note that by our definition of primary, (p) does not ramify in O_K). We

use R to denote a quadratic residue (mod ϖ) and N to denote a quadratic nonresidue (mod ϖ). Note first that

$$E_K := \sum_{x \bmod \varpi} \tilde{e}_K\left(\frac{x}{\varpi}\right) = 1 + \sum_{R \bmod \varpi} \widetilde{e}_K\left(\frac{R}{\varpi}\right) + \sum_{N \bmod \varpi} \widetilde{e}_K\left(\frac{N}{\varpi}\right).$$

As $(\varpi, \overline{\varpi}) = 1$,

$$\sum_{x \bmod \varpi} \tilde{e}_K\left(\frac{x}{\varpi}\right) = \sum_{x \bmod \varpi} \widetilde{e}_K\left(\frac{x\overline{\varpi}}{\varpi\overline{\varpi}}\right)$$

$$= \sum_{x \bmod \varpi} \widetilde{e}_K\left(\frac{x\overline{\varpi}}{p}\right) = \sum_{x \bmod \varpi} \widetilde{e}_K\left(\frac{x}{p}\right).$$

On summing over a set of representatives (mod p) instead of (mod ϖ) and noting that $\omega_K - \overline{\omega}_K = \sqrt{D_K}$,

$$E_K \frac{N(p)}{N(\varpi)} = \sum_{x \bmod p} \widetilde{e}_K\left(\frac{x}{p}\right) = \sum_{y,z \in \mathbb{Z}/p\mathbb{Z}} \widetilde{e}_K\left(\frac{y + z\omega_K}{p}\right) = \sum_{z \in \mathbb{Z}/p\mathbb{Z}} e\left(\frac{z}{p}\right) = 0.$$

We conclude that $E_K = 0$ and it follows that

$$g_K(\varpi) = \sum_{R \bmod \varpi} \widetilde{e}_K\left(\frac{R}{\varpi}\right) - \sum_{N \bmod \varpi} \widetilde{e}_K\left(\frac{N}{\varpi}\right)$$

$$= 1 + 2\sum_{R \bmod \varpi} \widetilde{e}_K\left(\frac{R}{\varpi}\right) = \sum_{v \bmod \varpi} \widetilde{e}_K\left(\frac{v^2}{\varpi}\right).$$

Now $g_K(\varpi)$ can be further rewritten as

$$g_K(\varpi) = \sum_{v \bmod \varpi} \widetilde{e}_K\left(\frac{v^2\overline{\varpi}}{\varpi\overline{\varpi}}\right) = \sum_{v \bmod \varpi} \widetilde{e}_K\left(\frac{v^2\overline{\varpi}}{p}\right).$$

On setting $v = x + y\omega_K$, $\varpi = a + b\omega_K$ so that $\overline{\varpi} = a + b\overline{\omega}_K$, we sum over a set of representatives (mod p) instead of (mod ϖ) to deduce that

$$g_K(\varpi)\frac{N(p)}{N(\varpi)} = \sum_{v \bmod p} \widetilde{e}_K\left(\frac{v^2\overline{\varpi}}{p}\right)$$

$$= \begin{cases} \displaystyle\sum_{x,y \in \mathbb{Z}/p\mathbb{Z}} e\left(\frac{-bx^2 + 2axy - bdy^2}{p}\right) & \text{if } d \equiv 2, 3 \text{ (mod 4)}, \\[1.5em] \displaystyle\sum_{x,y \in \mathbb{Z}/p\mathbb{Z}} e\left(\frac{-bx^2 + 2axy + (a + b(1 - d)/4)y^2}{p}\right) & \text{if } d \equiv 1 \text{ (mod 4)}. \end{cases} \tag{2-10}$$

We are thus led to consider the evaluation of the following exponential sum:

$$\sum_{x,y\in\mathbb{Z}/M\mathbb{Z}} e\left(\frac{fx^2 + gxy + hy^2}{M}\right),$$

where $M, f, g, h \in \mathbb{Z}$ with $2|g, 2 \nmid M$.

Let A be any rational integer coprime to M which is represented by the binary quadratic form $fx^2 + gxy + hy^2$. Then there exist integers r and t such that

$$A = fr^2 + grt + ht^2 \not\equiv 0 \pmod{M}.$$

Define rational integers s and u by

$$s = -gr - 2ht, \quad u = 2fr + gt.$$

Then

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix}^T \begin{pmatrix} f & g/2 \\ g/2 & h \end{pmatrix}\begin{pmatrix} r & s \\ t & u \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & (4fh - g^2)A \end{pmatrix}$$

in $\mathrm{GL}(2, \mathbb{Z}/M\mathbb{Z})$.

