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Rearrangements of affine subspaces in vector spaces over finite fields

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We consider the L^p norms of sums of characteristic functions of affine subspaces of a vector space V over a finite field under certain restrictions on p, $\dim V$ and the dimensions of the subspaces involved. We investigate the conditions under which these norms are increased when the affine subspaces are replaced by their parallel translates passing through 0. Applications to extremal configurations for Kakeya maximal-type inequalities are given and open questions are raised.

1. Introduction

Let V be a finite-dimensional vector space over a finite field $\mathbb{F}$. If U is an affine subspace of V, we denote by U^* the translate (rearrangement) of U which contains the origin, i.e. the *linear* subspace of V that is parallel to U. The cardinality of U is denoted by $|U|$.

It is trivial that $(U_1 \cap U_2)^* \subseteq U_1^* \cap U_2^*$ and hence that, for all positive integers m, $|U_1 \cap \cdots \cap U_m| \leqslant |U_1^* \cap \cdots \cap U_m^*|$. This leads immediately (upon multiplying out) to the following.

Proposition 1.1. *Let V be a finite-dimensional vector space over a finite field and let $\{V_\alpha \colon \alpha \in \mathcal{A}\}$ be a collection of affine subspaces of V. Suppose that $p \in \mathbb{N}$. Then, for any collection of non-negative coefficients $\{x_\alpha\}$,*

$$\sum_{v\in V}\biggl(\sum_\alpha x_\alpha \chi_{V_\alpha}(v)\biggr)^p \leqslant \sum_{v\in V}\biggl(\sum_\alpha x_\alpha \chi_{V_\alpha^*}(v)\biggr)^p.$$

In this paper we extend this inequality to certain non-integral values of p.

Proposition 1.2. *Let V be a finite-dimensional vector space over a finite field and let $\{V_\alpha \colon \alpha \in \mathcal{A}\}$ be a collection of affine subspaces of V. Suppose that $p \geqslant \dim V$. Then, for any collection of non-negative coefficients $\{x_\alpha\}$,*

$$\sum_{v\in V}\biggl(\sum_\alpha x_\alpha \chi_{V_\alpha}(v)\biggr)^p \leqslant \sum_{v\in V}\biggl(\sum_\alpha x_\alpha \chi_{V_\alpha^*}(v)\biggr)^p. \tag{1.1}$$

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Such inequalities and certain analogues in Euclidean space are of interest in the theory of Besicovitch–Kakeya-type maximal operators. For example, if $\dim V = 2$ and the V_α are lines, one in each direction of V, then the inequality of proposition 1.2 gives extremizers for the dual form of the Kakeya maximal operator on the spaces $\ell^p(V)$, $p \geqslant 2$. From this the sharp constant for the Kakeya maximal operator itself on $\ell^q(V)$, $q \leqslant 2$, may be obtained by direct calculation. For details see the discussion after proposition 2.5.

When $\dim V = 1$ or 2, the condition $p \geqslant \dim V$ in proposition 1.2 cannot be relaxed: see proposition 2.4.

2. Proofs and discussion

For the purposes of the proof of proposition 1.2 we shall in fact show something slightly stronger.

Proposition 2.1. *Let V be an n-dimensional affine subspace of an m-dimensional vector space W over a finite field $\mathbb{F}$. Let $\{V_\alpha \colon \alpha \in \mathcal{A}\}$ be a collection of affine subspaces of V. Suppose that $p \geqslant n$. Then, for any collection of non-negative coefficients $\{x_\alpha\}$,*

$$\sum_{v \in V} \bigg(\sum_\alpha x_\alpha \chi_{V_\alpha}(v) \bigg)^p \leqslant \sum_{v \in V^*} \bigg(\sum_\alpha x_\alpha \chi_{V_\alpha^*}(v) \bigg)^p.$$

Proof. We proceed by induction on n. When $n = 1$ the V_α correspond to points of V or V itself, and we may assume that they are distinct. So the inequality becomes

$$\sum_{v \in V} (x_v + x_V)^p \leqslant \bigg(\sum_{v \in V} x_v + x_V \bigg)^p + x_V^p(|\mathbb{F}| - 1).$$

Both sides are homogeneous of degree p in the variables x_v (with $v \in V$) and x_V, so we may assume that $x_V = 1$. We also note that when all the x_v are zero there is equality. So it suffices to show that, for each $v \in V$,

$$\frac{\partial}{\partial x_v} \sum_{v \in V} (x_v + 1)^p \leqslant \frac{\partial}{\partial x_v} \bigg(\sum_{v \in V} x_v + 1 \bigg)^p,$$

or that, for each $v \in V$,

$$(x_v + 1)^{p-1} \leqslant \bigg(\sum_{u \in V} x_u + 1 \bigg)^{p-1},$$

which is manifestly true if $p \geqslant 1$.

