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Special polynomials associated with rational solutions of the defocusing nonlinear Schrödinger equation and the fourth Painlevé equation

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Rational solutions and rational-oscillatory solutions of the defocusing nonlinear Schrödinger equation are expressed in terms of special polynomials associated with rational solutions of the fourth Painlevé equation. The roots of these special polynomials have a regular, symmetric structure in the complex plane. The rational solutions verify results of Nakamura and Hirota [*J. Phys. Soc. Japan*, **54** (1985) 491–499] whilst the rational-oscillatory solutions appear to be new solutions of the defocusing nonlinear Schrödinger equation.

1 Introduction

The nonlinear Schrödinger (NLS) equation

$$\mathrm{i}u_t + u_{xx} + 2\sigma|u|^2u = 0, \qquad \sigma = \pm 1, \tag{1.1}$$

where subscripts denote partial derivatives, is one of the most important nonlinear partial differential equations (PDEs). In 1972, Zakharov & Shabat [91] developed the inverse scattering method of solution for it. There has been considerable interest in PDEs solvable by inverse scattering, the *soliton equations*, since the discovery in 1967 by Gardner *et al.* [35] of the method for solving the initial value problem for the Korteweg-de Vries (KdV) equation

$$u_t + 6uu_x + u_{xxx} = 0. \tag{1.2}$$

In fact the NLS equation (1.1) was the second equation to be solved by inverse scattering after the KdV equation (1.2).

Prior to the discovery that the NLS equation (1.1) was solvable by inverse scattering, it had been considered by researchers in water waves [14, 15, 90] (see also Ablowitz & Segur [4, 5]). In 1973, Hasegawa & Tappert [40, 41] discussed the relevance of the NLS equation (1.1) in optical fibres and their associated solitary wave solutions. They did computer simulations to demonstrate the stability of these solitary waves and discussed how the NLS equation (1.1) described the instabilities of wave packets in fibre optics.

Hasegawa and Tappert showed that optical fibres could sustain envelope solitons – both bright and dark solitons. Bright solitons, which decay as $|x| \to \infty$, arise with anomalous (positive) dispersion for (1.1) with $\sigma = 1$, the *focusing NLS equation.* Dark solitons, which do not decay as $|x| \to \infty$, arise with normal (negative) dispersion for (1.1) with $\sigma = -1$, the *defocusing NLS equation.* These solitons propagate in the longitudinal dimension having a single mode guided in the direction perpendicular to the propagation direction. Whilst Hasegawa and Tappert were working on the subject, Zakharov & Shabat [91] published their paper on the solution of the initial value problem of the NLS equation (1.1). Although the Zakharov and Shabat paper was published in 1972, Hasegawa and Tappert were unaware of it until they had completed their studies. The results of Zakharov & Shabat [91] actually confirmed the conjecture by Hasegawa & Tappert [40, 41] of stable bright nonlinear pulse transmission of envelop light waves in optical fibres. In the early 1980s, Mollenauer *et al.* [60, 61] showed that solitons could be produced in laboratory experiments. They observed the narrowing of the light wave pulse as the input power was increased, thereby verifying the soliton phenomenon in the fibre. Optical solitons in fibres form as the nonlinearity balances the *dispersive* spreading of a guided wavepacket. See elsewhere [1, 11, 39, 52] for further details on the application of the NLS equation (1.1) in optical solitons.

The idea of studying the motion of poles of solutions of the KdV equation (1.2) is due to Kruskal [53]; see also Thickstun [81]. Airault *et al.* [10] studied the motion of the poles of rational solutions of the KdV equation (1.2) and related the motion to an integrable many-body problem, the Calogero-Moser system with constraints [2, 8, 21]. Studies of rational solutions of other soliton equations include for the classical Boussinesq system [74], the Kadomtsev-Petviashvili equation [72, 73] and the NLS equation (1.1) [42, 43, 64].

Ablowitz & Segur [3] demonstrated a close relationship between completely integrable PDEs solvable by inverse scattering and the Painlevé equations. For example, $\mathrm{P_{II}}$

$$w'' = 2w^3 + zw + \alpha, \tag{1.3}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$, α is an arbitrary constant, arises as a scaling reduction of the KdV equation (1.2) [3] and $\mathrm{P_{IV}}$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{1.4}$$

where α and β are arbitrary constants, arises as a scaling reduction of the NLS equation (1.1), see §3.1. Consequently, some special solutions can of the KdV equation (1.2) and the NLS equation (1.1) can be expressed in terms of solutions of $\mathrm{P_{II}}$ and $\mathrm{P_{IV}}$, respectively.

The six Painlevé equations ($\mathrm{P_{I}}$–$\mathrm{P_{VI}}$), were discovered by Painlevé, Gambier and their colleagues whilst studying which second order ordinary differential equations of the form $w'' = F\left(z, w, w'\right)$, where F is rational in w' and w and analytic in z, have the property that the solutions have no movable branch points; now known as the *Painlevé property* (cf. [44, Chap. 14]). The general solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known elementary functions and so require the introduction of new transcendental functions [44, 84]. Indeed, the Painlevé equations can be thought of as nonlinear analogues of the classical special functions [22, 45, 84]. However, it is well known that P_{II}–P_{VI} have rational solutions, algebraic solutions, and solutions expressed in terms of the classical special functions [9, 13, 30, 31, 32, 37, 38, 50, 57, 58, 59, 62, 63, 68, 69, 70, 71, 78].

Vorob'ev [87] and Yablonskii [89] expressed the rational solutions of P_{II} (1.3) in terms of certain special polynomials, which are now known as the *Yablonskii–Vorob'ev polynomials*. Okamoto [68] derived analogous special polynomials, which are now known as the *Okamoto polynomials*, related to some of the rational solutions of P_{IV} (1.4). Subsequently these were generalized by Noumi & Yamada [67] so that all rational solutions of P_{IV} can be expressed in terms of logarithmic derivatives of special polynomials – see § 2.2 and § 2.4. Clarkson & Mansfield [27] investigated the locations of the roots of the Yablonskii–Vorob'ev polynomials in the complex plane and showed that these roots have a very regular, approximately triangular structure. An earlier study of the distribution of the roots of the Yablonskii–Vorob'ev polynomials is given by Kametaka *et al.* [48] – see also Iwasaki *et al.* [45, p. 255, p. 339]. The structure of the (complex) roots of the special polynomials associated with rational solutions of P_{IV} is described in Clarkson [24], which either have an approximate rectangular structure and or are a combination of approximate rectangular and triangular structures. The term "approximate" is used since the patterns are not exact triangles and rectangles as the roots lie on arcs rather than straight lines.

In this paper, our interest is in the special polynomials and associated rational and rational-oscillatory solutions of the defocusing NLS equation

$$\mathrm{i}u_t = u_{xx} - 2|u|^2u. \tag{1.5}$$

As already mentioned, P_{IV} arises as a scaling reduction of the defocusing NLS equation, and so these rational and rational-oscillatory solutions of (1.5) are expressed in terms the special polynomials associated with rational solutions of P_{IV} (1.4). Specifically, it is shown that the defocusing NLS equation (1.5) has rational solutions in the form

$$u_n(x,t) = ng_n(x,t)/f_n(x,t), \tag{1.6}$$

where $g_n(x,t)$ and $f_n(x,t)$ are monic polynomials in x of degrees n^2-1 and n^2, respectively, and rational-oscillatory solutions in the form

$$\widetilde{u}_n(x,t) = \frac{\widetilde{g}_n(x,t)}{6t\widetilde{f}_n(x,t)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{1.7 a}$$

$$\widehat{u}_n(x,t) = \frac{\widehat{g}_n(x,t)}{6t\widehat{f}_n(x,t)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{1.7 b}$$

where $\widetilde{g}_n(x,t)$, $\widetilde{f}_n(x,t)$, $\widehat{g}_n(x,t)$ and $\widehat{f}_n(x,t)$ are monic polynomials in x of degrees $3n^2-2n+1$, $3n^2-2n$, $3n^2+2n+1$ and $3n^2+2n$ respectively, with coefficients that are polynomials in t. The polynomials $g_n(x,t)$ and $f_n(x,t)$ are expressed in terms of the generalized Hermite polynomials and the polynomials $\widetilde{g}_n(x,t)$, $\widetilde{f}_n(x,t)$, $\widehat{g}_n(x,t)$ and $\widehat{f}_n(x,t)$ are expressed in terms of the generalized Okamoto polynomials. The rational solutions verify results of Nakamura & Hirota [64] and Hone [43] whilst the rational-oscillatory solutions appear to be new solutions of (1.5). Plots of the zeroes and poles of the rational solution (1.6) and the rational-oscillatory solutions (1.7) are given. The polynomials $g_n(x,t)$ and

$f_n(x,t)$ are expressed in terms of Wronskians and the polynomials $\widetilde{g}_n(x,t)$, $\widetilde{f}_n(x,t)$, $\widehat{g}_n(x,t)$ and $\widehat{f}_n(x,t)$ in terms of Schur polynomials. Further, we show that the defocusing NLS equation (1.5) has generalized rational solutions in the form

$$u_n(x,t) = nG_n(x,t;\boldsymbol{\kappa}_{2n-1})/F_n(x,t;\boldsymbol{\kappa}_{2n-1}), \tag{1.8}$$

where $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ and $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ are monic polynomials in x of degrees n^2-1 and n^2, respectively, with coefficients that are polynomials in t and the parameters $\boldsymbol{\kappa}_{2n-1} = (\kappa_3, \kappa_4, \ldots, \kappa_{2n-1})$, which are arbitrary constants. The generalized rational solutions (1.8) generalize results of Hone [42, 43]. Additionally the $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ and $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ are expressed in terms of Wronskians.

This paper is organized as follows. In § 2 we review the special polynomials associated with rational solutions of $\mathrm{P_{IV}}$ (1.4) and discuss some of their properties. In § 3 we use the special polynomials discussed in § 2 to derive special polynomials and associated rational solutions of the form (1.6) and rational-oscillatory solutions of the form (1.7) for the defocusing NLS equation (1.5). In § 4 we discuss the generalized rational solutions of the form (1.8) for the defocusing NLS equation (1.5). Finally, in § 5 we discuss our results.

2 Special polynomials associated with rational solutions of $\mathrm{P_{IV}}$

2.1 Rational solutions of $\mathrm{P_{IV}}$

Rational solutions of $\mathrm{P_{IV}}$ (1.4) are summarized in the following theorem.

Theorem 2.1 $\mathrm{P_{IV}}$ *has rational solutions if and only if the parameters α and β are given by either*

$$\alpha = m, \qquad \beta = -2(2n-m+1)^2, \tag{2.1}$$

or

$$\alpha = m, \qquad \beta = -2(2n-m+\tfrac{1}{3})^2, \tag{2.2}$$

with $m,n \in \mathbb{Z}$. Further the rational solutions for these parameters are unique.

Proof See Lukashevich [54], Gromak [36] and Murata [62]; see also [13, 37, 85]. □

Simple rational solutions of $\mathrm{P_{IV}}$ are

$$w_1(z;\pm 2,-2) = \pm 1/z, \qquad w_2(z;0,-2) = -2z, \qquad w_3(z;0,-\tfrac{2}{9}) = -\tfrac{2}{3}z. \tag{2.3}$$

It is known that there are three sets of rational solutions of $\mathrm{P_{IV}}$, which have the solutions (2.3) as the simplest members. These sets are known as the "$-1/z$ hierarchy", the "$-2z$ hierarchy" and the "$-\frac{2}{3}z$ hierarchy", respectively (cf. [13]). The "$-1/z$ hierarchy" and the "$-2z$ hierarchy" form the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (2.1) and the "$-\frac{2}{3}z$ hierarchy" forms the set with parameters given by (2.2). The rational

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solutions of $\mathrm{P_{IV}}$ with parameters given by (2.1) lie at the vertexes of the "Weyl chambers" and those with parameters given by (2.2) lie at the centres of the "Weyl chamber" [85].