Moreover,

$$ru - st = 2A \not\equiv 0 \pmod{M}.$$

Thus,

$$\begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}/M\mathbb{Z}).$$

The above matrix hence induces an invertible change of variables, so that

$$\sum_{x,y\in\mathbb{Z}/M\mathbb{Z}} e\left(\frac{fx^2 + gxy + hy^2}{M}\right) = \sum_{x,y\in\mathbb{Z}/M\mathbb{Z}} e\left(\frac{Ax^2 + (4fh - g^2)Ay^2}{M}\right).$$

We can now apply the discussions above to study the expression for $g_K(\varpi)$ in (2-10). Note that by taking $x = 1, y = 0$, we see that we can take A to be $-b$ in our case and in either case

$$pg_K(\varpi) = \sum_{x,y\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{-bx^2 - 4pby^2}{p}\right)$$

$$= p\sum_{x\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{-bx^2}{p}\right) = p\left(\frac{-b}{p}\right)_{\mathbb{Z}}\tau(\chi_p),$$

where

$$\tau(\chi_p) = \sum_{x\in\mathbb{Z}/p\mathbb{Z}} e\left(\frac{x^2}{p}\right).$$

is the quadratic Gauss sum in $\mathbb{Q}$. It follows from (2-8) and (2-9) that

$$g_K(\varpi) = \left(\frac{-1}{p}\right)\tau(\chi_p).$$

It follows from [7, Ch. 2] that $\tau(\chi_p) = p^{1/2} = N(\varpi)^{1/2}$ when $p \equiv 1 \pmod 4$ and $\tau(\chi_p) = ip^{1/2} = iN(\varpi)^{1/2}$ when $p \equiv 3 \pmod 4$.

Now, if $\varpi = p$ is an odd rational prime, we compute $g(p)$ directly. As in [7, Ch. 2] (note that in this case, we still have $\sum_{x \bmod p} \tilde{e}(x/p) = 0$),

$$g(p) = \sum_{x \bmod p} \tilde{e}\left(\frac{x^2}{p}\right).$$

We now write $x = a + b\omega_K$ with $a, b \pmod p$ in $\mathbb{Z}$ to see that

$$g(p) = \sum_{x \bmod p} \tilde{e}\left(\frac{x^2}{p}\right) = \sum_{b=1}^{p}\sum_{a=1}^{p} e\left(\frac{2ab - b^2}{p}\right) = p = N(p)^{1/2}.$$

Recall that $N(p) = p^2 \equiv 1 \pmod 4$. The proof is complete. $\qquad\square$

Now we define more generally, for any $n, k \in O_K, (n, 2) = 1$,

$$g_K(k, n) = \sum_{x \bmod n} \left(\frac{x}{n}\right)\tilde{e}\left(\frac{kx}{n}\right)$$

and

$$G_K(k, n) = \left(\frac{1-i}{2} + \left(\frac{-1}{n}\right)\frac{1+i}{2}\right)g_K(k, n).$$

Note that

$$g_K(k, n) = \left(\frac{1+i}{2} + \left(\frac{-1}{n}\right)\frac{1-i}{2}\right)G_K(k, n). \tag{2-11}$$

The notion of $G_K(k, n)$ is motivated by the work of Soundararajan [37, Section 2.2]. For our purpose in this paper, we only need know the explicit values of $G_K(k, n)$ at primary primes n, which are given in the following lemma, whose proof is omitted as it is similar to that of [15, Lemma 2.4].

LEMMA 2.2

(i) *We have*

$$G_K(rs, n) = \overline{\left(\frac{s}{n}\right)}G_K(r, n), \quad (s, n) = 1. \tag{2-12}$$

(ii)   Let ϖ be a primary prime in O_K. Suppose that ϖ^h is the largest power of ϖ
       dividing k. (If $k = 0$, then set $h = \infty$.) Then, for $l \geq 1$,

$$
G_K(k, \varpi^l) = \begin{cases}
0 & \text{if } l \leq h \text{ is odd,} \\
\varphi(\varpi^l) = \#(O_K/(\varpi^l))^* & \text{if } l \leq h \text{ is even,} \\
-N(\varpi)^{l-1} & \text{if } l = h + 1 \text{ is even,} \\
\left(\dfrac{-k\varpi^{-h}}{\varpi}\right) N(\varpi)^{l-1/2} & \text{if } l = h + 1 \text{ is odd,} \\
0 & \text{if } l \geq h + 2.
\end{cases}
$$

2.4. The explicit formula.   Our approach in this paper relies on the following
explicit formula, which essentially converts a sum over zeros of an *L*-function to
a sum over primes. As it is similar to that of [15, Lemma 2.7], we omit the proof
here.

