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Removable Circuits in Binary Matroids

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We show that, if M is a connected binary matroid of cogirth at least five which does not have both an F_7-minor and an F_7^*-minor, then M has a circuit C such that $M - C$ is connected and $r(M - C) = r(M)$.

1. Introduction

We shall consider the problem of finding sufficient conditions for the existence of a circuit in a given matroid M whose deletion leaves the rank or connectivity of M unchanged. The existence of such a circuit in graphs has been considered by various authors. The most general result for simple graphs can be deduced from a theorem of Mader [6, Satz 1].

Theorem 1.1. *Let k be a positive integer and G be a simple k-connected graph of minimum degree at least $k + 2$. Then G has a circuit C such that $G - E(C)$ is k-connected.*

Stronger results for the special case when G is simple and $k = 2$ can be found in Jackson [4], Thomassen and Toft [11], and Lemos and Oxley [5].

It seems natural to ask if Theorem 1.1 can be extended to graphs which may contain multiple edges. Sinclair [10] has obtained the following results for small values of k.

Theorem 1.2. *Let $k \in \{1, 2\}$ and G be a k-connected graph of minimum degree at least $f(k)$, where $f(1) = 3$ and $f(2) = 5$. Then G has a circuit C such that $G - E(C)$ is k-connected.*

A proof of a slightly stronger result when $k = 2$ is given in Lemma 2.1 of this paper. Examples constructed by N. Robertson and later B. Jackson (see [4]), show that the value of $f(2)$ given in Theorem 1.2 cannot be reduced from five to four. However, this reduction is valid for graphs which do not contain a vertex of degree four incident with two edge-disjoint 2-circuits [10], for planar graphs [1], and, more generally, graphs with no Petersen minor [2].

Oxley asked in [8, Problem 14.4.8] if the following partial extension of Theorem 1.1 when $k = 2$ is valid for binary matroids: does every connected binary matroid of girth at least three and cogirth at least four have a circuit C such that $M - C$ is connected? Lemos and Oxley [5] subsequently constructed a cographic matroid of cogirth four which shows that the answer to Oxley's question is no. It remains an open problem, however, to decide if there exists an integer $t \geqslant 5$ such that all connected binary matroids M of cogirth at least t have a circuit C such that $M - C$ is connected. We shall show in Theorem 3.1 that this assertion is true with $t = 5$ for binary matroids M which do not have both an F_7- and an F_7^*-minor. This gives a partial generalization of Theorem 1.2 for the case when $k = 2$. Our proof uses the decomposition theory of Seymour in [9], which implies that a 3-connected, vertically 4-connected binary matroid which does not have both an F_7-minor and an F_7^*-minor is either graphic or cographic, or is isomorphic to R_{10}, F_7 or F_7^*. We shall first show that our result holds for graphic and cographic matroids. We then proceed by contradiction and show that a smallest counterexample to the result would be vertically 4-connected. It then only remains to check that the result holds for matroids obtained from R_{10}, F_7 or F_7^* by parallel extensions.

2. Graphs

We shall consider finite graphs which may contain multiple edges, but no loops. We consider a connected graph G to be 2-connected if $G - v$ is connected for all $v \in V(G)$. We shall use $E_G(v)$ to denote the set of edges of G incident with a vertex v and put $d_G(v) = |E_G(v)|$. We will suppress the subscript G when it is clear to which graph we are referring. Given a circuit C of G, put $|C| = |E(C)|$.

We first obtain, in Lemma 2.1 below, a slight extension of the case $k = 2$ of Theorem 1.2. We need this extension for our inductive proof on matroids. Lemma 2.1 itself follows from a result of Sinclair [10]. We include a proof in this paper for the sake of completeness.

Lemma 2.1. *Let G be a 2-connected graph on n vertices and C_0 be a circuit of G such that $|C_0| \leqslant 3$ and $n > |C_0|$. Suppose that for all $v \in V(G) - V(C_0)$ we have $d_G(v) \geqslant 5$. Then $G - E(C_0)$ has a circuit C such that $G - E(C)$ is 2-connected.*

Proof. Suppose the lemma is false and let G be a counterexample. The hypotheses of the lemma imply that we may choose a circuit C in $G - E(C_0)$. Let $H = G - E(C)$, let B_0 be the block of H which contains C_0, and B be an end-block of H distinct from B_0. We may suppose that C has been chosen such that $|E(B)|$ is minimal. Let e be an edge of B chosen such that, if B contains a cut-vertex x of H, then e is incident with x. Since $d_G(v) \geqslant 5$ for all $v \in V(G) - V(C_0)$, at most one vertex of $B - e$ has degree less than two.

