Robotica (2016) volume 34, pp. 957–972. © Cambridge University Press 2014
doi:10.1017/S0263574714001994

Analysis of kinematics and statics for a novel 6-DoF parallel mechanism with three planar mechanism limbs

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(Accepted June 23, 2014. First published online: July 31, 2014)

SUMMARY
A novel 6-degree-of-freedom (DoF) parallel manipulator with three planar mechanism limbs is proposed and its kinematics and statics are analyzed systematically. First, the characteristics of the proposed manipulator are analyzed and the degree of freedom is calculated. Second, the formulae for solving the displacement, the velocity, and the acceleration are derived. Third, an analytic example is given for solving the kinematics and statics of this manipulator, and the analytic solved results are analyzed and verified by the simulation mechanism. Finally, a workspace is constructed and analyzed based on a comparison between the proposed manipulator and another 6-DoF parallel manipulator.

KEYWORDS: Parallel manipulator; Planar mechanism limbs; Kinematics.

Nomenclature:

Symbol	*Description*
PM	Parallel manipulator
DoF	Degree of freedom
B, m	The base, moving platform
O, o	The center point of B, m
$\{B\}$	Coordinate O-XYZ on B
$\{m\}$	Coordinate o-xyz on m at o
P, R	Prismatic joint, revolute joint
U, S	Universal joint, spherical joint
B_i, b_i	The vertices of B and m
μ	The number of redundant
r_i, r_{ij}	Virtual leg and active leg of PM
$\boldsymbol{\delta}_i, \boldsymbol{\delta}_{ij}$	The unit vectors of r_i, r_{ij} $i = 1, 2, 3; j = 1, 2$
L, l	The sides of B, m
e, E	The distance from b_i to o, B_i to O
b_{ij}	Connection joints between g_i and r_{ij}
α, β, γ	Euler angles of m about (Z, Y_1, Z_2)
r_{vi}, q	Vertical rod, $q = 3^{1/2}$
$\boldsymbol{v}, \boldsymbol{\omega}$	Linear and angular velocity of m at o
$\boldsymbol{a}, \boldsymbol{\varepsilon}$	Linear and angular acceleration of m
V, A	General velocity and acceleration of m
n_l, n_j	The number of links and the number of joints

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$g_i, \boldsymbol{g}_i, g$	Upper beam, its unit vector, and its haft length
$G_i, \boldsymbol{G}_i, G$	Lower beam, its unit vector, and its haft length
X_o, Y_o, Z_o	The position components of o in $\{B\}$
v_{ri}, ω_{ri}	Scalar velocity along r_i and angular velocity of r_i
v_{rij}, ω_{rij}	Scalar velocity along r_{ij} and angular velocity of r_{ij}
$\boldsymbol{v}_{bi}, \boldsymbol{v}_{bij}$	Velocity of g_i at b_i and b_{ij}
$x_l, x_m, x_n y_l, y_m, y_n z_l, z_m, z_n$	Nine orientation parameters of m
$\boldsymbol{\omega}_{gi}, \boldsymbol{\varepsilon}_{gi}$	Angular velocity and acceleration of g_j
$\boldsymbol{v}_r, \boldsymbol{a}_r$	General input velocity, general input acceleration
$\mathbf{J}, \mathbf{H}$	Jacobian, Hessian matrix of PM with planar limb
$\parallel, \perp, \mid$	Parallel, perpendicular, collinear constraint
$\boldsymbol{\omega}_{rij}, \boldsymbol{\varepsilon}_{rij}$	Angular velocity and acceleration of r_{ij}
$\boldsymbol{F}, \boldsymbol{T}$	The central force and torque applied on m at o
$\boldsymbol{F}_{ij}$	Active force along r_{ij}
$\omega_{i1}, \boldsymbol{R}_{i1}$	Scalar angular velocity of G_i about B, its unit vector
$\omega_{i2}, \boldsymbol{R}_{i2}$	Scalar angular velocity of r_i about G_i, its unit vector
$\omega_{i3}, \boldsymbol{R}_{i3}$	Scalar angular velocity of g_i about r_i, its unit vector
$\omega_{i4}, \boldsymbol{R}_{i4}$	Scalar angular velocity of r_{vi} about m, its unit vector
$\omega_{i5}, \boldsymbol{R}_{i5}$	Scalar angular velocity of g_i about r_{vi}, its unit vector

1. Introduction

Currently, various 6-DoF parallel mechanisms (PMs) have been applied in the fields of industrial robots, micromanipulators, parallel machine tools, damping platform, rehabilitation robot, simulator of real 3-dimensional (3D) earthquakes, walking leg, heavy-duty forging manipulator, and haptic device, because the 6-DoF PMs have high stiffness, good dexterity, compact size, and high power to weight ratio.[1–3] In this aspect, Patricia Ben-Horin and Shoham[3] and Zhang[4] synthesized a class of PMs with six limbs and several spherical (*S*) joints. Huang *et al.*[5] analyzed the structure and the property of the singularity loci of a 3/6 Gough–Stewart PM with six *S* joints. Aginaga *et al.*[6] developed a revolute joint-universal joint-spherical joint (6-RUS PM) with three *S* joints and analyzed its static stiffness. Li *et al.*[7] determined the maximal singularity-free zones in the 6D workspace of the 3/6 Gough–Stewart PM with six *S* joints. Tong *et al.*[9] optimized a class of the generalized symmetric Gough–Stewart PMs with six *S* joints. Shim *et al.*[10] proposed a decoupled three prismatic joint - revolute joint-prismatic joint-spherical joint (PRPS-type PM) with three *S* joints. Lee and Park[11] introduced a 6-DoF PM which makes use of two stacked PMs and a central axis.

However, a spherical joint *S* is composed of a ball rod and a globe bearing; therefore, it has following disadvantages: (1) The capability of the pulling force bearing is lower due to a small effective surface of the force bearing between the ball and the globe bearing in the pulling direction. (2) The rotation range of the ball rod in the globe bearing is small. (3) The precision of the spherical joint *S* is lowed under alternately heavy loads or a long time service because a backlash between the ball and the globe bearing cannot be removed. Hence, the applications of 6-DoF PMs with *S* joints are limited. For this reason, the spatial PMs with the planar mechanism limbs have attracted much attention. The planar mechanism limb only includes revolute joints *R* and prismatic joint *P*, which has the following merits: (1) The planar mechanism of the limbs with *R* joint and *P* joint is simple in structure and easy in manufacturing; (2) *R* joint has a larger capability of pulling force bearing than that of *S* joint; (3) The precision of *R* joint is higher than that of *S* joint under large cyclic loading because the backlash of *R* joint can be eliminated more easily than that of *S* joint by a preload; (4) The workspace of the proposed manipulator can be increased because the rotation range of *R* joint is larger than that of *S* joint before interference.

In this aspect, Wu and Gosselin[12] developed a PM with three limbs formed by a planar four-bar linkage for its spatial dynamic balancing. Gogu[13] proposed a family of *T2R1*-type PMs with bifurcated planar-spatial motion of the moving platform. Yoon and Ryu[14] designed a locomotion interface with two planar PMs that allows human walking. Yu *et al.*[15] and Yang *et al.*[16] proposed some PMs with decoupled-motion architecture. In the aspect of the kinestatic analysis of conventional PMs, the principle of virtual work,[17,18] the spatial vector analytic approach,[19–23] and the combination

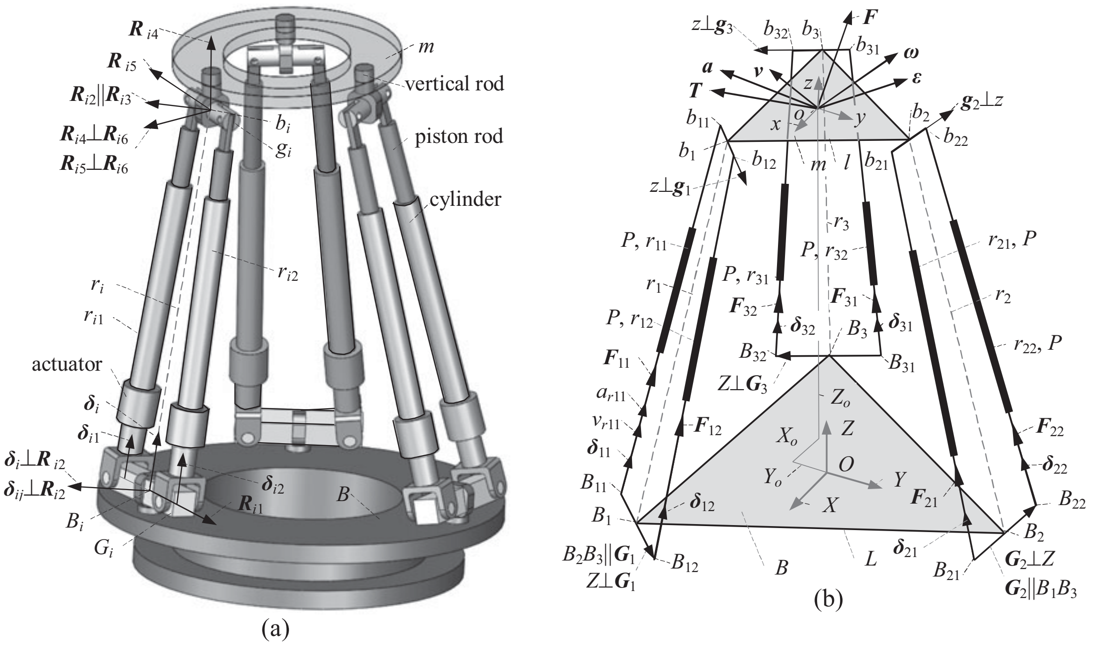

Fig. 1. (a) A 3D model of novel PM with three planar mechanism limbs, and (b) its kinematics and statics model.

of virtual work theory with Computer-Aided Design (CAD) variation geometry[24] have been applied to the study of kinematics and statics of PMs with $3 \sim 6$ active limbs. Up to now, there is no effort toward the kinestatic analysis of 6-DoF PMs with planar mechanism limbs. For this reason, this paper focuses on kinematic and static analysis of a novel 6-DoF PM with planar mechanism limbs. Its structure characteristics, kinematics, and statics are studied systematically.

2. Characteristics of Novel PM with Three Planar Mechanism Limbs and Its DoF

A 3D model of a novel PM with three planar mechanism limbs is constructed using the advanced CAD software, see Fig. 1a. This PM includes a platform m, a base B, and three identical planar mechanism limbs r_i ($i = 1, 2, 3$). In fact, m is a regular triangle $\Delta b_1 b_2 b_3$ with three vertices b_i, three sides $l_i = l$, and a central point o; B is a regular triangle $\Delta B_1 B_2 B_3$ with three vertices B_i, three sides $L_i = L$, and a central point O. Each of r_i includes a vertical rod r_{vi}, an upper beam g_i, a lower beam G_i, and two linear active legs r_{ij}. Each of r_{ij} comprises a linear actuator, a cylinder, and a piston. In each of r_i, r_{vi} connects with m by a vertical revolute joint R_{i4} at b_i, and it connects with the middle of g_i by a horizontal revolute joint R_{i5}; the middle of G_i connects with B by a horizontal revolute joint R_{i2} at B_i; the two ends of r_{ij} connect with the two ends of g_i and G_i by revolute joints; g_i, G_i, and r_{ij} form a closed planar mechanism with two linear actuators. Let $\perp$ be a perpendicular constraint, $\|$ be a parallel constraint, $|$ be a collinear constraint, $\{m\}$ be a coordinate frame o-xyz fixed on m at o, and $\{B\}$ is a coordinate frame O-XYZ fixed on B at O. In this PM, (g_i, G_i, r_i, and r_{ij}) are in the same plane; in addition, following geometric conditions $z \perp m$, $R_{i4}||z$, $R_{i5}||m$, $g_i||m$, $R_{i1}||B$, $G_i||B$, $b_1b_3||x$, $ob_2|y$, $b_1b_3 = b_1b_2 = b_2b_3 = l$, $L_i = L$, $ob_i = e$, ($i = 1, 2, 3$, $j = 1, 2$) are satisfied.

