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Unistructurality of cluster algebras

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Abstract

We prove that any skew-symmetrizable cluster algebra is unistructural, which is a conjecture by Assem, Schiffler and Shramchenko. As a corollary, we obtain that a cluster automorphism of a cluster algebra $\mathcal{A}(\mathcal{S})$ is just an automorphism of the ambient field $\mathcal{F}$ which restricts to a permutation of the cluster variables of $\mathcal{A}(\mathcal{S})$.

1. Introduction

A cluster algebra $\mathcal{A}(\mathcal{S})$ is a subalgebra of an ambient field $\mathcal{F}$ generated by certain combinatorially defined generators (i.e., cluster variables), which are grouped into overlapping clusters. Roughly speaking, a cluster algebra is a commutative algebra with an extra combinatorial structure. Assem, Schiffler and Shramchenko made the following conjecture.

Conjecture 1.1 [ASS14, Conjecture 1.2]. The set of cluster variables uniquely determines the cluster algebra structure, that is, any cluster algebra is unistructural (see Definition 2.8 for details).

This is a very interesting conjecture in the following sense. We know that a cluster algebra $\mathcal{A}(\mathcal{S})$ is a commutative algebra with an extra combinatorial structure. As an algebra, $\mathcal{A}(\mathcal{S})$ is generated by the set of cluster variables. So the algebraic structure of $\mathcal{A}(\mathcal{S})$ is uniquely determined by the set of cluster variables. The above conjecture predicts that the combinatorial structure of $\mathcal{A}(\mathcal{S})$ can be also uniquely determined by the set of cluster variables.

Conjecture 1.1 has been affirmed for cluster algebras of Dynkin type or rank 2 in [ASS14] and for cluster algebras of type $\tilde{A}$ in [Baz16]. Recently, Bazier-Matte and Plamondon have affirmed Conjecture 1.1 for cluster algebras from surfaces without punctures in [BP20]. Note that all cluster algebras considered above are *with trivial coefficients*.

In this paper, we will affirm Conjecture 1.1 for any skew-symmetrizable cluster algebras *with general coefficients* and the following is our main theorem.

Theorem 1.2. *Any skew-symmetrizable cluster algebra is unistructural.*

This paper is organized as follows. In § 2 some basic definitions, notation and known results are introduced. In § 3 the proof of Theorem 1.2 is given.

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2. Preliminaries

2.1 Cluster algebras

Recall that $(\mathbb{P}, \oplus, \cdot)$ is a *semifield* if $(\mathbb{P}, \cdot)$ is an abelian multiplicative group endowed with a binary operation of auxiliary addition $\oplus$ which is commutative, associative and such that multiplication distributes over auxiliary addition.

Let $\mathrm{Trop}(y_1, \ldots, y_m)$ be a free abelian group generated by $\{y_1, \ldots, y_m\}$. We define the addition $\oplus$ in $\mathrm{Trop}(y_1, \ldots, y_m)$ by $\prod_i y_i^{a_i} \oplus \prod_i y_i^{b_i} = \prod_i y_i^{\min(a_i, b_i)}$. Then $(\mathrm{Trop}(y_1, \ldots, y_m), \oplus)$ is a semifield, which is called a *tropical semifield*.

The multiplicative group of any semifield $\mathbb{P}$ is torsion-free [FZ02], hence its group ring $\mathbb{Z}\mathbb{P}$ is a domain. We take an ambient field $\mathcal{F}$ to be the field of rational functions in n independent variables with coefficients in $\mathbb{Z}\mathbb{P}$.

An integer matrix $B_{n\times n} = (b_{ij})$ is called *skew-symmetrizable* if there is a positive integer diagonal matrix S such that SB is skew-symmetric, where S is said to be a *skew-symmetrizer* of B.

Definition 2.1. (i) A *labeled seed* in $\mathcal{F}$ is a triplet $(\mathbf{x}, \mathbf{y}, B)$ such that:

- $\mathbf{x} = (x_1, \ldots, x_n)$ is an n-tuple such that $X = \{x_1, \ldots, x_n\}$ is a free generating set of $\mathcal{F}$ over $\mathbb{Z}\mathbb{P}$. We call $\mathbf{x}$ (respectively, X) the *labeled cluster* (respectively, (*unlabeled*) *cluster*) and $x_1, \ldots, x_n$ the *cluster variables* of $(\mathbf{x}, \mathbf{y}, B)$.
- $\mathbf{y} = \{y_x\}_{x\in\mathbf{x}}$ is a subset of $\mathbb{P}$. We call $y_{x_1}, \ldots, y_{x_n}$ the *coefficients* of $(\mathbf{x}, \mathbf{y}, B)$.
- $B = (b_{x_i,x_j})$ is an $\mathbf{x} \times \mathbf{x}$ skew-symmetrizable matrix, called an *exchange matrix*.

(ii) Let $(\mathbf{x}, \mathbf{y}, B)$ be a labeled seed. The triplet $(X, \mathbf{y}, B)$ is called an (*unlabeled*) *seed*, where X is the (unlabeled) cluster of the labeled seed $(\mathbf{x}, \mathbf{y}, B)$.

Let $(\mathbf{x}, \mathbf{y}, B)$ be a (labeled or unlabeled) seed in $\mathcal{F}$. One can associate *binomials* $F_{x_1}, \ldots, F_{x_n}$ defined by

$$F_{x_k} = \frac{y_{x_k}}{1 \oplus y_{x_k}} \prod_{b_{x_i,x_k}>0} x_i^{b_{x_i,x_k}} + \frac{1}{1 \oplus y_{x_k}} \prod_{b_{x_i,x_k}<0} x_i^{-b_{x_i,x_k}}.$$

We call $F_{x_1}, \ldots, F_{x_n}$ the *exchange binomials* of $(\mathbf{x}, \mathbf{y}, B)$.

Definition 2.2. Let $(\mathbf{x}, \mathbf{y}, B)$ be a (labeled or unlabeled) seed in $\mathcal{F}$, and $F_{x_1}, \ldots, F_{x_n}$ be the exchange binomials of $(\mathbf{x}, \mathbf{y}, B)$. Define the *mutation* of $(\mathbf{x}, \mathbf{y}, B)$ at x_k as a new triple $\mu_{x_k}(\mathbf{x}, \mathbf{y}, B) = (\mathbf{x}', \mathbf{y}', B')$ in $\mathcal{F}$ given by:

$$x'_i = \begin{cases} x_i & \text{if } x_i \neq x_k, \\ F_{x_k}/x_k & \text{if } x_i = x_k; \end{cases}$$

$$y'_{x'_i} = \begin{cases} y_{x_k}^{-1} & \text{if } x_i = x_k, \\ y_{x_i} y_{x_k}^{\max(b_{x_k,x_i},0)} (1 \oplus y_{x_k})^{-b_{x_k,x_i}} & \text{otherwise;} \end{cases}$$

$$b'_{x'_i,x'_j} = \begin{cases} -b_{x_i,x_j} & \text{if } x_i = x_k \text{ or } x_j = x_k, \\ b_{x_i,x_j} + \mathrm{sgn}(b_{x_i,x_k}) \max(b_{x_i,x_k} b_{x_k,x_j}, 0) & \text{otherwise.} \end{cases}$$

It can be seen that $\mu_{x_k}(\mathbf{x}, \mathbf{y}, B)$ is also a (labeled or unlabeled) seed and

$$\mu_{x'_k}(\mu_{x_k}(\mathbf{x}, \mathbf{y}, B)) = (\mathbf{x}, \mathbf{y}, B).$$

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Let $\mathbb{T}_n$ be the n-regular tree, and label the edges of $\mathbb{T}_n$ by $1, \ldots, n$ such that the n different edges adjacent to the same vertex of $\mathbb{T}_n$ receive different labels.