Thus we may choose a circuit C' contained in $B - e$. Using the minimality of $|E(B)|$ and the fact that G is 2-connected, we see that each end-block of $H - E(C')$ is incident with C and each component of $H - E(C')$ is incident with at least two vertices of C. Thus $G - E(C') = (H - E(C')) \cup E(C)$ is 2-connected. This contradicts the choice of G as a counterexample to the theorem. □

Given a graph G and $U \subseteq V(G)$, we use $N_G(U)$ to denote the set of vertices of $V(G) - U$ adjacent to a vertex of U and $G[U]$ to denote the subgraph of G induced by U. For $S \subseteq E(G)$, let G/S be the graph obtained from G by contracting the edges in S, and $V(S)$ the set of vertices of G incident with S.

We next show, in Lemma 2.2 below, that the case $k = 2$ of Theorem 1.2 can be extended to cographic matroids.

Lemma 2.2. *Let G be a 2-connected graph on n vertices and X_0 be a cocircuit of G such that $|X_0| \leqslant 3$ and $|E(G)| \geqslant n + |X_0| - 1$. Suppose that $G - X_0$ has girth at least five. Then there exists $v \in V(G) - V(X_0)$ such that $G/E(v)$ is 2-connected.*

Proof. Suppose the lemma is false and let G be a counterexample. The hypotheses of the lemma imply that we may choose a vertex v in $V(G) - V(X_0)$. Let $H = G/E(v)$ and x be the vertex of H corresponding to $N_G(v) \cup \{v\}$. Then x is the unique cut-vertex of H. Since $X_0 \cap E(v) = \emptyset$, X_0 is a cocircuit of H and hence is contained in a block B of H. Let $U = V(B) - x$. We may suppose that v has been chosen such that $|U|$ is maximal. Note that $N_G(U) \subseteq N_G(v)$. Furthermore, since G is 2-connected, $|N_G(U)| \geqslant 2$ and $G[U \cup N_G(U) \cup \{v\}]$ is 2-connected. Choose $v' \in V(H) - V(B)$. Then $v' \in V(G) - V(X_0)$. Let $H' = G/E(v')$ and x' be the vertex of H corresponding to $N_G(v') \cup \{v'\}$. Let B' be the block of H' containing X_0 and $U' = V(B') - x'$. Then $U \cup (N_G(U) - N_G(v'))$ is properly contained in $V(B')$. By the maximality of $|U|$ we must have $N_G(U) \subseteq N_G(v')$. Now the facts that $N_G(U) \subseteq \{v\} \cup N_G(v)$ and $|N_G(U)| \geqslant 2$ imply that $E(v) \cup E(v')$ contains a circuit of G of length at most four. This contradicts the fact that $G - X_0$ has girth at least five. □

3. Binary matroids

We shall use the following operation on binary matroids from Seymour [9]. Given binary matroids M_1 and M_2, let $M_1 \triangle M_2$ be the binary matroid with $E(M) = E(M_1) \triangle E(M_2)$ and circuits all minimal non-empty subsets of $E(M)$ of the form $C_1 \triangle C_2$, where C_i is a circuit of M_i. We refer the reader to [8] for other definitions on matroids. Our main result is as follows.

Theorem 3.1. *Let M be a connected binary matroid which does not have both an F_7-minor and an F_7^*-minor. Let C_0 be a circuit of M such that $|C_0| \leqslant 3$ and $r(M) > r(C_0)$. Suppose $|X| \geqslant 5$ for all cocircuits X of M such that $X \cap C_0 = \emptyset$. Then $M - C_0$ has a circuit C such that $M - C$ is connected and $r(M - C) = r(M)$.*

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Proof. We proceed by contradiction. Suppose the theorem is false and let M be a counterexample chosen such that $r(M)$ is as small as possible.

Claim 1. *M is vertically 3-connected.*

Proof. Suppose that M has a vertical 2-separation (S_1, S_2). Choose (S_1, S_2) such that $|S_1 \cap C_0|$ is minimal. Since $r(S_i) \geqslant 2$ we have $|S_i| \geqslant 2$. By [9, 2.6], $M = M_1' \triangle M_2'$ for minors M_1' and M_2' of M such that $2 \leqslant r(M_i') < r(M)$ and $E(M_1') \cap E(M_2') = \{f\}$. Let M_i be the parallel extension of M_i' at f by a new element g, for $1 \leqslant i \leqslant 2$. Then $C_0' = \{f, g\}$ is a 2-circuit of M_i and $E(M_i) - C_0' = S_i$. Since M is connected, each M_i is connected. Since M_i' is a minor of M, M_i is binary and does not have both an F_7-minor and an F_7^*-minor. Since $C_0' \cap E(M) = \emptyset$ we have $C_0' \cap C_0 = \emptyset$. Since $|C_0| \leqslant 3$, $|C_0 \cap E(M_1)| \leqslant 1$.