LEMMA 2.3. *Let $\phi(x)$ be an even Schwartz function whose Fourier transform $\hat{\phi}(u)$ is
compactly supported. For any square-free $c \in O_K, (c, 2) = 1, X \leq N(c) \leq 2X$,*

$$
S(\chi^{(-4c_K c)}, \phi) = \int_{-\infty}^{\infty} \phi(t)\, dt - \frac{1}{2} \int_{-\infty}^{\infty} \hat{\phi}(u)\, du
$$
$$
- 2S(\chi^{(-4c_K c)}, X; \hat{\phi}) + O\left(\frac{\log\log 3X}{\log X}\right),
$$

with the implicit constant depending on ϕ, where

$$
S(\chi^{(-4c_K c)}, X; \hat{\phi}) = \frac{1}{\log X} \sum_{\varpi \text{ primary}} \frac{\log N(\varpi)}{\sqrt{N(\varpi)}} \chi^{(-4c_K c)}(\varpi) \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right).
$$

We note that in the expression of $S(\chi^{(-4c_K c)}, X; \hat{\phi})$, the sum is only over primary
primes as there are only finitely many prime ideals that have no primary prime
generators and hence they only contribute to the error term.

We shall also need the following result.

LEMMA 2.4. *Suppose that the GRH is true. For any nonprincipal Hecke character χ
of trivial infinite type with modulus n, we have for $x \geq 1$,*

$$
S(x, \chi) = \sum_{\substack{N(\varpi) \leq x \\ \varpi \text{ primary}}} \chi(\varpi) \log N(\varpi) \ll \min\{x, x^{1/2} \log^3(x) \log N(n)\}.
$$

PROOF.   The x in the minimum is trivial and the other term comes from the standard
analysis of $L'(s, \chi)/L(s, \chi)$. This lemma is the same as [15, Lemma 2.5] and the proof
is given there.   □

2.5. Poisson summation. Our proof of Theorem 1.1 requires the following application of Poisson summation.

LEMMA 2.5. *Let* $n \in O_K, (n, 2) = 1$ *and* $(\cdot/n)$ *be the quadratic symbol* (mod n). *For any Schwartz class function W, we have for all* $a > 0$,

$$\sum_{m \in O_K} \left(\frac{m}{n}\right) W\left(\frac{aN(m)}{X}\right) = \frac{X}{aN(n)} \sum_{k \in O_K} G_K(k, n) \widetilde{W}_K\left(\sqrt{\frac{N(k)X}{aN(n)}}\right), \qquad (2\text{-}13)$$

where

$$\widetilde{W}_K(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(x + y\omega_K)) \widetilde{e}(-t(x + y\omega_K)) \, dx \, dy, \quad t \geq 0.$$

PROOF. We first recall the following Poisson summation formula for O_K (see the proof of [13, Lemma 4.1]), which is itself an easy consequence of the classical Poisson summation formula in two dimensions:

$$\sum_{j \in O_K} f(j) = \sum_{k \in O_K} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x + y\omega_K) \widetilde{e}(-k(x + y\omega_K)) \, dx \, dy.$$

We then have

$$\sum_{m \in O_K} \left(\frac{m}{n}\right) W\left(\frac{aN(m)}{X}\right) = \sum_{r \,(\mathrm{mod}\, n)} \left(\frac{r}{n}\right) \sum_{j \in O_K} W\left(\frac{aN(r + jn)}{X}\right)$$

$$= \sum_{r \,(\mathrm{mod}\, n)} \left(\frac{r}{n}\right) \sum_{k \in O_K} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W\left(\frac{aN(r + (x + y\omega_K)n)}{X}\right)$$
$$\times \widetilde{e}(-k(x + y\omega_K)) \, dx \, dy.$$