Assume that we have proved the result whenever V is an affine subspace (of some ambient space) of dimension less than n. Let V now be an n-dimensional affine subspace of W, let V_α ($\alpha \in \mathcal{A}$) be affine subspaces of V (which we may assume are distinct) and let us consider

$$\Psi_p(\boldsymbol{x}) = \sum_{v \in V^*} \bigg(\sum_\alpha x_\alpha \chi_{V_\alpha^*}(v) \bigg)^p - \sum_{v \in V} \bigg(\sum_\alpha x_\alpha \chi_{V_\alpha}(v) \bigg)^p,$$

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where $\boldsymbol{x} = (x_\alpha)_{\alpha\in\mathcal{A}}$. By homogeneity we may assume that the index α_V, corresponding to the whole space V, satisfies $x_{\alpha_V} = 1$. We also note that, when all the other x_α are zero, $\Psi_p(\boldsymbol{x}) = 0$. So it suffices to show that for each α corresponding to a *proper* affine subspace of V we have

$$\frac{\partial}{\partial x_\alpha}\Psi_p(\boldsymbol{x}) \geqslant 0.$$

Now

$$\begin{aligned}\frac{1}{p}\frac{\partial}{\partial x_\alpha}\Psi_p(\boldsymbol{x}) &= \sum_{v\in V^*}\left(\sum_\beta x_\beta\chi_{V_\beta^*}(v)\right)^{p-1}\chi_{V_\alpha^*}(v) - \sum_{v\in V}\left(\sum_\beta x_\beta\chi_{V_\beta}(v)\right)^{p-1}\chi_{V_\alpha}(v)\\ &= \sum_{v\in V_\alpha^*}\left(\sum_\beta x_\beta\chi_{V_\beta^*\cap V_\alpha^*}(v)\right)^{p-1} - \sum_{v\in V_\alpha}\left(\sum_\beta x_\beta\chi_{V_\beta\cap V_\alpha}(v)\right)^{p-1}\\ &\geqslant \sum_{v\in V_\alpha^*}\left(\sum_\beta x_\beta\chi_{(V_\beta\cap V_\alpha)^*}(v)\right)^{p-1} - \sum_{v\in V_\alpha}\left(\sum_\beta x_\beta\chi_{V_\beta\cap V_\alpha}(v)\right)^{p-1}.\end{aligned}$$

Since the dimension of V_α is at most $n-1$, the inductive hypothesis applies when $p-1 \geqslant n-1$. Thus, for $p \geqslant n$ we have

$$\frac{\partial}{\partial x_\alpha}\Psi_p(\boldsymbol{x}) \geqslant 0,$$

and hence $\Psi_p(\boldsymbol{x}) \geqslant 0$, completing the inductive step. □

Remark 2.2. As the reader may readily verify, the proof can be adapted to show that for $p \geqslant n$ the difference of the right-hand side and the left-hand side of (1.1) is not only non-negative but also a non-decreasing function of p.

As an immediate application we have the following.

Proposition 2.3. *Let V be a vector space over a finite field and let $\{V_\alpha : \alpha\in\mathcal{A}\}$ be a collection of affine subspaces of V, each of which has dimension at most n. Suppose that $p \geqslant n+1$. Then, for any collection of non-negative coefficients $\{x_\alpha\}$,*

$$\sum_{v\in V}\left(\sum_\alpha x_\alpha\chi_{V_\alpha}(v)\right)^p \leqslant \sum_{v\in V}\left(\sum_\alpha x_\alpha\chi_{V_\alpha^*}(v)\right)^p.$$