Suppose $C_0 \cap E(M_1) = \{e\}$. Then $C_0 = C_1 \triangle C_2$ for some circuits C_i of M_i, $1 \leqslant i \leqslant 2$. Thus $|C_1| = 2$ and e is parallel to f and g in M_1. Let $h \in S_1 - e$ and Y be a circuit of M which meets both S_1 and S_2. Then $Y = Y_1 \triangle Y_2$ for some circuits Y_i of M_i such that $|Y_i \cap C_0'| = 1$, $1 \leqslant i \leqslant 2$. Thus $Y_1 - C_0' + e$ is a circuit of both M_1 and M, and $r(S_1 - e) = r(S_1) \geqslant 2$. Similarly, since $e \in C_0 \subseteq S_2 + e$, we have $r(S_2 + e) = r(S_2) \geqslant 2$. Thus $(S_1 - e, S_2 + e)$ is a vertical 2-separation of M. This contradicts the minimality of $|S_1 \cap C_0|$. Hence we must have $C_0 \cap S_1 = \emptyset$.

Let X_1 be a cocircuit of M_1 such that $X_1 \cap C_0' = \emptyset$. Then X_1 is a cocircuit of M such that $X_1 \cap C_0 = \emptyset$ so, by a hypothesis of the theorem, we have $|X_1| \geqslant 5$. Using the minimality of $r(M)$ we deduce that $M_1 - C_0'$ has a circuit C such that $M_1 - C$ is connected and $r(M_1 - C) = r(M_1)$. Since $M - C = (M_1 - C) \triangle M_2$, we have that C is a circuit of $M - C_0$ such that $M - C$ is connected and $r(M - C) = r(M)$. This contradicts the choice of M. Thus M has no vertical 2-separation and hence M is vertically 3-connected. □

Claim 2. *M is vertically 4-connected.*

Proof. Suppose that M has a vertical 3-separation (S_1, S_2). Choose (S_1, S_2) such that $|S_1 \cap C_0|$ is minimal. Since $|C_0| \leqslant 3$, $|C_0 \cap S_1| \leqslant 1$. We first show that $|S_i| \geqslant 4$ for $1 \leqslant i \leqslant 2$.

Suppose $|S_i| = 3$ for some $i \in \{1, 2\}$. Since $r(S_i) \geqslant 3$ we must have $r(S_i) = 3$. Since $r(S_1) + r(S_2) - r(M) = 2$ we have $r(S_j) = r(M) - 1$, for $j = 3 - i$. Thus the closure of S_j is a hyperplane of M. The complement of this hyperplane will be a cocircuit X_0 of M contained in S_i. Since $|X_0| \leqslant |S_i| = 3$, it follows from a hypothesis of the theorem that $X_0 \cap C_0 \neq \emptyset$. Since M is binary we must have $|X_0 \cap C_0| = 2$. Since S_i is independent we must have $|C_0| = 3$ and $|S_j \cap C_0| = 1$. By the minimality of $|S_1 \cap C_0|$, we must have $i = 2$. Choosing $e_0 \in S_1 \cap C_0$ we have $r(S_1 - e) \leqslant r(S_1)$ and, since $e_0 \in C_0 \subseteq S_2 + e_0$, $r(S_2 + e_0) = r(S_2) = 3$. Thus $(S_1 - e_0, S_2 + e_0)$ is either a vertical 2-separation of M, contradicting Claim 1, or it is a vertical 3-separation of M, contradicting the minimality of $|S_1 \cap C_0|$. Thus $|S_i| \geqslant 4$ for $i \in \{1, 2\}$.

By [9, 2.9], $M = M_1 \triangle M_2$ for minors M_1 and M_2 of M such that $3 \leqslant r(M_i) < r(M)$, $E(M_1) \cap E(M_2) = C_0'$ for some 3-circuit $C_0' = \{f, g, h\}$ of M_i, and $E(M_i) - C_0' = S_i$ for $1 \leqslant i \leqslant 2$. Since M is connected, each M_i is connected. Since M_i is a minor of M, M_i is binary and does not have both an F_7- and an F_7^*-minor. Since $C_0' \cap E(M) = \emptyset$ we have $C_0' \cap C_0 = \emptyset$.