Definition 2.3. A *cluster pattern* $\mathcal{S}$ is an assignment of a labeled seed $(\mathbf{x}_t, \mathbf{y}_t, B_t)$ to every vertex t of the n-regular tree $\mathbb{T}_n$, such that $(\mathbf{x}_{t'}, \mathbf{y}_{t'}, B_{t'}) = \mu_{x_{k;t}}.(\mathbf{x}_t, \mathbf{y}_t, B_t)$ for any edge $t \overset{k}{\longrightarrow} t'$, where $x_{k;t}$ is the kth cluster variable of $\mathbf{x}_t$.

Let $\mathcal{S}$ be a cluster pattern, and $(\mathbf{x}_t, \mathbf{y}_t, B_t)$ be the labeled seed at $t \in \mathbb{T}_n$. We always denote

$$\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t}), \quad X_t = \{x_{1;t}, \ldots, x_{n;t}\}, \quad \mathbf{y}_t = \{y_{1;t}, \ldots, y_{n;t}\} \quad \text{and} \quad B_t = (b^t_{ij}),$$

where $y_{i;t}$ (respectively, b^t_{ij}) should be understood as $y_{x_{i;t};t}$ (respectively, $b^t_{x_{i;t},x_{j;t}}$) and the mutation $\mu_{x_{k;t}}(\mathbf{x}_t, \mathbf{y}_t, B_t)$ will be denoted by $\mu_k(\mathbf{x}_t, \mathbf{y}_t, B_t)$. We will also use $(X_t, \mathbf{y}_t, B_t)$ to denote the (unlabeled) seed at t.

- Let $\mathcal{S}$ be a cluster pattern. The *cluster algebra* $\mathcal{A}(\mathcal{S})$ associated with $\mathcal{S}$ is the $\mathbb{ZP}$-subalgebra of the field $\mathcal{F}$ generated by all cluster variables of $\mathcal{S}$.
- If $\mathcal{S}$ is a cluster pattern with coefficients in $\mathrm{Trop}(y_1, \ldots, y_m)$, the corresponding cluster algebra $\mathcal{A}(\mathcal{S})$ is said to be a *cluster algebra of geometric type*.
- If $\mathcal{S}$ is a cluster pattern with coefficients in $\mathrm{Trop}(y_1, \ldots, y_n)$ and there exists a labeled seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ such that $y_{i;t_0} = y_i$ for $i = 1, \ldots, n$, then the corresponding cluster algebra $\mathcal{A}(\mathcal{S})$ is called a *cluster algebra with principal coefficients at* t_0.

In [FZ02] Fomin and Zelevinsky proved cluster variables enjoying the Laurent phenomenon and conjectured that the Laurent phenomenon has positivity. The positivity of the Laurent phenomenon was proved first by Lee and Schiffler in [LS15] for skew-symmetric cluster algebras, later by Gross *et al.* in [GHKK18] for skew-symmetrizable cluster algebras, and recently by Davison in [Dav18] for skew-symmetric quantum cluster algebras.

Theorem 2.4. *Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra with a labeled seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$.*

(i) *(Laurent phenomenon [FZ02, Theorem 3.1]) Any cluster variable $x_{i;t}$ of $\mathcal{A}(\mathcal{S})$ is a $\mathbb{ZP}$-linear combination of Laurent monomials in $\mathbf{x}_{t_0}$.*

(ii) *(Positive Laurent phenomenon [GHKK18]) Any cluster variable $x_{i;t}$ of $\mathcal{A}(\mathcal{S})$ is an $\mathbb{NP}$-linear combination of Laurent monomials in $\mathbf{x}_{t_0}$.*

Recall that the *exchange graph* $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ of a cluster algebra $\mathcal{A}(\mathcal{S})$ is a graph such that:

- the set of vertices of $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ is in bijection with the set of (unlabeled) seeds of $\mathcal{A}(\mathcal{S})$;
- two vertices are joined by an edge if and only if the corresponding two seeds are obtained from each other by a single mutation.

The *cluster complex* of a cluster algebra $\mathcal{A}(\mathcal{S})$ is the simplicial complex whose set of vertices is the set of cluster variables and whose simplices are the subsets of clusters.

Theorem 2.5 [GSV08, Theorem 5]. *Suppose that every seed in a cluster algebra $\mathcal{A}(\mathcal{S})$ is uniquely determined by its cluster. Then two clusters are adjacent in the exchange graph of $\mathcal{A}(\mathcal{S})$ if and only if they have exactly $n-1$ common variables.*

Proposition 2.6 [CL18, Proposition 6.1]. *Let $\mathcal{A}(\mathcal{S}(1)), \mathcal{A}(\mathcal{S}(2))$ be two skew-symmetrizable cluster algebras having the same exchange matrix at t_0. Denoted by $(\mathbf{x}_t(k), \mathbf{y}_t(k), B_t(k))$ the labeled seed of $\mathcal{A}(\mathcal{S}(k))$ at $t \in \mathbb{T}_n$, $k = 1, 2$. The following statements hold.*

(i) $x_{i;t_1}(1) = x_{j;t_2}(1)$ *if and only if* $x_{i;t_1}(2) = x_{j;t_2}(2)$, *where* $t_1, t_2 \in \mathbb{T}_n$ *and* $i, j \in \{1, 2, \ldots, n\}$.

(ii) *If there exists a permutation σ of $\{1, \ldots, n\}$ such that*

$$x_{i;t_1}(1) = x_{\sigma(i);t_2}(1)$$

for $i = 1, \ldots, n$, then $y_{i;t_1}(1) = y_{\sigma(i);t_2}(1)$ and $b_{ij}^{t_1}(1) = b_{\sigma(i)\sigma(j)}^{t_2}(1)$ for any i and j.

Corollary 2.7. *Let $\mathcal{A}(\mathcal{S})$ be a cluster algebra and $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ be the exchange graph of $\mathcal{A}(\mathcal{S})$. Then:*

(i) *every seed of $\mathcal{A}(\mathcal{S})$ is uniquely determined by its cluster;*

(ii) $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ *coincides with the dual graph of the cluster complex of $\mathcal{A}(\mathcal{S})$;*

(iii) $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ *is uniquely determined by the set of cluster variables of $\mathcal{A}(\mathcal{S})$ and the set of clusters of $\mathcal{A}(\mathcal{S})$;*

(iv) $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ *is uniquely determined by the initial exchange matrix of $\mathcal{A}(\mathcal{S})$, that is, $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ does not depend on the choice of coefficients of $\mathcal{A}(\mathcal{S})$.*

Proof. (i) This follows from Proposition 2.6(ii) directly.

(ii) By (i) and Theorem 2.5, we know that the exchange graph $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ is just the graph such that:

(a) the set of vertices of $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ is in bijection with the set of clusters of $\mathcal{A}(\mathcal{S})$;

(b) two vertices are joined by an edge if and only if the corresponding two clusters differ by a single cluster variable.