We change variables in the integral, writing

$$\sqrt{N\left(\frac{n}{k}\right)} \frac{k}{n} \frac{(r + (x + y\omega_K)n)}{\sqrt{X/a}} = u + v\omega_K$$

with $u, v \in \mathbb{R}$. (If $k = 0$, we omit the factors involving k/n.) The Jacobian of this transformation being $aN(n)/X$, we find that

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W\left(\frac{aN(r + (x + y\omega_K)n)}{X}\right) \widetilde{e}(-k(x + y\omega_K)) \, dx \, dy$$

$$= \frac{X}{aN(n)} \widetilde{e}\left(\frac{kr}{n}\right) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} W(N(u + v\omega_K))$$
$$\times \widetilde{e}\left(-(u + v\omega_K) \sqrt{N\left(\frac{k}{n}\right)\frac{X}{a}}\right) du \, dv,$$

whence

$$\sum_{m \in O_K} \left(\frac{m}{n}\right) W\left(\frac{aN(m)}{X}\right)$$

$$= \frac{X}{aN(n)} \sum_{k \in O_K} \widetilde{W}_K\left(\sqrt{\frac{N(k)X}{aN(n)}}\right) \sum_{r \pmod n} \left(\frac{r}{n}\right) \overline{e}\left(\frac{kr}{n}\right).$$

The inner sum of the last expression above is $g_K(k, n)$ by definition. We apply (2-11) to write $g_K(k, n)$ in terms of $G_K(k, n)$, using that $G_K(k, n) = (-1/n)G_K(-k, n)$ (by (2-12)), and recombining the k and $-k$ terms to see that (2-13) holds. This completes the proof of the lemma. $\qquad\square$

From Lemma 2.5, we readily deduce the following result.

COROLLARY 2.6. *Let $n \in O_K, (n, 2) = 1$. For any Schwartz class function W,*

$$\sum_{\substack{c \in O_K \\ (c,2)=1}} \left(\frac{c}{n}\right) W\left(\frac{N(c)}{X}\right)$$

$$= \left(\frac{c_K}{n}\right) \frac{X}{N(n)} \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k}} G_K(k, n) \widetilde{W}_K\left(\sqrt{\frac{N(k)X}{N(c_K)N(n)}}\right).$$

PROOF. By Lemma 2.5,

$$\sum_{\substack{c \in O_K \\ (c,2)=1}} \left(\frac{c}{n}\right) W\left(\frac{N(c)}{X}\right)$$

$$= \sum_{\substack{(m) \\ m|c_K}} \mu_K(m)\left(\frac{m}{n}\right) \sum_c \left(\frac{c}{n}\right) W\left(\frac{N(m)N(c)}{X}\right)$$

$$= \sum_{\substack{(m) \\ m|c_K}} \mu_K(m)\left(\frac{m}{n}\right) \frac{X}{N(m)N(n)} \sum_{k \in O_K} G_K(k, n) \widetilde{W}_K\left(\sqrt{\frac{N(k)X}{N(m)N(n)}}\right). \qquad (2\text{-}14)$$

Using (2-12),

$$G_K\left(\frac{c_K}{m}k, n\right) = \left(\frac{c_K/m}{n}\right) G_K(k, n).$$

Substituting this back into the last expression in (2-14) gives the desired result. $\qquad\square$

We require some simple estimates on $\widetilde{W}_K(t)$ and its derivatives. It is easy to see that

$$
\widetilde{W}_K(t) = \begin{cases} \displaystyle\int_{\mathbb{R}^2} W\left(x^2 + xy + y^2\frac{1-d}{4}\right)e(ty)\,dx\,dy & \text{if } d \equiv 1 \pmod 4, \\[2em] \displaystyle\int_{\mathbb{R}^2} W(x^2 - dy^2)e(ty)\,dx\,dy & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}
$$