In a comprehensive study of properties of solutions of $\mathrm{P_{IV}}$, Okamoto [68] introduced two sets of polynomials associated with rational solutions of $\mathrm{P_{IV}}$, analogous to the Yablonskii–Vorob'ev polynomials mentioned above. Noumi and Yamada [67] generalized Okamoto's results and introduced the *generalized Hermite polynomials* $H_{m,n}$, defined in Theorem 2.2, and the *generalized Okamoto polynomials* $Q_{m,n}$, defined in Theorem 2.4; see also Clarkson [24]. Noumi & Yamada [67] expressed both the generalized Hermite polynomials and the generalized Okamoto polynomials in terms of Schur functions related to the so-called modified Kadomtsev-Petviashvili (mKP) hierarchy. Kajiwara & Ohta [49] also expressed rational solutions of $\mathrm{P_{IV}}$ in terms of Schur functions by expressing the solutions in the form of determinants; see also §2.3 and 2.5. We note that Noumi & Yamada [67] obtained their results on rational solutions of $\mathrm{P_{IV}}$ by considering the symmetric representation of $\mathrm{P_{IV}}$ given by the symmetric $\mathrm{P_{IV}}$ ($\mathrm{sP_{IV}}$) system

$$\varphi_1' + \varphi_1(\varphi_2 - \varphi_3) + 2\mu_1 = 0, \tag{2.4 a}$$

$$\varphi_2' + \varphi_2(\varphi_3 - \varphi_1) + 2\mu_2 = 0, \tag{2.4 b}$$

$$\varphi_3' + \varphi_3(\varphi_1 - \varphi_2) + 2\mu_3 = 0, \tag{2.4 c}$$

where μ_1, μ_2 and μ_3 are constants, with the constraints

$$\mu_1 + \mu_2 + \mu_3 = 1, \qquad \varphi_1 + \varphi_2 + \varphi_3 = -2z. \tag{2.4 d}$$

Eliminating φ_2 and φ_3, then $w = \varphi_1$ satisfies $\mathrm{P_{IV}}$ with $\alpha = \mu_3 - \mu_2$ and $\beta = -2\mu_1^2$. We remark that $\mathrm{sP_{IV}}$ (2.4) was derived earlier by Bureau [18, 19]; other studies of the $\mathrm{sP_{IV}}$ system (2.4) include [7, 33, 65, 66, 76, 77, 83, 86, 88].

2.2 Generalized Hermite polynomials

Here we consider the generalized Hermite polynomials $H_{m,n}$ which are defined in the following theorem.

Theorem 2.2 *Suppose* $H_{m,n}$ *satisfies the recurrence relations*

$$2mH_{m+1,n}H_{m-1,n} = H_{m,n}H''_{m,n} - \left(H'_{m,n}\right)^2 + 2mH_{m,n}^2, \tag{2.5 a}$$

$$2nH_{m,n+1}H_{m,n-1} = -H_{m,n}H''_{m,n} + \left(H'_{m,n}\right)^2 + 2nH_{m,n}^2, \tag{2.5 b}$$

with $H_{0,0} = H_{1,0} = H_{0,1} = 1$ *and* $H_{1,1} = 2z$, *then*

$$w_{m,n}^{[1]} = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(H_{m+1,n}/H_{m,n}\right)\right\}, \tag{2.6 a}$$

$$w_{m,n}^{[2]} = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(H_{m,n}/H_{m,n+1}\right)\right\}, \tag{2.6 b}$$

$$w_{m,n}^{[3]} = -2z + \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(H_{m,n+1}/H_{m+1,n}\right)\right\}, \tag{2.6 c}$$

where $w^{[j]}_{m,n} = w(z\,;\alpha^{[j]}_{m,n},\beta^{[j]}_{m,n})$ *for* $j = 1,2,3$, *are solutions of* $\mathrm{P_{IV}}$, *respectively for*

$$\alpha^{[1]}_{m,n} = 2m+n+1, \qquad \beta^{[1]}_{m,n} = -2n^2, \tag{2.7 a}$$
$$\alpha^{[2]}_{m,n} = -(m+2n+1), \qquad \beta^{[2]}_{m,n} = -2m^2, \tag{2.7 b}$$
$$\alpha^{[3]}_{m,n} = n-m, \qquad \beta^{[3]}_{m,n} = -2(m+n+1)^2. \tag{2.7 c}$$

Proof See Theorem 4.4 in Noumi & Yamada [67]; also Theorem 3.1 in Clarkson [24]. We remark that, in terms of the rational solutions (2.6), the rational solution of $\mathrm{sP_{IV}}$ (2.4) is given by $\varphi_j = w^{[j]}_{m,n}$, $j = 1,2,3$, with $\mu_1 = n$, $\mu_2 = -m-n$ and $\mu_3 = m+1$. □

The polynomials $H_{m,n}$ defined by (2.5) are called the *generalized Hermite polynomials* since $H_{m,1}(z) = H_m(z)$ and $H_{1,m}(z) = \mathrm{i}^{-m}H_m(\mathrm{i}z)$, where $H_m(z)$ is the standard Hermite polynomial defined by

$$H_m(z) = (-1)^m \exp(z^2)\frac{\mathrm{d}^m}{\mathrm{d}z^m}\left\{\exp(-z^2)\right\}$$

or alternatively through the generating function

$$\sum_{m=0}^{\infty} \frac{H_m(z)\,\xi^m}{m!} = \exp(2\xi z - \xi^2)$$

(cf. [6, 12, 80]). The rational solutions of $\mathrm{P_{IV}}$ defined by (2.6) include all solutions in the "$-1/z$" and "$-2z$" hierarchies, i.e. the set of rational solutions of $\mathrm{P_{IV}}$ with parameters given by (2.1), and can be expressed in terms of determinants whose entries are Hermite polynomials [49, 67]; see also § 2.3. These rational solutions of $\mathrm{P_{IV}}$ are special cases of the special function solutions which are expressible in terms of parabolic cylinder functions $D_\nu(\xi)$; for further details see, for example, Clarkson [24].

The polynomial $H_{m,n}$ has degree mn with integer coefficients [67]; in fact $H_{m,n}(\frac{1}{2}\zeta)$ is a monic polynomial in ζ with integer coefficients. Further $H_{m,n}$ possesses the symmetry

$$H_{n,m}(z) = \mathrm{i}^{-mn}H_{m,n}(\mathrm{i}z). \tag{2.8}$$

Additional bilinear equations satisfied by of the generalized Hermite polynomials $H_{m,n}$ are given in the following theorem.

Theorem 2.3 *The generalized Hermite polynomials* $H_{m,n}$ *satisfy the following relations*

$$\mathrm{D}_z^2 H_{m,n}\bullet H_{m,n} + 8mnH_{m+1,n-1}H_{m-1,n+1} = 0, \tag{2.9 a}$$
$$\left\{\mathrm{D}_z^2 - 2z\mathrm{D}_z + 2(m-n+1)\right\} H_{m,n}\bullet H_{m+1,n-1} = 0, \tag{2.9 b}$$

where D_z *is the Hirota operator defined by*

$$\mathrm{D}_z F(z)\bullet G(z) = \left[\left(\frac{\mathrm{d}}{\mathrm{d}z_1} - \frac{\mathrm{d}}{\mathrm{d}z_2}\right) F(z_1)G(z_2)\right]_{z_1=z_2=z}. \tag{2.10}$$

Proof The relation (2.9 *a*) follows from Proposition 4.2 and Theorem 4.4 in Noumi & Yamada [67], and (2.9 *b*) from Lemma 5.7 in Noumi & Yamada [67]. □

Setting $m = n$ in (2.9) yields

$$\mathrm{D}_z^2 H_{n,n} \bullet H_{n,n} + 8n^2 H_{n+1,n-1} H_{n-1,n+1} = 0, \tag{2.11 a}$$

$$(\mathrm{D}_z^2 - 2z\mathrm{D}_z + 2) H_{n,n} \bullet H_{n+1,n-1} = 0. \tag{2.11 b}$$

Examples of generalized Hermite polynomials and plots of the locations of their roots in the complex plane are given in Clarkson [24]. Plots of the complex roots of $H_{20,20}$ and $H_{21,19}$ are given in Figure 1. These plots, which are invariant under reflections in the real and imaginary z-axes, take the form of $m \times n$ "rectangles", though these are only approximate rectangles since the roots lie on arcs rather than straight lines as can be seen by looking at the actual values of the roots.

2.3 Determinantal representation of the generalized Hermite polynomials

The generalized Hermite polynomials $H_{m,n}$ can be expressed in determinantal form as follows:

$$H_{m,n} = c_{m,n} \mathscr{W}(H_m, H_{m+1}, \ldots, H_{m+n-1}), \qquad c_{m,n} = \prod_{j=1}^{n-1} \frac{(\frac{1}{2})^j}{j!},$$

where H_n is the Hermite polynomial and $\mathscr{W}(\varphi_1, \varphi_2, \ldots, \varphi_n)$ is the standard Wronskian defined by

$$\mathscr{W}(\varphi_1, \varphi_2, \ldots, \varphi_n) = \begin{vmatrix} \varphi_1 & \varphi_2 & \cdots & \varphi_n \\ \varphi_1^{(1)} & \varphi_2^{(1)} & \cdots & \varphi_n^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_1^{(n-1)} & \varphi_2^{(n-1)} & \cdots & \varphi_n^{(n-1)} \end{vmatrix}, \tag{2.12}$$

with $\varphi_j^{(m)} \equiv \mathrm{d}^m \varphi_j / \mathrm{d}z^m$. An alternative representation is

$$H_{m,n} = \widetilde{c}_{m,n} \begin{vmatrix} H_m & H_{m+1} & \cdots & H_{m+n-1} \\ H_{m+1} & H_{m+2} & \cdots & H_{m+n} \\ \vdots & \vdots & \ddots & \vdots \\ H_{m+n-1} & H_{m+n} & \cdots & H_{m+2n-2} \end{vmatrix}, \qquad \widetilde{c}_{m,n} = \prod_{j=1}^{n-1} \frac{(-\frac{1}{2})^j}{j!},$$

since H_n satisfies the recurrence relation $H_{n+1} = 2zH_n - H_n'$ [6, 12, 80]. The generalized Hermite polynomials $H_{m,n}$ can also be expressed in terms of Schur polynomials which are defined in Definition 2.7. For further details see Kajiwara & Ohta [49] and Noumi & Yamada [67].

2.4 Generalized Okamoto polynomials

Here we consider the generalized Okamoto polynomials $Q_{m,n}$ which were introduced by Noumi & Yamada [67] and are defined in Theorem 2.4 below. Following Clarkson [24],

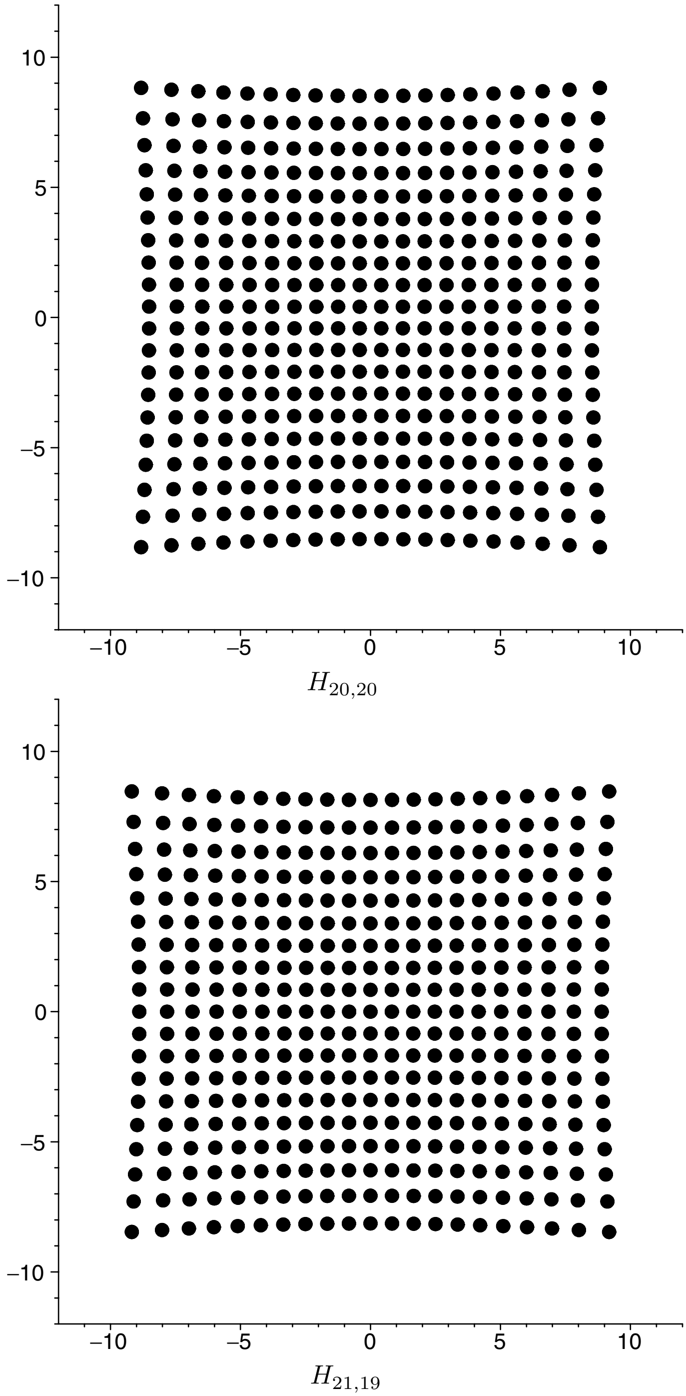

FIGURE 1. Roots of the generalized Hermite polynomials $H_{20,20}$, $H_{21,19}$.