So $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ coincides with the dual graph of the cluster complex of $\mathcal{A}(\mathcal{S})$.

(iii) By the definition of cluster complex, we know the cluster complex of $\mathcal{A}(\mathcal{S})$ is uniquely determined by the set of cluster variables of $\mathcal{A}(\mathcal{S})$ and the set of clusters of $\mathcal{A}(\mathcal{S})$. By (ii), we get that $\mathbf{EG}(\mathcal{A}(\mathcal{S}))$ is uniquely determined by the set of cluster variables of $\mathcal{A}(\mathcal{S})$ and the set of clusters of $\mathcal{A}(\mathcal{S})$.

(iv) By Proposition 2.6(i), we know that the cluster complex of $\mathcal{A}(\mathcal{S})$ is uniquely determined by the initial exchange matrix of $\mathcal{A}(\mathcal{S})$. Then the result follows from (ii). □

Corollary 2.7(i) for cluster algebras of geometry type can be found in [GSV08, Theorem 5], and Corollary 2.7(iv) for skew-symmetric cluster algebras is a result in [CKLP13, Corollary 5.5].

For a cluster algebra $\mathcal{A}(\mathcal{S})$, we denote by $\mathcal{X}(\mathcal{S})$ the set of cluster variables of $\mathcal{A}(\mathcal{S})$.

Definition 2.8 [ASS14]. A cluster algebra $\mathcal{A}(\mathcal{S})$ is *unistructural* if for any cluster algebra $\mathcal{A}(\mathcal{S}')$, $\mathcal{X}(\mathcal{S}) = \mathcal{X}(\mathcal{S}')$ implies that the two cluster algebras have the same set of clusters and $\mathbf{EG}(\mathcal{A}(\mathcal{S})) = \mathbf{EG}(\mathcal{A}(\mathcal{S}'))$.

2.2 The enough g-pairs property

Let $\mathcal{A}(\mathcal{S})$ be a cluster algebra with principal coefficients at t_0. One can give a $\mathbb{Z}^n$-grading of $\mathbb{Z}[x_{1;t_0}^{\pm 1}, \ldots, x_{n;t_0}^{\pm 1}, y_1, \ldots, y_n]$ as follows:

$$\deg(x_{i;t_0}) = \mathbf{e}_i, \quad \deg(y_j) = -\mathbf{b}_j,$$

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where $\mathbf{e}_i$ is the ith column vector of I_n, and $\mathbf{b}_j$ is the jth column vector of B_{t_0}, $i, j = 1, 2, \ldots, n$. As shown in [FZ07], every cluster variable $x_{i;t}$ of $\mathcal{A}(\mathcal{S})$ is homogeneous with respect to this $\mathbb{Z}^n$-grading. The *g-vector* $g(x_{i;t})$ of a cluster variable $x_{i;t}$ is defined to be its degree with respect to the $\mathbb{Z}^n$-grading, and we write $g(x_{i;t}) = (g^t_{1i}, g^t_{2i}, \ldots, g^t_{ni})^\top \in \mathbb{Z}^n$. Let $\mathbf{x}_t$ be a labeled cluster of $\mathcal{A}(\mathcal{S})$. The matrix $G_t = (g(x_{1;t}), \ldots, g(x_{n;t}))$ is called the *G-matrix* of $\mathbf{x}_t$.

Denote $\mathbf{x}_t^{\mathbf{a}} := \prod_{i=1}^n x_{i;t}^{a_i}$ for $\mathbf{a} \in \mathbb{Z}^n$, which is a Laurent monomial in $\mathbf{x}_t$. If $\mathbf{a} \in \mathbb{N}^n$, then $\mathbf{x}_t^{\mathbf{a}}$ is called a *cluster monomial* in $\mathbf{x}_t$.

Clearly, any Laurent monomial $\mathbf{x}_t^{\mathbf{a}}$ is also homogeneous with respect to the $\mathbb{Z}^n$-grading. The degree of $\mathbf{x}_t^{\mathbf{a}}$ is $g(\mathbf{x}_t^{\mathbf{a}}) := G_t\mathbf{a}$, which is called the *g-vector of* $\mathbf{x}_t^{\mathbf{a}}$.

Theorem 2.9.

(i) *[GHKK18, CL18] Different cluster monomials have different g-vectors.*

(ii) *[NZ12, CL18] Each G-matrix has determinant* ± 1.

Let I be a subset of $\{1, \ldots, n\}$. We say that $(k_1, \ldots, k_s)$ is an *I-sequence*, if $k_j \in I$ for $j = 1, \ldots, s$.

Definition 2.10. Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra of rank n with initial seed at t_0, and $I = \{i_1, \ldots, i_p\}$ be a subset of $\{1, 2, \ldots, n\}$.

(i) We say that a labeled seed $(\mathbf{x}_t, \mathbf{y}_t, B_t)$ of $\mathcal{A}(\mathcal{S})$ is *connected with the initial labeled seed* $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ *by an I-sequence*, if there exists an I-sequence $(k_1, \ldots, k_s)$ such that

$$(\mathbf{x}_t, \mathbf{y}_t, B_t) = \mu_{k_s} \cdots \mu_{k_2}\mu_{k_1}(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0}).$$

(ii) We say that a labeled cluster $\mathbf{x}_t$ of $\mathcal{A}(\mathcal{S})$ is *connected with* $\mathbf{x}_{t_0}$ *by an I-sequence*, if there exists a labeled seed containing the labeled cluster $\mathbf{x}_t$ such that this labeled seed is connected with the initial labeled seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ by an I-sequence.

Clearly, if the labeled cluster $\mathbf{x}_t$ is connected with $\mathbf{x}_{t_0}$ by an I-sequence, then $x_{i;t} = x_{i;t_0}$ for $i \notin I$.

For $I = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, n\}$, we assume that $i_1 < i_2 < \cdots < i_p$. Let

$$\pi_I : \mathbb{R}^n \to \mathbb{R}^{|I|} = \mathbb{R}^p$$

be the canonical projection given by $\pi_I(\mathbf{m}) = (m_{i_1}, \ldots, m_{i_p})^\top$, for $\mathbf{m} = (m_1, \ldots, m_n)^\top \in \mathbb{R}^n$.

Definition 2.11. Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra of rank n with principal coefficients at t_0, and I be a subset of $\{1, \ldots, n\}$.

(i) For two labeled clusters $\mathbf{x}_t, \mathbf{x}_{t'}$ of $\mathcal{A}(\mathcal{S})$, the pair $(\mathbf{x}_t, \mathbf{x}_{t'})$ is called a *g-pair along I*, if it satisfies the following conditions:

- $\mathbf{x}_{t'}$ is connected with $\mathbf{x}_{t_0}$ by an I-sequence;
- for any cluster monomial $\mathbf{x}_t^{\mathbf{v}}$ in $\mathbf{x}_t$, there exists a cluster monomial $\mathbf{x}_{t'}^{\mathbf{v}'}$ in $\mathbf{x}_{t'}$ with $v'_i = 0$ for $i \notin I$ such that
$$\pi_I(g(\mathbf{x}_t^{\mathbf{v}})) = \pi_I(g(\mathbf{x}_{t'}^{\mathbf{v}'})),$$
where $g(\mathbf{x}_t^{\mathbf{v}})$ and $g(\mathbf{x}_{t'}^{\mathbf{v}'})$ are g-vectors of the cluster monomials $\mathbf{x}_t^{\mathbf{v}}$ and $\mathbf{x}_{t'}^{\mathbf{v}'}$, respectively.