We change variables in the integral, writing

$$
x + \frac{y}{2} = x', \qquad \frac{\sqrt{-d}}{2}y = y' \quad \text{if } d \equiv 1 \pmod 4
$$

and

$$
x = x', \qquad \sqrt{-d}y = y' \quad \text{if } d \equiv 2, 3 \pmod 4.
$$

The Jacobians of these transformations are $\sqrt{-d}/2$ when $d \equiv 1 \pmod 4$ and $\sqrt{-d}$ otherwise. So, we find that if $d \equiv 1 \pmod 4$, then

$$
\widetilde{W}_K(t) = \frac{2}{\sqrt{-d}} \int_{\mathbb{R}^2} W(x'^2 + y'^2)\cos\left(\frac{4\pi ty'}{\sqrt{-d}}\right)dx'\,dy'
$$

and, if $d \equiv 2, \ 3 \pmod 4$, then

$$
\widetilde{W}_K(t) = \frac{1}{\sqrt{-d}} \int_{\mathbb{R}^2} W(x'^2 + y'^2)\cos\left(\frac{4\pi ty'}{\sqrt{-d}}\right)dx'\,dy'.
$$

We deduce from the above that $\widetilde{W}_K(t) \in \mathbb{R}$ for any $t \geq 0$. Suppose that $W(t)$ is a nonnegative smooth function supported on $(1, 2)$ satisfying

$$W(t) = 1 \text{ for } t \in (1 + 1/U, 2 - 1/U) \text{ and } W^{(j)}(t) \ll_j U^j \quad \text{for all integers } j \geq 0.$$

The following estimations for $\widetilde{W}_K(t)$ and its derivatives are from [14, Section 2.8] (beware of the difference between the supports of the functions):

$$
\widetilde{W}_K^{(\mu)}(t) \ll_j \min\{1, \ U^{j-1}t^{-j}\} \tag{2-15}
$$

for all integers $\mu \geq 0, j \geq 1$ and all $t > 0$.

We also have

$$
\widetilde{W}_K(0) = A_K + O\left(\frac{1}{U}\right), \tag{2-16}
$$

where

$$
A_K = \begin{cases} \displaystyle\frac{2\pi}{\sqrt{-d}} & \text{if } d \equiv 1 \pmod 4, \\[2em] \displaystyle\frac{\pi}{\sqrt{-d}} & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}
$$

As $\Phi(t)$ satisfies the assumptions on $W(t)$, the estimations (2-15)–(2-16) are valid for $\widetilde{\Phi}_K(t)$. In what follows, we shall hence use these estimations for $\widetilde{\Phi}_K(t)$ without further justification.

3. Proof of Theorem 1.1

3.1. Evaluation of $C_K(X)$. We have

$$\sideset{}{^*}\sum_{\substack{N(c)\leq X \\ (c,2)=1}} 1 = \sum_{\substack{N(c)\leq X \\ (c,2)=1}} \mu_K^2(c) = \sum_{\substack{N(c)\leq X \\ (c,2)=1}} \sum_{\substack{(l) \\ l^2|c \\ (l,2)=1}} \mu_K(l)$$

$$= \sum_{\substack{(l) \\ N(l)\leq \sqrt{X} \\ (l,2)=1}} \mu_K(l) \sum_{\substack{N(c)\leq X/N(l^2) \\ (c,2)=1}} 1,$$

where the '$*$' on the sum over c means that the sum is restricted to square-free elements c of O_K.

Using analyses similar to those in the study of the Gauss circle problem, we have, with $\theta = 131/416$ (see [20]), that

$$\sum_{N(a)\leq x} 1 = A_K x + O(x^\theta).$$

This implies that

$$\sum_{\substack{N(c)\leq X/N(l^2) \\ (c,2)=1}} 1 = A_K \Big(\sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \Big) \frac{X}{N(l^2)} + O\Big(\Big(\frac{X}{N(l^2)}\Big)^\theta\Big).$$

We then deduce that

$$\sideset{}{^*}\sum_{\substack{N(c)\leq X \\ (c,2)=1}} 1 = X A_K \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{(l) \\ (l,2)=1}} \frac{\mu_K(l)}{N(l^2)} + O(X^{1/2}).$$

We conclude from this that as $X \to \infty$,

$$\sideset{}{^*}\sum_{(c,2)=1} \Phi\Big(\frac{N(c)}{X}\Big) \sim \#C_K(X) \sim X A_K \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{(l) \\ (l,2)=1}} \frac{\mu_K(l)}{N(l^2)}.$$

