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THE FROOT-STEIN MODEL REVISITED

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ABSTRACT

We investigate the model of Froot & Stein (1998), a model which has very strong implications for risk management. We argue that their conclusions are too strong and need to be qualified. We also argue that their analysis is incorrect and incomplete. Specifically, there are some unusual consequences of their model, which may be linked to the chosen pricing formula.

KEYWORDS

Capital Budgeting; CAPM; Hedging; Risk Management; Sharpe Ratio

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1. Introduction

A number of models have been developed to characterise the optimal risk selection strategy for financial institutions. One particularly celebrated model is due to Froot, Scharfstein & Stein (1993) and Froot & Stein (1998), henceforth FSS and FS. FS won the best paper award in the *Journal of Financial Economics* in 1998, and both papers are quite widely cited in top finance journals.

Together these articles present a model using shareholder value as a measure to rate different strategies in a near perfect market. FSS introduces a model of the dynamics of the hedging position along with future financing and investment opportunities. One basic element of this model is that the company has some technological advantage, allowing a higher return on capital than the market would offer, although this advantage levels off as a function of capital. FS includes the dynamics of the market into this set-up, allowing a valuation of a company in the market. One major conclusion of these articles is that the production technology of a company should fully hedge all financial risk if the company operates with the shareholders'

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interests at heart [see FS, Proposition 1, p63 and Proposition 2, p64]. Of course, these papers address more general issues, like the overall capital budgeting and allocation decisions of companies, but much of the precise conclusions rely on these earlier results.

The FS result implies that, for example, in the insurance industry underwriting is what creates value, not exposure to financial risks. While this point of view is gaining acceptance in the insurance industry, it certainly was not reflected in the practical strategies observed throughout the 1990s (see Hancock *et al.*, 2001). The consequence of FS, i.e. shunning all financial exposure, has generated a lot of interest in the insurance industry, due to its huge implications for the practical management of that industry.

FS make use of the Capital Asset Pricing Model — henceforth the CAPM — for pricing risk, and of the empirically well-documented facts that payoff is marginally decreasing in the capital invested, and that capital does not flow frictionlessly in the market. Most financial analysts agree with the assumptions in the FS approach separately. What Froot & Stein have done is to combine the three assumptions into one model and draw some strong conclusions.

From a common sense point of view, the FS conclusion seems too strong. Should companies really never have a positive net investment in stocks, no matter what the relationship is between the risk premium and the volatility of stocks? Larger volatility, *ceteris paribus*, makes it less favourable for the firm to invest in stocks relative to what individuals can obtain by themselves, because the firm loses more eventually. On the other hand, a larger risk premium should, *ceteris paribus*, make it more favourable for the firm to invest in stocks relative to what individuals can obtain by themselves, because of the firm's technological advantage. There is a trade-off between risk and return for the firm as for an individual, except that there is an extra element of loss aversion to the firm. It is not clear, though, that this should lead to the total hedging outcome found by FS.

In this paper we argue that the FS framework is logically inconsistent and incomplete in its specification. This is a common problem with models involving non-linearity, and it is even argued that some versions of the CAPM itself suffer from a similar defect (see Cochrane, 2001, p165). For this reason we ignore this problem, and proceed to analyse the model according to the rules followed by the authors. We show that under their rules the main results of FS [Propositions 1 and 2] are incorrect as stated. First, an additional condition on market parameters is needed to ensure that the perfect hedging strategy is a local maximum rather than a local minimum in the bank's optimisation problem. This condition is that the Sharpe ratio of the market return be less than one, i.e. the risk premium be less than the standard deviation. When that condition is violated, it is optimal to hold some stocks. Empirically, we perhaps expect to see Sharpe ratios less than one, at least for long horizons and for usual market indices. Using 100 years of data, Dimson *et al.* (2002) obtain numbers in the 0.1-0.4 range. On the other hand, Lo (2002) points to some issues in the construction of the Sharpe ratio and finds some funds producing Sharpe ratios over one. So this case is empirically relevant. Second, we show that, even when the necessary condition for perfect hedging to be a local maximum is satisfied, the globally optimal strategy, which can be to go extremely short in the market portfolio and long in the risk free rate, can be nonsensical. This is obviously not a desirable outcome, and to fix it requires either a different pricing model than the CAPM or the imposition of non-linear constraints regarding the outside shareholder option, which would preclude simple and explicit results such as they obtained. This is best understood as a nasty side effect of the implicit linearisation used by the authors in solving the model.