(ii) $\mathcal{A}(\mathcal{S})$ is said to have *the enough g-pairs property* if for any subset I of $\{1, \ldots, n\}$ and any labeled cluster $\mathbf{x}_t$ of $\mathcal{A}(\mathcal{S})$, there exists a labeled cluster $\mathbf{x}_{t'}$ such that $(\mathbf{x}_t, \mathbf{x}_{t'})$ is a g-pair along I.

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Note that $\pi_I(g(\mathbf{x}_t^{\mathbf{v}})) = \pi_I(g(\mathbf{x}_{t'}^{\mathbf{v}'}))$ just means that the two g-vectors $g(\mathbf{x}_t^{\mathbf{v}})$ and $g(\mathbf{x}_{t'}^{\mathbf{v}'})$ coincide at the components indexed by I.

Example 2.12. Let $\mathcal{A}(\mathcal{S})$ be the principal coefficients cluster algebra at the seed $((x_1, x_2), (y_1, y_2), B)$, where $B = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. By calculation, we know that $\mathcal{A}(\mathcal{S})$ has five different cluster variables appearing in the following labeled clusters:

$$(x_1, x_2) \xrightarrow{\mu_2} (x_1, x_3) \xrightarrow{\mu_1} (x_4, x_3) \xrightarrow{\mu_2} (x_4, x_5) \xrightarrow{\mu_1} (x_2, x_5) \xrightarrow{\mu_2} (x_2, x_1),$$

where $x_3 = (y_2x_1 + 1)/x_2$, $x_4 = (y_1y_2x_1 + y_1 + x_2)/x_1x_2$, $x_5 = (y_1 + x_2)/x_1$. The following are the G-matrices of the corresponding labeled clusters:

$$G_{(x_1,x_2)} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad G_{(x_1,x_3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad G_{(x_4,x_3)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$
$$G_{(x_4,x_5)} = \begin{pmatrix} -1 & -1 \\ 0 & 1 \end{pmatrix}, \quad G_{(x_2,x_5)} = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}, \quad G_{(x_2,x_1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We can check that the pair of labeled clusters $((x_4, x_5), (x_1, x_2))$ is a g-pair along an $I = \{2\}$-sequence.

Firstly, (x_1, x_2) is connected with the initial labeled cluster (x_1, x_2) by an $I = \{2\}$-sequence.

Secondly, for any cluster monomial $x_4^a x_5^b$ $(a, b \geqslant 0)$ in (x_4, x_5), its g-vector is the vector

$$g(x_4^a x_5^b) = (-a - b, b)^\top.$$

We can choose the cluster monomial $x_1^0 x_2^b = x_2^b$ in (x_1, x_2), and we know that its g-vector is

$$g(x_2^b) = (0, b)^\top.$$

It is easy to see that the two g-vectors $g(x_4^a x_5^b)$ and $g(x_2^b)$ coincide at the components indexed by $I = \{2\}$, that is, $\pi_I(g(x_4^a x_5^b)) = \pi_I(g(x_2^b))$.

Theorem 2.13 [CL18]. *Any skew-symmetrizable cluster algebra $\mathcal{A}(\mathcal{S})$ with principal coefficients at t_0 has the enough g-pairs property.*

2.3 Compatibility degree on the set of cluster variables

Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra, and $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be a labeled seed of $\mathcal{A}(\mathcal{S})$. By the Laurent phenomenon, any cluster variable x of $\mathcal{A}(\mathcal{S})$ has the form $x = \sum_{\mathbf{v} \in V} c_{\mathbf{v}} \mathbf{x}_{t_0}^{\mathbf{v}}$, where V is a finite subset of $\mathbb{Z}^n$, $0 \neq c_{\mathbf{v}} \in \mathbb{ZP}$. Let $-d_i$ be the minimal exponent of $x_{i;t_0}$ appearing in the expansion $x = \sum_{\mathbf{v} \in V} c_{\mathbf{v}} \mathbf{x}_{t_0}^{\mathbf{v}}$, where $i = 1, \ldots, n$. Then x has the form

$$x = \frac{f(x_{1;t_0}, \ldots, x_{n;t_0})}{x_{1;t_0}^{d_1} \cdots x_{n;t_0}^{d_n}},$$

where $f \in \mathbb{ZP}[x_{1;t_0}, \ldots, x_{n;t_0}]$ with $x_{j;t_0} \nmid f$ for $j = 1, \ldots, n$. The vector

$$d^{t_0}(x) := (d_1, \ldots, d_n)^\top$$

is called the *denominator vector* (or *d-vector* for short) of the cluster variable x with respect to $\mathbf{x}_{t_0}$.

Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra, and $\mathcal{X}(\mathcal{S})$ be the set of cluster variables of $\mathcal{A}(\mathcal{S})$. In [CL18], we proved that there exists a well-defined function

$$d : \mathcal{X}(\mathcal{S}) \times \mathcal{X}(\mathcal{S}) \to \mathbb{Z}_{\geqslant -1},$$

which is called the *compatibility degree* of $\mathcal{A}(\mathcal{S})$. For any two cluster variables $x_{i;t}$ and $x_{j;t_0}$, the value of $d(x_{j;t_0}, x_{i;t})$ is defined by the following steps:

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- choose an (unlabeled) cluster X_{t_0} containing the cluster variable $x_{j;t_0}$;
- compute the d-vector of $x_{i;t}$ with respect to X_{t_0}, say, $d^{t_0}(x_{i;t}) = (d_1, \ldots, d_n)^\top$;
- $d(x_{j;t_0}, x_{i;t}) := d_j$, which is called the *compatibility degree of $x_{i;t}$ with respect to $x_{j;t_0}$*.

Remark 2.14 [CL18, Theorem 6.3]. The compatibility degree has the following properties.

(1) The value $d(x_{j;t_0}, x_{i;t})$ does not depend on the choice of X_{t_0}, thus the compatibility degree function is well defined.

(2) $d(x_{j;t_0}, x_{i;t}) = -1$ if and only if $d(x_{i;t}, x_{j;t_0}) = -1$, and if and only if $x_{j;t_0} = x_{i;t}$.

(3) $d(x_{j;t_0}, x_{i;t}) = 0$ if and only if $d(x_{i;t}, x_{j;t_0}) = 0$, and if and only if $x_{j;t_0} \neq x_{i;t}$ and there exists a cluster $X_{t'}$ containing both $x_{j;t_0}$ and $x_{i;t}$.

(4) By (2), (3) and $d(x_{j;t_0}, x_{i;t}) \geqslant -1$, we know that $d(x_{j;t_0}, x_{i;t}) \leqslant 0$ if and only if $d(x_{i;t}, x_{j;t_0}) \leqslant 0$, and if and only if there exists a cluster $X_{t'}$ containing both $x_{j;t_0}$ and $x_{i;t}$.

(5) By (4), we know that $d(x_{j;t_0}, x_{i;t}) > 0$ if and only if $d(x_{i;t}, x_{j;t_0}) > 0$, if and only if there exists no cluster $X_{t'}$ containing both $x_{j;t_0}$ and $x_{i;t}$.