Using arguments similar to those in [15, Section 3.1] and Lemma 2.3, we see that Theorem 1.1 follows from

$$
\begin{aligned}
&\lim_{X\to\infty} \frac{S_K(X,Y;\hat{\phi},\Phi)}{X\log X} \\
&= -\frac{A_K}{4} \sum_{\substack{(m)\\ m\mid c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{(l)\\ (l,2)=1}} \frac{\mu_K(l)}{N(l^2)} \int_{-\infty}^{\infty} (1-\chi_{[-1,1]}(t))\hat{\phi}(t)\,dt,
\end{aligned} \tag{3-1}
$$

where ϕ is any Schwartz function with $\hat{\phi}$ supported in $(-2+\varepsilon, 2-\varepsilon)$ for any $0 < \varepsilon < 1$ and

$$
\begin{aligned}
&S_K(X,Y;\hat{\phi},\Phi) \\
&= \sideset{}{^*}\sum_{(c,2)=1} \sum_{\substack{\varpi \text{ primary}\\ N(\varpi)\leq Y}} \frac{\chi^{(-4c_K c)}(\varpi)\log N(\varpi)}{\sqrt{N(\varpi)}} \hat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\Phi\Big(\frac{N(c)}{X}\Big).
\end{aligned}
$$

Here $Y = X^{2-\epsilon}$ and we shall write the condition $N(\varpi) \leq Y$ explicitly throughout this section.

3.2. Expressions $S_{K,M}(X, Y; \hat{\phi}, \Phi)$ and $S_{K,R}(X, Y; \hat{\phi}, \Phi)$. Let $Z = \log^5 X$ and write $\mu_K^2(c) = M_{K,Z}(c) + R_{K,Z}(c)$, where

$$
M_{K,Z}(c) = \sum_{\substack{(l),\, l^2\mid c\\ N(l)\leq Z}} \mu_K(l) \quad \text{and} \quad R_{K,Z}(c) = \sum_{\substack{(l),\, l^2\mid c\\ N(l)>Z}} \mu_K(l).
$$

Define

$$
\begin{aligned}
&S_{K,M}(X,Y;\hat{\phi},\Phi) \\
&= \sum_{(c,2)=1} M_{K,Z}(c) \sum_{\substack{\varpi \text{ primary}\\ N(\pi)\leq Y}} \frac{\log N(\varpi)}{\sqrt{N(\varpi)}} \chi^{(-4c_K c)}(\varpi)\hat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\Phi\Big(\frac{N(c)}{X}\Big)
\end{aligned}
$$

and

$$
\begin{aligned}
&S_{K,R}(X,Y;\hat{\phi},\Phi) \\
&= \sum_{(c,2)=1} R_{K,Z}(c) \sum_{\substack{\varpi \text{ primary}\\ N(\pi)\leq Y}} \frac{\log N(\varpi)}{\sqrt{N(\varpi)}} \chi^{(-4c_K c)}(\varpi)\hat{\phi}\Big(\frac{\log N(\varpi)}{\log X}\Big)\Phi\Big(\frac{N(c)}{X}\Big),
\end{aligned}
$$

so that $S_K(X, Y; \hat{\phi}, \Phi) = S_{K,M}(X, Y; \hat{\phi}, \Phi) + S_{K,R}(X, Y; \hat{\phi}, \Phi)$.

As the proof of Theorem 1.1 is similar to that of [15, Theorem 1.1], we focus on obtaining the (secondary) main term and are brief on the error terms in what follows.

We write $S_{K,M}(X, Y; \hat{\phi}, \Phi)$ as

$$S_{K,M}(X, Y; \hat{\phi}, \Phi) = \sum_{\varpi \text{ primary}} \frac{\log N(\varpi)}{\sqrt{N(\varpi)}} \left(\frac{-c_K}{\varpi}\right) \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right)$$

$$\times \sum_{\substack{(l) \\ (l,2)=1 \\ N(l) \leq Z}} \mu_K(l) \left(\frac{l^2}{\varpi}\right) \sum_{\substack{c \in O_K \\ (c,2)=1}} \left(\frac{c}{\varpi}\right) \Phi\left(\frac{N(cl^2)}{X}\right).$$

Applying Corollary 2.6,

$$\sum_{\substack{c \in O_K \\ (c,2)=1}} \left(\frac{c}{\varpi}\right) \Phi\left(\frac{N(cl^2)}{X}\right)$$

$$= \left(\frac{c_K}{\varpi}\right) \frac{X}{N(l^2\varpi)} \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k}} G_K(k, \varpi) \widetilde{\Phi}_K\left(\sqrt{\frac{N(k)X}{N(c_K l^2 \varpi)}}\right).$$