In this PM, the number of links, $n_l = 23$, including base B, a moving platform m, six cylinders, six piston rods, three lower beams, three upper beams, and three vertical rods. The number of joints, $n_J = 27$, including six prismatic joints and 21 revolute joints. The number of redundant constraints is $\mu = 9$, correspond to three planar mechanism limbs. The number of located degrees of freedom of joints is $\sum f_k = 6 \times 1 + 21 \times 1 = 27$, including six prismatic joints and 21 revolute joints. Thus, degree of freedom of this PM with three planar mechanism limbs is calculated based on the revised Grübler–Kutzbach formula[1,2] given below,

$$M = 6(n_l - n_J - 1) + \sum f_k + \mu = 6 \times (23 - 27 - 1) + 27 + 3 \times 3 = 6. \quad (1)$$

https://doi.org/10.1017/S0263574714001994 Published online by Cambridge University Press

3. Displacement Analysis of PM with Three Planar Mechanism Limbs

The displacement analysis of active legs is a foundation for deriving the velocity of PM with three planar mechanism limbs. The coordinates of b_i in $\{m\}$ and B_i in $\{B\}$ are expressed as follows[18]:

$$\boldsymbol{B}_i = \frac{E}{2}\begin{bmatrix} \pm q \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 \\ E \\ 0 \end{bmatrix}, \quad \boldsymbol{b}_i^m = \frac{e}{2}\begin{bmatrix} \pm q \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{b}_2^m = \begin{bmatrix} 0 \\ e \\ 0 \end{bmatrix}, \quad \boldsymbol{o} = \begin{bmatrix} X_o \\ Y_o \\ Z_o \end{bmatrix}, \quad \begin{matrix} (i = 1, 3) \\ e = ql/3 \\ E = qL/3 \end{matrix}. \tag{2}$$

Here, $q = 3^{1/2}$, E is the distance from B_i $(i = 1, 3)$ to O, e is the distance from b_i to o. "±" is replaced by "+" as $i = 1$; "±" is replaced by "–" as $i = 3$. This condition is also available for Eqs. (3), (4), and (7) with "±".

Let φ be one of the three Euler angles (α, β, γ). Set $s_\varphi = \sin\varphi$, $c_\varphi = \cos\varphi$. b_i and z of m in $\{B\}$ are expressed by Eq. (A1) in the Appendix. The vectors of b_i on m in $\{B\}$ are expressed by the transformed matrix (A1) in the Appendix based on Eq. (2) as follows:

$$\boldsymbol{b}_i = \frac{1}{2}\begin{bmatrix} \pm qex_l - ey_l + 2X_o \\ \pm qex_m - ey_m + 2Y_o \\ \pm qex_n - ey_n + 2Z_o \end{bmatrix}, \quad \boldsymbol{b}_2 = \begin{bmatrix} ey_l + X_o \\ ey_m + Y_o \\ ey_n + Z_o \end{bmatrix}, \quad (i = 1, 3). \tag{3}$$

Let $\boldsymbol{r}_i$ $(i = 1, 2, 3)$ be the vector from B_i to b_i and $\boldsymbol{e}_i$ be the vector from o to b_i. They are derived from Eqs. (2) and (3) as follows:

$$\boldsymbol{r}_i = \frac{1}{2}\begin{bmatrix} \pm qex_l - ey_l + 2X_o - \pm qE \\ \pm qex_m - ey_m + 2Y_o + E \\ \pm qex_n - ey_n + 2Z_o \end{bmatrix}, \quad \boldsymbol{r}_2 = \begin{bmatrix} ey_l + X_o \\ ey_m + Y_o - E \\ ey_n + Z_o \end{bmatrix}, \quad \boldsymbol{e}_i = \frac{e}{2}\begin{bmatrix} \pm qx_l - y_l \\ \pm qx_m - y_m \\ \pm qx_n - y_n \end{bmatrix},$$
$$\boldsymbol{e}_2 = e\begin{bmatrix} y_l \\ y_m \\ y_n \end{bmatrix}, \quad (i = 1, 3). \tag{4}$$

Let $\boldsymbol{G}_{0i}$ and $\boldsymbol{G}_i$ be the vector of the lower beam G_i and its unit vector. These are derived from Eq. (2) as follows:

$$\boldsymbol{G}_{01} = \boldsymbol{B}_2 - \boldsymbol{B}_3 = \frac{E}{2}\begin{bmatrix} q \\ 3 \\ 0 \end{bmatrix}, \quad \boldsymbol{G}_{02} = \boldsymbol{B}_1 - \boldsymbol{B}_3 = E\begin{bmatrix} q \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{G}_{03} = \boldsymbol{B}_1 - \boldsymbol{B}_2 = \frac{E}{2}\begin{bmatrix} q \\ -3 \\ 0 \end{bmatrix},$$
$$\boldsymbol{G}_i = \begin{bmatrix} G_{ix} \\ G_{iy} \\ G_{iz} \end{bmatrix} = \frac{\boldsymbol{G}_{0i}}{|\boldsymbol{G}_{0i}|}, \quad (i = 1, 2, 3). \tag{5}$$

Let $\boldsymbol{g}_{0i}$ and $\boldsymbol{g}_i$ be the vector and the unit vector of the upper beam g_i. These are derived from Eqs. (3) and (5) as follows:

$$\boldsymbol{g}_{0i} = \begin{cases} \boldsymbol{z} \times (\boldsymbol{G}_{0i} \times \boldsymbol{r}_i) = (\boldsymbol{z} \cdot \boldsymbol{r}_i)\boldsymbol{G}_{0i} - (\boldsymbol{z} \cdot \boldsymbol{G}_{0i})\boldsymbol{r}_i, \ as \ \arccos\langle \boldsymbol{G}_{0i}, \boldsymbol{g}_{0i} \rangle \le 90^\circ, \\ -\boldsymbol{z} \times (\boldsymbol{G}_{0i} \times \boldsymbol{r}_i) = -(\boldsymbol{z} \cdot \boldsymbol{r}_i)\boldsymbol{G}_{0i} + (\boldsymbol{z} \cdot \boldsymbol{G}_{0i})\boldsymbol{r}_i, \ as \ \arccos\langle \boldsymbol{G}_{0i}, \boldsymbol{g}_{0i} \rangle > 90^\circ, \end{cases} \tag{6}$$
$$\boldsymbol{g}_i = [g_{ix} \quad g_{iy} \quad g_{iz}]^{\mathrm{T}} = \boldsymbol{g}_{0i} / |\boldsymbol{g}_{0i}| \ (i = 1, 2, 3).$$

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here,

$$
\begin{aligned}
\boldsymbol{z} \cdot \boldsymbol{r}_i &= \frac{1}{2}[z_l(\pm qex_l - ey_l + 2X_o - \pm qE) + z_m(\pm qex_m - ey_m + 2Y_o + E) \\
&\quad + z_n(\pm qex_n - ey_n + 2Z_o)], \\
\boldsymbol{z} \cdot \boldsymbol{r}_2 &= z_l(ey_l + X_o) + z_m(ey_m + Y_o - E) + z_n(ey_n + Z_o), (i = 1, 3), \\
\boldsymbol{z} \cdot \boldsymbol{G}_{01} &= E(qz_l + 3z_m)/2, \boldsymbol{z} \cdot \boldsymbol{G}_{02} = Eqz_l, \boldsymbol{z} \cdot \boldsymbol{G}_{03} = E(qz_l - 3z_m)/2.
\end{aligned}
\tag{7}
$$

Let $B_{i1}B_i = B_iB_{i2} = G$, $b_ib_{i1} = b_ib_{i2} = g$, $\boldsymbol{r}_{ij}$ be the vector from B_{ij} to b_{ij}. $\boldsymbol{r}_{ij}$ are expressed as follows:

$$
\begin{aligned}
&\boldsymbol{B}_{i1}\boldsymbol{B}_i = \boldsymbol{B}_i\boldsymbol{B}_{i2} = G\boldsymbol{G}_i, \boldsymbol{g}_{i1} = \boldsymbol{b}_i\boldsymbol{b}_{i1} = -g\boldsymbol{g}_i, \boldsymbol{g}_{i2} = \boldsymbol{b}_i\boldsymbol{b}_{i2} = g\boldsymbol{g}_i, i = 1, 2, 3 \\
&\begin{cases} \boldsymbol{r}_{i1} = \boldsymbol{B}_{i1}\boldsymbol{B}_i + \boldsymbol{B}_i\boldsymbol{b}_i + \boldsymbol{g}_{i1} = \boldsymbol{r}_i + G\boldsymbol{G}_i - g\boldsymbol{g}_i \\ \boldsymbol{r}_{i2} = \boldsymbol{B}_i\boldsymbol{b}_i - \boldsymbol{B}_i\boldsymbol{B}_{i2} + \boldsymbol{g}_{i2} = \boldsymbol{r}_i - G\boldsymbol{G}_i + g\boldsymbol{g}_i \end{cases}.
\end{aligned}
\tag{8}
$$

Let $\boldsymbol{\delta}_i$ be the unit vector of $\boldsymbol{r}_i$, $\boldsymbol{\delta}_{ij}$ be the unit vector of $\boldsymbol{r}_{ij}$. The formulae for solving r_i, r_{ij}, $\boldsymbol{r}_{ij}$, $\boldsymbol{\delta}_i$ and $\boldsymbol{\delta}_{ij}$ are derived from Eqs. (5), (6), (7), and (8) as follows:

$$
\begin{aligned}
&\boldsymbol{r}_{ij} = \boldsymbol{r}_i + (-1)^j \boldsymbol{m}_i, \quad r_i = (r_{ix}^2 + r_{iy}^2 + r_{iz}^2)^{1/2}, \quad r_{ij} = (r_{ijx}^2 + r_{ijy}^2 + r_{ijz}^2)^{1/2}, \\
&\boldsymbol{m}_i = g\boldsymbol{g}_i - G\boldsymbol{G}_i = g\frac{\boldsymbol{g}_{0i}}{|\boldsymbol{g}_{0i}|} - G\frac{\boldsymbol{G}_{0i}}{|\boldsymbol{G}_{0i}|}, \boldsymbol{\delta}_i = \frac{\boldsymbol{r}_i}{r_i}, \boldsymbol{\delta}_{ij} = \frac{\boldsymbol{r}_{ij}}{r_{ij}}, \quad (i = 1, 2, 3, j = 1, 2).
\end{aligned}
\tag{9}
$$