We say that $x_{i;t}$ and $x_{j;t_0}$ are *d-compatible* if $d(x_{j;t_0}, x_{i;t}) \leqslant 0$, that is, if there exists a cluster $X_{t'}$ containing both $x_{j;t_0}$ and $x_{i;t}$. A subset M of $\mathcal{X}(\mathcal{S})$ is a *d-compatible set* if any two cluster variables in this set are d-compatible.

There is another type of compatible sets, which we call c-compatible sets. A subset M of $\mathcal{X}(\mathcal{S})$ is a *c-compatible set* if there exists a cluster $X_{t'}$ such that $M \subseteq X_{t'}$. Roughly speaking, c-compatibility is just compatibility with respect to clusters.

Theorem 2.15 [CL18, Theorem 7.4]. *Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra, and $\mathcal{X}(\mathcal{S})$ be the set of cluster variables of $\mathcal{A}(\mathcal{S})$. Then:*

(i) *a subset M of $\mathcal{X}(\mathcal{S})$ is a d-compatible set if and only if it is a c-compatible set, that is, M is a subset of some cluster of $\mathcal{A}(\mathcal{S})$;*

(ii) *a subset M of $\mathcal{X}(\mathcal{S})$ is a maximal d-compatible set if and only it is a maximal c-compatible set, that is, M is a cluster of $\mathcal{A}(\mathcal{S})$.*

3. Proof of Theorem 1.2

Lemma 3.1. *Let $\mathcal{A}(\mathcal{S}^{\mathrm{pr}})$ be a skew-symmetrizable cluster algebra with principal coefficients at t_0, and $x_{i;t}$ be any cluster variable. Let $\mathbf{x}_{t'}$ be a labeled cluster such that $(\mathbf{x}_t, \mathbf{x}_{t'})$ is a g-pair along an $I = \{1, 2, \ldots, k-1, k+1, \ldots, n\}$-sequence ($\mathbf{x}_{t'}$ exists thanks to Theorem 2.13), and $d^{t'}(x_{i;t}) = (d_1, \ldots, d_n)^\top$ be the d-vector of $x_{i;t}$ with respect to $\mathbf{x}_{t'}$. Finally, let $\mathbf{r} = (r_1, \ldots, r_n)^\top \in \mathbb{Z}^n$ be the vector such that $G_{t'}\mathbf{r} = g(x_{i;t})$ ($\mathbf{r}$ exists thanks to Theorem 2.9(ii)), and $F = \lambda \mathbf{x}_{t'}^{\mathbf{r}}$ be the Laurent monomial with exponent vector $\mathbf{r}$ appearing in the Laurent expansion of $x_{i;t}$ with respect to the labeled cluster $\mathbf{x}_{t'}$. Then the following statements hold.*

(i) *$\lambda = 1$.*

(ii) *$x_{k;t'} = x_{k;t_0}$ and $r_j \geqslant 0$ for any j different from k.*

(iii) (a) *If $r_k > 0$, then $x_{i;t} = x_{k;t'} = x_{k;t_0}$;*

(b) *if $r_k = 0$, then $x_{i;t}$ is in $\mathbf{x}_{t'}$ and is different from $x_{k;t'} = x_{k;t_0}$;*

(c) *if $x_{i;t}$ is not in $\mathbf{x}_{t'}$, then $r_k < 0$.*

(iv) $$d_k = \begin{cases} -1 & \text{if } r_k > 0, \\ 0 & \text{if } r_k = 0, \\ \text{a positive integer} & \text{if } r_k < 0. \end{cases}$$

Proof. This lemma is essentially due to [CL18, Lemma 5.2], but with different presentation. For the convenience of the reader, we give the proof.

Without loss of generality, we can assume that $k = n$.

(i) This is a direct result of [CL18, Theorem 3.1], which says that the Laurent expansion of $x_{i;t}$ with respect to $\mathbf{x}_{t'}$ has the form

$$x_{i;t} = \mathbf{x}_{t'}^{\mathbf{r}}\Big(1 + \sum_{0\neq \mathbf{v}\in\mathbb{N}^n,\ \mathbf{u}\in\mathbb{Z}^n} c_{\mathbf{v}}\mathbf{y}^{\mathbf{v}}\mathbf{x}_{t'}^{\mathbf{u}}\Big),$$

where $c_{\mathbf{v}} \geqslant 0$ and $\mathbf{r}$ satisfies $g(x_{i;t}) = G_{t'}\mathbf{r}$.

(ii) Denote $g(x_{i;t}) = (g_1, \ldots, g_n)^\top$. Since $(\mathbf{x}_t, \mathbf{x}'_t)$ is a g-pair along $I = \{1, \ldots, n-1\}$, we know that for the cluster variable $x_{i;t}$ (as a cluster monomial in $\mathbf{x}_t$), there exists a cluster monomial $\mathbf{x}_{t'}^{\mathbf{v}'}$ in $\mathbf{x}_{t'}$ with $v'_j = 0$ for $j \notin I$, that is, $v'_n = 0$ such that

$$\pi_I(g(x_{i;t})) = \pi_I(g(\mathbf{x}_{t'}^{\mathbf{v}'})) = \pi_I(G_{t'}\mathbf{v}').$$

Because $(\mathbf{x}_t, \mathbf{x}'_t)$ is a g-pair along $I = \{1, \ldots, n-1\}$, we know that $\mathbf{x}_{t'}$ is connected with $\mathbf{x}_{t_0}$ by an $I = \{1, \ldots, n-1\}$-sequence, and thus $x_{n;t'} = x_{n;t_0}$. So the G-matrix of $\mathbf{x}_{t'}$ has the form $G_{t'} = \begin{pmatrix} G(t') & 0 \\ * & 1 \end{pmatrix}$. Thus $\pi_I(g(x_{i;t})) = \pi_I(g(\mathbf{x}_{t'}^{\mathbf{v}'})) = \pi_I(G_{t'}\mathbf{v}')$ just means that

$$(g_1, \ldots, g_{n-1})^\top = G(t')(v'_1, \ldots, v'_{n-1})^\top.$$

By $g(x_{i;t}) = (g_1, \ldots, g_{n-1}, g_n)^\top = G_{t'}\mathbf{r}$, we get that

$$(g_1, \ldots, g_{n-1})^\top = G(t')(r_1, \ldots, r_{n-1})^\top.$$

It is known from Theorem 2.9(ii) that $\det(G(t')) = \det(G_{t'}) = \pm 1$, so we can get

$$(r_1, \ldots, r_{n-1})^\top = (v'_1, \ldots, v'_{n-1})^\top \in \mathbb{N}^{n-1}.$$

That is, $r_j \geqslant 0$ for any j different from n.

(iii) We begin with parts (a) and (b). If $r_n \geqslant 0$, then $\mathbf{r} \in \mathbb{N}^n$, and $\mathbf{x}_{t'}^{\mathbf{r}}$ is a cluster monomial in $\mathbf{x}_{t'}$ having the same g-vector with the cluster variable $x_{i;t}$. By Theorem 2.9(i), we get that $x_{i;t} = \mathbf{x}_{t'}^{\mathbf{r}}$. Then by [CL18, Lemma 5.1], $x_{i;t}$ is a cluster variable in $\mathbf{x}_{t'}$. More precisely, if $r_n > 0$, then $x_{i;t} = x_{n;t'} = x_{n;t_0}$. If $r_n = 0$, then $x_{i;t} = x_{j;t'}$ for some $j \neq n$.

