NORMAL OPERATORS ON BANACH SPACES

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(Received 17 March, 1978)

1. The main result and its consequences. A (bounded, linear) operator H on a Banach space $\mathscr{X}$ is said to be hermitian if $\|\exp(itH)\| = 1$ for all real t. An operator N on $\mathscr{X}$ is said to be normal if $N = H + iK$, where H and K are commuting hermitian operators. These definitions generalize those familiar concepts of operators on Hilbert spaces. Also, the normal derivations defined in [**1**] are normal operators. For more details about hermitian operators and normal operators on general Banach spaces, see [**4**]. The main result concerning normal operators in the present paper is the following theorem.

THEOREM A. *Suppose that N in $\mathscr{L}(\mathscr{X})$ is normal. Then $\|Nx + w\| \geq \|w\|$ for each x in $\mathscr{X}$ and each w in* $\ker N$. *(In other words, the kernel of N is orthogonal to the range of N.)*

The proof of Theorem A will be presented in the next section. Granting this theorem for the moment, we can deduce the following corollaries.

COROLLARY 1. *If N is a normal operator on $\mathscr{X}$ and λ, μ are distinct eigenvalues of N, then $\|x + y\| \geq \|x\|$ for $x \in \ker(N - \lambda)$ and $y \in \ker(N - \mu)$. In other words, eigenspaces corresponding to distinct eigenvalues are mutually orthogonal.*

Proof. Since N is normal, so is $N - \lambda$. Hence, by Theorem A, $\|x + y\| = \|(N - \lambda)((\lambda - \mu)^{-1}y) + x\| \geq \|x\|$.

COROLLARY 2. *If N is a normal operator on a separable space, then there are at most countably many eigenvalues.*

Proof. This follows from Corollary 1 and a topological consideration.

The next corollary is a special case of Proposition 1 in [**7**].

COROLLARY 3. *If N is a normal operator and $N^2x = 0$, then $Nx = 0$.*

Proof. Let $w = Nx$. Then $Nw = 0$ and hence, by Theorem A, $0 = \|N(-x) + w\| \geq \|w\|$.

COROLLARY 4. *Let N be a normal operator on $\mathscr{X}$. If 0 is in the spectrum $\sigma(N)$ of N and the range $N\mathscr{X}$ is closed, then 0 is an isolated point in $\sigma(N)$, $\mathscr{X} = \ker N \oplus N\mathscr{X}$ and $\|P\| = 1$ where P is the projection from $\mathscr{X}$ onto $\ker N$ along $N\mathscr{X}$.*

The proof of this corollary is similar to that of Proposition 3 in [**9**] and hence omitted.

COROLLARY 5. *Let N be a normal operator on $\mathscr{X}$. If $\{w_n\}$ is a sequence of unit vectors*

† The author would like to thank the National Research Council of Canada for the financial support.

Glasgow Math. J. **20** (1979) 163–168.

in $\mathscr{X}$ such that $\lim_n \|Nw_n\| = 0$, *then, for each bounded sequence* $\{x_n\}$ *in* $\mathscr{X}$,

$$\limsup_n \|Nx_n - w_n\| \geq 1.$$

Proof. First let us recall the "Berberian–Quigley extension". (See [**6**].) Let $\ell^\infty(\mathscr{X})$ be the Banach space of all bounded sequences in $\mathscr{X}$ with sup-norm and let $c_0(\mathscr{X})$ be the subspace of $\ell^\infty(\mathscr{X})$ consisting of those $\{x_n\}$ with $\lim_n \|x_n\| = 0$. Let $\mathscr{X}^0$ be the quotient space $\ell^\infty(\mathscr{X})/c_0(\mathscr{X})$. For every T in $\mathscr{L}(\mathscr{X})$, the mapping $\{x_n\} \mapsto \{Tx_n\}$ sends $\ell^\infty(\mathscr{X})$ or $c_0(\mathscr{X})$ into itself and hence it induces an operator T^0 on $\mathscr{X}^0$ with $\|T^0\| = \|T\|$. It is easy to see that if T is hermitian or normal, then so is T^0. Now the corollary follows from an application of Theorem A to N^0.