4. Velocity Analysis of 6-DoF PM with three Planar Mechanism Limbs

4.1. Basic kinematic equations

The velocity analysis provides a theoretical foundation for the derivation of statics and acceleration of a 6-DoF PM with three planar mechanism limbs. Let $\boldsymbol{V}$, $\boldsymbol{v}$, $\boldsymbol{\omega}$, $\boldsymbol{A}$, a, and ε be the general forward velocity, the translational velocity, the angular velocity, the general forward acceleration, the translational acceleration, and the angular acceleration of m at o respectively. They are expressed as follows:

$$
\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix}, \quad \boldsymbol{A} = \begin{bmatrix} \boldsymbol{a} \\ \varepsilon \end{bmatrix}, \quad \boldsymbol{v} = \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}, \quad \boldsymbol{\omega} = \begin{bmatrix} \omega_x \\ \omega_y \\ \omega_z \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \end{bmatrix}. \tag{10}
$$

Let $\boldsymbol{\zeta}$ be a vector. Its skew-symmetric matrix $\hat{\boldsymbol{\zeta}}$ or $s(\boldsymbol{\zeta})$ satisfies $\hat{\boldsymbol{\zeta}} = \boldsymbol{\zeta}\times = s(\boldsymbol{\zeta})$ and $\hat{\boldsymbol{\zeta}}^T = -\hat{\boldsymbol{\zeta}}$.[1] Let $\boldsymbol{v}_{bi}$ be the velocity of m at b_i, $\boldsymbol{v}_{bij}$ be the velocity vector of the upper beam g_i at b_{ij}, $\boldsymbol{\omega}_{gi}$ be the angular velocity of g_i; $\boldsymbol{\omega}_{ri}$ be the angular velocity of r_i, $\boldsymbol{\omega}_{rij}$ be the angular velocity of r_{ij}, ω_{gij} be the scalar angular velocity of g_i about r_{ij} at b_{ij}, v_{ri} be the scalar velocity along r_i, and v_{rij} be the input scalar velocity along r_{ij}. Let ω_{i1} and $\boldsymbol{R}_{i1}$ be the scalar angular velocity of the lower beam G_i about B at B_i and its unit vector respectively; ω_{i2} and $\boldsymbol{R}_{i2}$ be the scalar angular velocity of r_i about G_i at B_i and its unit vector respectively; ω_{i3} and $\boldsymbol{R}_{i3}$ be the scalar angular velocity of g_i about r_i at b_i and its unit vector respectively, and there is $\boldsymbol{R}_{i3}||\boldsymbol{R}_{i2}$. Let ω_{i4} and $\boldsymbol{R}_{i4}$ be the scalar angular velocity of vertical rod r_{vi} about m at b_i and its unit vector respectively. Let ω_{i5} and $\boldsymbol{R}_{i5}$ be the scalar angular velocity of g_i about r_{vi} at b_i and its unit vector respectively. Their relative basic kinematic equations and the geometry-constrained equations are expressed by Eqs. (A2) to (A7) in the Appendix.

4.2. Angular velocity ω_{ri} of virtual leg r_i and linear velocity v_{ri} along virtual leg r_i

The angular velocity $\boldsymbol{\omega}_{ri}$ of the virtual leg r_i and the linear velocity $\boldsymbol{v}_{ri}$ along r_i must be derived before solving the general input velocity and the general forward velocity. Their derivations are explained as follows:

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Let $D_1 = \boldsymbol{r}_i \cdot (\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) = -\boldsymbol{R}_{i2} \cdot (\boldsymbol{R}_{i1} \times \boldsymbol{r}_i), \mathbf{D}_2 = \boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^{\mathrm{T}} - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^{\mathrm{T}}$. From Eq. (A7), we derive the following equations:

$$\omega_{i1} = \frac{-\boldsymbol{R}_{i2}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\mathbf{e}}_i\boldsymbol{\omega})}{-D_1} = \boldsymbol{J}_{\omega i1}\boldsymbol{V}, \quad \boldsymbol{J}_{\omega i1} = \frac{\boldsymbol{R}_{i2}^{\mathrm{T}}}{D_1}[\,\hat{\boldsymbol{\delta}}_i^2 \quad -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\,], \quad \omega_{i2} = \frac{-\boldsymbol{R}_{i1}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega})}{D_1}. \quad (11)$$

$\boldsymbol{\omega}_{ri}$ is derived from Eqs. (A3) and (11) as given below:

$$\boldsymbol{\omega}_{ri} = \omega_{i1}\boldsymbol{R}_{i1} + \omega_{i2}\boldsymbol{R}_{i2} = \frac{\mathbf{D}_2\hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\mathbf{e}}_i\boldsymbol{\omega})}{D_1} = \boldsymbol{J}_{\omega ri}\boldsymbol{V}, \quad \boldsymbol{J}_{\omega ri} = \frac{\mathbf{D}_2}{D_1}[\,\hat{\boldsymbol{\delta}}_i^2 \quad -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\,]. \quad (12)$$

v_{ri} is derived from Eq. (A5) as given below:

$$v_{ri} = \boldsymbol{v}_{bi} \cdot \boldsymbol{\delta}_i = (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i) \cdot \boldsymbol{\delta}_i = \boldsymbol{\delta}_i \cdot \boldsymbol{v} + (\boldsymbol{\delta}_i \times \boldsymbol{e}_i) \cdot \boldsymbol{\omega} = \boldsymbol{J}_{vi}\boldsymbol{V}, \boldsymbol{J}_{vi} = [\,\boldsymbol{\delta}_i^{\mathrm{T}} \quad -(\hat{\boldsymbol{e}}_i\boldsymbol{\delta}_i)^{\mathrm{T}}\,]_{1\times 6}. \quad (13)$$

4.3. Angular velocity $\boldsymbol{\omega}_{gi}$ of upper beam g_i and angular velocity $\boldsymbol{\omega}_{rij}$ of active leg r_{ij}

The angular velocity $\boldsymbol{\omega}_{gi}$ of the upper beam g_i and the angular velocity $\boldsymbol{\omega}_{rij}$ of the active leg r_{ij} must also be derived before solving the general input and forward velocities. Their derivations are explained as follows.

Dot multiply both sides of Eq. (A4) by $\boldsymbol{R}_{i6}$; $\boldsymbol{\omega}_{i3}$ and $\boldsymbol{\omega}_{gi}$ are derived based on ($\boldsymbol{R}_{i6} \perp \boldsymbol{R}_{i4}$, $\boldsymbol{R}_{i6} \perp \boldsymbol{R}_{i5}$) and Eq. (12) as follows:

$$\begin{aligned}
&(\boldsymbol{\omega} + \boldsymbol{\omega}_{i4}\boldsymbol{R}_{i4} + \boldsymbol{\omega}_{i5}\boldsymbol{R}_{i5}) \cdot \boldsymbol{R}_{i6} = (\boldsymbol{\omega}_{ri} + \boldsymbol{\omega}_{i3}\boldsymbol{R}_{i3}) \cdot \boldsymbol{R}_{i6}, \Rightarrow \boldsymbol{\omega}_{i3} = \frac{\boldsymbol{R}_{i6}^{\mathrm{T}}(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri})}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}}, \mathbf{D}_3 = \frac{\boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}}}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}},\\
&\boldsymbol{\omega}_{gi} = \boldsymbol{\omega}_{ri} + \boldsymbol{\omega}_{i3}\boldsymbol{R}_{i3} = \boldsymbol{\omega}_{ri} + \mathbf{D}_3(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri}) = (\mathbf{E}_{3\times 3} - \mathbf{D}_3)\boldsymbol{\omega}_{ri} + \mathbf{D}_3\boldsymbol{\omega} \qquad (14)\\
&= (\mathbf{E}_{3\times 3} - \mathbf{D}_3)\boldsymbol{J}_{\omega ri}\boldsymbol{V} + [\,\mathbf{0}_{3\times 3} \quad \mathbf{D}_3\,]\boldsymbol{V} = \mathbf{J}_{\omega gi}\boldsymbol{V}, \boldsymbol{J}_{\omega gi} = (\mathbf{E}_{3\times 3} - \mathbf{D}_3)J_{\omega ri} + [\,\mathbf{0}_{3\times 3} \quad \mathbf{D}_3\,].
\end{aligned}$$

Dot multiply both sides of Eq. (A4) by $\boldsymbol{\delta}_{ij}$ based on $\boldsymbol{\delta}_{ij} \perp \boldsymbol{R}_{i2}$, it leads to

$$\boldsymbol{\omega}_{gi} \cdot \boldsymbol{\delta}_{ij} = \boldsymbol{\omega}_{rij} \cdot \boldsymbol{\delta}_{ij} + \boldsymbol{\omega}_{gij}\boldsymbol{R}_{i2} \cdot \boldsymbol{\delta}_{ij} \Rightarrow \boldsymbol{\omega}_{gi} \cdot \boldsymbol{\delta}_{ij} = \boldsymbol{\omega}_{rij} \cdot \boldsymbol{\delta}_{ij}. \quad (15)$$

Cross multiply both sides of Eq. (11d) by $\boldsymbol{\delta}_{ij}$, it leads to

$$\boldsymbol{\delta}_{ij} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i + \boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij}) = \boldsymbol{\delta}_{ij} \times (v_{rij}\boldsymbol{\delta}_{ij} + r_{ij}\boldsymbol{\omega}_{rij} \times \boldsymbol{\delta}_{ij}) = r_{ij}\boldsymbol{\omega}_{rij} - r_{ij}\boldsymbol{\delta}_{ij}(\boldsymbol{\omega}_{rij} \cdot \boldsymbol{\delta}_{ij}). \quad (16)$$

Substitute Eq. (15) into Eq. (16), $\boldsymbol{\omega}_{rij}$ is derived as given below:

$$\begin{aligned}
\boldsymbol{\omega}_{rij} &= \frac{\boldsymbol{\delta}_{ij}}{r_{ij}} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i + \boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij}) + \boldsymbol{\delta}_{ij}(\boldsymbol{\omega}_{rij} \cdot \boldsymbol{\delta}_{ij})\\
&= \frac{\boldsymbol{\delta}_{ij} \times (\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i)}{r_{ij}} + \frac{\boldsymbol{\delta}_{ij} \times (\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij})}{r_{ij}} + \boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{ij}^{\mathrm{T}}\boldsymbol{\omega}_{gi}\\
&= \boldsymbol{J}_{\omega ij}\boldsymbol{V}, \boldsymbol{J}_{\omega ij} = \frac{1}{r_{ij}}\left[\,\hat{\boldsymbol{\delta}}_{ij} \quad -\hat{\boldsymbol{\delta}}_{ij}\hat{\mathbf{e}}_i\,\right] + \left(\boldsymbol{\delta}_{ij}\boldsymbol{\delta}_{ij}^{\mathrm{T}} - \frac{\hat{\boldsymbol{\delta}}_{ij}\hat{\boldsymbol{g}}_{ij}}{r_{ij}}\right)\boldsymbol{J}_{\omega gi}, (i = 1, 2, 3,\ j = 1, 2). (17)
\end{aligned}$$

4.4. General input velocity V_r, forward velocity V, and statics model

The statics model provides a theoretical foundation for determining actuator, establishing stiffness model, and solving elastic deformation for a 6-DoF PM with three planar mechanism limbs.