Part (c) follows from (a) and (b).

(iv) This follows from (iii) and the definition of the components of d-vectors. □

The above lemma is about cluster algebras with principal coefficients, and we can turn it into a lemma about cluster algebras with general coefficients with the help of separation formulas in [FZ07] by Fomin and Zelevinsky.

Lemma 3.2 [FZ07, Theorem 3.7]. *Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra with initial labeled seed $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$, and $\mathcal{A}(\mathcal{S}^{\mathrm{pr}})$ be a cluster algebra with principal coefficients at $(\mathbf{x}_{t_0}^{\mathrm{pr}}, \mathbf{y}_{t_0}^{\mathrm{pr}}, B_{t_0}^{\mathrm{pr}})$ such that $\mathbf{x}_{t_0}^{\mathrm{pr}} = \mathbf{x}_{t_0}$ and $B_{t_0}^{\mathrm{pr}} = B_{t_0}$. Let $x_{i;t}$ be a cluster variable of $\mathcal{A}(\mathcal{S})$ and*

$$x_{i;t}^{\mathrm{pr}} = x_{i;t}^{\mathrm{pr}}(x_{1;t_0}, \ldots, x_{n;t_0}; y_1, \ldots, y_n)$$

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be the corresponding cluster variable of $\mathcal{A}(\mathcal{S}^{\mathrm{pr}})$. *Then the Laurent expansion of* $x_{i;t}$ *with respect to the initial cluster* $\mathbf{x}_{t_0}$ *can be obtained from the Laurent expansion of* $x_{i;t}^{\mathrm{pr}}$ *with respect to* $\mathbf{x}_{t_0}^{\mathrm{pr}} = \mathbf{x}_{t_0}$ *by the formula*

$$x_{i;t} = \frac{x_{i;t}^{\mathrm{pr}}|_{\mathcal{F}}(x_{1;t_0}, \ldots, x_{n;t_0}; y_{1;t_0}, \ldots, y_{n;t_0})}{F|_{\mathbb{P}}(y_{1;t_0}, \ldots, y_{n;t_0})},$$

where $F(y_1, \ldots, y_n) = x_{i;t}^{\mathrm{pr}}|_{x_{1;t_0}=\cdots=x_{n;t_0}=1}$.

Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra, and $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ be a labeled seed of $\mathcal{A}(\mathcal{S})$. By the Laurent phenomenon, any cluster variable $x_{i;t}$ of $\mathcal{A}(\mathcal{S})$ has the form $x_{i;t} = \sum_{\mathbf{v}\in V} c_{\mathbf{v}} \mathbf{x}_{t_0}^{\mathbf{v}}$, where V is a finite subset of $\mathbb{Z}^n$ and $0 \neq c_{\mathbf{v}} \in \mathbb{ZP}$. We call V the *support set* of $x_{i;t}$ with respect to $\mathbf{x}_{t_0}$ and denote it by $V^{t_0}(x_{i;t}) := V$.

COROLLARY 3.3. *Let* $\mathcal{A}(\mathcal{S})$ *be a skew-symmetrizable cluster algebra, and* $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$ *be a labeled seed of* $\mathcal{A}(\mathcal{S})$. *For any cluster variable* $x_{i;t}$, *the support set* $V^{t_0}(x_{i;t})$ *only depends on the exchange matrix at* t_0, *and does not depend on the choice of coefficients of the cluster algebra.*

Proof. Let $\mathcal{A}(\mathcal{S}^{\mathrm{pr}})$ be a cluster algebra with principal coefficients at $(\mathbf{x}_{t_0}^{\mathrm{pr}}, \mathbf{y}_{t_0}^{\mathrm{pr}}, B_{t_0}^{\mathrm{pr}})$ such that $\mathbf{x}_{t_0}^{\mathrm{pr}} = \mathbf{x}_{t_0}$ and $B_{t_0}^{\mathrm{pr}} = B_{t_0}$, and $x_{i;t}^{\mathrm{pr}}$ be cluster variable of $\mathcal{A}(\mathcal{S}^{\mathrm{pr}})$ corresponding to $x_{i;t}$. Denote by $V^{t_0}(x_{i;t}^{\mathrm{pr}})$ the support set of $x_{i;t}^{\mathrm{pr}}$ with respect to $\mathbf{x}_{t_0}^{\mathrm{pr}} = \mathbf{x}_{t_0}$. By Lemma 3.2, we know that $V^{t_0}(x_{i;t}^{\mathrm{pr}}) = V^{t_0}(x_{i;t})$. By the arbitrariness of the choice of coefficients of $\mathcal{A}(\mathcal{S})$, we know the support set $V^{t_0}(x_{i;t})$ only depends on the exchange matrix at t_0. □

PROPOSITION 3.4. *Let* $\mathcal{A}(\mathcal{S})$ *be a skew-symmetrizable cluster algebra with an initial labeled seed* $(\mathbf{x}_{t_0}, \mathbf{y}_{t_0}, B_{t_0})$, *and* $\mathcal{A}(\mathcal{S}^{\mathrm{pr}})$ *be a cluster algebra with principal coefficients at* $(\mathbf{x}_{t_0}^{\mathrm{pr}}, \mathbf{y}_{t_0}^{\mathrm{pr}}, B_{t_0}^{\mathrm{pr}})$ *such that* $\mathbf{x}_{t_0}^{\mathrm{pr}} = \mathbf{x}_{t_0}$ *and* $B_{t_0}^{\mathrm{pr}} = B_{t_0}$. *For any cluster variable* $x_{i;t}$ *of* $\mathcal{A}(\mathcal{S})$, *let*

$$x_{i;t}^{\mathrm{pr}} = x_{i;t}^{\mathrm{pr}}(x_{1;t_0}, \ldots, x_{n;t_0}; y_1, \ldots, y_n)$$

be the corresponding cluster variable of $\mathcal{A}(\mathcal{S}^{\mathrm{pr}})$. *Let* $\mathbf{x}_{t'}^{\mathrm{pr}}$ *be the labeled cluster such that* $(\mathbf{x}_t^{\mathrm{pr}}, \mathbf{x}_{t'}^{\mathrm{pr}})$ *is a g-pair along an* $I = \{1, 2, \ldots, k-1, k+1, \ldots, n\}$*-sequence* ($\mathbf{x}_{t'}^{\mathrm{pr}}$ *exists thanks to Theorem 2.13*), *and* $\mathbf{x}_{t'}$ *be the corresponding labeled cluster of* $\mathcal{A}(\mathcal{S})$. *Let* $d^{t'}(x_{i;t}) = (d_1, \ldots, d_n)^\top$ *be the d-vector of* $x_{i;t}$ *with respect to* $\mathbf{x}_{t'}$. *Finally, let* $\mathbf{r} = (r_1, \ldots, r_n)^\top \in \mathbb{Z}^n$ *be the vector such that* $G_{t'}\mathbf{r} = g(x_{i;t}^{\mathrm{pr}})$ ($\mathbf{r}$ *exists thanks to Theorem 2.9(ii)*), *and* $F = \lambda \mathbf{x}_{t'}^{\mathbf{r}}$ *be the Laurent monomial with exponent vector* $\mathbf{r}$ *appearing in the Laurent expansion of* $x_{i;t}$ *with respect to the labeled cluster* $\mathbf{x}_{t'}$. *Then the following statements hold.*

(i) $\lambda \neq 0$.