COROLLARY 6 (See [**1**] and [**2**].) *Let a_1 and a_2 be normal operators on a Hilbert space $\mathscr{H}$. Define $N \in \mathscr{L}(\mathscr{L}(\mathscr{H}))$ by $Nx = a_1x - xa_2$. Then we have*: (1) $\|Nx + w\| \geq \|w\|$ *for all x in $\mathscr{L}(\mathscr{H})$ and w in* $\ker N$, (2) $\|x + y\| \geq \|x\|$ *if $Nx = \lambda x$, $Ny = \mu y$ and $\lambda \neq \mu$*, (3) $N^2x = 0$ *implies* $Nx = 0$, *and* (4) *if, furthermore, the range of N is closed, then the spectra of both a_1 and a_2 are finite.*

Proof. Suppose $a_j = h_j + ik_j$ $(j = 1,2)$, where h_j and k_j are commuting hermitian operators on $\mathscr{H}$. Then $Nx = Hx + iKx$, where $Hx = h_1x - xh_2$ and $Kx = k_1x - xk_2$. Note that both H and K are hermitian and $HKx = h_1k_1x + xh_2k_2 - h_1xh_2 - k_1xk_2 = KHx$. Now (1), (2) and (3) follows from Theorem A, Corollary 1 and Corollary 3 respectively and (4) follows from Corollary 4 and Rosenblum's theorem [**8**].

REMARK. Corollary 6 still holds if $\mathscr{L}(\mathscr{X})$ is replaced by a Banach algebra and a_1 and a_2 are assumed to be normal elements in it.

2. Proof of the main result. Now we proceed to prove Theorem A. It depends on the construction of certain projections in $\mathscr{L}(\mathscr{X}^*)$ (where $\mathscr{X}^*$ is the dual space of $\mathscr{X}$) which resemble conditional expectations in the theory of C*-algebra.

To begin with, let V be a power bounded operator on $\mathscr{X}$, say $\|V^n\| \leq M$ for every positive integer n. Let glim be a generalized (Banach) limit. For each ϕ in $\mathscr{X}^*$, the map $E\phi : x \to \operatorname{glim}_n \langle \phi, V^n x\rangle$ is a bounded linear functional on $\mathscr{X}$ with $\|E\phi\| \leq M\|\phi\|$. Thus we obtain an operator E in $\mathscr{L}(\mathscr{X}^*)$. Note that, in case $\|V\| \leq 1$, we have $\|E\phi\| \leq \|\phi\|$ for all ϕ in $\mathscr{X}^*$. We list some properties of E in the following lemma.

LEMMA 1. *Suppose that $V \in \mathscr{L}(\mathscr{X})$ is power bounded and E is an operator in $\mathscr{L}(\mathscr{X}^*)$ defined as above. Then we have*:

(1) *If $A \in \mathscr{L}(\mathscr{X})$ and $AV = VA$, then $A^*E = EA^*$.*
(2) $V^*E = EV^* = E$.
(3) *For $w \in \mathscr{X}$, $Vw = w$ if and only if $\langle E\phi, w\rangle = \langle \phi, w\rangle$ for all ϕ in $\mathscr{X}^*$.*
(4) *For $\phi \in \mathscr{X}^*$, $V^*\phi = \phi$ if and only if $E\phi = \phi$.*
(5) $E^2 = E$.

In particular, E is a projection from $\mathscr{X}^*$ onto $\{\phi \in \mathscr{X}^* : V^*\phi = \phi\}$.

Proof. (1) For $x \in \mathscr{X}, \phi \in \mathscr{X}^*$, we have $\langle EA^*\phi, x\rangle = \operatorname{glim}_n \langle A^*\phi, V^n x\rangle = \operatorname{glim}_n \langle \phi, V^n Ax\rangle = \langle E\phi, Ax\rangle = \langle A^*E\phi, x\rangle$ and hence $EA^* = A^*E$.

(2) For $x \in \mathscr{X}$, $\phi \in \mathscr{X}^*$, we have $\langle V^*E\phi, x\rangle = \langle E\phi, Vx\rangle = \operatorname{glim}_n \langle \phi, V^{n+1}x\rangle = \operatorname{glim}_n \langle \phi, V^n x\rangle = \langle E\phi, x\rangle$. Hence $V^*E = E$.