Dot multiply both sides of Eq. (11d) by δ_{ij}, v_{rij} is derived from Eqs. (14) and (17) as follows:

$$\begin{aligned}\boldsymbol{v}_{rij} &= \boldsymbol{v}_{bij}\cdot\boldsymbol{\delta}_{ij} = (\boldsymbol{v}_{bi} + \boldsymbol{\omega}_{gi}\times\boldsymbol{g}_{ij})\cdot\boldsymbol{\delta}_{ij} = (\boldsymbol{v} + \boldsymbol{\omega}\times\boldsymbol{e}_i)\cdot\boldsymbol{\delta}_{ij} + (\boldsymbol{J}_{\omega gi}\boldsymbol{V}\times\boldsymbol{g}_{ij})\cdot\boldsymbol{\delta}_{ij}\\ &= \boldsymbol{J}_{ij}\boldsymbol{V},\ \boldsymbol{J}_{ij} = \begin{bmatrix}\boldsymbol{\delta}_{ij}^{\mathrm{T}} & (\hat{\mathbf{e}}_i\boldsymbol{\delta}_{ij})^{\mathrm{T}}\end{bmatrix}_{1\times 6} + (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\boldsymbol{J}_{\omega gi},\ (i = 1, 2, 3,\ j = 1, 2).\end{aligned} \tag{18}$$

Let F_{rij} (i = 1, 2, 3; j = 1, 2) be the active force applied on and along r_{ij}, and (F, T) be the workload wrench applied on m at o. The general input velocity $\boldsymbol{V}_r$, the general forward velocity V, and the statics are derived based on the principle of virtual work,[1] and the derived velocity formula is as follows:

$$\begin{aligned}&(\boldsymbol{V}_r)_{6\times 1} = \mathbf{J}_{6\times 6}\boldsymbol{V},\\ &\boldsymbol{V} = \mathbf{J}^{-1}V_r,\\ &\boldsymbol{f}^{\mathrm{T}}\boldsymbol{V}_r + [\boldsymbol{F}^{\mathrm{T}}\quad \boldsymbol{T}^{\mathrm{T}}]\boldsymbol{V} = 0,\\ &\quad \boldsymbol{f} = -(\mathbf{J}^T)^{-1}\begin{bmatrix}\boldsymbol{F}\\ \boldsymbol{T}\end{bmatrix},\quad \boldsymbol{V}_r = \begin{bmatrix}v_{r11}\\ v_{r12}\\ v_{r21}\\ v_{r22}\\ v_{r31}\\ v_{r32}\end{bmatrix},\quad \boldsymbol{f} = \begin{bmatrix}F_{r11}\\ F_{r12}\\ F_{r21}\\ F_{r22}\\ F_{r31}\\ F_{r32}\end{bmatrix},\quad \mathbf{J} = \begin{bmatrix}\boldsymbol{J}_{11}\\ \boldsymbol{J}_{12}\\ \boldsymbol{J}_{21}\\ \boldsymbol{J}_{22}\\ \boldsymbol{J}_{31}\\ \boldsymbol{J}_{32}\end{bmatrix}.\\ &i = 1, 2, 3; j = 1, 2,\end{aligned} \tag{19a}$$

$$\begin{aligned}&\boldsymbol{J}_{ij} = \begin{bmatrix}\boldsymbol{\delta}_{ij}^{\mathrm{T}} & (\hat{\mathbf{e}}_i\boldsymbol{\delta}_{ij})^{\mathrm{T}}\end{bmatrix}_{1\times 6} + (\hat{\mathbf{e}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\boldsymbol{J}_{\omega gi},\ \boldsymbol{J}_{\omega ri} = \frac{\mathbf{D}_2}{D_1}\begin{bmatrix}\hat{\boldsymbol{\delta}}_i^2 & -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\end{bmatrix},\\ &\boldsymbol{J}_{\omega gi} = (\mathbf{E}_{3\times 3} - \mathbf{D}_3)J_{\omega ri} + [\mathbf{0}_{3\times 3}\quad \mathbf{D}_3],\\ &D_1 = \boldsymbol{R}_i\cdot(\boldsymbol{R}_{i1}\times\boldsymbol{R}_{i2}),\quad \mathbf{D}_2 = \boldsymbol{R}_{i1}\boldsymbol{R}_{i2}^{\mathrm{T}} - \boldsymbol{R}_{i2}\boldsymbol{R}_{i1}^{\mathrm{T}},\quad \mathbf{D}_3 = \boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}}/(\boldsymbol{R}_{i3}\cdot\boldsymbol{R}_{i6}).\end{aligned} \tag{19b}$$

Here, $\mathbf{J}$ is a 6 × 6 Jacobian matrix of PM with three planar mechanism limbs.

When given the general input velocity $\boldsymbol{V}_r$ of active legs r_{ij} (i = 1, 2, 3; j = 1, 2), the general forward velocity V of the moving platform can be solved using Eq. (19). When given the workload wrench (F, T) applied on m, the active forces F_{rij} applied on and along r_{ij} can be solved using Eq. (19).

4.5. Analysis of structure singularity

When $r_{ij}|g_i$ ($i = 1, 2, 3, j = 1, 2$) are satisfied, each of r_{ij} has two kinematic solutions. This case is called as a structure singularity of PM with three planar mechanism limbs. In this case, $\boldsymbol{r}_{ij}\perp\boldsymbol{z}$ is satisfied. From Eq. (9) and $\boldsymbol{g}_i\perp\boldsymbol{z}$, it leads to

$$\boldsymbol{r}_{ij}\cdot\boldsymbol{z} = \boldsymbol{r}_i\cdot\boldsymbol{z} + (-1)^j\boldsymbol{m}_i\cdot\boldsymbol{z} \Rightarrow \boldsymbol{r}_i\cdot\boldsymbol{z} = (-1)^j G\boldsymbol{G}_i\cdot\boldsymbol{z}. \tag{20}$$

Next, from Eqs. (7) and (20), a structure singularity equation is derived as given below:

$$z_l X_o + z_m Y_o + z_n Z_o = E(qz_l + z_m) \Rightarrow z\cdot o = E(qz_l + z_m). \tag{21}$$

The structure singularity can be determined based on Eq. (21) and must be avoided. Therefore, the angle between active leg r_{ij} and the upper beam g_i must be less than 180° for avoiding the structure singularity.

5. Acceleration Analysis of 6-DoF PM with Three Planar Mechanism Limbs

A standard acceleration formula is a basis of the analysis of dynamics and the control of a 6-DoF PM with three planar mechanism limbs. Let a_{rij} be the input scalar acceleration along r_{ij}, $\boldsymbol{a}_{bi}$ be the translational acceleration of m at b_i, and $\boldsymbol{\varepsilon}_{gi}$ be the angular vector acceleration of g_i.

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Differentiate Eq. (18) with respect to time, a_{rij} is derived using Eq. (A5) in the Appendix, and Eqs. (14) and (17) as follows:

$$a_{rij} = [\boldsymbol{\delta}_{ij}^{\mathrm{T}} \quad (\boldsymbol{e}_i \times \boldsymbol{\delta}_{ij})^{\mathrm{T}}]\boldsymbol{A} + (\boldsymbol{g}_{ij} \times \boldsymbol{\delta}_{ij})^{\mathrm{T}}\boldsymbol{\varepsilon}_{gi} + \boldsymbol{V}^{\mathrm{T}\,1}\mathbf{h}_{ij}\boldsymbol{V}, (i = 1, 2, 3,\ j = 1, 2),$$
$$^{1}\mathbf{h}_{ij} = \begin{bmatrix} \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \\ \mathbf{0}_{3\times3} & \hat{\boldsymbol{e}}_i\hat{\boldsymbol{\delta}}_{ij} \end{bmatrix} - \begin{bmatrix} \mathbf{E}_{3\times3} \\ \hat{\boldsymbol{e}}_i \end{bmatrix}\hat{\boldsymbol{\delta}}_{ij}\boldsymbol{J}_{\omega ij} + \boldsymbol{J}_{\omega gi}^{\mathrm{T}}\hat{\mathbf{g}}_{ij}(\hat{\boldsymbol{\delta}}_{ij}\boldsymbol{J}_{\omega gi} - \hat{\boldsymbol{\delta}}_{ij}\boldsymbol{J}_{\omega ij}). \tag{22}$$

Here, $^{3}\mathbf{h}_{ij}$ is the sub-Hessian matrix. A detail derivation of a_{rij} is given by Eq. (A8) in the Appendix.

Differentiate Eq. (14) with respect to time, and $\boldsymbol{\varepsilon}_{gi}$ is derived as given below:

$$\boldsymbol{\varepsilon}_{gi} = (\mathbf{E}_{3\times3} - \mathbf{D}_3)'\boldsymbol{\omega}_{ri} + (\mathbf{E}_{3\times3} - \mathbf{D}_3)\boldsymbol{\varepsilon}_{ri} + \mathbf{D}_3'\boldsymbol{\omega} + \mathbf{D}_3\boldsymbol{\varepsilon} = \mathbf{D}_3'(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri}) + (\mathbf{E}_{3\times3} - \mathbf{D}_3)\boldsymbol{\varepsilon}_{ri} + \mathbf{D}_3\boldsymbol{\varepsilon}. \tag{23}$$

From Eq. (23), it leads to

$$(\boldsymbol{g}_{ij} \times \boldsymbol{\delta}_{ij})^{\mathrm{T}}\boldsymbol{\varepsilon}_{gi} = (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\boldsymbol{\varepsilon}_{gi} = (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}'_3(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri}) + \mathbf{c}_{ij}\boldsymbol{\varepsilon}_{ri} + (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3\boldsymbol{\varepsilon}$$
$$= [\mathbf{0}_{1\times3} \quad (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3]\boldsymbol{A} + (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}'_3(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri}) + \mathbf{c}_{ij}\boldsymbol{\varepsilon}_{ri}, (\mathbf{c}_{ij})_{1\times3} = (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}(\mathbf{E}_{3\times3} - \mathbf{D}_3). \tag{24}$$

Based on Eqs. (A2) and (A3) in the Appendix and Eq. (14), the differentiation of $\boldsymbol{R}_{i3}$, $(\boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^{T})$, $(\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6})$, and $\mathbf{D}_3$ with respect to time are derived as follows:

$$\boldsymbol{R}'^{\mathrm{T}}_{i3} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{J}_{\omega ri}^{\mathrm{T}}\frac{\hat{\boldsymbol{\delta}}_i\hat{\mathbf{R}}_{i1}\hat{\mathbf{R}}_{i3}^2}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|}, (\boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}})' = \boldsymbol{R}'_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}} + \boldsymbol{R}_{i3}(\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5}\boldsymbol{\omega}_{gi})^{\mathrm{T}},$$
$$(\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6})' = \boldsymbol{R}'^{\mathrm{T}}_{i3}\boldsymbol{R}_{i6} + \boldsymbol{R}_{i3}^{\mathrm{T}}(\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5}\boldsymbol{\omega}_{gi}),$$
$$\mathbf{D}'_3 = \frac{\boldsymbol{R}'_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}} + \boldsymbol{R}_{i3}(\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega}_{gi})^{\mathrm{T}}}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}} - \mathbf{D}_3\frac{\boldsymbol{R}'_{i3} \cdot \boldsymbol{R}_{i6} + \boldsymbol{R}_{i3} \cdot (\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5}\boldsymbol{\omega}_{gi})}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}}. \tag{25}$$

A detailed derivation of Eq. (25) is given by Eqs. (A9), (A10), and (A11) in the Appendix.