Suppose $C_0 \cap S_1 = \{e\}$. Since $e \in C_0 \subseteq S_2 + e$ we have $r(S_2 + e) = r(S_2)$. Thus

$$r(S_1 - e) + r(S_2 + e) \leqslant r(S_1) + r(S_2) = r(M) + 2.$$

Claim 1 implies that equality must hold and hence $r(S_1 - e) = r(S_1) \geqslant 3$. Thus $(S_1 - e, S_2 + e)$ is a vertical 3-separation of M. This contradicts the minimality of $|S_1 \cap C_0|$ and hence we must have $C_0 \cap E(M_1) = \emptyset$.

Let X_1 be a cocircuit of M_1 such that $X_1 \cap C_0' = \emptyset$. Then X_1 intersects all circuits of M_1 in an even number of elements. Since $M = M_1 \triangle M_2$ and $E(M_1) \cap E(M_2) = C_0'$, it follows that X_1 intersects all circuits of M in an even number of elements. Thus X_1 contains a cocircuit of M. Since $X_1 \cap C_0 = \emptyset$, it follows from a hypothesis of the theorem that $|X_1| \geqslant 5$. Using the minimality of $r(M)$ we deduce that $M_1 - C_0'$ has a circuit C such that $M_1 - C$ is connected and $r(M_1 - C) = r(M_1)$. Since $M - C = (M_1 - C) \triangle M_2$ it follows that C is a circuit of M such that $M - C$ is connected and $r(M - C) = r(M)$. This contradicts the choice of M. Thus M has no vertical 3-separation and hence M is vertically 4-connected. □

We are now ready to complete the proof of the theorem. Let M' be the simple matroid obtained by replacing all parallel classes of M by single elements. By Claims 1 and 2, M' is a 3-connected vertically 4-connected binary matroid. By [9, 7.6 and 14.3], M' is either graphic or cographic, or is isomorphic to R_{10}, F_7 or F_7^*. Thus M is either graphic or cographic, or can be obtained from R_{10}, F_7 or F_7^* by a sequence of parallel extensions. If the latter alternative holds, then, since R_{10}, F_7 and F_7^* have many cocircuits of size four, $M - C_0$ must contain a circuit C of size two. The 3-connectivity of M' now implies that $M - C$ is connected and $r(M - C) = r(M)$. Hence M is graphic or cographic. Lemmas 2.1 and 2.2 now give a contradiction to the choice of M as a counterexample to the theorem. □

4. Closing remarks

Remark 1. It follows from Theorem 1.2 that every connected graph G of minimum degree at least three has a circuit C such that $G - E(C)$ is connected. Thus every graphic matroid M of cogirth at least three has a circuit C such that $r(M) = r(M - C)$. The same result holds for a cographic matroid M of cogirth at least three. (This can be seen by considering the graph G for which M is the cographic matroid. Then G has girth at least three and the set of edges incident with any non-cut-vertex of G will give the required circuit C of M.) The result does not extend to regular matroids of cogirth at least three since it does not hold for R_{10} (which has cogirth four). However, if M is a binary matroid which does not have both an F_7- and an F_7^*-minor, and has cogirth at least five, then we may apply Theorem 3.1 to a component of M to deduce that M has a circuit C such that $r(M) = r(M - C)$.

One may hope that all binary matroids M of sufficiently high girth have a circuit C such that $r(M) = r(M - C)$. This is not the case. To see this note that $r(M) = r(M - C)$ if and only if C does not contain any cocircuit of M. Thus, if M is identically self-dual then no such circuit can exist. The assertion now follows since there exist identically self-dual binary matroids of arbitrarily high cogirth. The column matroid of the parity check matrix of the binary Reed–Muller code $R(s, 2s + 1)$, for example, is identically self-dual and has cogirth 2^{s+1}.

Remark 2. It is not true that there exists an integer t such that every connected matroid M of cogirth at least t has a circuit C such that $M - C$ is connected. This can be seen by considering the uniform matroid $U_{m,2m}$. It is still conceivable, however, that this may hold for binary matroids.

Problem 1. *Does there exist an integer t such that every connected binary matroid M of cogirth at least t has a circuit C such that $M - C$ is connected?*

A related result for arbitrary matroids has been obtained by Lemos and Oxley [5, Theorem 4.1].

Theorem 4.1. *Let M be a connected matroid satisfying $|E(M)| \geqslant 3r(M) \geqslant 3$. Then M has a circuit C such that $M - C$ is connected.*

Remark 3. We could also ask for sufficient conditions for the existence of a cocircuit in a matroid M the deletion of which preserves the connectivity of M. The following result of Seymour (see [7, Lemma 6]) is in the spirit of this paper. It is a matroid analogue of an earlier graph-theoretic result of Kaugars (see [3, p. 31]).

Lemma 4.2. *Let M be a connected binary matroid of girth and cogirth at least three. Then M has a cocircuit X such that $M - X$ is connected.*

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