Proof. We have

$$\begin{aligned}\sum_{v\in V}\left(\sum_\alpha x_\alpha\chi_{V_\alpha}(v)\right)^p &= \sum_\alpha x_\alpha\sum_{v\in V}\chi_{V_\alpha}(v)\left(\sum_\beta x_\beta\chi_{V_\beta}(v)\right)^{p-1}\\ &= \sum_\alpha x_\alpha\sum_{v\in V_\alpha}\left(\sum_\beta x_\beta\chi_{V_\beta\cap V_\alpha}(v)\right)^{p-1}\\ &\leqslant \sum_\alpha x_\alpha\sum_{v\in V_\alpha^*}\left(\sum_\beta x_\beta\chi_{(V_\beta\cap V_\alpha)^*}(v)\right)^{p-1}\end{aligned}$$

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$$\leqslant \sum_{\alpha} x_\alpha \sum_{v \in V_\alpha^*} \bigg(\sum_{\beta} x_\beta \chi_{V_\beta^* \cap V_\alpha^*}(v) \bigg)^{p-1}$$

$$= \sum_{v \in V} \bigg(\sum_{\alpha} x_\alpha \chi_{V_\alpha^*}(v) \bigg)^{p}$$

by proposition 2.1 applied to each V_α; this is valid when $p-1 \geqslant n$. □

There exists the following converse to proposition 2.3 when $n = 0$ or 1.

PROPOSITION 2.4. *Let $n = 0$ or 1. Let V be a vector space over a finite field and let $\{V_\alpha : \alpha \in \mathcal{A}\}$ be the collection of n-dimensional affine subspaces of V. If (1.1) holds, then either $p = 1$ or $p \geqslant n + 1$.*

Proof. First suppose that $n = 0$. Zero-dimensional affine subspaces of V are just points, so (1.1) becomes simply

$$\sum_{v \in V} x_v^p \leqslant \bigg(\sum_{v \in V} x_v \bigg)^p;$$

this can only hold for $p \geqslant 1$.

Now suppose that $n = 1$ and $p > 1$. According to [4], when $\mathrm{char}(\mathbb{F}) \neq 2$, there exists a configuration of $|\mathbb{F}| + 1$ lines in $\mathbb{F}^2$, one in each direction (i.e. a Besicovitch–Kakeya set in the finite field setting), whose union $\mathcal{E}$ has cardinality $\frac{1}{2}(|\mathbb{F}|^2 + |\mathbb{F}|)$. It is readily verified that each point of $\mathcal{E}$ belongs to exactly two of the lines. Place this configuration on a two-dimensional plane contained in V and let the V_α be the corresponding lines. Then

$$\sum_{v} \bigg(\sum_{\alpha} \chi_{V_\alpha}(v) \bigg)^p = 2^{p-1}(|\mathbb{F}|^2 + |\mathbb{F}|),$$

while, on the other hand,

$$\sum_{v} \bigg(\sum_{\alpha} \chi_{V_\alpha^*}(v) \bigg)^p = (|\mathbb{F}| + 1)^p + |\mathbb{F}|^2 - 1.$$

These expressions are equal for $p = 1$ and $p = 2$, and the latter is larger than the former if $p \geqslant 2$. However, if $1 < p < 2$, the former is asymptotically $2^{p-1}|\mathbb{F}|^2$ for large $|\mathbb{F}|$, while the latter is (the smaller) $|\mathbb{F}|^2$. Hence, (1.1) cannot hold for $1 < p < 2$. □

The next case of interest is when $\{V_\alpha : \alpha \in \mathcal{A}\}$ is the collection of all two-dimensional affine hyperplanes in $\mathbb{F}^3$. We already know that (1.1) holds for $p = 1$, $p = 2$ and $p \geqslant 3$ in this setting. Taking the Cartesian product of the planar configuration $\mathcal{E}$ as above with $\mathbb{F}$ shows that it cannot hold when $1 < p < 2$. We do not, however, know whether (1.1) holds for $2 < p < 3$ in this case. One may suspect that a possible counter-example to (1.1) might consist of a collection of hyperplanes, exactly one 'perpendicular' to each direction in $\mathbb{F}^3$. Nevertheless, if such a set is to be a counter-example, it must be so for somewhat more subtle reasons than emerged in the proof of proposition 2.4; the following result shows that the left- and right-hand sides of (1.1) are asymptotically equal when $|\mathbb{F}|$ is large.