(ii) $x_{k;t'} = x_{k;t_0}$ *and* $r_j \geqslant 0$ *for any* j *different from* k.

(iii) (a) *If* $r_k > 0$, *then* $x_{i;t} = x_{k;t'} = x_{k;t_0}$;

(b) *if* $r_k = 0$, *then* $x_{i;t}$ *is in* $\mathbf{x}_{t'}$ *and is different from* $x_{k;t'} = x_{k;t_0}$;

(c) *if* $x_{i;t}$ *is not in* $\mathbf{x}_{t'}$, *then* $r_k < 0$.

(iv) $d_k = \begin{cases} -1 & \text{if } r_k > 0, \\ 0 & \text{if } r_k = 0, \\ \text{a positive integer} & \text{if } r_k < 0. \end{cases}$

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Proof. (i) By Lemma 3.1(i), we know that $\mathbf{r}$ is in the support set $V^{t'}(x^{\mathrm{pr}}_{i;t})$. By Corollary 3.3, $V^{t'}(x_{i;t}) = V^{t'}(x^{\mathrm{pr}}_{i;t})$. So $\lambda \neq 0$.

(ii) By Lemma 3.1(i), we know that $x^{\mathrm{pr}}_{k;t'} = x^{\mathrm{pr}}_{k;t_0}$ and $r_j \geqslant 0$ for any j different from k. By Lemma 3.2, we can get $x_{k;t'} = x_{k;t_0}$.

(iii) Parts (a) and (b) follow from Lemma 3.1(iii) (a), (b) and Lemma 3.2.

Part (c) follows from (iii) (a) and (b).

(iv) This follows from (iii) and the definition of the components of d-vectors. □

Proof of Theorem 1.2. Let $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S}')$ be two skew-symmetrizable cluster algebras having the same set of cluster variables, that is, $\mathcal{X}(\mathcal{S}) = \mathcal{X}(\mathcal{S}')$. We need to show that $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S}')$ have the same set of clusters and $\mathbf{EG}(\mathcal{A}(\mathcal{S})) = \mathbf{EG}(\mathcal{A}(\mathcal{S}'))$.

We first show that $x, z \in \mathcal{X}(\mathcal{S}) = \mathcal{X}(\mathcal{S}')$ are d-compatible in $\mathcal{A}(\mathcal{S})$ if and only if they are d-compatible in $\mathcal{A}(\mathcal{S}')$.

Denote by $\mathbf{x}_t = (x_{1;t}, \ldots, x_{n;t})$ the labeled cluster of $\mathcal{A}(\mathcal{S})$ at the vertex $t \in \mathbb{T}_n$ and by $\mathbf{z}_u = (z_{1;u}, \ldots, z_{m;u})$ the labeled cluster of $\mathcal{A}(\mathcal{S}')$ at the vertex $u \in \mathbb{T}_m$. We know that $m = n$, because they are both the transcendence degree of $\mathcal{F}$ over $\mathbb{QP}$.

Let $x, z \in \mathcal{X}(\mathcal{S}) = \mathcal{X}(\mathcal{S}')$ be two cluster variables which are d-compatible in $\mathcal{A}(\mathcal{S}')$. Assume by contradiction that x and z are not d-compatible in $\mathcal{A}(\mathcal{S})$, that is, there exists no cluster of $\mathcal{A}(\mathcal{S})$ containing both x and z. For the cluster variables z and x, by viewing (z, x) as $(x_{i;t}, x_{k;t_0})$ in Proposition 3.4 and applying Proposition 3.4(iii)(c) for $\mathcal{A}(\mathcal{S})$, we can find a cluster $\mathbf{x}_{t'}$ of $\mathcal{A}(\mathcal{S})$ containing x (say, $x = x_{1;t'}$) such that there exists a nonzero Laurent monomial F appearing in the Laurent expansion of z with respect to $\mathbf{x}_{t'}$ such that the exponent of $x_{j;t'}$ in F is nonnegative for any $j \neq 1$ and the exponent of $x = x_{1;t'}$ in F is negative (because there exists no cluster of $\mathcal{A}(\mathcal{S})$ containing both x and z). We can assume that $F = c x_{1;t'}^{-v_1} \prod_{i=2}^{n} x_{i;t'}^{v_i}$ with $v_1 > 0$, $v_2, \ldots, v_n \geqslant 0$ and $0 \neq c \in \mathbb{NP}$. Thus the Laurent expansion of z with respect to $\mathbf{x}_{t'}$ can be written as

$$z = F + \tilde{F}(x_{1;t'}, \ldots, x_{n;t'}) = c x_{1;t'}^{-v_1} \prod_{i=2}^{n} x_{i;t'}^{v_i} + \tilde{F}(x_{1;t'}, \ldots, x_{n;t'}),$$

where $\tilde{F}$ is a Laurent polynomial with positive coefficients.

Since $x = x_{1;t'}$ and z are d-compatible in $\mathcal{A}(\mathcal{S}')$, there exists a cluster $\mathbf{z}_u$ of $\mathcal{A}(\mathcal{S}')$ such that $\mathbf{z}_u$ contains both $x = x_{1;t'}$ and z. Without loss of generality, we can assume that $x = x_{1;t'} = z_{1;u}$ and $z = z_{2;u}$. Consider the Laurent expansion of $x_{i;t'}$ with respect to $\mathbf{z}_u$,

$$x_{i;t'} = \frac{g_i(z_{1;u}, \ldots, z_{m;u})}{z_{1;u}^{d_{1i}} \cdots z_{m;u}^{d_{mi}}},$$

where g_i is a polynomial in $z_{1;u}, \ldots, z_{m;u}$ with positive coefficients and $z_{l;u} \nmid g_i$ for any l. By Remark 2.14, and $x_{i;t'} \neq x_{1;t'} = x = z_{1;u}$ for any $i = 2, \ldots, n$, we know that $d_{1i} \geqslant 0$ for $i = 2, \ldots, n$. So for each $i = 2, \ldots, n$, there exists a Laurent monomial G_i appearing in the expansion of $x_{i;t'}$ with respect to $\mathbf{z}_u$ such that the exponent of $z_{1;u} = x$ is nonpositive in G_i, otherwise all the exponents of $z_{1;u} = x$ appearing in the Laurent expansion of $x_{i;t'}$ are positive, and this will lead to $d_{1i} < 0$, which contradicts $d_{1i} \geqslant 0$. So $x_{i;t'}$ has the form

$$x_{i;t'} = G_i + \tilde{G}_i(z_{1;u}, \ldots, z_{m;u}) = c_i z_{1;u}^{-a_{1i}} \prod_{l=2}^{m} z_{l;u}^{a_{li}} + \tilde{G}_i(z_{1;u}, \ldots, z_{m;u}),$$

where $\tilde{G}_i$ is a Laurent polynomial with positive coefficients, and $a_{1i} \geqslant 0$, $0 \neq c_i \in \mathbb{NP}$ for some semifield $\mathbb{P}$.