Next, item $(\hat{\boldsymbol{e}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}'_3$ in Eq. (24) is derived based on Eq. (25) as follows:

$$(\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}'_3 = \boldsymbol{R}'^{\mathrm{T}}_{i3}\frac{\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij}\boldsymbol{R}_{i6}^{\mathrm{T}} - \boldsymbol{R}_{i6}(\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}} + (\boldsymbol{\omega}^{\mathrm{T}}\hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5} - \boldsymbol{\omega}_{gi}^{\mathrm{T}}\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4})\frac{(\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\boldsymbol{R}_{i3} - \boldsymbol{R}_{i3}(\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}}. \tag{26}$$

Thus, item $(\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3'(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri})$ in Eq. (24) is transformed into a standard model as follows:

$$(\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}'_3(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri}) = \boldsymbol{V}^{\mathrm{T}}\mathbf{d}_{ij}(\boldsymbol{\omega} - \boldsymbol{\omega}_{ri}) = \boldsymbol{V}^{\mathrm{T}\,2}\mathbf{h}_{ij}\boldsymbol{V}, {}^{2}\mathbf{h}_{ij} = [\mathbf{0}_{3\times3} \quad \mathbf{d}_{ij}] - \mathbf{d}_{ij}\boldsymbol{J}_{\omega ri}. \tag{27}$$

Here, $\mathbf{d}_{ij}$ is a 6×6 matrix. Its expression is given by (A12) in the Appendix.

Differentiate $\boldsymbol{\omega}_{ri}$ in Eq. (12) with respect to time, it leads to

$$\boldsymbol{\varepsilon}_{ri} = \frac{(\boldsymbol{R}_{i1}\boldsymbol{R}'^{\mathrm{T}}_{i2} - \boldsymbol{R}'_{i2}\boldsymbol{R}_{i1}^{\mathrm{T}})}{D_1}(\hat{\boldsymbol{\delta}}_i^2\boldsymbol{v} - \hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\boldsymbol{\omega}) - \frac{\boldsymbol{\omega}_{ri}}{D_1}[(v_{ri}\boldsymbol{\delta}_i^{\mathrm{T}} + r_i\boldsymbol{\omega}_{ri}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i)(\boldsymbol{R}_{i1} \times \boldsymbol{R}_{i2}) + r_i\boldsymbol{\delta}_i^{\mathrm{T}}\hat{\mathbf{R}}_{i1}\boldsymbol{R}'_{i2}]$$
$$+ \frac{\mathbf{D}_2}{D_1}\{\hat{\boldsymbol{\delta}}_i^2\boldsymbol{a} - \hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\boldsymbol{\varepsilon} + [\boldsymbol{\delta}_i(\boldsymbol{\omega}_{ri}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i) - (\hat{\boldsymbol{\delta}}_i\boldsymbol{\omega}_{ri})\boldsymbol{\delta}_i^{\mathrm{T}}](\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega}) - \hat{\boldsymbol{\delta}}_i^2[(\boldsymbol{\omega} \times \boldsymbol{e}_i) \times \boldsymbol{\omega}]\}. \tag{28}$$

Thus, item $\mathbf{c}_{ij}\boldsymbol{\varepsilon}_{ri}$ in Eq. (27) is transformed into a standard model as given below:

$$
\begin{aligned}
\mathbf{c}_{ij}\boldsymbol{\varepsilon}_{ri} &= \frac{\mathbf{c}_{ij}\mathbf{D}_2}{D_1}[\hat{\boldsymbol{\delta}}_i^2 \quad -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i]\boldsymbol{A} + \boldsymbol{V}^{\mathrm{T}3}\mathbf{h}_{ij}\boldsymbol{V},\\
{}^3\mathbf{h}_{ij} &= \frac{\mathbf{J}_{\omega ri}^{\mathrm{T}}}{D_1}\left\{\frac{\hat{\boldsymbol{\delta}}_i\hat{\mathbf{R}}_{i1}\hat{\mathbf{R}}_{i3}^2}{|\boldsymbol{R}_{i1}\times\boldsymbol{\delta}_i|}(\mathbf{c}_{ij}\boldsymbol{R}_{i1}\mathbf{E}_{3\times 3} - \mathbf{c}_{ij}^{\mathrm{T}}\mathbf{R}_{i1}^{\mathrm{T}})[\hat{\boldsymbol{\delta}}_i^2 \quad -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i]\right.\\
&\quad + r_i\hat{\boldsymbol{\delta}}_i\left(\frac{\hat{\mathbf{R}}_{i1}\hat{\mathbf{R}}_{i3}^2}{|\boldsymbol{R}_{i1}\times\boldsymbol{\delta}_i|}\hat{\mathbf{R}}_{i1}\boldsymbol{\delta}_i - \boldsymbol{R}_{i1}\times\boldsymbol{R}_{i2}\right)\mathbf{c}_{ij}\mathbf{J}_{\omega ri}\\
&\quad \left. - \mathbf{c}_{ij}^{\mathrm{T}}\boldsymbol{\delta}_i^{\mathrm{T}}(\boldsymbol{R}_{i1}\times\boldsymbol{R}_{i2})\mathbf{J}_{vi} + \hat{\boldsymbol{\delta}}_i(\mathbf{c}_{ij}\mathbf{D}_2\boldsymbol{\delta}_i + \mathbf{D}_2^{\mathrm{T}}\mathbf{c}_{ij}^{\mathrm{T}}\boldsymbol{\delta}_i^{\mathrm{T}})[\mathbf{E}_{3\times 3} \quad -\hat{\boldsymbol{e}}_i]\right\}\\
&\quad + \frac{1}{D_1}\begin{bmatrix}\mathbf{0}_{3\times 3} & \mathbf{0}_{3\times 3}\\ \mathbf{0}_{3\times 3} & \hat{\boldsymbol{e}}_i s(\hat{\boldsymbol{\delta}}_i^2\mathbf{D}_2^{\mathrm{T}}\mathbf{c}_{ij}^{\mathrm{T}})\end{bmatrix}.
\end{aligned}
\tag{29}
$$

Here, ${}^3\mathbf{h}_{ij}$ is the sub-Hessian matrix. The detail derivation of Eq. (29) is shown by Eq. (A13) in the Appendix.

From Eqs. (23) to (27), it leads to

$$
\begin{aligned}
(\boldsymbol{g}_{ij}\times\boldsymbol{\delta}_{ij})\cdot\boldsymbol{\varepsilon}_{gi} &= (\hat{\boldsymbol{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}'_3(\boldsymbol{\omega}-\boldsymbol{\omega}_{ri}) + (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}(\mathbf{E}_{3\times 3}-\mathbf{D}_3)\boldsymbol{\varepsilon}_{ri} + [\mathbf{0}_{1\times 3} \quad (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3]\boldsymbol{A}\\
&= \left[\frac{\mathbf{c}_{ij}\mathbf{D}_2\hat{\boldsymbol{\delta}}_i^2}{D_1} \quad (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3 - \frac{\mathbf{c}_{ij}\mathbf{D}_2\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i}{D_1}\right]\boldsymbol{A} + \boldsymbol{V}^{\mathrm{T}}({}^2\mathbf{h}_{ij} + {}^3\mathbf{h}_{ij})\boldsymbol{V}.
\end{aligned}
\tag{30}
$$

Finally, a standard formula for solving the general input acceleration A_r is derived from Eqs. (23), (27), and (30) as follows:

$$
\begin{aligned}
&a_{rij} = (\boldsymbol{J}_{ij})_{1\times 6}\boldsymbol{A} + \boldsymbol{V}^{\mathrm{T}}(h_{ij})_{6\times 6}\boldsymbol{V}, \\
&\boldsymbol{A}_r = \mathbf{J}_{6\times 6}\boldsymbol{A} + \boldsymbol{V}^{\mathrm{T}}\mathbf{H}\boldsymbol{V}, \qquad \boldsymbol{A}_r = \begin{bmatrix} a_{r11}\\ a_{r12}\\ a_{r21}\\ a_{r22}\\ a_{r31}\\ a_{r32}\end{bmatrix}_{6\times 1}, \quad \mathbf{J} = \begin{bmatrix}(J_{11})_{1\times 6}\\ (J_{12})_{1\times 6}\\ (J_{21})_{1\times 6}\\ (J_{22})_{1\times 6}\\ (J_{31})_{1\times 6}\\ (J_{32})_{1\times 6}\end{bmatrix},\\
&\boldsymbol{A} = \mathbf{J}^{-1}(A_r - \boldsymbol{V}^{\mathrm{T}}\mathbf{H}\boldsymbol{V}),\\
&\mathbf{h}_{ij} = \sum_{k=1}^{3}{}^k(h_{ij})_{6\times 6},\\
&\mathbf{H} = \begin{bmatrix}(\mathbf{h}_{11})_{6\times 6}\\ (\mathbf{h}_{12})_{6\times 6}\\ (\mathbf{h}_{21})_{6\times 6}\\ (\mathbf{h}_{22})_{6\times 6}\\ (\mathbf{h}_{31})_{6\times 6}\\ (\mathbf{h}_{32})_{6\times 6}\end{bmatrix}, \quad \begin{matrix} i = 1, 2, 3\\ j = 1, 2,\end{matrix}\\
&\boldsymbol{J}_{ij} = \left[\boldsymbol{\delta}_{ij}^{T} + \frac{(\hat{\boldsymbol{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}(\mathbf{E}_{3\times 3}-\mathbf{D}_3)\mathbf{D}_2\hat{\boldsymbol{\delta}}_i^2}{D_1} \quad (\hat{\mathbf{e}}_i\boldsymbol{\delta}_{ij})^{T} + (\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3 - \frac{(\hat{\mathbf{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}(\mathbf{E}_{3\times 3}-\mathbf{D}_3)\mathbf{D}_2\hat{\boldsymbol{\delta}}_i^2\hat{\mathbf{e}}_i}{D_1}\right].
\end{aligned}
\tag{31}
$$

Here, $\mathbf{H}$ is the Hessian matrix of a 6-DoF PM with planar mechanism limbs, and $\mathbf{h}_{ij}$ ($i = 1, 2, 3; j = 1, 2$) are the 6×6 sub-Hessian matrices of $\mathbf{H}$. ${}^k\mathbf{h}_{ij}$ ($k = 1, 2, 3$) are the 6×6 sub-Hessian matrices of $\mathbf{h}_{ij}$.

When given the general input velocity $\boldsymbol{V}_r$ and the acceleration $\boldsymbol{A}_r$ of active legs r_{ij} ($i = 1, 2, 3; j = 1, 2$), the general forward acceleration A of the moving platform can be solved using Eq. (31).

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Table I. Given parameters of a 6-DoF PM of with three planar mechanism limbs.

Symbols (unit)	l (mm)	L (mm)	$2g$ (mm)	$2G$ (mm)	F (kN)	T (N· m)
Value	120	240	30	40	$[0\ 0\ -1]^T$	$[0\ 0\ 10]^T$

6. Analytical Examples of Kinematics and Statics and Numerical Solved Results

In order to verify all derived equations, an analytical numerical example and the solved results are given as follows. The relative given parameters are listed in Table I.