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we reindex these polynomials by setting $Q_{m,n}^{[NY]} = Q_{m-n,n}$, i.e. $Q_{m+n,n}^{[NY]} = Q_{m,n}$, where $Q_{m+n,n}^{[NY]}$ is the polynomial defined Noumi & Yamada [67], since we feel that $Q_{m,n}$ is more natural.

Theorem 2.4 *Suppose $Q_{m,n}$ satisfies the recurrence relations*

$$Q_{m+1,n}Q_{m-1,n} = \tfrac{9}{2}\left[Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right] + \left[2z^2 + 3(2m+n-1)\right]Q_{m,n}^2, \qquad (2.13\,a)$$

$$Q_{m,n+1}Q_{m,n-1} = \tfrac{9}{2}\left[Q_{m,n}Q''_{m,n} - \left(Q'_{m,n}\right)^2\right] + \left[2z^2 + 3(1-m-2n)\right]Q_{m,n}^2, \qquad (2.13\,b)$$

with $Q_{0,0} = Q_{1,0} = Q_{0,1} = 1$ and $Q_{1,1} = \sqrt{2}\,z$, then

$$\widetilde{w}_{m,n}^{[1]} = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(Q_{m+1,n}/Q_{m,n}\right)\right\}, \qquad (2.14\,a)$$

$$\widetilde{w}_{m,n}^{[2]} = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(Q_{m,n}/Q_{m,n+1}\right)\right\}, \qquad (2.14\,b)$$

$$\widetilde{w}_{m,n}^{[3]} = -\tfrac{2}{3}z + \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left(Q_{m,n+1}/Q_{m+1,n}\right)\right\}, \qquad (2.14\,c)$$

where $\widetilde{w}_{m,n}^{[j]} = w(z;\widetilde{\alpha}_{m,n}^{[j]},\widetilde{\beta}_{m,n}^{[j]})$ for $j = 1,2,3$, are solutions of P_{IV}, respectively for

$$\widetilde{\alpha}_{m,n}^{[1]} = 2m+n, \qquad \widetilde{\beta}_{m,n}^{[1]} = -2(n-\tfrac{1}{3})^2, \qquad (2.15\,a)$$

$$\widetilde{\alpha}_{m,n}^{[2]} = -(m+2n), \qquad \widetilde{\beta}_{m,n}^{[2]} = -2(m-\tfrac{1}{3})^2, \qquad (2.15\,b)$$

$$\widetilde{\alpha}_{m,n}^{[3]} = n-m, \qquad \widetilde{\beta}_{m,n}^{[3]} = -2(m+n+\tfrac{1}{3})^2. \qquad (2.15\,c)$$

Proof See Theorem 4.3 in Noumi & Yamada [67]; also Theorem 4.1 in Clarkson [24]. We remark that, in terms of the rational solutions (2.14), the rational solution of sP_{IV} (2.4) is given by $\varphi_j = \widetilde{w}_{m,n}^{[j]}$, $j = 1,2,3$, with $\mu_1 = n - \frac{1}{3}$, $\mu_2 = -m-n+\frac{2}{3}$ and $\mu_3 = m + \frac{2}{3}$. □

The polynomials $Q_{m,n}$ defined by (2.13) are called the *generalized Okamoto polynomials* since Okamoto [68] defined the polynomials in the special cases when $n = 0$ and $n = 1$. The rational solutions of P_{IV} defined by (2.14) include all solutions in the "$-\frac{2}{3}z$" hierarchy, i.e. the set of rational solutions of P_{IV} with parameters given by (2.2), which also can be expressed in the form of determinants [49, 67]; see also § 2.5.

The polynomial $Q_{m,n}$ has degree $d_{m,n} = m^2 + n^2 + mn - m - n$ with integer coefficients [67]; in fact $Q_{m,n}(\frac{1}{2}\sqrt{2}\,\zeta)$ is a monic polynomial in ζ with integer coefficients. Further $Q_{m,n}$ possesses the symmetries

$$Q_{n,m}(z) = \exp(-\tfrac{1}{2}\pi \mathrm{i} d_{m,n})\, Q_{m,n}(\mathrm{i}z), \qquad (2.16\,a)$$

$$Q_{1-m-n,n}(z) = \exp(-\tfrac{1}{2}\pi \mathrm{i} d_{m,n})\, Q_{m,n}(\mathrm{i}z). \qquad (2.16\,b)$$

Note that $d_{m,n} = m^2 + n^2 + mn - m - n$ satisfies $d_{m,n} = d_{n,m} = d_{1-m-n,n}$.

Additional bilinear equations satisfied by the generalized Okamoto polynomials $Q_{m,n}$ are given in the following theorem.

Theorem 2.5 *The generalized Okamoto polynomials* $Q_{m,n}$ *satisfy the following relations:*

$$\left\{D_z^2 + \tfrac{8}{9}z^2 - \tfrac{4}{3}(m-n)\right\} Q_{m,n} \bullet Q_{m,n} = \tfrac{4}{9}Q_{m+1,n-1}Q_{m-1,n+1}, \qquad (2.17\,a)$$

$$\left\{D_z^2 + \tfrac{2}{3}zD_z + \tfrac{2}{3}(m-n+1)\right\} Q_{m,n} \bullet Q_{m+1,n-1} = 0, \qquad (2.17\,b)$$

Proof The relation (2.17 *a*) follows from Theorem 4.3 in Noumi & Yamada [67], and (2.17 *b*) from Lemma 5.7 in Noumi & Yamada [67]. □

Setting $m = n$ in (2.17) yields

$$(D_z^2 + \tfrac{8}{9}z^2)Q_{n,n} \bullet Q_{n,n} = \tfrac{4}{9}Q_{n+1,n-1}Q_{n-1,n+1}, \qquad (2.18\,a)$$

$$(D_z^2 + \tfrac{2}{3}zD_z + \tfrac{2}{3})Q_{n,n} \bullet Q_{n+1,n-1} = 0. \qquad (2.18\,b)$$

Examples of generalized Okamoto polynomials and plots of the locations of their complex roots are given in Clarkson [24]. Plots of the complex roots of $Q_{10,10}$ and $Q_{11,9}$ are given in Figure 2 and plots of the complex roots of $Q_{-8,-8}$ and $Q_{-9,-7}$ are given in Figure 3. The roots of the polynomial $Q_{m,n}$, with $m, n \geqslant 1$, take the form of $m \times n$ "rectangle" with an "equilateral triangle", which have either $m-1$ or $n-1$ roots, on each of its sides. The roots of the polynomial $Q_{-m,-n}$, with $m, n \geqslant 1$, take the form of $m \times n$ "rectangle" with an "equilateral triangle", which now have either m or n roots, on each of its sides. These are only approximate rectangles and equilateral triangles as can be seen by looking at the actual values of the roots. We remark that as for the generalized Hermite polynomials above, the plots are invariant under reflections in the real and imaginary z-axes.

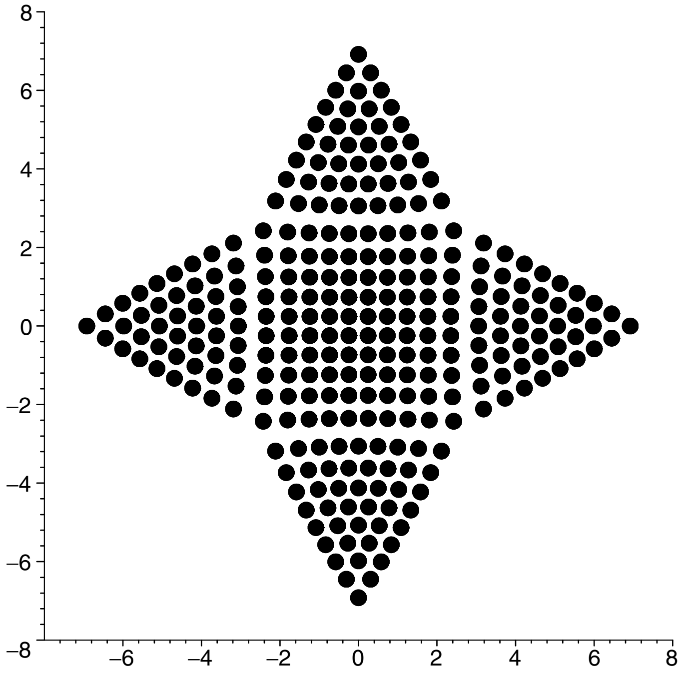
Due to the symmetries (2.16), the roots of the polynomials $Q_{-m,n}$ and $Q_{m,-n}$, with $m, n \geqslant 1$ take similar forms as these polynomials they can be expressed in terms of $Q_{M,N}$ and $Q_{-M,-N}$ for suitable $M, N \geqslant 1$. Specifically, the roots of the polynomial $Q_{-m,n}$, with $m \geqslant n \geqslant 1$, has the form of a $n \times (m-n+1)$ "rectangle" with an "equilateral triangle", which have either $n-1$ or $n-m-1$ roots, on each of its sides. Also the roots of the polynomial $Q_{-m,n}$ with $n > m \geqslant 1$, has the form of a $m \times (n-m-1)$ "rectangle" with an "equilateral triangle", which have either m or $n-m-1$ roots, on each of its sides. Further, we note that $Q_{-m,m} = Q_{m,1}$ and $Q_{1-m,m} = Q_{m,0}$, for all $m \in \mathbb{Z}$, where $Q_{m,0}$ and $Q_{m,0}$ are the original polynomials introduced by Okamoto [68]. Analogous results hold for $Q_{m,-n}$, with $m, n \geqslant 1$.

2.5 Determinantal representation of the generalized Okamoto polynomials

To describe the determinantal representation of the generalized Okamoto polynomials, we first recall the definition of the *Schur polynomial* $S_{\boldsymbol{\lambda}}(\boldsymbol{x})$, where $\boldsymbol{x} = (x_1, x_2, \ldots)$, associated with the partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Definition 2.6 A *partition* $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, or a *Young diagram*, is a sequence of descending non-negative integers such that

$$\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0.$$



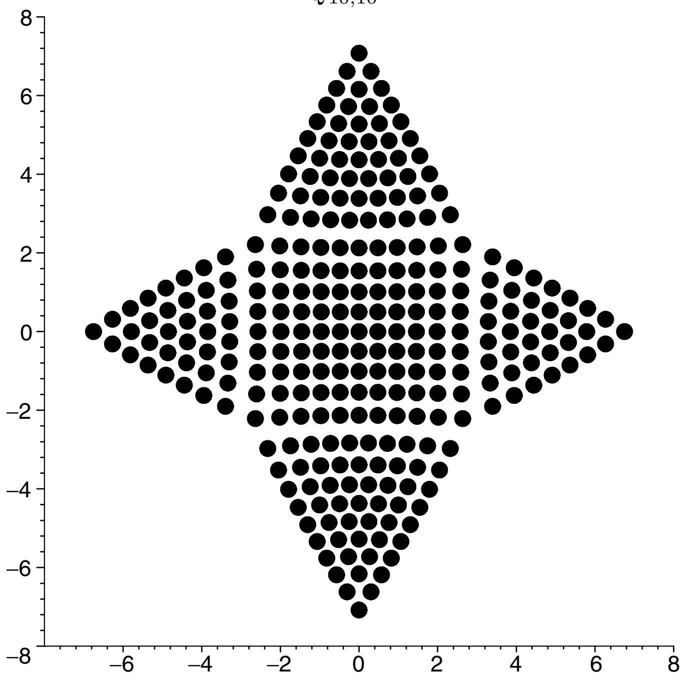


FIGURE 2. Roots of the generalized Okamoto polynomials $Q_{10,10}$ and $Q_{11,9}$.