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Substituting $x_{1;t'} = x = z_{1;u}$ and $x_{i;t'} = c_i z_{1;u}^{-a_{1i}} \prod_{l=2}^{m} z_{l;u}^{a_{li}} + \tilde{G}_i(z_{1;u}, \ldots, z_{m;u})$ for $i \geqslant 2$ into

$$z = c x_{1;t'}^{-v_1} \prod_{i=2}^{n} x_{i;t'}^{v_i} + \tilde{F}(x_{1;t'}, \ldots, x_{n;t'}),$$

we obtain the expansion of $z = z_{2;u}$ with respect to $\mathbf{z}_u$, which has the form

$$\begin{aligned} z_{2;u} = z &= c z_{1;u}^{-v_1} \prod_{i=2}^{n} \left(c_i z_{1;u}^{-a_{1i}} \prod_{l=2}^{m} z_{l;u}^{a_{li}} \right)^{v_i} + R(z_{1;u}, \ldots, z_{m;u}) \\ &= c \prod_{i=2}^{n} c_i z_{1;u}^{-(v_1 + v_2 a_{12} + \cdots + v_n a_{1n})} \prod_{l=2}^{m} z_{l;u}^{v_2 a_{l2} + \cdots + v_n a_{ln}} + R(z_{1;u}, \ldots, z_{m;u}), \end{aligned}$$

where R can be written as $R = r_1(z_{1;u}, \ldots, z_{m;u})/r_2(z_{1;u}, \ldots, z_{m;u})$ with $r_1, r_2 \in \mathbb{NP}[z_{1;u}, \ldots, z_{m;u}]$. Thus we get that

$$z_{2;u} - c \prod_{i=2}^{n} c_i z_{1;u}^{-(v_1 + v_2 a_{12} + \cdots + v_n a_{1n})} \prod_{l=2}^{m} z_{l;u}^{v_2 a_{l2} + \cdots + v_n a_{ln}} = \frac{r_1(z_{1;u}, \ldots, z_{m;u})}{r_2(z_{1;u}, \ldots, z_{m;u})}. \tag{1}$$

Note that there is a $\mathbb{Q}$-algebra homomorphism $\varphi : \mathbb{QP}(z_{1;u}, \ldots, z_{m;u}) \to \mathbb{Q}(z_{1;u}, \ldots, z_{m;u})$ given by

$$\varphi(a) = \begin{cases} 1 & \text{if } a \in \mathbb{P}, \\ z_{p;u} & \text{if } a = z_{p;u} \text{ for some } p = 1, \ldots, m. \end{cases}$$

So the equality (1) in $\mathbb{QP}(z_{1;u}, \ldots, z_{m;u})$ induces an equality in $\mathbb{Q}(z_{1;u}, \ldots, z_{m;u})$ by the action of the homomorphism φ. The new equality is

$$z_{2;u} - \varphi\left(c \prod_{i=2}^{n} c_i \right) z_{1;u}^{-(v_1 + v_2 a_{12} + \cdots + v_n a_{1n})} \prod_{l=2}^{m} z_{l;u}^{v_2 a_{l2} + \cdots + v_n a_{ln}} = \frac{\varphi(r_1)(z_{1;u}, \ldots, z_{m;u})}{\varphi(r_2)(z_{1;u}, \ldots, z_{m;u})}, \tag{2}$$

where $\varphi(r_1), \varphi(r_2) \in \mathbb{N}[z_{1;u}, \ldots, z_{m;u}]$ and $\varphi(c \prod_{i=2}^{n} c_i) \geqslant 1$.

Since $v_1 > 0$ and $v_i, a_{1i} \geqslant 0$, we get that $v_1 + v_2 a_{12} + \cdots + v_n a_{1n} > 0$. Take $z_{1;u} = 1/2 = 2^{-1}$ and $z_{2;u} = z_{3;u} = \cdots = z_{m;u} = 1$, the left-hand side of equality (2) is

$$1 - \varphi\left(c \prod_{i=2}^{n} c_i \right) 2^{(v_1 + v_2 a_{12} + \cdots + v_n a_{1n})} \leqslant 1 - 2^{(v_1 + v_2 a_{12} + \cdots + v_n a_{1n})} < 0,$$

but the right-hand side of equality (2) is nonnegative in this case, by $\varphi(r_1), \varphi(r_2) \in \mathbb{N}[z_{1;u}, \ldots, z_{m;u}]$. This is a contradiction. So if x and z are d-compatible in $\mathcal{A}(\mathcal{S}')$, they must be d-compatible in $\mathcal{A}(\mathcal{S})$.

Similarly, we can show that if x and z are d-compatible in $\mathcal{A}(\mathcal{S})$, they must be d-compatible in $\mathcal{A}(\mathcal{S}')$.

Thus $x, z \in \mathcal{X}(\mathcal{S}) = \mathcal{X}(\mathcal{S}')$ are d-compatible in $\mathcal{A}(\mathcal{S})$ if and only if they are d-compatible in $\mathcal{A}(\mathcal{S}')$. So a subset $M \subseteq \mathcal{X}(\mathcal{S}) = \mathcal{X}(\mathcal{S}')$ is a maximal d-compatible set in $\mathcal{A}(\mathcal{S})$ if and only if it is a maximal d-compatible set in $\mathcal{A}(\mathcal{S}')$. That is, a subset $M \subseteq \mathcal{X}(\mathcal{S}) = \mathcal{X}(\mathcal{S}')$ is a cluster in $\mathcal{A}(\mathcal{S})$ if and only if it is a cluster in $\mathcal{A}(\mathcal{S}')$, by Theorem 2.15. Hence, $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S}')$ have the same set of clusters.

Hence, $\mathcal{A}(\mathcal{S})$ and $\mathcal{A}(\mathcal{S}')$ have the same set of cluster variables and have the same set of clusters. By Corollary 2.7(iii), we know that $\mathbf{EG}(\mathcal{A}(\mathcal{S})) = \mathbf{EG}(\mathcal{A}(\mathcal{S}'))$. This completes the proof. □

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Let $\mathcal{A}(\mathcal{S})$ be a cluster algebra with coefficients in $\mathbb{ZP}$. A *cluster automorphism* of $\mathcal{A}(\mathcal{S})$ is a $\mathbb{ZP}$-automorphism of the algebra $\mathcal{A}(\mathcal{S})$ mapping a cluster to a cluster and commuting with mutations.

Corollary 3.5. *Let $\mathcal{A}(\mathcal{S})$ be a skew-symmetrizable cluster algebra. Then $f : \mathcal{A}(\mathcal{S}) \to \mathcal{A}(\mathcal{S})$ is a cluster automorphism if and only if f is an automorphism of the ambient field $\mathcal{F}$ which restricts to a permutation of the set of cluster variables.*

Proof. It is known that this result is true for unistructural cluster algebras by [ASS14, Theorem 1.4]. Then it follows from Theorem 1.2. □

We know that the definition of compatibility degree function on the set of cluster variables mainly depends on the cluster structure of a cluster algebra. By Theorem 1.2, the cluster structure of a cluster algebra is uniquely determined by the set of cluster variables, so the compatibility degree function is an intrinsic function on the set of cluster variables. It would be interesting to give a geometric or categorial explanation for compatibility degree functions in general. Note that for cluster algebras of finite type or from surfaces, the compatibility degree functions have nice explanations and one can refer to [FZ03a, FZ03b, CP15, Zhu07, FST08].

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