The analytic results are obtained from a data file, which is generated from a compiled program. The simulation results are also obtained from a data file, which is generated from a simulation mechanism in the advanced CAD software. The solved and verified processes are explained as follows: (1) Give the translational acceleration a_{rij} of the active legs r_{ij} ($i = 1, 2, 3; j = 1, 2$) and the workload wrench; see Fig. 2a and Table I. (2) Solve the displacement and the translational velocity of r_{ij} ($i = 1, 2, 3; j = 1, 2$), the solutions are shown in Figs. 2b and c. (3) Solve the angular velocity of active legs r_{ij}, the displacement, the translational velocity, and the translational acceleration of the moving platform m based on the displacement of r_{ij}, the solutions are shown in Figs. 2d–g. (4) Solve the orientations, the angular velocity, and the angular acceleration of the moving platform m, the solutions are shown in Figs. 2h–j. (5) Solve the active forces of the active legs r_{ij}, the solutions are shown in Fig. 2k. (6) Construct a simulation mechanism using the advanced CAD software, and verify all the analytically solved results.

The characteristics of a 6-DoF PM with three planar mechanism limbs are found from the solved results as follows:

1. When $r_{ij}(i = 1, 2, 3; j = 1, 2)$ are varied within $280 \sim 450$ mm, the displacement components of m are varied within ($0 \sim 220$, $0 \sim 160$, $220 \sim 250$) mm for (X_o, Y_o, Z_o) respectively; see Fig. 2e. It implies that the novel 6-DoF PM with three planar mechanism limbs has a quite large position workspace.
2. When r_{ij} ($i = 1, 2, 3; j = 1, 2$) are varied within $280 \sim 450$ mm, the orientation components of m are varied within $(100 \sim 50, 0 \sim 20, -110 \sim 20)^\circ$ for (α, β, γ) respectively; see Fig. 2e. It implies that the novel 6-DoF PM with three planar mechanism limbs has a quite large orientation workspace.
3. When the displacement, the linear velocity, and the linear acceleration of the active legs r_{ij} are varied smoothly, the displacement, the linear velocity, and the linear acceleration of m are varied smoothly in a large range; the orientations, the angular velocity, and the angular acceleration of m are also varied smoothly in a large range. The active forces of r_{ij} are varied smoothly in a large range. It implies that the novel 6-DoF PM with three planar mechanism limbs has good characteristics of kinematics and statics.

7. Workspace and Comparisons

When given the same parameters and under condition of m moving in three translations (see Table II), the workspace volume V_W of the 6-DoF PM with three planar mechanism limbs and the workspace volumes $V_{W\theta}$ of a 6–6 Stewart PM[2,4] are solved based on relative analytical formulae, see Figs. 3a–c. Here, θ is the limited rotational angle of the S joint, $\theta = \pm$ (30, 35, 40, 45, 50)°; see Fig. 3d. The solved relative data of the curve family are input into the advanced CAD software, and the 3D surface of the workspaces is formed from a family of curves; see Fig. 3. The solutions of the workspace volumes of the two PMs are listed in Table II. It is known from the solved results that the workspace volume $V_{W\theta}$ of the 6–6 Stewart PM is enlarged with the increase of θ. Generally, since θ should be less than $\pm 35^\circ$ before interference occurs in the S joint, $V_W = 71.905$ mm^3 is larger than $V_{W35} = 33.585$ mm^3.

The construction procedures of the workspace volumes are explained as follows:

Step 1. Set the constrained conditions as follows: The extensions of the active leg r_{ij} of PM are in the range of 0.5→0.75 m, the rotation angles of the spherical joint are in the range of $\pm\theta$.

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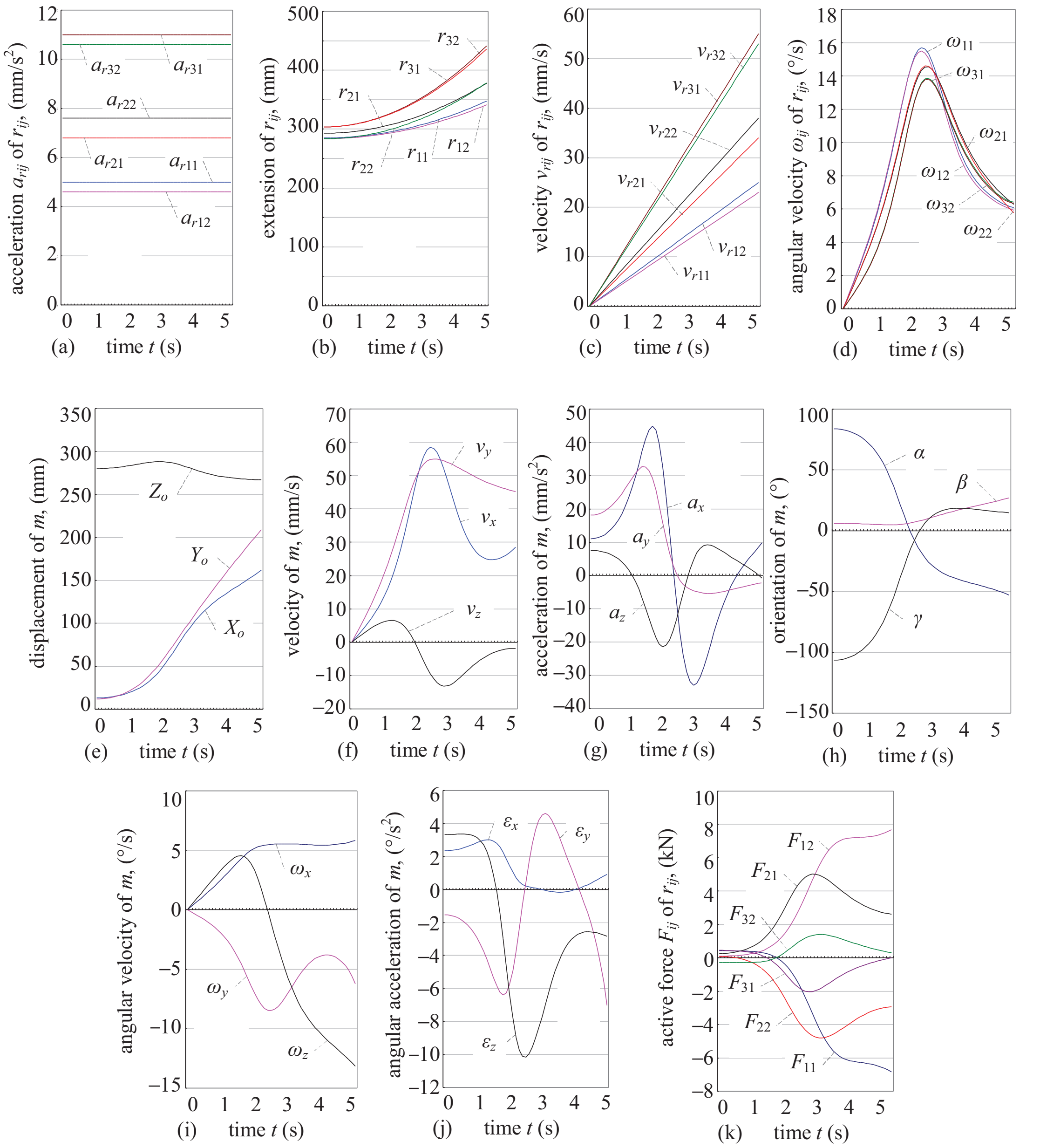

Fig. 2. Analytically solved results of kinematics and statics.

Table II. Solved workspace volumes of two PMs under given parameters, and m moving only in translations.

Different PMs	Given parameters					Solved results of volume workspace (mm^3)	
	e (mm)	r_{ij} (mm)	E (mm)	$2g$ (mm)	θ (°)		
6-DoF PM with 3 planar mechanism limbs	100	500 ∼ 750	250	25		V_W	71.905
6–6 Stewart PM	100	500 ∼ 750	250	0	±30	V_{W30}	20.793
					±35	V_{W35}	33.585
					±40	V_{W40}	48.745
					±45	V_{W45}	58.947
					±50	V_{W50}	77.163

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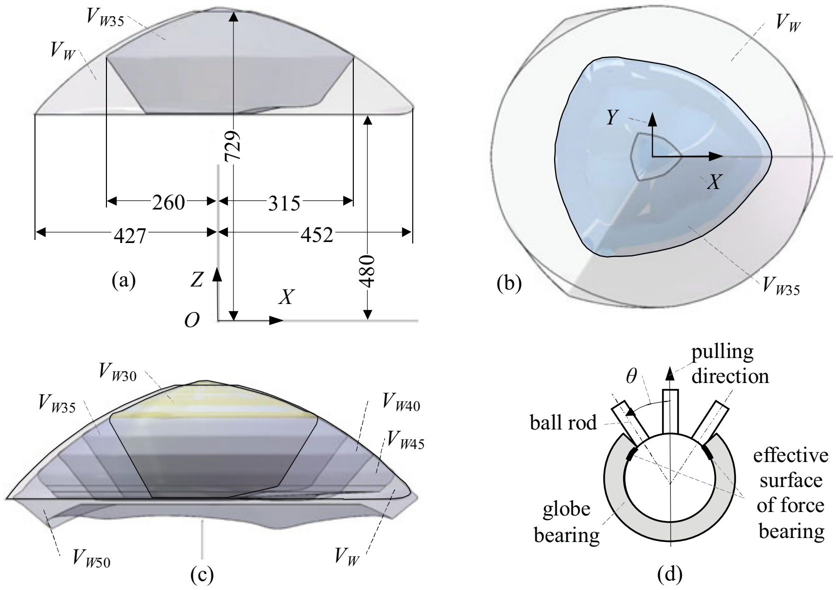

Fig. 3. Workspace volumes of two PMs under given parameters and different θ. (a) Front view of V_W and V_{W35}; (b) top view of V_W and V_{W35}; (c) front view of V_W and $V_{W\theta}$; (d) structure of S joint with limited angle θ.

Step 2. Search all the positions that can be reached by the center of m using the inverse displacement formulae in Eq. (9) in Matlab.

The search principle for compiling program is described as follows:

- Set Euler angles (α, β, γ) of m to be 0, then solve the center coordinate of m.
- Judge the constrained conditions of the coordinate of m at o using the inverse displacement formulae in Eq. (9). If the coordinate of m at o satisfies all the constrained conditions, it is located in the workspace of PM with a fixed orientation; otherwise, it is not located in the workspace.
- Solve all the boundaries of the workspace.

Step 3. Transfer all the solved position data into the advanced CAD software, generate a family of similar spatial curves, and construct lower and upper boundary surfaces from the family of similar spatial curves.

Step 4. Generate the workspace volumes of the PM by lower and upper boundary surfaces; see Fig. 3.

Step 5. Based on the displacement formulae of the existing 6–6 Stewart PM,[2,4] repeat Steps 1 to 4, and construct a workspace of the existing 6–6 Stewart PM.

Step 6. Assemble all the 3D workspaces V_w and $V_{W\theta}$, $\theta = \pm$ (30, 35, 40, 45, 50)° with the same [B]; see Fig. 3d. The measured volumes of every 3D workspace are listed in Table II.