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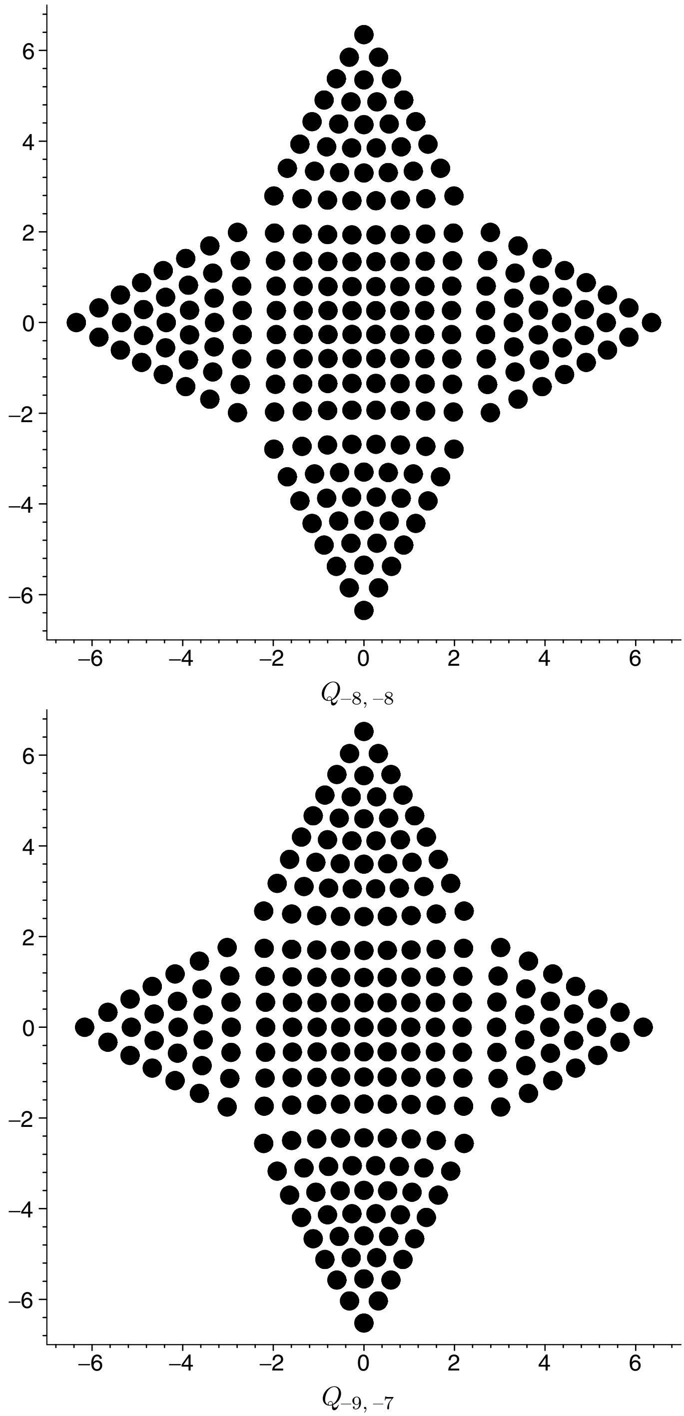

FIGURE 3. Roots of the generalized Okamoto polynomials $Q_{-8,-8}$ and $Q_{-9,-7}$.

(Some definitions require that a partition is an infinite sequence of descending non-zero numbers $(\lambda_1, \lambda_2, \dots)$ with $\lambda_j = 0$ for j sufficiently large.) The numbers λ_j are called the *parts* of $\boldsymbol{\lambda}$. The number of non-zero parts λ_j is called the *length*, denoted by $\ell(\boldsymbol{\lambda})$; here $\ell(\boldsymbol{\lambda}) = n$. The sum of the parts is called the *weight*, denoted by $|\boldsymbol{\lambda}|$; here $|\boldsymbol{\lambda}| = \sum_{j=1}^{n} \lambda_j$.

Definition 2.7 The *Schur polynomial* $S_{\boldsymbol{\lambda}}(\boldsymbol{x})$, with $\boldsymbol{x} = (x_1, x_2, \dots)$, for the partition $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is defined by the determinant

$$S_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \begin{vmatrix} \varphi_{\lambda_1}(\boldsymbol{x}) & \varphi_{\lambda_1+1}(\boldsymbol{x}) & \dots & \varphi_{\lambda_1+n-1}(\boldsymbol{x}) \\ \varphi_{\lambda_2-1}(\boldsymbol{x}) & \varphi_{\lambda_2}(\boldsymbol{x}) & \dots & \varphi_{\lambda_2+n-2}(\boldsymbol{x}) \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{\lambda_n-n+1}(\boldsymbol{x}) & \varphi_{\lambda_n-n+2}(\boldsymbol{x}) & \dots & \varphi_{\lambda_n}(\boldsymbol{x}) \end{vmatrix}, \tag{2.19}$$

where the polynomials $\varphi_m(\boldsymbol{x})$ are defined by the generating function

$$\sum_{m=0}^{\infty} \varphi_m(\boldsymbol{x})\, \xi^m = \exp\left(\sum_{j=1}^{\infty} x_j \xi^j \right), \tag{2.20}$$

with $\varphi_m(\boldsymbol{x}) = 0$ for $m < 0$.

We remark that the entries $\varphi_{\lambda_1}, \varphi_{\lambda_2}, \dots, \varphi_{\lambda_n}$ on the diagonal of $S_{\boldsymbol{\lambda}}(\boldsymbol{x})$ correspond to the partition $\boldsymbol{\lambda}$. Further, the Schur polynomial $S_{\boldsymbol{\lambda}}(\boldsymbol{x})$ is a τ-function of the KP hierarchy [47]. From the definition (2.20) it follows that

$$\frac{\partial^j \varphi_m}{\partial {x_1}^j} = \varphi_{m-j} = \frac{\partial \varphi_m}{\partial x_j},$$

and so the Schur polynomial defined by (2.19) can be written as the Wronskian

$$S_{\boldsymbol{\lambda}}(\boldsymbol{x}) = \mathscr{W}_1(\varphi_{\lambda_n}, \varphi_{\lambda_{n-1}+1}, \dots, \varphi_{\lambda_2+n-2}, \varphi_{\lambda_1+n-1}),$$

where $\mathscr{W}_1(\varphi_{\lambda_n}, \varphi_{\lambda_{n-1}+1}, \dots, \varphi_{\lambda_2+n-2}, \varphi_{\lambda_1+n-1})$ is the Wronskian with respect to x_1.

For the generalized Okamoto polynomials we choose

$$\boldsymbol{x} = (2\sqrt{2}\, z, 6, 0, 0, \dots), \tag{2.21}$$

i.e. $x_1 = 2\sqrt{2}\, z$, $x_2 = 6$ and $x_j = 0$, for $j \geqslant 3$, and have representations

$$Q_{m,n}(z) = c_{m,n} S_{\boldsymbol{\lambda}(m,n)}(\boldsymbol{x}), \qquad Q_{-m,-n}(z) = c_{-m,-n} S_{\boldsymbol{\lambda}(-m,-n)}(\boldsymbol{x}),$$

where $\boldsymbol{x}$ is given by (2.21) and the partitions are given by

$$\begin{aligned} \boldsymbol{\lambda}(m,n) = (&2m+n-2, 2m+n-4, \dots, n+4, n+2, n, \\ &n-1, n-1, n-2, n-2, \dots, 2, 2, 1, 1), \end{aligned} \tag{2.22 a}$$

$$\begin{aligned} \boldsymbol{\lambda}(-m,-n) = (&2m+n, 2m+n-2, \dots, n+4, n+2, n, \\ &n, n-1, n-1, n-2, n-2, \dots, 2, 2, 1, 1), \end{aligned} \tag{2.22 b}$$

with $c_{m,n}$ and $c_{-m,-n}$ constants chosen so that $Q_{m,n}(\frac{1}{2}\sqrt{2}\,\zeta)$ and $Q_{-m,-n}(\frac{1}{2}\sqrt{2}\,\zeta)$ are monic polynomials, respectively; for further details see Kajiwara & Ohta [49] and Noumi & Yamada [67], also Clarkson [24]. Note that for $m, n \geqslant 1$ then the lengths of the partitions are

$$\ell(\boldsymbol{\lambda}(m,n)) = m + 2(n-1), \qquad \ell(\boldsymbol{\lambda}(-m,-n)) = m + 2n,$$

and also

$$\begin{aligned} |\boldsymbol{\lambda}(m,n)| &= d_{m,n} = m^2 + n^2 + mn - m - n, \\ |\boldsymbol{\lambda}(-m,-n)| &= d_{-m,-n} = m^2 + n^2 + mn + m + n. \end{aligned}$$

2.6 Rational solutions of the $\mathrm{P_{IV}}$ Hamiltonian

Here we express rational solutions of the ordinary differential equation satisfied by the Hamiltonian for $\mathrm{P_{IV}}$ in terms of the generalized Hermite and Okamoto polynomials.

The Hamiltonian for $\mathrm{P_{IV}}$ is [68]

$$\mathscr{H}_{\mathrm{IV}}(q, p, z\,; \theta_0, \theta_\infty) = 2qp^2 - (q^2 + 2zq + 2\theta_0)p + \theta_\infty q,$$

then from Hamilton's equation we have

$$q' = \frac{\partial \mathscr{H}_{\mathrm{IV}}}{\partial p} = 4qp - q^2 - 2zq - 2\theta_0, \tag{2.23 a}$$

$$p' = -\frac{\partial \mathscr{H}_{\mathrm{IV}}}{\partial q} = -2p^2 + 2pq + 2zp - \theta_\infty. \tag{2.23 b}$$

Eliminating p in (2.23), then $q = w$ satisfies $\mathrm{P_{IV}}$ with $\alpha = 1 - \theta_0 + 2\theta_\infty$ and $\beta = -2\theta_0^2$, and eliminating q in (2.23), then $w = -2p$ satisfies $\mathrm{P_{IV}}$ with $\alpha = -1 + 2\theta_0 - \theta_\infty$ and $\beta = -2\theta_\infty^2$. The Hamiltonian function $\sigma(z\,; \theta_0, \theta_\infty) = \mathscr{H}_{\mathrm{IV}}(q, p, z\,; \theta_0, \theta_\infty)$ satisfies

$$\left(\sigma''\right)^2 = 4\left(z\sigma' - \sigma\right)^2 - 4\sigma'\left(\sigma' + 2\theta_0\right)\left(\sigma' + 2\theta_\infty\right), \tag{2.24}$$

[68, 46]. This is equivalent to equation SD-I.c in the classification of second-order, second-degree ordinary differential equations with the Painlevé property due to Cosgrove & Scoufis [28], an equation first derived and solved by Chazy [20] and rederived by Bureau [17]. Further, equation (2.24) arises in various applications including random matrix theory [33, 82]. It was shown [25] that rational solutions of (2.24) have the form

$$\sigma_{m,n} = \frac{\mathrm{d}}{\mathrm{d}z} \ln H_{m,n}, \qquad \theta_0 = -n, \qquad \theta_\infty = m, \tag{2.25 a}$$

$$\widetilde{\sigma}_{m,n} = \frac{4z^3}{27} - \tfrac{2}{3}(m-n)z + \frac{\mathrm{d}}{\mathrm{d}z} \ln Q_{m,n}, \qquad \theta_0 = -n + \tfrac{1}{3}, \qquad \theta_\infty = m - \tfrac{1}{3}. \tag{2.25 b}$$

where $H_{m,n}$ and $Q_{m,n}$ are the generalized Hermite and Okamoto polynomials, respectively. Using this Hamiltonian formalism, it was shown [25] that $H_{m,n}$ and $Q_{m,n}$, which are defined by differential-difference equations (2.5) and (2.13), respectively, also satisfy fourth order bilinear ordinary differential equations and homogeneous difference equations. It seems reasonable to expect that these ordinary differential equations will be useful for the derivation of properties of $H_{m,n}$ and $Q_{m,n}$ since there are more methods for studying properties of ordinary differential equations than differential-difference equations.