2. The Froot and Stein Model

The FS model has three periods. In the first period the firm chooses its capital structure, and in the second period it chooses its financial investments/hedging position. In the final period, the firm realises its financial investments and then invests the proceeds plus some external financing in a subsequent project. This project has concave payoff to the total investment. The external financing carries convex costs. Our objective is to analyse the hedging position, so we do not consider any 'new products' as they did. We focus on just two time periods, zero and one, and two decisions: the hedging decision and the external financing decision.[1]

Assumption 1

Let at time zero the company's internal capital be K, and let it be exposed to an initial risk, which at time one will generate capital w, where w is a non-degenerate random variable with known distribution at time zero.

One could think of w as the underwriting plus investment result for an insurance company. At time one the company receives capital w, now holding the realised internal capital $K \cdot (1-\tau) + w$, where τ is a deadweight cost for holding K between time zero and time one, which could be, for example, taxes.

In the final period there are investment opportunities, and these are valued as follows.

Assumption 2

Denote by $F(x)$ the NPV at time one of present and future cash flows x in the company. Let F be given by $F(x) = f(x) - x$, where $f(x)$, a thrice differentiable function, known at time zero, is the PV at time one of the present and future cashflows. Let $f' \geq 1$ and $f'' \leq 0$. Also assume that $f(0) = 0$.

[1] We do not analyse the capital selection issue as do FS.

Funding of investments can be achieved by raising external capital at time one to cover a shortfall in the internal capital, i.e. when w comes out low.

Assumption 3

Assume that the company at time one can raise capital $e \geq 0$ by repaying in the future an amount with PV of $C(e) + e$, where C is a thrice differentiable function with $C(0) = 0$, $C' \geq 0$, and $C'' > 0$.

This means that it is proportionally more expensive to loan larger amounts than smaller ones, and that it is a negative NPV transaction. This assumption states that external funding is not frictionless for a company. Define:

$$W = K \cdot (1 - \tau) + w.$$

Then the level of internal capital after external funding I is $I = W + e$. The PV and NPV of the company at time one are, respectively, $f(I) + (-C(e) - e)$ and $f(I) - I + (-C(e) - e) + e = F(I) - C(e)$.

The company's problem in the final period is to derive the optimal external funding, i.e. maximise the NPV at time one by solving the following optimisation problem:

$$\max_e F(I) - C(e) \qquad \text{subject to} \qquad W = K \cdot (1 - \tau) + w$$
$$I = W + e \qquad e \geq 0.$$

There exists a unique solution to this optimisation problem, and the solution is described by a value function $P(W)$. It retains the same properties as F, namely that it is concave and increasing, as argued in FSS. A proof of this can be found in Høgh (2003, Lemma 6). If P is concave, the marginal return on investments must be decreasing, and the optimal level of investments must be increasing in the level of internal capital W.

At time zero the firm has to decide on its hedging policy, based on its valuation of the payoffs at this earlier time. The outcome w is composed of tradable and non-tradable risks.

Assumption 4

Assume that w can be expressed as $w = w^T + w^N$, where w^T is tradable and w^N is non-tradable in the market. Assume that w^N and w^T are normally distributed, w^N containing only non-systematic risk, and that w^T is the financial systematic risk exposure. The trading choice set consists of just the market portfolio with return r_M and the risk free asset with return R only, so that:

$$w^T = w^T(\alpha) = V \cdot [1 + (\alpha \cdot r_M + (1 - \alpha) \cdot R)]$$

where V is the total value invested at time zero.

The stochastic variable w^N represents the part of the risk in the company's production technology for which there exists no combination of tradeable assets that fully or partially hedges that risk. Note that $W = W(\alpha) = w^N + w^T(\alpha) + K \cdot (1 - \tau)$ is a function of the scalar α. This parameter represents the hedging decision that has to be chosen so as to maximise the present value of the firm at time zero.

The FS analysis is divided in two main arguments. First, FS say: "Without any real loss of generality, we can assume that prices are determined by a simple one-factor model'', i.e. the CAPM can be invoked to imply that the present value PV of P, where P is the NPV function at time one, is:

$$PV = Q(\alpha) \tag{1}$$

where:

$$Q(\alpha) = \frac{\mathrm{E}(P(W)) - \gamma \cdot \mathrm{cov}(P(W), r_M)}{1 + R}$$

and $W = \hat{I} - \hat{e}$, where $\hat{e}$ is the optimal external funding and $\hat{I}$ is the optimal level of investment, while:

$$\gamma = \frac{\mathrm{E}(r_M) - R}{\mathrm{var}(r_M)}.$$

Second, they conclude that Q has a single optimum at $\alpha = 0$. They interpret this as implying the following [FS, pp63-64].