Proposition 2.5. *Let $\{V_\alpha\colon \alpha \in \mathcal{A}\}$ be any collection of two-dimensional affine hyperplanes in $\mathbb{F}^3$ with exactly one 'perpendicular' to each direction in $\mathbb{F}^3$. Then for $2 < p < 3$ we have*

$$\left|\left(\frac{\sum_v(\sum_\alpha \chi_{V_\alpha}(v))^p}{\sum_v(\sum_\alpha \chi_{V_\alpha^*}(v))^p}\right)^{1/p} - 1\right| \leqslant C_p|\mathbb{F}|^{1-3/p}.$$

Proof. Let $H(v) = \sum_\alpha \chi_{V_\alpha}(v)$ and $H^*(v) = \sum_\alpha \chi_{V_\alpha^*}(v)$. Then $H^*(0) = |\mathbb{F}|^2+|\mathbb{F}|+1$ and, for $v \neq 0$, $H^*(v) = |\mathbb{F}|+1$. With $\|\cdot\|_p$ denoting the usual l^p norm with respect to *normalized* counting measure on V and $\mathbb{E}$ the corresponding normalized integral, we have

$$\mathbb{E}H = \mathbb{E}H^* := \bar{H} = |\mathbb{F}|(1 + |\mathbb{F}|^{-1} + |\mathbb{F}|^{-2}) \sim |\mathbb{F}|$$

by direct calculation of $\mathbb{E}H^*$. By direct calculation of $\mathbb{E}H^{*2}$ and the assumption that there is exactly one hyperplane in each direction we have

$$\mathbb{E}H^2 = \mathbb{E}H^{*2} \sim |\mathbb{F}|^2 \sim \bar{H}^2.$$

Crucially, direct calculation of $\mathbb{E}(H^* - \bar{H})^2$ gives the improved estimate

$$\mathbb{E}(H - \bar{H})^2 = \mathbb{E}H^2 - \bar{H}^2 = \mathbb{E}H^{*2} - \bar{H}^2 = \mathbb{E}(H^* - \bar{H})^2 \leqslant C|\mathbb{F}| \sim C\bar{H} \ll \bar{H}^2.$$

Here, C is independent of $|\mathbb{F}|$, and the improvement over the trivial estimate is manifested in the last inequality. Finally, direct calculation of $\mathbb{E}H^{*p}$ for $2 < p \leqslant 3$ gives

$$\mathbb{E}H^3 \leqslant \mathbb{E}H^{*3} \sim |\mathbb{F}|^3 \sim \bar{H}^3$$

and (for $2 < p < 3$)

$$\mathbb{E}H^{*p} \sim |\mathbb{F}|^p \sim \bar{H}^p.$$

So, by Hölder's inequality,

$$\mathbb{E}(H - \bar{H})^p \leqslant (\mathbb{E}(H - \bar{H})^2)^{3-p}(\mathbb{E}(H - \bar{H})^3)^{p-2} \leqslant C\bar{H}^{3-p}\bar{H}^{3(p-2)} = C\bar{H}^{2p-3}$$

(where C now depends only upon p), and, in particular,

$$\mathbb{E}(H^* - \bar{H})^p \leqslant C\bar{H}^{2p-3}$$

too.

Hence,

$$\|H - H^*\|_p \leqslant \|H - \bar{H}\|_p + \|\bar{H} - H^*\|_p \leqslant C\bar{H}^{(2p-3)/p} \sim C\bar{H}|\mathbb{F}|^{1-3/p}.$$