We can now recast $S_{K,M}(X, Y; \hat{\phi}, \Phi)$ as

$$S_{K,M}(X, Y; \hat{\phi}, \Phi)$$

$$= X \sum_{\substack{(l) \\ (l,2)=1 \\ N(l) \leq Z}} \frac{\mu_K(l)}{N(l^2)} \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k}} \sum_{\varpi \text{ primary}} \frac{\log N(\varpi)}{N(\varpi)^{3/2}} \left(\frac{-l^2}{\varpi}\right)$$

$$\times G_K(k, \varpi) \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right) \widetilde{\Phi}_K\left(\sqrt{\frac{N(k)X}{N(c_K l^2 \varpi)}}\right)$$

$$= X \sum_{\substack{(l) \\ (l,2)=1 \\ N(l) \leq Z}} \frac{\mu_K(l)}{N(l^2)} \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k \\ k \neq 0}} \sum_{\varpi \text{ primary}} \frac{\log N(\varpi)}{N(\varpi)} \left(\frac{kl^2}{\varpi}\right)$$

$$\times \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right) \widetilde{\Phi}_K\left(\sqrt{\frac{N(k)X}{N(c_K l^2 \varpi)}}\right),$$

as one deduces easily from Lemma 2.1 that $G_K(0, \varpi) = 0$ and, when $k \neq 0$,

$$G_K(k, \varpi) = \left(\frac{-k}{\varpi}\right) N(\varpi)^{1/2}.$$

3.3. Estimation of $S_{K,M}(X, Y; \hat{\phi}, \Phi)$, the second main term. We define $\chi^{(kl)}$ to be $(kl^2/\cdot)$. Similar to our discussions in Section 2.2, when $k \neq 0$ is not a square, χ_{kl^2} can be regarded as a nonprincipal Hecke character modulo $4kl^2$ of trivial infinite type. Just

as the estimations in [15, Section 3.6], we deduce via Lemma 2.4 that the contribution of $k \neq 0$, $\square$ in $S_{K,M}(X, Y; \hat{\phi}, \Phi)$ is $o(X \log X)$.

We denote the contribution of the terms $k = \square$, $k \neq 0$ in $S_{K,M}(X, Y; \hat{\phi}, \Phi)$ by $S_{K,M,\square}(X, Y; \hat{\phi}, \Phi)$. Thus,

$$S_{K,M}(X, Y; \hat{\phi}, \Phi) = S_{K,M,\square}(X, Y; \hat{\phi}, \Phi) + o(X \log X). \tag{3-2}$$

In what follows, we show that $S_{K,M,\square}(X, Y; \hat{\phi}, \Phi)$ gives rise to a second main term. Before we proceed, we need the following result.

LEMMA 3.1. *For $y > 0$,*

$$\sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k \\ k \neq 0}} \widetilde{\Phi}_K\left(\frac{N(k)}{y}\right) = -\widetilde{\Phi}_K(0) \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} + O\left(\frac{U^2}{y^{1/2}}\right).$$

PROOF. Note that

$$\sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k \\ k \neq 0}} \widetilde{\Phi}_K\left(\frac{N(k)}{y}\right)$$

$$= \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k}} \widetilde{\Phi}_K\left(\frac{N(k)}{y}\right) - \widetilde{\Phi}_K(0) \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)}$$

and

$$\sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k}} \widetilde{\Phi}_K\left(\frac{N(k)}{y}\right)$$

$$= \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{k \in O_K} \widetilde{\Phi}_K\left(\frac{N(c_K/m)N(k)}{y}\right).$$

By taking $n = 1$ in (2-13), we immediately obtain that for $m|c_K$,

$$\sum_{k \in O_K} \widetilde{\Phi}\left(\frac{N(c_K/m)N(k)}{y}\right) = \frac{y}{N(c_K/m)} \sum_{j \in O_K} \breve{\Phi}\left(\sqrt{\frac{N(j)y}{N(c_K/m)}}\right),$$

where

$$\breve{\Phi}(t) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{\Phi}(N(u + v\omega_K))\widetilde{e}\left(-t(u + v\omega_K)\right) du\, dv, \quad t \geq 0.$$

It follows that

$$
\sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{k \in O_K} \widetilde{\Phi}_K\left(\frac{N(c_K/m)N(k)}{y}\right)
$$
$$
= \frac{y}{N(c_K)} \sum_{\substack{(m) \\ m|c_K}} \mu_K(m) \sum_{j \in O_K} \check{\Phi}\left(\sqrt{\frac{N(j)y}{N(c_K/m)}}\,\right). \tag{3-3}
$$