(3) If $Vw = w$, then $\langle E\phi, w\rangle = \operatorname{glim}_n \langle \phi, V^n w\rangle = \langle \phi, w\rangle$. Conversely, suppose that $\langle E\phi, w\rangle = \langle \phi, w\rangle$ for all ϕ in $\mathscr{X}^*$. Then, by (2), we have $\langle \phi, Vw\rangle = \langle V^*\phi, w\rangle = \langle EV^*\phi, w\rangle = \langle E\phi, w\rangle = \langle \phi, w\rangle$ for all ϕ in $\mathscr{X}^*$. Hence $Vw = w$.

(4) If $V^*\phi = \phi$, then $\langle E\phi, x\rangle = \operatorname{glim} \langle \phi, V^n x\rangle = \operatorname{glim} \langle V^{*n}\phi, x\rangle = \langle \phi, x\rangle$ for all x and hence $E\phi = \phi$. Conversely, if $E\phi = \phi$, then, by (2), we have $V^*\phi = V^*E\phi = E\phi = \phi$.

(5) follows from (2) and (4).

As an aside, we give a different proof of Sinclair's result [**9**; Proposition 1] by applying the above lemma (and without using Kakutani's fixed point theorem.) First we need a technical lemma.

LEMMA 2. *If $T \in \mathscr{L}(\mathscr{X})$, $x \in \mathscr{X}$, $(\exp T)x = x$ and $|\exp \lambda - 1| < 1$ for all λ in $\sigma(T)$, then $Tx = 0$.*

Proof. The lemma follows by applying the expansion $T = -\sum_{n=1}^{\infty} n^{-1}(I - \exp T)^n$ to the vector x.

COROLLARY 7 (Sinclair [**9**]). *Let $T \in \mathscr{L}(\mathscr{X})$. If 0 is in the boundary of the closed convex hull of the (spatial) numerical range of T, then $\|Tx + w\| \geq \|w\|$ for $x \in \mathscr{X}$ and $w \in \ker T$.*

Proof. By multiplying T by a suitable constant which is small enough in modulus, we may assume that $\operatorname{Re} \lambda \leq 0$ and $|\exp \lambda - 1| < 1$ for λ in the closed convex hull of the numerical range of T. Let $V = \exp T$. Then, by [**4**, Theorem 3.6], $\|V\| \leq 1$. Let $w \in \ker T$. Then $Vw = w$. By the spectral mapping theorem, $\sigma(V^*) = \sigma(V) = \exp \sigma(T) \subseteq \{\lambda : |\lambda - 1| < 1\}$. By Lemma 1, there exists a projection E in $\mathscr{L}(\mathscr{X}^*)$ with $\{\phi \in \mathscr{X}^* : V^*\phi = \phi\}$ as its range such that $\|E\| \leq 1$, $EV^* = V^*E = E$ and $\langle E\phi, w\rangle = \langle \phi, w\rangle$ for all $\phi \in \mathscr{X}^*$. If $E\phi = \phi$, then $\exp(T^*)\phi = V^*E\phi = E\phi = \phi$ and hence, by Lemma 2, $T^*\phi = 0$. Therefore $T^*E = 0$. If $\|\phi\| \leq 1$, then $\|E\phi\| \leq 1$ and hence

$$\|Tx + w\| \geq |\langle E\phi, Tx + w\rangle| = |\langle T^*E\phi, x\rangle + \langle E\phi, w\rangle| = |\langle \phi, w\rangle|.$$

Therefore $\|Tx + w\| \geq \sup\{|\langle \phi, w\rangle| : \|\phi\| \leq 1\} = \|w\|$.

The next lemma is already known. (See [**7**].) However, for the convenience of the reader, a proof of it is included here.

LEMMA 3. *If $N = H + iK$ is a normal operator, where H and K are commuting hermitian operators and $Nx = 0$, then $Hx = Kx = 0$.*

https://doi.org/10.1017/S0017089500003888 Published online by Cambridge University Press

Proof. From the assumption we have $Kx = iHx$. Since $HK = KH$, by induction, we have $K^n x = (iH)^n x$ for $n = 1, 2, 3, \ldots$. Hence $\exp(\lambda K)x = \exp(i\lambda H)x$ for every complex number λ. Now, suppose $\lambda = \alpha + i\beta$, where α and β are real. Then we have

$$\exp(\lambda K)x = \exp(i\beta K)\exp(\alpha K)x = \exp(i\beta K)\exp(i\alpha H)x.$$

Hence $\|\exp(\lambda K)x\| \leq \|x\|$. By Liouville's theorem, $\exp(\lambda K)x$ is a constant function. Differentiate this function at $\lambda = 0$. We obtain $\exp(\lambda K)Kx = 0$ and hence $Kx = 0$.