Proposition FS

The firm will always wish to fully hedge its exposure to any tradeable risks.

This is, perhaps, the main mathematical result of their paper, and is extremely strong and far reaching.[2] In Theorem 1 we show that this proposition is false — their conclusions do not logically follow from their premises. We show that the optimal value for Q is not always $\alpha = 0$. This value can be a local maximum when certain conditions on the distribution of the risky asset holds, but when those conditions do not hold it can even be a (local) minimum. In such cases there can be a local maximum at some positive α. In addition, sometimes one even gets the implausible outcome that the global maximum occurs at $\alpha = -\infty$.

[2] The remaining analysis of the FS paper, like choice of capital and 'new products' relies on this result quite heavily.

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3. The Main Result

Define the Sharpe ratio on the market portfolio:

$$s_M = \frac{\mathrm{E}(r_M) - R}{\mathrm{s.d.}(r_M)}.$$

We can now state the following result, which is proved in the Appendix.

Proposition 1

Suppose that P is a thrice differentiable function. Then the first derivative of Q with respect to α is given by:

$$\frac{\partial}{\partial\alpha}Q(\alpha) = \alpha V^2 \frac{\mathrm{E}[P''(W)]}{1+R}[\mathrm{var}(r_M) - [\mathrm{E}(r_M) - R]^2] - \alpha^2 V^3 \frac{\mathrm{E}[P'''(W)]}{1+R}\mathrm{var}(r_M)\mathrm{E}(r_M - R). \tag{2}$$

(1) If $\mathrm{E}[P''(W)] < 0$, then $\alpha = 0$ is a local maximum for Q if $s_M < 1$ or a local minimum for Q if $s_M > 1$.
(2) Let $\hat{\alpha}$ be a critical point different from zero for Q. Then $\hat{\alpha}$ is found by solving:

$$\hat{\alpha} = \frac{\mathrm{E}[P''(W)]|_{\alpha=\hat{\alpha}}}{\mathrm{E}[P'''(W)]|_{\alpha=\hat{\alpha}}} \cdot \frac{1 - s_M^2}{V \cdot \mathrm{E}(r_M - R)}. \tag{3}$$

Proposition 1 says that the critical point $\alpha = 0$, i.e. the point where W contains no systematic risk, is not always even a local maximum.[3] Whether $\alpha = 0$ is a maximum or a minimum does not depend on the technology (i.e. P) at all, but does depend on the prospect of the market, specifically on whether $s_M < 1$ or $s_M > 1$. When $s_M > 1$, the value $\alpha = 0$ is actually a minimum, contrary to what FS claim. In that case, there can be a local maximum at some $\hat{\alpha} > 0$, provided that $\mathrm{E}[P'''(W)]|_{\alpha=\hat{\alpha}} > 0$, which can be expected for an increasing and concave function. The precise location of the solution $\hat{\alpha}$, when $\hat{\alpha} \neq 0$, depends on the technology through P. A risk averse individual would always seek some positive investment in risky assets, whereas the FS firm sometimes does and sometimes does not invest in risky assets, with the decision hinging on the prospect of the market. Finally, even if $s_M < 1$ and $\alpha = 0$ is a local maximum, $\alpha = 0$ may still not be the global maximum, as we show in the following example.

[3] This analysis relies on elementary calculus results that can be found in the textbook by Spiegel (1963).

Example 1

Suppose that $P(W)$ is given by:

$$P(W) = -\beta_1 \cdot e^{-\beta_2 \cdot W} + \beta_3 \qquad \beta_1 > 0 \quad \beta_2 > 0$$

which is consistent with the model of Section 2. Then the relation between $\mathrm{E}[P''(W)]$ and $\mathrm{E}[P'''(W)]$ is especially simple, since $P''(W) = -\beta_1 \cdot \beta_2^2 \cdot e^{-\beta_2 \cdot W}$ and $P'''(W) = \beta_1 \cdot \beta_2^3 \cdot e^{-\beta_2 \cdot W}$. This implies that $-\beta_2 \cdot \mathrm{E}[P''(W)] = \mathrm{E}[P'''(W)] \neq 0$. Therefore, for $s_M < 1$, $\alpha = 0$ is a local minimum for $Q(\alpha)$, and for $s_M > 1$, $\alpha = 0$ is a local maximum. The solution to equation (3), i.e. $\hat{\alpha} \neq 0$, is determined by V, β_2 and the market parameters only, since equation (3), in this case, is equal to:

$$\hat{\alpha} = -\frac{1}{V \cdot \beta_2} \cdot \frac{\mathrm{var}(r_M) - [\mathrm{E}(r_M) - R]^2}{\mathrm{var}(r_M) \cdot \mathrm{E}(r_M - R)} =: a.$$

There is one critical point different from $\alpha = 0$ for $Q(\alpha)$. For $V > 0$, $\mathrm{E}(r_M) > R$, and $P(W) = -e^{-W} + 1$, the function Q has the appearance shown in Figure 1.