8. Conclusions

The proposed novel 6-DoF PM with three planar mechanism limbs has the following merits:

- The planar mechanism of limb only includes revolute joints and prismatic joint; therefore it is simple in structure and easy to manufacture.
- The revolute joint has a larger capability of pulling force bearing than that of a spherical joint.
- The revolute joint has a higher precision than a spherical joint under large cyclic loading because the backlash in the revolute joint can be eliminated more easily than that in the spherical joint or the universal joint.

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- The workspace can be increased because the revolute joint has a larger rotation range than a spherical joint before interference.

The formulae for solving the standard Jacobian/Hessian matrices and the kinematics/statics, and determining the structure singularity are established for the 6-DoF PM with three planar mechanism limbs. The derived formulae are verified by the simulation results. When given the general input velocities and accelerations of the six active legs, the general forward velocity and acceleration of the moving platform can be solved using the established kinematics model. When given the workload wrench applied on the platform, the active forces applied on and along the six active legs can be solved using the established statics model.

The solved results show that the proposed 6-DoF PM with three planar mechanism limbs has quite large position workspace and orientation workspace, and good kinematic and static characteristics, and the workspace volume is much larger than that of the 6–6 Stewart PM.

The studied results in this paper provide a theoretical foundation for determining/selecting actuators, establishing the dynamics and stiffness models, solving the elastic deformation, and the control of a 6-DoF PM with three planar mechanism limbs in the future.

Acknowledgements

The authors would like to acknowledge Project (51175447) supported by National Natural Science Foundation of China (NSFC), and Key planned project of Hebei China application foundation No. 11962127D.

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Appendix

The coordinates of b_i and z of m in $\{B\}$ are expressed as follows:[18]

$$\boldsymbol{b}_i = \mathbf{R}_m^B \boldsymbol{b}_i^m + \boldsymbol{o}(i = 1, 2, 3),$$
$$\mathbf{R}_m^B = \begin{bmatrix} x_l & y_l & z_l \\ x_m & y_m & z_m \\ x_n & y_n & z_n \end{bmatrix} = \begin{bmatrix} c_\alpha c_\beta c_\gamma - s_\alpha s_\gamma & -c_\alpha c_\beta s_\gamma - s_\alpha c_\gamma & c_\alpha s_\beta \\ s_\alpha c_\beta c_\gamma + c_\alpha s_\gamma & -s_\alpha c_\beta s_\gamma + c_\alpha c_\gamma & s_\alpha s_\beta \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix},$$
$$\boldsymbol{z} = \mathbf{R}_m^B \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} z_l \\ z_m \\ z_n \end{bmatrix}. \tag{A1}$$

Here, $\mathbf{R}_m^B$ is the rotation matrix from $\{m\}$ to $\{B\}$ of order ZYZ (about Z by α, about Y_1 by β, and about Z_2 by γ); $x_l, x_m, x_n y_l, y_m, y_n z_l, z_m, z_n$ are nine orientation parameters of m.[18] Let φ be one of three Euler angles (α, β, γ). $s_\varphi = \sin\varphi$, $c_\varphi = cos\varphi$.

The relative basic kinematic equations and the geometry-constrained equations are expressed by Eqs. (A2), (A3), (A4), and (A5) as follows:[1]

$$\boldsymbol{R}_{i1} = \boldsymbol{G}_i, \boldsymbol{R}_{i2} = \boldsymbol{R}_{i3} = \frac{\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|}, \boldsymbol{R}_{i4} = \boldsymbol{z}, \boldsymbol{R}_{i5} = \frac{\boldsymbol{g}_{i1}}{|\boldsymbol{g}_{i1}|},$$
$$\boldsymbol{R}_{i6} = \boldsymbol{R}_{i4} \times \boldsymbol{R}_{i5}, \boldsymbol{R}'_{i4} = \boldsymbol{\omega} \times \boldsymbol{R}_{i4}, \boldsymbol{R}'_{i5} = \boldsymbol{\omega}_{gi} \times \boldsymbol{R}_{i5}, \tag{A2}$$
$$\boldsymbol{\omega}_{ri} = \boldsymbol{\omega}_{i1}\boldsymbol{R}_{i1} + \boldsymbol{\omega}_{i2}\boldsymbol{R}_{i2}, \tag{A3}$$
$$\boldsymbol{\omega}_{gi} = \boldsymbol{\omega} + \boldsymbol{\omega}_{i4}\boldsymbol{R}_{i4} + \boldsymbol{\omega}_{i5}\boldsymbol{R}_{i5} = \boldsymbol{\omega}_{ri} + \boldsymbol{\omega}_{i3}\boldsymbol{R}_{i3} = \boldsymbol{\omega}_{rij} + \boldsymbol{\omega}_{gij}\boldsymbol{R}_{i2}, \tag{A4}$$
$$\boldsymbol{v}_{bi} = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i = v_{ri} \cdot \boldsymbol{\delta}_i + \boldsymbol{\omega}_{ri} \times \boldsymbol{r}_i, v_{ri} = \boldsymbol{v}_{bi} \cdot \boldsymbol{\delta}_i, \boldsymbol{r}_i = r_i \boldsymbol{\delta}_i,$$
$$\boldsymbol{v}_{bij} = \boldsymbol{v}_{bi} + \boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij} = \boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i + \boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij} = \boldsymbol{v}_{rij} + \boldsymbol{\omega}_{rij} \times \boldsymbol{r}_{ij}, v_{rij} = \boldsymbol{v}_{bij} \cdot \boldsymbol{\delta}_{ij}, \boldsymbol{r}_{ij} = r_{ij}\boldsymbol{\delta}_{ij}. \tag{A5}$$

Cross multiplying both sides of the first Eq. (A3) by $\boldsymbol{r}_i$, based on Eq. (A5), leads to

$$\begin{aligned} \boldsymbol{\omega}_{i1}\boldsymbol{R}_{i1} \times \boldsymbol{r}_i + \boldsymbol{\omega}_{i2}\boldsymbol{R}_{i2} \times \boldsymbol{r}_i &= \boldsymbol{\omega}_{ri} \times \boldsymbol{r}_i = \boldsymbol{v}_{bi} - v_{ri} \cdot \boldsymbol{\delta}_i = (\boldsymbol{\delta}_i \cdot \boldsymbol{\delta}_i)\boldsymbol{v}_{bi} - (\boldsymbol{v}_{bi} \cdot \boldsymbol{\delta}_i)\boldsymbol{\delta}_i \\ &= -\boldsymbol{\delta}_i \times (\boldsymbol{\delta}_i \times \boldsymbol{v}_{bi}) = -\hat{\boldsymbol{\delta}}_i^2 \boldsymbol{v}_{bi} = -\hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{e}_i) = -\hat{\boldsymbol{\delta}}_i^2(\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega}). \end{aligned} \tag{A6}$$

Dot multiply both sides of Eq. (A6) by $\boldsymbol{R}_{i2}$ and $\boldsymbol{R}_{i1}$ respectively, it leads to

$$\boldsymbol{\omega}_{i1}(\boldsymbol{R}_{i1} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i2} = \boldsymbol{R}_{i2}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i^2(-\boldsymbol{v} + \hat{\boldsymbol{e}}_i\boldsymbol{\omega}), \boldsymbol{\omega}_{i2}(\boldsymbol{R}_{i2} \times \boldsymbol{r}_i) \cdot \boldsymbol{R}_{i1} = \boldsymbol{R}_{i1}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i^2(-\boldsymbol{v} + \hat{\boldsymbol{e}}_i\boldsymbol{\omega}). \tag{A7}$$

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The detailed derivation of a_{rij} is given as follows:

$$\begin{aligned}
a_{rij} &= v'_{rij} = [(\boldsymbol{v}_{bi}+\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij}) \cdot \boldsymbol{\delta}_{ij}]' = (\boldsymbol{v}_{bi}+\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij})' \cdot \boldsymbol{\delta}_{ij} + (\boldsymbol{v}_{bi}+\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij}) \cdot \boldsymbol{\delta}'_{ij} \\
&= [\boldsymbol{a}_{bi} + \boldsymbol{\varepsilon}_{gi} \times \boldsymbol{g}_{ij} + \boldsymbol{\omega}_{gi} \times (\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij})] \cdot \boldsymbol{\delta}_{ij} + (\boldsymbol{v}_{bi}+\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij}) \cdot \boldsymbol{\delta}'_{ij} \\
&= \boldsymbol{a}_{bi} \cdot \boldsymbol{\delta}_{ij} + \boldsymbol{\delta}_{ij} \cdot (\boldsymbol{\varepsilon}_{gi} \times \boldsymbol{g}_{ij}) + \boldsymbol{\delta}_{ij} \cdot [\boldsymbol{\omega}_{gi} \times (\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij})] + (\boldsymbol{v}_{bi}+\boldsymbol{\omega}_{gi} \times \boldsymbol{g}_{ij}) \cdot \boldsymbol{\delta}'_{ij} \\
&= \boldsymbol{\delta}_{ij}^{\mathrm{T}}[\boldsymbol{a}+\boldsymbol{\varepsilon} \times \boldsymbol{e}_i + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{e}_i)] + (\boldsymbol{g}_{ij} \times \boldsymbol{\delta}_{ij})^T \boldsymbol{\varepsilon}_{gi} + \boldsymbol{\omega}_{gi}^T \hat{\mathbf{g}}_{ij} \hat{\boldsymbol{\delta}}_{ij} \boldsymbol{\omega}_{gi} + \boldsymbol{v}_{bi}^T \boldsymbol{\delta}'_{ij} - (\hat{\mathbf{g}}_{ij} \boldsymbol{\omega}_{gi})^T \boldsymbol{\delta}'_{ij} \\
&= [\boldsymbol{\delta}_{ij}^{\mathrm{T}} \quad (\boldsymbol{e}_i \times \boldsymbol{\delta}_{ij})^{\mathrm{T}}]\boldsymbol{A} + [\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{e}_i)] \cdot \boldsymbol{\delta}_{ij} + (\boldsymbol{g}_{ij} \times \boldsymbol{\delta}_{ij})^T \boldsymbol{\varepsilon}_{gi} + \boldsymbol{\omega}_{gi}^T \hat{\mathbf{g}}_{ij} (\hat{\boldsymbol{\delta}}_{ij} \boldsymbol{\omega}_{gi} + \boldsymbol{\delta}'_{ij}) + \boldsymbol{v}_{bi}^T \boldsymbol{\delta}'_{ij} \\
&= [\boldsymbol{\delta}_{ij}^{\mathrm{T}} \quad (\boldsymbol{e}_i \times \boldsymbol{\delta}_{ij})^{\mathrm{T}}]\boldsymbol{A} + (\boldsymbol{g}_{ij} \times \boldsymbol{\delta}_{ij})^{\mathrm{T}} \boldsymbol{\varepsilon}_{gi} + \boldsymbol{V}^{\mathrm{T}1}\mathbf{h}_{ij} \boldsymbol{V}, (i = 1, 2, 3,\ j = 1, 2),
\end{aligned} \tag{A8}$$

$$\boldsymbol{\delta}'_{ij} = \boldsymbol{\omega}_{rij} \times \boldsymbol{\delta}_{ij} = -\hat{\boldsymbol{\delta}}_{ij} \boldsymbol{\omega}_{rij}.$$