Therefore,

$$|\|H\|_p - \|H^*\|_p| \leqslant C\bar{H}|\mathbb{F}|^{1-3/p}$$

and so

$$\left|1 - \frac{\|H\|_p}{\|H^*\|_p}\right| \leqslant C\frac{\bar{H}}{\|H^*\|_p}|\mathbb{F}|^{1-3/p} \sim C|\mathbb{F}|^{1-3/p},$$

as required. □

We now return to the connection with the Kakeya maximal operator, which relates to the case that $\{V_\alpha\colon \alpha \in \mathcal{A}\}$ is an arbitrary collection of one-dimensional affine subspaces of the d-dimensional space V, one in each of the $(|\mathbb{F}|^d - 1)/(|\mathbb{F}| - 1)$ distinct directions in V. The dual form of the Kakeya maximal theorem of Ellenberg *et al*. [3] is the inequality

$$\sum_{v\in V}\bigg(\sum_{\alpha} x_\alpha \chi_{V_\alpha}(v)\bigg)^{d/(d-1)} \leqslant C|\mathbb{F}|\sum_{\alpha} x_\alpha^{d/(d-1)}, \tag{2.1}$$

where the constant C depends only on d. Since

$$\begin{aligned}\sum_{v\in V}\bigg(\sum_{\alpha} x_\alpha \chi_{V_\alpha^*}(v)\bigg)^{p} &= \bigg(\sum_{\alpha} x_\alpha\bigg)^{p} + (|\mathbb{F}|-1)\sum_{\alpha} x_\alpha^p \\ &\leqslant \bigg\{\bigg(\frac{|\mathbb{F}|^d-1}{|\mathbb{F}|-1}\bigg)^{p/p'} + |\mathbb{F}| - 1\bigg\}\sum_{\alpha} x_\alpha^p\end{aligned}$$

with equality when all the x_α are equal, the best possible constant C in (2.1) is at least

$$|\mathbb{F}|^{-1}\bigg\{\bigg(\frac{|\mathbb{F}|^d-1}{|\mathbb{F}|-1}\bigg)^{1/(d-1)} + |\mathbb{F}| - 1\bigg\} = \bigg(\frac{1-|\mathbb{F}|^{-d}}{1-|\mathbb{F}|^{-1}}\bigg)^{1/(d-1)} + 1 - |\mathbb{F}|^{-1},$$

which is exactly 2 when $d = 2$ and is $2 - o(1)$ for $d \geqslant 3$. Were it to be true that the rearrangement inequality (1.1) held for one-dimensional affine subspaces and $p = d/(d-1)$, it would follow that C would be given by the previously displayed value. (One may indeed conjecture that this is true: 'small' two-dimensional Kakeya sets do not provide counter-examples to it. If it were true, it would also follow that Kakeya sets in $\mathbb{F}^d$ must have cardinality at least

$$|\mathbb{F}|^d\bigg\{1 + \frac{(1-|\mathbb{F}|^{-1})^{d/(d-1)}}{(1-|\mathbb{F}|^{-d})^{1/(d-1)}}\bigg\}^{-(d-1)} = \frac{1}{2^{d-1}}|\mathbb{F}|^d(1+o(1));$$

this would improve the lower bound $2^{-d}|\mathbb{F}|^d$ established in [2].)

When $d = 2$ this analysis does hold. Considering the case $p > 2$ and arbitrary $d \geqslant 2$ in a similar manner, it shows that

$$\begin{aligned}\sum_{v\in V}\bigg(\sum_{\alpha} x_\alpha \chi_{V_\alpha}(v)\bigg)^{p} &\leqslant \sum_{v\in V}\bigg(\sum_{\alpha} x_\alpha \chi_{V_\alpha^*}(v)\bigg)^{p} \\ &= \bigg(\sum_{\alpha} x_\alpha\bigg)^{p} + (|\mathbb{F}|-1)\sum_{\alpha} x_\alpha^p \\ &\leqslant \bigg\{\bigg(\frac{|\mathbb{F}|^d-1}{|\mathbb{F}|-1}\bigg)^{p/p'} + |\mathbb{F}| - 1\bigg\}\sum_{\alpha} x_\alpha^p\end{aligned}$$

with equality in the third line when all the x_α are equal. Thus, lines passing through the origin with equal weights (a 'bush') give an extremal configuration for the d-dimensional dual Kakeya inequality when $p \geqslant 2$.

REMARK 2.6. The thesis [5] contains the case $n = 1$ of proposition 2.3 and its application to extremals for the dual formulation of the finite field Kakeya maximal function inequality. The motivation for that, and the present work, comes from [1], where it was shown that 'multi-bushes' are quasi-extremals in (all but the endpoint case of) a suitable model version (the 'Gaussian' version) of the corresponding multilinear Kakeya maximal problem in Euclidean space.

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