REMARK. From the proof of the above lemma we see that, besides the commutativity of H and K, all we need is the boundedness of $\alpha \mapsto \exp(i\alpha H)$ and $\beta \mapsto \exp(i\beta K)$ ($\alpha, \beta \in \mathbf{R}$). Hence this lemma can be generalized for those $H + iK$, where H and K are commuting pre-hermitian operators. By an argument similar to that of Corollary 6, we can deduce Berkson, Dowson and Elliott's extension of Fuglede's theorem [**3**; Theorem 1] from this fact.

Proof of Theorem A. By Lemma 3, we have $Hw = Kw = 0$. Let glim be a generalized limit. Define E, $F \in \mathscr{L}(\mathscr{X}^*)$ by the identities $\langle E\phi, x\rangle = \operatorname{glim}_n \langle \phi, \exp(inH)x\rangle$ and $\langle F\phi, x\rangle = \operatorname{glim}_n \langle \phi, \exp(inK)x\rangle$. Then, by Lemma 1, E and F are projections satisfying $\|E\| \leq 1$, $\|F\| \leq 1$ and $\langle E\phi, w\rangle = \langle \phi, w\rangle = \langle F\phi, w\rangle$, for all ϕ in $\mathscr{X}^*$. Furthermore, $(\exp iH)^* E = E(\exp iH)^* = E$ and $(\exp iK)^* F = F(\exp iK)^* = F$. By Lemma 2, it is easy to see that $H^*E = 0$ and $K^*F = 0$. Note that, since $KH = HK$, we have $K^*E = EK^*$ and hence $K^*EF = EK^*F = 0$. For $\phi \in \mathscr{X}^*$ with $\|\phi\| \leq 1$, we have

$$\begin{aligned} \|Nx + w\| &\geq |\langle EF\phi, Nx + w\rangle| \\ &= |\langle H^*EF\phi, x\rangle + i\langle K^*EF\phi, x\rangle + \langle EF\phi, w\rangle| \\ &= |\langle \phi, w\rangle| . \end{aligned}$$

Hence $\|Nx + w\| \geq \|w\|$. The proof is complete.

REMARK. We have mentioned at the beginning of this section that the projection E in Lemma 1 resembles conditional expectations in the theory of C*-algebra. To make this statement more clear, we consider the following special case. Let $\mathscr{H}$ be a Hilbert space $\mathscr{X} = \mathscr{L}(\mathscr{H})$ and let h be a hermitian operator on $\mathscr{H}$. Define H and V in $\mathscr{L}(\mathscr{X})$ by $Hx = hx - xh$ and $Vx = \exp(iH)x = \exp(ih)x\exp(-ih)$. Let glim be a generalized limit and P, in $\mathscr{L}(\mathscr{X})$, be the projection defined in such a way that $\langle (Px)\xi, \eta\rangle = \operatorname{glim}\langle (V^n x)\xi, \eta\rangle$, where ξ, η are in $\mathscr{H}$. It is easy to check that P is a conditional expectation from $\mathscr{L}(\mathscr{H})$ onto the von Neumann algebra $\{x \in \mathscr{L}(\mathscr{H}) : xh = hx\}$, the commutant of h. Define $E \in \mathscr{L}(\mathscr{X}^*)$ in the same way as that in the beginning of this section, i.e., $\langle E\phi, x\rangle = \operatorname{glim}\langle \phi, V^n x\rangle$. Then we have $P^*\phi = E\phi$ if ϕ is of the form $\phi(x) = \langle x\xi, \eta\rangle$ for some vectors ξ, η in $\mathscr{H}$. Thus E is "almost" the dual of P. One can check that Lemma 1 still holds if E is replaced by P^*.

3. Compact normal operators. Many results concerning compact hermitian operators (see [**5**, §28]) can be generalized to compact normal operators.

PROPOSITION 1. *Let $T \in \mathcal{L}(\mathcal{X})$ be compact and normal, let λ_n be the non-zero (distinct) eigenvalues of T arranged such that $|\lambda_1| \geq |\lambda_2| \geq |\lambda_3| \geq \ldots$ and let P_n be the spectral projection corresponding to λ_n. Then the following statements hold.* (1) *Each eigenvalue of T has ascent* 1. (2) $\|P_n\| = 1$. (3) *If each P_n is hermitian, then $T = \sum \lambda_n P_n$, a norm-convergent series.* (4) *If $\lim_n n\lambda_n = 0$, then $T = \sum \lambda_n P_n$.*

Proof. (1) and (2) follows from Corollary 4 in §1. One can modify the proof of Theorem 28.1 in [**5**, p. 82] to obtain (3) and (4).