We have, when $t > 0$, using (2-15) and via integration by parts,

$$
\check{\Phi}(t) \ll \frac{U^2}{t^3}.
$$

The lemma follows immediately from this bound and (3-3). □

Now, by a change of variables $k \mapsto k^2$ and noting that $k^2 = k_1^2$ if and only if $k = \pm k_1$,

$$
S_{K,M,\square}(X, Y; \hat{\phi}, \Phi)
$$
$$
= \frac{X}{2} \sum_{\substack{(l) \\ (l,2)=1 \\ N(l) \le Z}} \frac{\mu_K(l)}{N(l^2)} \sum_{\substack{(\varpi,l)=1 \\ \varpi \text{ primary}}} \frac{\log N(\varpi)}{N(\varpi)} \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right)
$$
$$
\times \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k \\ k \ne 0 \\ (k,\varpi)=1}} \widetilde{\Phi}_K\left(N(k)\sqrt{\frac{X}{N(c_K l^2 \varpi)}}\,\right)
$$
$$
= \frac{X}{2} \sum_{\substack{(l) \\ (l,2)=1 \\ N(l) \le Z}} \frac{\mu_K(l)}{N(l^2)} \sum_{\substack{(\varpi,l)=1 \\ \varpi \text{ primary} \\ N(\varpi) \ge X/N(l^2)}} \frac{\log N(\varpi)}{N(\varpi)} \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right)
$$
$$
\times \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{k \in O_K \\ \frac{c_K}{m}|k \\ k \ne 0}} \widetilde{\Phi}_K\left(N(k)\sqrt{\frac{X}{N(c_K l^2 \varpi)}}\,\right) + o(X \log X)
$$
$$
= -\widetilde{\Phi}(0)\frac{X}{2} \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{(l) \\ (l,2)=1 \\ N(l) \le Z}} \frac{\mu_K(l)}{N(l^2)}
$$
$$
\times \sum_{\substack{(\varpi,l)=1 \\ \varpi \text{ primary} \\ N(\varpi) \ge X/N(l^2)}} \frac{\log N(\varpi)}{N(\varpi)} \hat{\phi}\left(\frac{\log N(\varpi)}{\log X}\right) + o(X \log X),
$$

where the second equality above follows by noting that removing the condition $(k, \varpi) = 1$ and then restricting the sum over ϖ to those satisfying $N(\varpi) \geq X/N(l^2)$ only gives rise to an error term of $o(X \log X)$; the last equality above follows from Lemma 3.1.

Applying the following consequence of the prime ideal theorem [31, Theorem 8.9], which asserts that for $x \geq 1$,

$$\sum_{\substack{N(\varpi) \leq x \\ \varpi \text{ primary}}} \frac{\log N(\varpi)}{N(\varpi)} = \log x + O(\log \log 3x),$$

we deduce via partial summation and (2-16),

$$\begin{aligned}
S_{K,M,\square}&(X, Y; \hat{\phi}, \Phi) \\
&= -\frac{X \log X}{4} A_K \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{(l) \\ (l,2)=1}} \frac{\mu_K(l)}{N(l^2)} \\
&\quad \times \int_{-\infty}^{\infty} (1 - \chi_{[-1,1]}(t)) \hat{\phi}(t) \, dt + o(X \log X).
\end{aligned} \tag{3-4}$$

3.4. Conclusion. We estimate $S_{K,R}(X, Y; \hat{\phi}, \Phi)$ as in [15, Section 3.3] to see that

$$S_{K,R}(X, Y; \hat{\phi}, \Phi) = o(X \log X).$$

We now combine the above estimation with (3-2) and (3-4) to obtain that

$$\begin{aligned}
S_K&(X, Y; \hat{\phi}, \Phi) \\
&= -\frac{X \log X}{4} A_K \sum_{\substack{(m) \\ m|c_K}} \frac{\mu_K(m)}{N(m)} \sum_{\substack{(l) \\ (l,2)=1}} \frac{\mu_K(l)}{N(l^2)} \\
&\quad \times \int_{-\infty}^{\infty} (1 - \chi_{[-1,1]}(t)) \hat{\phi}(t) \, dt + o(X \log X),
\end{aligned}$$

which implies (3-1); and this completes the proof of Theorem 1.1.

Acknowledgements

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