3 Rational and rational-oscillatory solutions of the nonlinear Schrödinger equation

3.1 Scaling reduction of the defocusing NLS equation

The defocusing NLS equation (1.5) has the scaling reduction

$$u(x,t) = t^{-1/2}U(\zeta), \qquad \zeta = xt^{-1/2}, \tag{3.1}$$

where $U(\zeta)$ satisfies

$$\frac{d^2U}{d\zeta^2} + \tfrac{1}{2}\mathrm{i}\left(U + \zeta\frac{dU}{d\zeta}\right) = 2|U|^2U. \tag{3.2}$$

Setting $U(\zeta) = R(\zeta)\exp\{\mathrm{i}\Theta(\zeta)\}$ in (3.2) and formally equating real and imaginary parts yields

$$\frac{d^2R}{d\zeta^2} - R\left(\frac{d\Theta}{d\zeta}\right)^2 = \tfrac{1}{2}R\zeta\frac{d\Theta}{d\zeta} + 2R^3, \tag{3.3 a}$$

$$2\frac{dR}{d\zeta}\frac{d\Theta}{d\zeta} + R\frac{d^2\Theta}{d\zeta^2} + \tfrac{1}{2}\zeta\frac{dR}{d\zeta} + \tfrac{1}{2}R = 0 \tag{3.3 b}$$

(see [16, 34] for further details). Multiplying (3.3 b) by R and integrating yields

$$\frac{d\Theta}{d\zeta} = -\tfrac{1}{4}\zeta - \frac{1}{4R^2}\int^{\zeta} R^2(s)\,ds,$$

where the constant of integration is set to zero, without loss of generality. Substituting this into (3.3 a) and setting $V(\zeta) = \int^{\zeta} R^2(s)\,ds$, yields

$$2\frac{dV}{d\zeta}\frac{d^3V}{d\zeta^3} = \left(\frac{d^2V}{d\zeta^2}\right)^2 - (\tfrac{1}{4}\zeta^2)\left(\frac{dV}{d\zeta}\right)^2 + \tfrac{1}{4}V^2 + 8\left(\frac{dV}{d\zeta}\right)^3.$$

which has first integral

$$\left(\frac{d^2V}{d\zeta^2}\right)^2 = -\tfrac{1}{4}\left(V - \zeta\frac{dV}{d\zeta}\right)^2 + 4\left(\frac{dV}{d\zeta}\right)^3 + K\frac{dV}{d\zeta}, \tag{3.4}$$

with K an arbitrary constant. Making the transformation $V(\zeta) = -\tfrac{1}{2}e^{-\pi\mathrm{i}/4}W(z)$, with $\zeta = 2e^{\pi\mathrm{i}/4}z$, in (3.4) yields

$$\left(W''\right)^2 = 4\left(zW' - W\right)^2 - 4\left(W'\right)^3 + 4\kappa^2W', \tag{3.5}$$

with $\kappa^2 = 4K = \tfrac{4}{9}(\alpha+1)^2$, is the special case of (2.24) with $\theta_0 = \pm\tfrac{1}{2}\kappa$ and $\theta_\infty = \mp\tfrac{1}{2}\kappa$, and so can be solved in terms of P_{IV}, as shown in §2.6. Hence (3.4) is solvable in terms of P_{IV} provided that $K = \tfrac{1}{9}(\alpha+1)^2$ and $\beta = -\tfrac{2}{9}(\alpha+1+2\mathrm{i}\mu)^2$. Therefore, from (2.25), rational

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solutions of (3.5) have the form

$$W_n = \frac{\mathrm{d}}{\mathrm{d}z}\ln H_{n,n}, \qquad \kappa = \pm 2n \tag{3.6 a}$$

$$\widetilde{W}_n = \frac{4z^3}{27} + \frac{\mathrm{d}}{\mathrm{d}z}\ln Q_{n,n}, \qquad \kappa = \pm 2\left(n - \tfrac{1}{3}\right), \tag{3.6 b}$$

and so rational solutions of (3.4) have the form

$$V_n(\zeta) = -\frac{\mathrm{d}}{\mathrm{d}\zeta}\ln H_{n,n}\left(\tfrac{1}{2}\mathrm{e}^{-\pi\mathrm{i}/4}\zeta\right), \tag{3.7 a}$$

$$\widetilde{V}_n(\zeta) = \frac{\zeta^3}{108} - \frac{\mathrm{d}}{\mathrm{d}\zeta}\ln Q_{n,n}\left(\tfrac{1}{2}\mathrm{e}^{-\pi\mathrm{i}/4}\zeta\right). \tag{3.7 b}$$

Hence the rational solutions of (3.4) and (3.5) have a "square pattern", i.e. $m = n$.

The first few rational solutions $V_n(\zeta)$ given by (3.7 a) are

$$V_1(\zeta) = -\frac{1}{\zeta}, \qquad V_2(\zeta) = -\frac{4\zeta^3}{\zeta^4 - 12}, \qquad V_3(\zeta) = -\frac{9(\zeta^8 - 40\zeta^4 - 240)}{\zeta(\zeta^8 - 72\zeta^4 - 2160)},$$
$$V_4(\zeta) = -\frac{16\zeta^3(\zeta^{12} - 180\zeta^8 - 3600\zeta^4 - 504000)}{\zeta^{16} - 240\zeta^{12} - 7200\zeta^8 - 2016000\zeta^4 + 6048000},$$

and so the associated solutions of (3.3) are

$$\begin{aligned}
R_1(\zeta) &= 1/\zeta, \qquad \Theta_1(\zeta) = 0,\\
R_2(\zeta) &= \frac{2\zeta\sqrt{\zeta^4 + 36}}{\zeta^4 - 12}, \qquad \Theta_2(\zeta) = \tfrac{1}{2}\mathrm{i}\ln\left(\frac{\zeta^2 - 6\mathrm{i}}{\zeta^2 + 6\mathrm{i}}\right),\\
R_3(\zeta) &= \frac{3\sqrt{\zeta^{16} + 16\zeta^{12} + 15840\zeta^8 - 172800\zeta^4 + 518400}}{\zeta(\zeta^8 - 72\zeta^4 - 2160)},\\
\Theta_3(\zeta) &= \tfrac{1}{2}\mathrm{i}\ln\left(\frac{\zeta^8 - 16\mathrm{i}\zeta^6 - 120\zeta^4 + 720}{\zeta^8 + 16\mathrm{i}\zeta^6 - 120\zeta^4 + 720}\right),\\
R_4(\zeta) &= \frac{4\zeta\sqrt{\rho_4(\zeta)}}{\zeta^{16} - 240\zeta^{12} - 7200\zeta^8 - 2016000\zeta^4 + 6048000},\\
\Theta_4(\zeta) &= \tfrac{1}{2}\mathrm{i}\ln\left[\varphi_4(\zeta)/\varphi_4^*(\zeta)\right],
\end{aligned}$$

with

$$\begin{aligned}
\varphi_4(\zeta) &= \zeta^{14} + 30\mathrm{i}\zeta^{12} - 540\zeta^{10} - 4200\mathrm{i}\zeta^8 + 10800\zeta^6 + 151200\mathrm{i}\zeta^4 + 504000\zeta^2 + 3024000\mathrm{i},\\
\varphi_4^*(\zeta) &= \zeta^{14} - 30\mathrm{i}\zeta^{12} - 540\zeta^{10} + 4200\mathrm{i}\zeta^8 + 10800\zeta^6 - 151200\mathrm{i}\zeta^4 + 504000\zeta^2 - 3024000\mathrm{i},\\
\rho_4(\zeta) &= \zeta^{28} - 180\zeta^{24} + 61200\zeta^{20} + 16056000\zeta^{16} - 1516320000\zeta^{12} + 8346240000\zeta^8\\
&\quad + 1168473600000\zeta^4 + 9144576000000 \equiv \varphi_4(\zeta)\varphi_4^*(\zeta),
\end{aligned}$$

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Hence we obtain the solutions of (3.6 *b*) given by

$$U_1(\zeta) = 1/\zeta, \tag{3.9 a}$$

$$U_2(\zeta) = 2\zeta(\zeta^2+6\mathrm{i})/(\zeta^4-12), \tag{3.9 b}$$

$$U_3(\zeta) = 3\frac{\zeta^8+16\mathrm{i}\zeta^6-120\zeta^4+720}{\zeta(\zeta^8-72\zeta^4-2160)}, \tag{3.9 c}$$

$$U_4(\zeta) = \frac{4\zeta\varphi_4(\zeta)}{\zeta^{16}-240\zeta^{12}-7200\zeta^8-2016000\zeta^4+6048000}, \tag{3.9 d}$$

with $\varphi_4(\zeta)$ as given above. Substituting these into the scaling reduction (3.1) yields rational solutions of the defocusing NLS equation (1.5) – see (3.15) for examples.

Similarly, the first few rational solutions $\widetilde{V}_n(\zeta)$ given by (3.7 *b*) are

$$\widetilde{V}_0(\zeta) = \frac{\zeta^3}{108}, \qquad \widetilde{V}_1(\zeta) = \frac{\zeta^3}{108}-\frac{1}{\zeta}, \qquad \widetilde{V}_2(\zeta) = \frac{\zeta^3}{108}-\frac{8\zeta^3(\zeta^4+252)}{\zeta^8+504\zeta^4-9072},$$

$$\widetilde{V}_3(\zeta) = \frac{\zeta^3}{108}-\frac{1}{\zeta}-\frac{20\zeta^3(\zeta^{16}+3024\zeta^{12}+1905120\zeta^8+594397440\zeta^4-37447038720)}{\widetilde{\eta}_3(\zeta)},$$

where

$$\widetilde{\eta}_3(\zeta) = \zeta^{20}+3780\zeta^{16}+3175200\zeta^{12}+1485993600\zeta^8 - 187235193600\zeta^4-6740466969600,$$

and so the associated solutions of (3.3) are

$$\widetilde{R}_0(\zeta) = \tfrac{1}{6}\zeta, \qquad \widetilde{\Theta}_0(\zeta) = -\tfrac{1}{6}\mathrm{i}\zeta^2,$$

$$\widetilde{R}_1(\zeta) = \frac{\sqrt{\zeta^4+36}}{6\zeta}, \qquad \widetilde{\Theta}_1(\zeta) = -\tfrac{1}{6}\mathrm{i}\zeta^2-\tfrac{1}{2}\mathrm{i}\ln\left(\frac{\zeta^2-6\mathrm{i}}{\zeta^2+6\mathrm{i}}\right),$$

$$\widetilde{R}_2(\zeta) = \frac{\zeta\sqrt{\zeta^{16}+1296\zeta^{12}+163296\zeta^8+45722880\zeta^4+2057529600}}{\zeta^8+504\zeta^4-9072},$$

$$\widetilde{\Theta}_2(\zeta) = -\tfrac{1}{6}\mathrm{i}\zeta^2-\tfrac{1}{2}\mathrm{i}\ln\left(\frac{\zeta^8-48\mathrm{i}\zeta^6-504\zeta^4-45360}{\zeta^8+48\mathrm{i}\zeta^6-504\zeta^4-45360}\right),$$

$$\widetilde{R}_3(\zeta) = \frac{\sqrt{\widetilde{\rho}_3(\zeta)}}{6\zeta\widetilde{\eta}_3(\zeta)}, \qquad \widetilde{\Theta}_3(\zeta) = -\tfrac{1}{6}\mathrm{i}\zeta^2-\tfrac{1}{2}\mathrm{i}\ln\left[\widetilde{\varphi}_3(\zeta)/\widetilde{\varphi}_3^*(\zeta)\right],$$

with

$$\begin{aligned}\widetilde{\varphi}_3(\zeta) = {} & \zeta^{22}-126\mathrm{i}\zeta^{20}-3780\zeta^{18}-98280\mathrm{i}\zeta^{16}-7711200\zeta^{14}+148599360\mathrm{i}\zeta^{12}\\ & +891596160\zeta^{10}+8915961600\mathrm{i}\zeta^8-187235193600\zeta^6+5617055808000\mathrm{i}\zeta^4\\ & +6740466969600\zeta^2-40442801817600\mathrm{i},\\ \widetilde{\varphi}_3^*(\zeta) = {} & \zeta^{22}+126\mathrm{i}\zeta^{20}-3780\zeta^{18}+98280\mathrm{i}\zeta^{16}-7711200\zeta^{14}-148599360\mathrm{i}\zeta^{12}\\ & +891596160\zeta^{10}-8915961600\mathrm{i}\zeta^8-187235193600\zeta^6-5617055808000\mathrm{i}\zeta^4\\ & +6740466969600\zeta^2+40442801817600\mathrm{i},\\ \widetilde{\rho}_3(\zeta) = {} & \zeta^{44}+8316\zeta^{40}+23632560\zeta^{36}+32291784000\zeta^{32}+20892155558400\zeta^{28}\\ & +6592176696268800\zeta^{24}+518751730353991680 0\zeta^{20}\\ & +131899506763091558400 0\zeta^{16}+1352199255007994265600 00\zeta^{12}\\ & +28306037738167346626560000\zeta^8-408905054714417465917440000\zeta^4\\ & +1635620218857669863669760000 \equiv \widetilde{\varphi}_3(\zeta)\widetilde{\varphi}_3^*(\zeta).\end{aligned}$$