The detailed derivations of the differentiations of $\boldsymbol{R}_{i3}$, $(\boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}})$, $(\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6})$, and $\mathbf{D}_3$ with respect to time are given as follows:

$$\begin{aligned}
\boldsymbol{R}'_{i3} &= \left(\frac{\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|}\right)' = \frac{(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|} - \frac{(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)\,|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|'}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|^2} = \frac{(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|} \\
&\quad - \frac{\boldsymbol{R}_{i3}[(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)^{\mathrm{T}}(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)]'}{2\,|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|^2} \\
&= \frac{(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|} - \frac{\boldsymbol{R}_{i3}[(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'^{\mathrm{T}}(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i) + (\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)^{\mathrm{T}}(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)']}{2\,|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|^2} \\
&= \frac{(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|} - \frac{\boldsymbol{R}_{i3}[(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'^{\mathrm{T}}\boldsymbol{R}_{i3} + \boldsymbol{R}_{i3}^{\mathrm{T}}(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)']}{2\,|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|} = \frac{(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)' - \boldsymbol{R}_{i3}\boldsymbol{R}_{i3}^{\mathrm{T}}(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|} \\
&= \frac{(\mathbf{E}_{3\times 3} - \boldsymbol{R}_{i3}\boldsymbol{R}_{i3}^{\mathrm{T}})(\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i)'}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|} = \frac{\hat{\mathbf{R}}_{i3}^2\hat{\mathbf{R}}_{i1}\hat{\boldsymbol{\delta}}_i}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|}\boldsymbol{\omega}_{ri},\ \boldsymbol{R}_{i3}'^{\mathrm{T}} = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{J}_{\omega ri}^{\mathrm{T}}\frac{\hat{\boldsymbol{\delta}}_i\hat{\mathbf{R}}_{i1}\hat{\mathbf{R}}_{i3}^2}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|}.
\end{aligned} \tag{A9}$$

$$\begin{aligned}
\boldsymbol{R}_{i3}(\boldsymbol{R}_{i6}'^{\mathrm{T}}) &= \boldsymbol{R}_{i3}(\boldsymbol{R}_{i4} \times \boldsymbol{R}_{i5})'^{\mathrm{T}} = \boldsymbol{R}_{i3}[(\boldsymbol{\omega} \times \boldsymbol{R}_{i4}) \times \boldsymbol{R}_{i5} + \boldsymbol{R}_{i4} \times (\boldsymbol{\omega}_{gi} \times \boldsymbol{R}_{i5})]^T, \\
\boldsymbol{R}_{i3}^T(\boldsymbol{R}_{i4} \times \boldsymbol{R}_{i5})' &= \boldsymbol{R}_{i3}^T[(\boldsymbol{\omega} \times \boldsymbol{R}_{i4}) \times \boldsymbol{R}_{i5} + \boldsymbol{R}_{i4} \times (\boldsymbol{\omega}_{gi} \times \boldsymbol{R}_{i5})] = \boldsymbol{R}_{i3}^T\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \boldsymbol{R}_{i3}^{\mathrm{T}}\hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5}\boldsymbol{\omega}_{gi},
\end{aligned} \tag{A10}$$

$$\begin{aligned}
(\boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^T)' &= \boldsymbol{R}'_{i3}\boldsymbol{R}_{i6}^T + \boldsymbol{R}_{i3}(\boldsymbol{R}_{i6}'^{\mathrm{T}}) = \boldsymbol{R}'_{i3}\boldsymbol{R}_{i6}^T + \boldsymbol{R}_{i3}(\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5}\boldsymbol{\omega}_{gi})^T, \\
(\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6})' &= (\boldsymbol{R}_{i3}^T)'\boldsymbol{R}_{i6} + \boldsymbol{R}_{i3}^T(\boldsymbol{R}_{i4} \times \boldsymbol{R}_{i5})' = \boldsymbol{R}_{i3}'^{\mathrm{T}}\boldsymbol{R}_{i6} + \boldsymbol{R}_{i3}^T(\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5}\boldsymbol{\omega}_{gi}).
\end{aligned}$$

$$\begin{aligned}
\mathbf{D}'_3 &= \left(\frac{\boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}}}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}}\right)' = \frac{1}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}}\left[(\boldsymbol{R}_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}})' - \mathbf{D}_3(\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6})'\right] \\
&= \frac{\boldsymbol{R}'_{i3}\boldsymbol{R}_{i6}^{\mathrm{T}} + \boldsymbol{R}_{i3}(\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega}_{gi})^T}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}} - \mathbf{D}_3\frac{\boldsymbol{R}'_{i3} \cdot \boldsymbol{R}_{i6} + \boldsymbol{R}_{i3} \cdot (\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\boldsymbol{\omega} - \hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5}\boldsymbol{\omega}_{gi})}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}}
\end{aligned} \tag{A11}$$

Formula for solving $\mathbf{d}_{ij}$ is represented as below:

$$\begin{aligned}
\mathbf{d}_{ij} &= \frac{1}{\boldsymbol{R}_{i3} \cdot \boldsymbol{R}_{i6}}\left\{\boldsymbol{J}_{\omega ri}^{\mathrm{T}}\frac{\hat{\boldsymbol{\delta}}_i\hat{\mathbf{R}}_{i1}\hat{\mathbf{R}}_{i3}^2}{|\boldsymbol{R}_{i1} \times \boldsymbol{\delta}_i|}[(\hat{\boldsymbol{g}}_{ij}\boldsymbol{\delta}_{ij})\boldsymbol{R}_{i6}^T - \boldsymbol{R}_{i6}(\hat{\boldsymbol{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3]\right. \\
&\quad \left. + \left(\begin{bmatrix}\mathbf{0}_{3\times 3} \\ \mathbf{E}_{3\times 3}\end{bmatrix}\hat{\mathbf{R}}_{i4}\hat{\mathbf{R}}_{i5} - \mathbf{J}_{\omega bi}^{\mathrm{T}}\hat{\mathbf{R}}_{i5}\hat{\mathbf{R}}_{i4}\right)[(\hat{\boldsymbol{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\boldsymbol{R}_{i3} - \boldsymbol{R}_{i3}(\hat{\boldsymbol{g}}_{ij}\boldsymbol{\delta}_{ij})^{\mathrm{T}}\mathbf{D}_3]\right\}.
\end{aligned} \tag{A12}$$

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The detail derivation of formulae (29) is given as follows:

$$
\begin{aligned}
\mathbf{c}_{ij}\boldsymbol{\varepsilon}_{ri} &= \frac{\mathbf{c}_{ij}(\boldsymbol{R}_{i1}\boldsymbol{R}'^{\mathrm{T}}_{i2} - \boldsymbol{R}'_{i2}\boldsymbol{R}^{\mathrm{T}}_{i1})}{D_1}(\hat{\boldsymbol{\delta}}_i^2\boldsymbol{v} - \hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\boldsymbol{\omega}) \\
&\quad - \frac{\mathbf{c}_{ij}\boldsymbol{\omega}_{ri}}{D_1}[(v_{ri}\boldsymbol{\delta}_i^{\mathrm{T}} + r_i\boldsymbol{\omega}_{ri}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i)(\boldsymbol{R}_{i1}\times\boldsymbol{R}_{i2}) + r_i\boldsymbol{\delta}_i^{\mathrm{T}}\hat{\mathbf{R}}_{i1}\boldsymbol{R}'_{i2}] \\
&\quad + \frac{\mathbf{c}_{ij}\mathbf{D}_2}{D_1}\{\hat{\boldsymbol{\delta}}_i^2\boldsymbol{a} - \hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i\boldsymbol{\varepsilon} + [\boldsymbol{\delta}_i(\boldsymbol{\omega}_{ri}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i) - (\hat{\boldsymbol{\delta}}_i\boldsymbol{\omega}_{ri})\boldsymbol{\delta}_i^{\mathrm{T}}](\boldsymbol{v} - \hat{\boldsymbol{e}}_i\boldsymbol{\omega}) - \hat{\boldsymbol{\delta}}_i^2[(\boldsymbol{\omega}\times\boldsymbol{e}_i)\times\boldsymbol{\omega}]\} \\
&= \frac{\mathbf{c}_{ij}\mathbf{D}_2}{D_1}[\hat{\boldsymbol{\delta}}_i^2 \quad -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i]\boldsymbol{A} + \frac{\boldsymbol{R}'^{\mathrm{T}}_{i2}(\mathbf{c}_{ij}\boldsymbol{R}_{i1}\mathbf{E}_{3\times3} - \mathbf{c}_{ij}^{\mathrm{T}}\boldsymbol{R}_{i1}^{\mathrm{T}})}{D_1}[\hat{\boldsymbol{\delta}}_i^2 \quad -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i]\boldsymbol{V} \\
&\quad - \frac{\boldsymbol{\omega}_{ri}^{\mathrm{T}}\mathbf{c}_{ij}^{\mathrm{T}}\boldsymbol{\delta}_i^{\mathrm{T}}(\boldsymbol{R}_{i1}\times\boldsymbol{R}_{i2})v_{ri} + \boldsymbol{\omega}_{ri}^{\mathrm{T}}r_i\hat{\boldsymbol{\delta}}_i(\boldsymbol{R}_{i1}\times\boldsymbol{R}_{i2})\mathbf{c}_{ij}\boldsymbol{\omega}_{ri} - \boldsymbol{R}'^{\mathrm{T}}_{i2}r_i\hat{\mathbf{R}}_{i1}\boldsymbol{\delta}_i\mathbf{c}_{ij}\boldsymbol{\omega}_{ri}}{D_1} \\
&\quad + \frac{\boldsymbol{\omega}_{ri}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i(\mathbf{c}_{ij}\mathbf{D}_2\boldsymbol{\delta}_i) + \boldsymbol{\omega}_{ri}^{\mathrm{T}}\hat{\boldsymbol{\delta}}_i\mathbf{D}_2^{\mathrm{T}}\mathbf{c}_{ij}^{\mathrm{T}}\boldsymbol{\delta}_i^{\mathrm{T}}}{D_1}[\mathbf{E}_{3\times3} \quad -\hat{\boldsymbol{e}}_i]\boldsymbol{V} + \frac{\boldsymbol{\omega}^{\mathrm{T}}\hat{\boldsymbol{e}}_i s(\hat{\boldsymbol{\delta}}_i^2\mathbf{D}_2^{\mathrm{T}}\mathbf{c}_{ij}^{\mathrm{T}})\boldsymbol{\omega}}{D_1} \\
&= \frac{\mathbf{c}_{ij}\mathbf{D}_2}{D_1}[\hat{\boldsymbol{\delta}}_i^2 \quad -\hat{\boldsymbol{\delta}}_i^2\hat{\boldsymbol{e}}_i]\boldsymbol{A} + \boldsymbol{V}^{\mathrm{T}3}\mathbf{h}_{ij}\boldsymbol{V}.
\end{aligned}
\tag{A13}
$$

Here, $s(\hat{\boldsymbol{\delta}}_i^2\mathbf{D}_2^{\mathrm{T}}\mathbf{c}_{ij}^{\mathrm{T}})$ is the skew-symmetric matrix of $(\hat{\boldsymbol{\delta}}_i^2\mathbf{D}_2^{\mathrm{T}}\mathbf{c}_{ij}^{\mathrm{T}})$.

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