The Q function attains its global maximum at $\alpha = -\infty$. If one also assumes that equation (1) holds, as do FS, i.e. that $Q = PV$, this says that the firm should borrow an infinite amount at the rate of return of the market portfolio and invest in the rate of return of the risk free asset. If the company

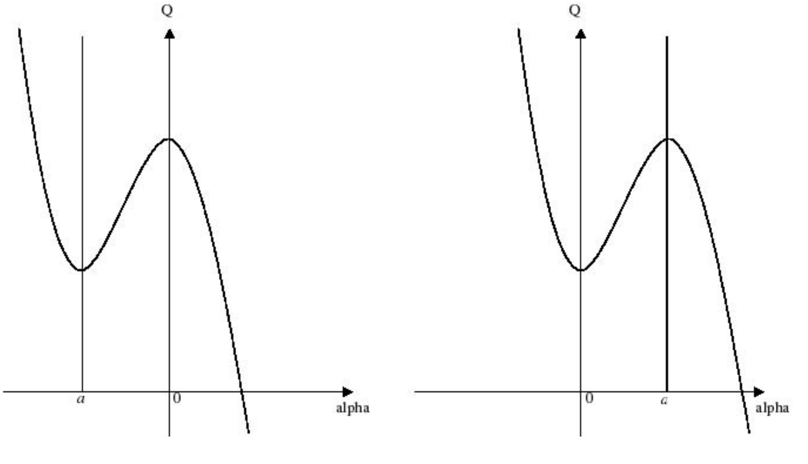

Figure 1. The function $Q(\alpha)$ for Example 1; in the first case, $s_M < 1$, while the second case has $s_M > 1$

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chooses to borrow a nearly infinite amount at the rate of return of the market portfolio and lend a nearly infinite amount at the rate of return of the risk free asset, then the resulting expected return is nearly $-\infty$, since $\mathrm{E}(r_M) \geq R$. This seems like an unrealistic consequence.

Why does this occur? To investigate this further, we have to consider the Q function more closely. With $P(W) = -e^{-W} + 1$, Q can be written as:

$$\begin{aligned}(1+R)Q(\alpha) &= \mathrm{E}[P(W)] - \gamma\,\mathrm{cov}(P(W), r_M)\\ &= \mathrm{E}[P(W)] - \mathrm{E}[P'(W)]\gamma\,\mathrm{cov}(W, r_M)\\ &= \mathrm{E}[P(W)] - \mathrm{E}[P'(W)]V\alpha\mathrm{E}(r_M - R)\\ &= \mathrm{E}[P(W)][1 + \alpha V\mathrm{E}(r_M - R)] - \alpha V\mathrm{E}(r_M - R).\end{aligned}$$

We can now see the reason why we observe the global maximum of Q for $\alpha = -\infty$:

(1) Since P is concave and increasing:

$$\mathrm{E}[P'(W)]V\mathrm{E}(r_M - R) > 0$$

so $\mathrm{E}[P(W)]$ increases in α. For α *sufficiently* negative, $\mathrm{E}[P(W)]$ is negative.

(2) $\alpha V\mathrm{E}(r_M - R)$ is increasing in α and is negative for α negative.

For α *sufficiently* negative, Q becomes a decreasing, convex function in α, and these attributes arise, among others, from P being concave. Hence, the explanation of the anomaly lies with the assumption that $PV(P) = Q$, which leads to a logical inconsistency. When we transform a symmetric random variable W by a concave transformation P, the random variable $P(W)$ cannot be symmetrically distributed. Hence, the CAPM cannot be applied to both payoffs W and $P(W)$, and so $PV(P) \neq Q$.

Furthermore, there is an additional constraint that, if $P(W) < 0$, the shareholders' limited liability put option is exercised. This constraint is ignored.