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Hence we obtain the solutions of (3.2) given by

$$\widetilde{U}_0(\zeta) = \tfrac{1}{6}\zeta \exp\left(-\tfrac{1}{6}\mathrm{i}\zeta^2\right), \tag{3.10a}$$

$$\widetilde{U}_1(\zeta) = \frac{\zeta^2 - 6\mathrm{i}}{6\zeta} \exp\left(-\tfrac{1}{6}\mathrm{i}\zeta^2\right), \tag{3.10b}$$

$$\widetilde{U}_2(\zeta) = \frac{\zeta(\zeta^8 - 48\mathrm{i}\zeta^6 - 504\zeta^4 - 45360)}{6(\zeta^8 + 504\zeta^4 - 9072)} \exp\left(-\tfrac{1}{6}\mathrm{i}\zeta^2\right), \tag{3.10c}$$

$$\widetilde{U}_3(\zeta) = \frac{\widetilde{\varphi}_3(\zeta)}{6\zeta\widetilde{\eta}_3(\zeta)} \exp\left(-\tfrac{1}{6}\mathrm{i}\zeta^2\right), \tag{3.10d}$$

with $\widetilde{\varphi}_3(\zeta)$ and $\widetilde{\eta}_3(\zeta)$ as given above. Substituting these into the scaling reduction (3.1) yields rational-oscillatory solutions of the defocusing NLS equation (1.5) – see (3.21) for examples.

3.2 Rational solutions of the defocusing NLS equation

Nakamura & Hirota [64] (see also Hone [43] and Boiti & Pempinelli [16]) state that the defocusing NLS equation (1.5) has rational solutions, which decay as $|x| \to \infty$, in the form

$$u_n(x,t) = n g_n(x,t)/f_n(x,t), \tag{3.11}$$

where $g_n(x,t)$ and $f_n(x,t)$ are monic polynomials in x of degrees n^2-1 and n^2, respectively, for $n \geqslant 1$. Further $f_n(x,t)$ is real for $x, t \in \mathbb{R}$. Actually, Nakamura and Hirota only show that the defocusing NLS equation (1.5) has solutions of the form (3.11) for $n = 1, 2, \ldots, 5$. Rational solutions of the defocusing NLS equation (1.5) are classified in the following theorem which verifies the results of Nakamura & Hirota [64] and Hone [43].

Theorem 3.1 *The defocusing NLS equation* (1.5) *has rational solutions of the form* (3.11) *where*

$$\begin{aligned} g_n(x,t) &= \exp\left\{\tfrac{1}{2}(n^2-1)(\ln t - \tfrac{1}{2}\pi\mathrm{i})\right\} H_{n+1,n-1}(z),\\ f_n(x,t) &= \exp\left\{\tfrac{1}{2}n^2(\ln t - \tfrac{1}{2}\pi\mathrm{i})\right\} H_{n,n}(z), \end{aligned} \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2t^{1/2}}, \tag{3.12}$$

and so

$$u_n(x,t) = \frac{n\,\mathrm{e}^{\pi\mathrm{i}/4}}{t^{1/2}} \frac{H_{n+1,n-1}(z)}{H_{n,n}(z)}, \qquad z = \frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2t^{1/2}}. \tag{3.13}$$

Proof Setting $U(\zeta) = ng(\zeta)/f(\zeta)$ in (3.2) yields the bilinear representation

$$\mathrm{D}_\zeta^2 f \bullet f + 2n^2 g g^* = \mu f^2, \tag{3.14a}$$

$$\left(\mathrm{D}_\zeta^2 - \tfrac{1}{2}\mathrm{i}\zeta \mathrm{D}_\zeta + \tfrac{1}{2}\mathrm{i}\right) f \bullet g = \mu f g, \tag{3.14b}$$

where μ is arbitrary. Then making the change of variables

$$g(\zeta) = \psi_n(z), \qquad g^*(\zeta) = \psi_n^*(z), \qquad f(\zeta) = \mathrm{e}^{-\pi\mathrm{i}/4}\phi_n(z),$$

with $\zeta = 2z e^{-\pi i/4}$ and letting $\mu = 0$ yields

$$\begin{aligned} &D_z^2 \phi_n \bullet \phi_n + 8n^2 \psi_n \psi_n^* = 0, \\ &(D_z^2 - 2z D_z + 2)\phi_n \bullet \psi_n = 0. \end{aligned}$$

which are equations (2.11), and so they have the solutions

$$\phi_n(z) = c_n H_{n,n}(z), \qquad \psi_n(z) = c_n H_{n+1,n-1}(z), \qquad \psi_n^*(z) = c_n H_{n-1,n+1}(z),$$

with c_n an arbitrary constant. Setting $g_n(x,t) = \psi_n(z)$ and $f_n(x,t) = t^{1/2} e^{-\pi i/4} \phi_n(z)$, with $z = \frac{1}{2} x t^{-1/2} e^{\pi i/4}$ and $c_n = \exp\left\{\frac{1}{2}(n^2-1)(\ln t - \frac{1}{2}\pi i)\right\}$, so that $g_n(x,t)$ and $f_n(x,t)$ are monic polynomials in x with coefficients that are polynomials in t, yields $g_n(x,t)$ and $f_n(x,t)$ as given in (3.12). Hence we obtain the rational solution of defocusing NLS equation (1.5) given by (3.13), as required. □

An alternative method of deriving the bilinear representation (3.14) is to make the scaling reduction

$$F(x,t) = t^{1/2} f(\zeta), \qquad G(x,t) = n g(\zeta), \qquad \zeta = x/t^{1/2}, \qquad \lambda = \mu/t,$$

in the bilinear representation of the defocusing NLS equation (1.5)

$$\begin{aligned} &D_x^2 F \bullet F + 2GG^* = \lambda F^2, \\ &(iD_t - D_x^2) F \bullet G = \lambda F G, \end{aligned}$$

which are obtained by making the transformation $u = G/F$ in (1.5).

The first few rational solutions given by (3.13) are

$$u_1(x,t) = 1/x, \tag{3.15a}$$

$$u_2(x,t) = \frac{2x(x^2+6it)}{x^4 - 12t^2}, \tag{3.15b}$$

$$u_3(x,t) = \frac{3(x^8 + 16itx^6 - 120t^2x^4 + 720t^4)}{x(x^8 - 72t^2x^4 - 2160t^4)}, \tag{3.15c}$$

$$u_4(x,t) = \frac{4g_4(x,t)}{x^{16} - 240t^2x^{12} - 7200t^4x^8 - 2016000t^6x^4 + 6048000t^8}, \tag{3.15d}$$

where

$$\begin{aligned} g_4(x,t) = x^{15} &+ 30itx^{13} - 540t^2x^{11} - 4200it^3x^9 + 10800t^4x^7 + 151200it^5x^5 \\ &+ 504000t^6x^3 + 3024000it^7x. \end{aligned}$$

Plots of the zeroes (+) and poles (∘) of $u_{15}(x,t)$, for fixed t, are given in Figure 4. We remark that $u_n(x,t)$ has n poles on the real x-axis for all t.

The polynomials $g_n(x,t)$, $f_n(x,t)$ can be expressed in terms of Wronskians using the determinantal representation of the generalized Hermite polynomials given in § 2.3. If we

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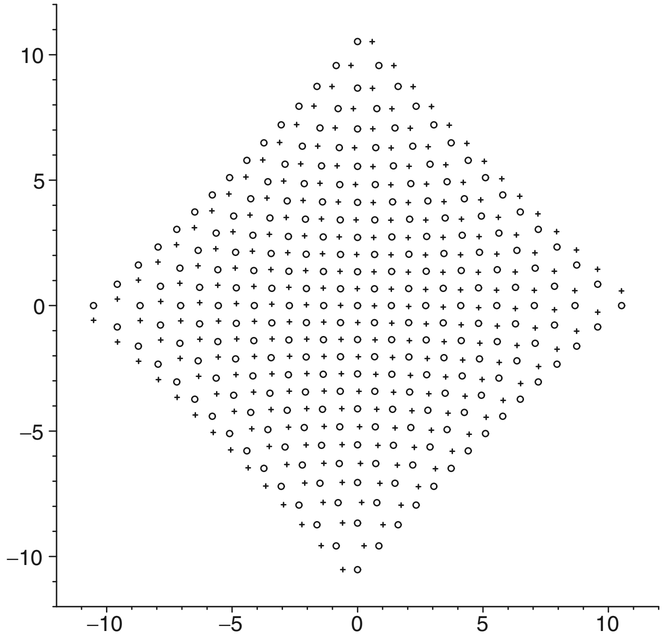

Figure 4. The zeroes (+) and poles (○) of the rational solutions $u_{15}(x,t)$.

define the polynomials $\varphi_n(x,t)$ through

$$\sum_{n=0}^{\infty} \varphi_n(x,t)\zeta^n = \exp(x\zeta - \mathrm{i}t\zeta^2), \tag{3.16}$$

then it is straightforward to show that

$$\begin{aligned} g_n(x,t) &= a_n \mathscr{W}(\varphi_{n-1}, \varphi_n, \ldots, \varphi_{2n-1}), \\ f_n(x,t) &= a_{n-1} \mathscr{W}(\varphi_n, \varphi_{n+1}, \ldots, \varphi_{2n-1}), \end{aligned} \qquad a_n = \prod_{m=1}^{n} \frac{1}{m!}, \tag{3.17}$$

where the constants a_n have been chosen so that $g_n(x,t)$ and $f_n(x,t)$ are monic polynomials in x with coefficients that are polynomials in t.

3.3 Rational-oscillatory solutions of the defocusing NLS equation

Analogously, using the rational solutions of (3.5) that are expressed in terms of the generalized Okamoto polynomials $Q_{m,n}$, i.e. (3.2), we obtain rational-oscillatory solutions of the defocusing NLS equation (1.5) in the form

$$\widetilde{u}_n(x,t) = \frac{\widetilde{g}_n(x,t)}{6t\widetilde{f}_n(x,t)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{3.18}$$

where $\widetilde{g}_n(x,t)$ and $\widetilde{f}_n(x,t)$ are monic polynomials in x of degrees $3n^2-2n+1$ and $3n^2-2n$, respectively, with coefficients that are polynomials in t for $n \geqslant 0$. Further $\widetilde{f}_n(x,t)$ is real for $x, t \in \mathbb{R}$.