Next we show that if the underlying space $\mathcal{X}$ is weakly complete, then the linear span of eigenspaces of T is dense in $\mathcal{X}$. First we need a technical lemma.

LEMMA 3. *Let $\mathcal{A}$ be a Banach algebra, $\mathcal{J}$ a closed two-sided ideal in $\mathcal{A}$ and $v: \mathcal{A} \to \mathcal{A}/\mathcal{J}$ the quotient map. Then the following two statements hold.*

(1) *If $h \in \mathcal{A}$ is hermitian, then so is $\nu(h)$.*

(2) *If $h, k \in \mathcal{A}$ are hermitian, $n = h + ik$ is normal and $n \in \mathcal{J}$, then $h, k \in \mathcal{J}$.*

Proof. (1) for each real t, we have $\|\exp(\pm it\nu(h))\| = \|\nu(\exp(\pm ith))\| \leq 1$. On the other hand, $\|\exp(-it\nu(h))\| \, \|\exp(it\nu(h))\| \geq \|\exp(-it\nu(h))\exp(it\nu(h))\| = 1$. Hence $\|\exp(it\nu(h))\| = 1$ for all real t. Therefore $\nu(h)$ is hermitian.

(2) By (1), $\nu(n) = \nu(h) + i\nu(k)$ is normal. By the assumption, $\nu(n) = 0$. Hence $\nu(h) = \nu(k) = 0$; i.e., h, $k \in \mathcal{J}$.

PROPOSITION 2. *Let $\mathcal{X}$ be a weakly complete Banach space, T a compact normal operator $\mathcal{X}$ and let λ_n, P_n be as in Proposition* 1. *Then $\mathcal{X}$ is the closed linear span of eigenvectors of T.*

Proof. Let $T = H + iK$, where H and K are commuting hermitian operators. Since T is compact, by Lemma 3, both H and K are compact. Let $\alpha_n = \operatorname{Re} \lambda_n$ and $\beta_n = \operatorname{Im} \lambda_n$. Suppose $x \in P_n\mathcal{X}$. Then $Kx = \lambda_n x$; that is, $((H - \alpha_n) + i(K - \beta_n))x = 0$ and hence, by Lemma 2 in §2, $(H - \alpha_n)x = 0$ and $(K - \beta_n)x = 0$. Thus, if α is a non-zero eigenvalue of H, $\mathcal{M}_\alpha$ is the eigenspace $\{x : Hx = \alpha x\}$ and $\operatorname{Re} \lambda_n = \alpha$, then $P_n\mathcal{X} \subseteq \mathcal{M}_\alpha$. Since T commutes with H, $\mathcal{M}_\alpha$ is invariant under T. It is not hard to see that T restricted to $\mathcal{M}_\alpha$ is a normal operator on a finite dimensional space $\mathcal{M}_\alpha$. If $x \in \mathcal{M}_\alpha$ is an eigenvector of T with $Tx = \lambda x$, then, by the same argument as before we obtain $Hx = (\operatorname{Re} \lambda)x$ and hence $\operatorname{Re} \lambda = \alpha$. We conclude that $\mathcal{M}_\alpha$ is the direct sum of all those $P_n\mathcal{X}$ with $\operatorname{Re} \lambda_n = \alpha$. Similarly, if β is a non-zero eigenvalue of K, we write $\mathcal{N}_\beta$ for $\{x \in \mathcal{X} : Kx = \beta x\}$, then $\mathcal{N}_\beta$ is the direct sum $\sum \{P_n\mathcal{X} : \operatorname{Im} \lambda_n = \beta\}$. Observe that $P_n\mathcal{X} = \mathcal{M}_{\alpha_n} \cap \mathcal{N}_{\beta_n}$.

By Theorem 28.5 of [**5**, p. 87], $\mathcal{X}$ is the closed linear span of the eigenspaces of H (or K). Now it is not difficult to see that $\mathcal{X}$ is the closed linear span of eigenspaces of T.

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