4. Conclusion

In the late 1990s, large returns on financial investments resulted in the insurance industry exposing itself to large financial risk and to cashflow underwriting. The economic approach to the valuation of companies introduced by FS apparently suggests a clear direction for what the insurance industry needs to do to improve the shareholder outcome (see Hancock *et al.*, 2001). It is therefore unfortunate that the analysis in the innovative paper of FS is incorrect and incomplete. We suggest three possible paths to follow:

(1) One could extend the two-moments CAPM to a three-moments CAPM, thereby eliminating the systematic error made when using the two-moments CAPM together with $P(W)$, since there are no longer any restrictions distribution-wise on P.
(2) One could use arbitrage pricing theory, like Black-Scholes, to find the present value of P inside a continuous time model instead.
(3) One could still use the two-moments CAPM as an approximation to the present value, but incorporate the shareholders' limited liability put option in the model setup of FS. That is to consider $\max[0, P(W)]$ instead of just $P(W)$, and so eliminate the optimum being sought, where shareholders, in reality, would have opted to default.

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APPENDIX

PROOF OF PROPOSITION 1

By the CAPM applied to W, Stein's Lemma (Cochrane, 2001, pp164-165), and interchanging differentiation and integration, the first derivative of $(1+R)\cdot PV\{P(W)\}$ with respect to α can be written as:

$$\begin{aligned}
(1+R)\frac{\partial}{\partial\alpha}Q(\alpha) &= \frac{\partial}{\partial\alpha}[\mathrm{E}[P(W)] - \gamma\cdot\mathrm{cov}(P(W), r_M)] \\
&= \frac{\partial}{\partial\alpha}[\mathrm{E}[P(W)] - \gamma\cdot\mathrm{E}[P'(W)]\cdot\mathrm{cov}(W, r_M)] \\
&= \frac{\partial}{\partial\alpha}[\mathrm{E}[P(W)] - \mathrm{E}[P'(W)]\cdot V\cdot\alpha\cdot\mathrm{E}(r_M - R)] \\
&= \mathrm{E}[P'(W)\cdot V\cdot(r_M - R)] \\
&\quad - \mathrm{E}[P''(W)\cdot V\cdot(r_M - R)]\cdot V\cdot\alpha\cdot\mathrm{E}(r_M - R) \\
&\quad - \mathrm{E}[P'(W)]\cdot V\cdot\mathrm{E}(r_M - R) \\
&= V^2\cdot\alpha\cdot\mathrm{E}[P''(W)]\cdot\mathrm{var}(r_M) \\
&\quad - V^2\cdot\mathrm{E}[P''(W)\cdot(r_M - R)]\cdot\alpha\cdot\mathrm{E}(r_M - R) \\
&= V^2\cdot\alpha\cdot\mathrm{E}[P''(W)]\cdot\mathrm{var}(r_M) \\
&\quad - V^2\cdot\alpha\cdot\mathrm{E}(r_M - R) \\
&\quad \cdot[\mathrm{E}[P'''(W)]\cdot\mathrm{cov}(W, r_M) + \mathrm{E}[P''(W)]\cdot\mathrm{E}(r_M - R)] \\
&= V^2\cdot\alpha\cdot\mathrm{E}[P''(W)]\cdot(\mathrm{var}(r_M) - [\mathrm{E}(r_M) - R]^2) \\
&\quad - V^3\cdot\alpha^2\cdot\mathrm{E}[P'''(W)]\cdot\mathrm{var}(r_M)\cdot\mathrm{E}(r_M - R).
\end{aligned}$$

This is as stated in equation (2). Clearly, $\alpha = 0$ is a critical point.

Since P is concave, $P'' \leq 0$ everywhere. Therefore:

$$\begin{aligned}
\frac{\partial^2}{\partial\alpha^2}Q(\alpha)|_{\alpha=0} &= \frac{\mathrm{E}[P''(W)]}{1+R}\cdot V^2\cdot(\mathrm{var}(r_M) - [\mathrm{E}(r_M) - R]^2) \\
&\begin{cases} > 0 & \text{if } \Delta < 0 \text{ and } \mathrm{E}[P''(W)]|_{\alpha=0} < 0 \\ = 0 & \text{if } \Delta = 0 \text{ and } \mathrm{E}[P''(W)]|_{\alpha=0} = 0 \\ < 0 & \text{if } \Delta > 0 \text{ and } \mathrm{E}[P''(W)]|_{\alpha=0} < 0 \end{cases}
\end{aligned} \tag{4}$$

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where $\Delta = \mathrm{var}(r_M) - [\mathrm{E}(r_M) - R]^2$. Critical points different from zero solve the equation:

$$\hat{\alpha} = \frac{\mathrm{E}[P''(W)]|_{\alpha=\hat{\alpha}}}{V \cdot \mathrm{E}[P'''(W)]|_{\alpha=\hat{\alpha}}} \cdot \frac{\mathrm{var}(r_M) - [\mathrm{E}(r_M) - R]^2}{\mathrm{var}(r_M) \cdot \mathrm{E}(r_M - R)}$$

which is as stated in (3) on dividing through.