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Theorem 3.2 *The defocusing NLS equation (1.5) has rational-oscillatory solutions of the form (3.18) where*

$$\begin{aligned}\widetilde{g}_n(x,t) &= \exp\left\{\tfrac{1}{2}(3n^2-2n+1)[\ln(2t)-\tfrac{1}{2}\pi\mathrm{i}]\right\} Q_{n+1,n-1}(z),\\ \widetilde{f}_n(x,t) &= \exp\left\{\tfrac{1}{2}(3n-2)n[\ln(2t)-\tfrac{1}{2}\pi\mathrm{i}]\right\} Q_{n,n}(z),\end{aligned} \qquad z=\frac{x\,\mathrm{e}^{\mathrm{i}\pi/4}}{2t^{1/2}}, \tag{3.19}$$

and so

$$\widetilde{u}_n(x,t)=\frac{\mathrm{e}^{-\pi\mathrm{i}/4}}{3\sqrt{2t}}\frac{Q_{n+1,n-1}(z)}{Q_{n,n}(z)}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \qquad z=\frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2t^{1/2}}. \tag{3.20}$$

Proof Making the change of variables

$$g(\zeta)=\widetilde{\psi}_n(z)\exp(-\tfrac{1}{6}\mathrm{i}\zeta^2), \qquad g^*(\zeta)=\widetilde{\psi}_n^*(z)\exp(\tfrac{1}{6}\mathrm{i}\zeta^2), \qquad f(\zeta)=\mathrm{e}^{\pi\mathrm{i}/4}\widetilde{\phi}_n(z),$$

with $\zeta=2z\mathrm{e}^{-\pi\mathrm{i}/4}$, and setting $\mu=\zeta^2/36$ in the bilinear equations (3.14) yields

$$\begin{aligned}(\mathrm{D}_z^2+\tfrac{8}{9}z^2)\widetilde{\phi}_n\bullet\widetilde{\phi}_n &= \tfrac{4}{9}\widetilde{\psi}_n\widetilde{\psi}_n^*,\\ (\mathrm{D}_z^2+2z\mathrm{D}_z+\tfrac{2}{3})\widetilde{\phi}_n\bullet\widetilde{\psi}_n &= 0,\end{aligned}$$

which are equations (2.18), and so they have the solutions

$$\widetilde{\phi}_n(z)=\widetilde{c}_nQ_{n,n}(z), \qquad \widetilde{\psi}_n(z)=\widetilde{c}_nQ_{n+1,n-1}(z), \qquad \widetilde{\psi}_n^*(z)=\widetilde{c}_nQ_{n-1,n+1}(z),$$

with $\widetilde{c}_n$ an arbitrary constant. Setting $\widetilde{g}_n(x,t)=\widetilde{\psi}_n(z)$ and $\widetilde{f}_n(x,t)=(2t)^{1/2}\,\mathrm{e}^{-\pi\mathrm{i}/4}\widetilde{\phi}_n(z)$, with $z=\frac{1}{2}xt^{-1/2}\,\mathrm{e}^{\pi\mathrm{i}/4}$ and $\widetilde{c}_n=\exp\left\{\frac{1}{2}(3n^2-2n+1)\left[\ln(2t)-\frac{1}{2}\pi\mathrm{i}\right]\right\}$, so that $\widetilde{g}_n(x,t)$ and $\widetilde{f}_n(x,t)$ are monic polynomials in x with coefficients that are polynomials in t, yields $\widetilde{g}_n(x,t)$ and $\widetilde{f}_n(x,t)$ as given in (3.19) and hence we obtain the rational solution of defocusing NLS equation (1.5) given by (3.20), as required. □

The first few rational-oscillatory solutions of the defocusing NLS equation (1.5) given in Theorem 3.2 are

$$\widetilde{u}_0(x,t)=\frac{x}{6t}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{3.21 a}$$

$$\widetilde{u}_1(x,t)=\frac{x^2-6\mathrm{i}t}{6xt}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{3.21 b}$$

$$\widetilde{u}_2(x,t)=\frac{x(x^8-48\mathrm{i}tx^6-504t^2x^4-45360t^4)}{6t(x^8+504t^2x^4-9072t^4)}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{3.21 c}$$

$$\widetilde{u}_3(x,t)=\frac{\widetilde{g}_3(x,t)}{6t\widetilde{f}_3(x,t)}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{3.21 d}$$

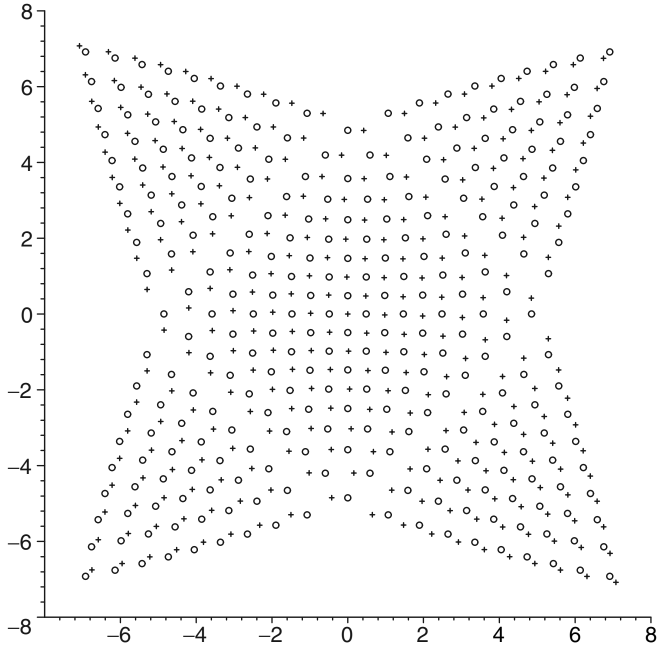

FIGURE 5. The zeroes (+) and poles (○) of the rational solutions $\widetilde{u}_{10}(x,t)$.

with

$$\begin{aligned}\widetilde{g}_3(x,t) &= x^{22} - 126itx^{20} - 3780t^2x^{18} - 98280it^3x^{16} - 7711200t^4x^{14} + 148599360it^5x^{12} \\ &\quad + 891596160t^6x^{10} + 8915961600it^7x^8 - 187235193600t^8x^6 \\ &\quad + 5617055808000it^9x^4 + 6740466969600t^{10}x^2 - 40442801817600it^{11}, \\ \widetilde{f}_3(x,t) &= x(x^{20} + 3780t^2x^{16} + 3175200t^4x^{12} + 1485993600t^6x^8 - 187235193600t^8x^4 \\ &\quad - 6740466969600t^{10}).\end{aligned}$$

The solution (3.21 a) is given by Hone [42, p. 123], otherwise we believe that these are new rational-oscillatory solutions of the defocusing NLS equation (1.5). Plots of the zeroes (+) and poles (○) of $\widetilde{u}_{10}(x,t)$, for fixed t, are given in Figure 5. We remark that $\widetilde{u}_n(x,t)$ has n poles on the real x-axis for all t.

Similarly, using the rational solutions that are expressed in terms of the generalized Okamoto polynomials $Q_{-m,-n}$, with $m,n>0$, we obtain further rational-oscillatory solutions in the form

$$\widehat{u}_n(x,t) = \frac{\widehat{g}_n(x,t)}{6t\widehat{f}_n(x,t)} \exp\left(-\frac{ix^2}{6t}\right), \tag{3.22}$$

where $\widehat{g}_n(x,t)$ and $\widehat{f}_n(x,t)$ are monic polynomials in x of degrees $3n^2+2n+1$ and $3n^2+2n$, respectively, with coefficients that are polynomials in t, for $n \geqslant 1$. Further $\widehat{f}_n(x,t)$ is real for $x,t \in \mathbb{R}$.

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Theorem 3.3 *The defocusing NLS equation (1.5) has rational-oscillatory solutions of the form* (3.22) *where*

$$\widehat{g}_n(x,t) = \exp\left\{\tfrac{1}{2}(3n^2+2n+1)[\ln(2t)-\tfrac{1}{2}\pi\mathrm{i}]\right\} Q_{-n-1,-n+1}(z),\qquad z=\frac{x\,\mathrm{e}^{\mathrm{i}\pi/4}}{2t^{1/2}},$$
$$\widehat{f}_n(x,t) = \exp\left\{\tfrac{1}{2}(3n+2)n[\ln(2t)-\tfrac{1}{2}\pi\mathrm{i}]\right\} Q_{-n,-n}(z),$$

and so

$$\widehat{u}_n(x,t) = \frac{\mathrm{e}^{-\pi\mathrm{i}/4}}{3\sqrt{2t}}\,\frac{Q_{-n-1,-n+1}(z)}{Q_{-n,-n}(z)}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right),\qquad z=\frac{x\,\mathrm{e}^{\pi\mathrm{i}/4}}{2t^{1/2}}, \tag{3.23}$$

where $n \geqslant 1$.

Proof The proof is analogous to that of Theorem 3.2 and so is left to the reader. □

The first few rational-oscillatory solutions defined by (3.23) are

$$\widehat{u}_1(x,t) = \frac{x^6-30\mathrm{i}tx^4-180t^2x^2+1080\mathrm{i}t^3}{6xt(x^4+180t^2)}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{3.24 a}$$
$$\widehat{u}_2(x,t) = \frac{\widehat{g}_2(x,t)}{6t\widehat{f}_2(x,t)}\exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{3.24 b}$$

where

$$\begin{aligned}\widehat{g}_2(x,t) &= x(x^{16}-96\mathrm{i}tx^{14}-2160t^2x^{12}-34560\mathrm{i}t^3x^{10}-2138400t^4x^8\\ &\quad +34214400\mathrm{i}t^5x^6+359251200t^6x^4+16166304000t^8),\\ \widehat{f}_2(x,t) &= x^{16}+2160t^2x^{12}+712800t^4x^8+256608000t^6x^4-2309472000t^8.\end{aligned}$$

Again, we believe that the rational-oscillatory solutions given in Theorem 3.3 are also new solutions of the defocusing NLS equation (1.5). Plots of the zeroes (+) and poles (∘) of $\widehat{u}_8(x,t)$, for fixed t, are given in Figure 6. We remark that $\widehat{u}u_n(x,t)$ has n poles on the real x-axis for all t.

The polynomials $\widetilde{g}_n(x,t)$, $\widetilde{f}_n(x,t)$, $\widehat{g}_n(x,t)$ and $\widehat{f}_n(x,t)$ can be expressed in terms of the Schur polynomial $S_\lambda(\boldsymbol{x})$ defined by (2.19). If we define the polynomials $\varphi_n(x,t)$ through

$$\sum_{n=0}^{\infty}\varphi_n(x,t)\zeta^n = \exp\left(x\zeta-3\mathrm{i}t\zeta^2\right), \tag{3.25}$$

then it is straightforward to show that

$$\widetilde{g}_n(x,t) = \widetilde{a}_nS_{\lambda(n+1,n-1)}(\boldsymbol{x}),\qquad \widetilde{f}_n(x,t) = \widetilde{b}_nS_{\lambda(n,n)}(\boldsymbol{x}), \tag{3.26}$$
$$\widehat{g}_n(x,t) = \widehat{a}_nS_{\lambda(-n-1,-n+1)}(\boldsymbol{x}),\qquad \widehat{f}_n(x,t) = \widehat{b}_nS_{\lambda(-n,-n)}(\boldsymbol{x}), \tag{3.27}$$

where $\boldsymbol{x} = (x,-3\mathrm{i}t^2,0,0,\dots)$ and the partitions are given by (2.22), for some constants $\widetilde{a}_n$, $\widetilde{b}_n$, $\widehat{a}_n$ and $\widehat{b}_n$ such that $\widetilde{g}_n(x,t)$, $\widetilde{f}_n(x,t)$, $\widehat{g}_n(x,t)$ and $\widehat{f}_n(x,t)$ are monic polynomials in x with coefficients that are polynomials in t.

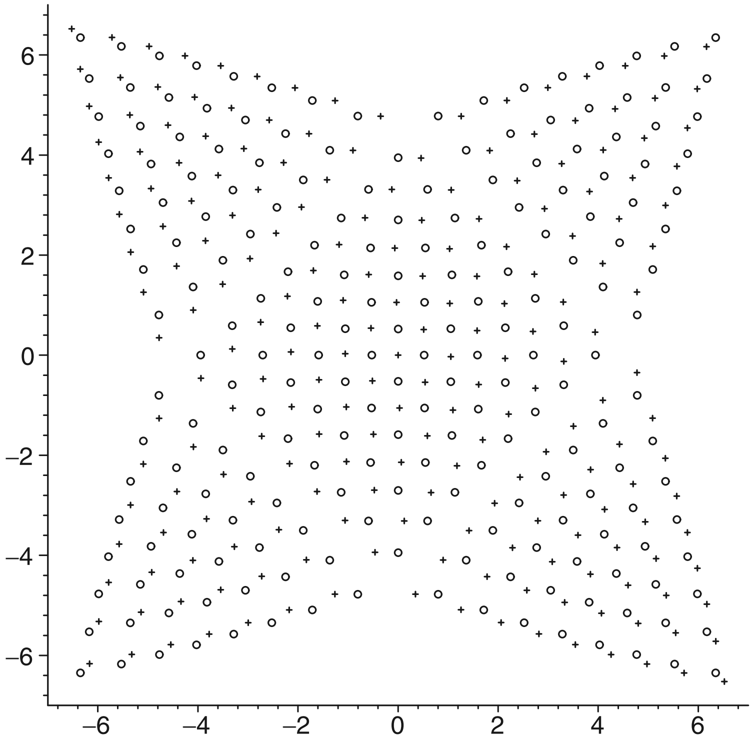

FIGURE 6. The zeroes (+) and poles (∘) of the rational solutions $\widehat{u}_8(x,t)$.

We remark that the defocusing NLS equation (1.5) has the rational-oscillatory solution

$$u(x,t) = \tfrac{1}{2}\rho e^{i(\kappa x-\omega t)}\left\{1 - \frac{4(1-i\rho^2 t)}{1-\rho^2(x-2\kappa t)^2+\rho^4 t^2}\right\}, \qquad \omega = \kappa^2 + \tfrac{1}{2}\rho^2,$$

with ρ and κ arbitrary constants, which is not of the form (3.20); for further details see Tajiri & Watanabe [79].

4 Generalized rational solutions of the nonlinear Schrödinger equation

Hone [42, 43] generalized the rational solution (3.13) showed that the defocusing NLS equation (1.5) has more general rational solutions of the form

$$u_n(x,t) = nG_n(x,t;\boldsymbol{\kappa}_{2n-1})/F_n(x,t;\boldsymbol{\kappa}_{2n-1}), \tag{4.1}$$

where $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ and $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ are monic polynomials in x of degrees n^2-1 and n^2, respectively, with coefficients that are polynomials in t and the parameters $\boldsymbol{\kappa}_{2n-1} = (\kappa_3,\kappa_4,\ldots,\kappa_{2n-1})$, with $(\kappa_3,\kappa_4,\ldots,\kappa_{2n-1})$ arbitrary constants. We remark that

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$F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ is real for $x,t\in\mathbb{R}$. The first few polynomials are

$$\begin{aligned}
G_2(x,t;\boldsymbol{\kappa}_3) &= x^3+6\mathrm{i}xt-\kappa_3,\\
F_2(x,t;\boldsymbol{\kappa}_3) &= x^4+2\kappa_3x-12t^2,\\
G_3(x,t;\boldsymbol{\kappa}_5) &= x^8+16\mathrm{i}x^6t+4\kappa_3x^5-10(12t^2-\mathrm{i}\kappa_4)x^4-4(20\mathrm{i}\kappa_3t+\kappa_5)x^3\\
&\quad+40\kappa_3^2x^2-4(60\kappa_3t^2+6\mathrm{i}\kappa_5t-5\mathrm{i}\kappa_4\kappa_3)x\\
&\quad+720t^4+120\mathrm{i}\kappa_4t^2-80\mathrm{i}\kappa_3^2t+4\kappa_5\kappa_3-5\kappa_4^2,\\
F_3(x,t;\boldsymbol{\kappa}_5) &= x^9+12\kappa_3x^6-72t^2x^5+6\kappa_5x^4-120\kappa_4tx^3+720\kappa_3t^2x^2\\
&\quad+3(720t^4-4\kappa_5\kappa_3+5\kappa_4^2)x-72\kappa_5t^2+120\kappa_3\kappa_4t-40\kappa_3^3.
\end{aligned}$$

Note that when $\boldsymbol{\kappa}_{2n-1}=\mathbf{0}$ then $F_n(x,t;\mathbf{0})=f_n(x,t)$ and $G_n(x,t;\mathbf{0})=g_n(x,t)$, where $f_n(x,t)$ and $g_n(x,t)$ are given by (3.12). Hone [42, 43] showed that the polynomials $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ and $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ can be derived recursively using Crum transformations analogous to the procedure used by Adler & Moser [8] to construct rational solutions of the KdV equation (1.2). Alternatively, the polynomials $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ and $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ can be expressed in terms of Wronskians by generalizing (3.16) and (3.17). If we define the polynomials $\Phi_n(x,t;\boldsymbol{\kappa}_n)$, with $\boldsymbol{\kappa}_n=(\kappa_3,\kappa_4,\ldots,\kappa_n)$, through

$$\sum_{n=0}^{\infty}\Phi_n(x,t;\boldsymbol{\kappa}_n)\zeta^n=\exp\left(x\zeta-\mathrm{i}t\zeta^2+\mathrm{i}\sum_{j=3}^{\infty}\frac{\kappa_j(-\mathrm{i}\zeta)^j}{j!}\right), \tag{4.2}$$

where κ_j, for $j\geqslant 3$, are arbitrary constants, then

$$\begin{aligned}
G_n(x,t;\boldsymbol{\kappa}_{2n-1}) &= a_n\mathscr{W}(\Phi_{n-1},\Phi_n,\ldots,\Phi_{2n-1}),\\
F_n(x,t;\boldsymbol{\kappa}_{2n-1}) &= a_{n-1}\mathscr{W}(\Phi_n,\Phi_{n+1},\ldots,\Phi_{2n-1}),
\end{aligned}\qquad a_n=\prod_{m=1}^{n}\frac{1}{m!}, \tag{4.3}$$

where the constants a_n have been chosen so that $G_n(x,t;\boldsymbol{\kappa}_{2n-1})$ and $F_n(x,t;\boldsymbol{\kappa}_{2n-1})$ are monic polynomials in x with coefficients that are polynomials in t.

We write the generalized rational solution (4.1) in the form

$$u_n(x,t)=n\frac{G_n(x,t;\boldsymbol{\kappa}_{2n-1})}{F_n(x,t;\boldsymbol{\kappa}_{2n-1})}\equiv\sum_{j=1}^{n^2}\frac{\psi_j(t;\boldsymbol{\kappa}_{2n-1})}{x-\varphi_j(t;\boldsymbol{\kappa}_{2n-1})},$$

to study the motion of the residues $\psi_j(t;\boldsymbol{\kappa}_{2n-1})$ and the poles $\varphi_j(t;\boldsymbol{\kappa}_{2n-1})$, for $j=1,2,\ldots,n^2$. Preliminary numerical simulations suggest the following conjecture, which it is anticipated can be verified by developing the ideas in Hone [42, 43], though we shall not pursue this further here.

Conjecture 4.1 *Generalized rational solutions of the defocusing NLS equation* (1.5) *have the form*

$$u(x,t)=\sum_{j=1}^{n}\frac{\alpha_j(t)}{x-a_j(t)}+\sum_{k=1}^{n(n-1)/2}\left\{\frac{\beta_k(t)}{x-b_k(t)}+\frac{\gamma_k(t)}{x-b_k^*(t)}\right\},$$

where $a_j(t)$ are real, $b_k^*(t)$ is the complex conjugate of $b_k(t)$ and

$$|\alpha_j(t)| = 1, \quad j = 1, 2, \ldots, n, \qquad \beta_k(t)\gamma_k^*(t) = 1, \quad k = 1, 2, \ldots, \tfrac{1}{2}n(n-1),$$

with $\gamma_k^*(t)$ the complex conjugate of $\gamma_k(t)$.

It is an open question whether, analogous to the generalized rational solutions (4.1), there are generalizations of the rational-oscillatory solutions (3.18) and (3.22) of the defocusing NLS equation (1.5) in the form

$$\widetilde{u}_n(x,t) = \frac{\widetilde{G}_n(x,t;\boldsymbol{\kappa}_n)}{6t\widetilde{F}_n(x,t;\boldsymbol{\kappa}_n)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{4.4}$$

$$\widehat{u}_n(x,t) = \frac{\widehat{G}_n(x,t;\boldsymbol{\kappa}_n)}{6t\widehat{F}_n(x,t;\boldsymbol{\kappa}_n)} \exp\left(-\frac{\mathrm{i}x^2}{6t}\right), \tag{4.5}$$

where $\widetilde{G}_n(x,t;\boldsymbol{\kappa}_n)$, $\widetilde{F}_n(x,t;\boldsymbol{\kappa}_n)$, $\widehat{G}_n(x,t;\boldsymbol{\kappa}_n)$ and $\widehat{F}_n(x,t;\boldsymbol{\kappa}_n)$ are monic polynomials in x of degrees $3n^2-2n+1$, $3n^2-2n$, $3n^2+2n+1$ and $3n^2+2n$, respectively, with coefficients that are polynomials in t and the parameters $\boldsymbol{\kappa}_n = (\kappa_3, \kappa_4, \ldots, \kappa_n)$, such that $\widetilde{G}_n(x,t;\mathbf{0}) = \widetilde{g}_n(x,t)$, $\widetilde{F}_n(x,t;\mathbf{0}) = \widetilde{f}_n(x,t)$, $\widehat{G}_n(x,t;\mathbf{0}) = \widehat{g}_n(x,t)$ and $\widehat{F}_n(x,t;\mathbf{0}) = \widehat{f}_n(x,t)$. It is shown above that the generalized rational solutions given by (4.2) and (4.3) are generalizations of (3.16) and (3.17). Hence it seems reasonable to expect that generalized rational-oscillatory solutions might be obtained by generalizing (3.25), (3.26) and (3.27). However if we define the polynomials $\Psi_n(x,t;\boldsymbol{\kappa}_n)$, with $\boldsymbol{\kappa}_n = (\kappa_3, \kappa_4, \ldots, \kappa_n)$, through

$$\sum_{n=0}^{\infty} \Psi_n(x,t;\boldsymbol{\kappa}_n)\zeta^n = \exp\left(x\zeta - 3\mathrm{i}t\zeta^2 + \mathrm{i}\sum_{j=3}^{\infty}\kappa_j(-\mathrm{i}\zeta)^j\right), \tag{4.6}$$

and consider the polynomials

$$\widetilde{G}_n(x,t;\boldsymbol{\kappa}_n) = \widetilde{a}_n S_{\boldsymbol{\lambda}(n+1,n-1)}(\boldsymbol{x}), \qquad \widetilde{F}_n(x,t;\boldsymbol{\kappa}_n) = \widetilde{b}_n S_{\boldsymbol{\lambda}(n,n)}(\boldsymbol{x}), \tag{4.7}$$

$$\widehat{G}_n(x,t;\boldsymbol{\kappa}_n) = \widehat{a}_n S_{\boldsymbol{\lambda}(-n-1,-n+1)}(\boldsymbol{x}), \qquad \widehat{F}_n(x,t;\boldsymbol{\kappa}_n) = \widehat{b}_n S_{\boldsymbol{\lambda}(-n,-n)}(\boldsymbol{x}), \tag{4.8}$$

where $\boldsymbol{x} = (x, -3\mathrm{i}t^2, -\kappa_3, \mathrm{i}\kappa_4, \ldots, (-\mathrm{i})^{j-1}\kappa_j, \ldots)$ and the partitions are given by (2.22), which is the "natural" generalization, then it seems that (4.4) and (4.5) are only solutions of the defocusing NLS equation (1.5) provided that $\boldsymbol{\kappa}_n = \mathbf{0}$.

5 Discussion

In this paper we have studied special polynomials associated with rational and rational-oscillatory solutions of the defocusing NLS equation (1.5) through special polynomials associated with rational solutions of $\mathrm{P_{IV}}$; the rational-oscillatory solutions seem to be new solutions of the defocusing NLS equation. The roots of these special polynomials are shown numerically to have a very symmetric structure in the complex plane.

The poles of rational solutions of the KdV equation (1.2) satisfy a dynamical system, a constrained Calogero-Moser system [8, 10, 21]. The zeroes and poles of the rational solutions of the defocusing NLS equation (1.5) given by (4.1) satisfy an dynamical system

[42, 43] which warrants further investigation, though we shall not pursue this further here. It is anticipated that zeroes and poles of the rational-oscillatory solutions of the defocusing NLS equation (1.5) given by (3.20) will also satisfy an interesting dynamical system, again we shall not pursue this further here. The motion of the poles of elliptic solutions of the KdV equation (1.2), which reduce to rational solutions in the limit, are discussed in Airault *et al.* [10] and Deconinck & Segur [29]. A study of the pole dynamics of elliptic solutions of the defocusing NLS equation (1.5) is another interesting open problem.

An explanation and interpretation of the numerical results for these special polynomials is an interesting open problem, as is whether they have applications, e.g. in numerical analysis? The classical orthogonal polynomials, such as Hermite, Laguerre, Legendre and Tchebychev polynomials which are associated with rational solutions classical special functions, play an important role in a variety of applications [6, 12, 80]. Hence it seems probable that the polynomials discussed here which are associated with rational solutions of nonlinear special functions, i.e. the Painlevé equations, and soliton equations will also arise in variety of applications.

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