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Mean values of derivatives of L-functions in function fields: III

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In this series of papers, we explore moments of derivatives of L-functions in function fields using classical analytic techniques such as character sums and approximate functional equation. The present paper is concerned with the study of mean values of derivatives of quadratic Dirichlet L-functions over function fields when the average is taken over monic and irreducible polynomials P in $\mathbb{F}_q[T]$. When the cardinality q of the ground field is fixed and the degree of P gets large, we obtain asymptotic formulas for the first moment of the first and the second derivative of this family of L-functions at the critical point. We also compute the full polynomial expansion in the asymptotic formulas for both mean values.

Keywords: function fields; irreducible polynomials; hyperelliptic curves; derivatives of L–functions; moments of L–functions; quadratic Dirichlet L–functions; random matrix theory

2010 *Mathematics subject classification:* Primary 11M38
Secondary 11M06; 11G20; 11M50; 14G10

1. Introduction

This is the third paper of a series of papers devoted to studying the mean values of derivatives of L-functions in function fields. In the first and second paper in this series [**1**, **4**], we considered the first moment of derivatives of L-functions in function fields when the average is taken over an ensemble of square-free monic polynomials in $\mathbb{F}_q[T]$. In this paper, we venture in compute mean values of derivatives of L-functions in function fields when the average is taken over monic and irreducible polynomials in $\mathbb{F}_q[T]$.

Hoffstein and Rosen [**9**] were the first to study the mean values of L-functions in function fields. In their beautiful paper, they established several results concerning the mean values of different families of L-functions in function fields. However, in their paper, they never considered mean values of L-functions associated with monic and irreducible polynomials. In this paper, we investigate averages over monic and irreducible polynomials.

It is well-known that averages taken over primes are much harder to compute than averages over square-free numbers. The same principle also applies to the function field setting, where averages over monic irreducible polynomials are more difficult to handle than averages over square-free polynomials. This is mainly due to the fact that the primes (or monic irreducibles) are a thinner set when compared with the set of square-free numbers (or square-free polynomials).

In [3], Andrade and Keating computed the first and the second moment of quadratic Dirichlet L-functions in function fields when the average is taken over monic and irreducible polynomials in $\mathbb{F}_q[T]$. This paper can be thought as an extension of the papers [3, 9] where we handle the more intricate case of averages of quadratic Dirichlet L-functions, that is, we handle averages over 'primes' in this setting.

The study of derivatives of L-functions is an important problem in analytic number theory and has its roots in the work of Ingham [10], who established asymptotic formulas for the second moment of derivatives of the Riemann zeta function. Developments in the study of moments of derivatives of the Riemann zeta function were lead by Conrey [5] and Gonek [8]. In [6], by computing moments of the derivative of characteristic polynomials in the unitary group $U(N)$, Conrey, Rubinstein and Snaith formulated a general conjecture for the moments of derivatives of the Riemann zeta function. For a summarized account of the results in this paragraph, we ask the reader to refer to [4, § 1].

The main object of this paper is to study moments of derivatives of L-functions in the function field setting. In this note, we establish the first moment of the first and the second derivative of quadratic Dirichlet L-functions associated with monic irreducible polynomials in $\mathbb{F}_q[T]$. In future work (part 4 in this series), we establish more general mean values of derivatives of L-functions associated with monic and irreducible polynomials.

2. Main theorems

The calculations in this paper will lead to the following theorems.

THEOREM 2.1. *Let $\mathbb{F}_q$ be a fixed finite field with q odd. Then*

$$\sum_{P\in\mathcal{P}_{2g+1,q}} L^{'}\left(\tfrac{1}{2},\chi_P\right) = (\log q)\frac{|P|}{\log_q|P|}\left(\left[\frac{g-1}{2}\right]\left(1+\left[\frac{g-1}{2}\right]\right)\right. \\ \left. -2g\left(\left[\frac{g-1}{2}\right]+1\right)-\left[\frac{g}{2}\right]\left(\left[\frac{g}{2}\right]+1\right)\right) \\ +O(|P|^{3/4}(\log_q|P|)). \tag{2.1}$$

Where $[x]$ indicates the integer part of x, $|P|=q^{2g+1}$,

$$\mathcal{P}_{2g+1,q}=\{P\in\mathbb{F}_q[T],\ \textit{monic and irreducible, and } deg(P)=2g+1\},$$

and $L(s,\chi_P)$ is the quadratic Dirichlet L-function associated with P where χ_P is the quadratic character defined by the Legendre symbol in $\mathbb{F}_q[T]$, that is,

$$\chi_P(f)=(P/f)\,.$$

Using that $2g+1=\log_q|P|$ the next result follows as a simple corollary of theorem 2.1.

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Corollary 2.2. *Let $\mathbb{F}_q$ be a fixed finite field of odd cardinality q. Using the same notation as in the theorem, we have,*

$$\sum_{P\in\mathcal{P}_{2g+1,q}} L^{'}(\tfrac{1}{2},\chi_P) \sim -1/4\log(q)|P|(\log_q|P|), \tag{2.2}$$

as $g\to\infty$.

The next result is more involving and it is about the second derivative.

Theorem 2.3. *Let $\mathbb{F}_q$ be a fixed finite field with q odd. Using the same notation as in the theorem 2.1, we have that*

$$\begin{aligned}\sum_{P\in\mathcal{P}_{2g+1,q}} & L^{''}\left(\tfrac{1}{2},\chi_P\right)\\ &= \frac{2}{3}(\log(q))^2\frac{|P|}{\log_q|P|}\left(\left[\frac{g}{2}\right]\left(1+\left[\frac{g}{2}\right]\right)\left(1+2\left[\frac{g}{2}\right]\right)\right.\\ &\quad \left.+\left(\left(1+\left[\frac{g-1}{2}\right]\right)\left(6g^2+\left[\frac{g-1}{2}\right]-6g\left[\frac{g-1}{2}\right]+2\left[\frac{g-1}{2}\right]^2\right)\right)\right)\\ &\quad + O(|P|^{3/4}(\log_q|P|)^2). \end{aligned}\tag{2.3}$$

As before, we deduce a corollary of theorem 2.3.

Corollary 2.4. *Let $\mathbb{F}_q$ be a fixed finite field of odd cardinality q. Using the same notation as in the theorem, we have,*

$$\sum_{P\in\mathcal{P}_{2g+1,q}} L^{''}\left(\tfrac{1}{2},\chi_P\right) \sim 1/6\log^2(q)^2|P|(\log_q|P|)^2, \tag{2.4}$$

as $g\to\infty$.

Remark 2.5. Note that the average values are taken over the family of polynomials in $\mathcal{P}_{2g+1}$, that is, over odd degree monic irreducible polynomials. We could also consider the case of even degree polynomials, that is, averages over the family $\mathcal{P}_{2g+2}$. In that case, the calculations are similar to the odd degree case and the only difference is the form of the approximate functional equation for $L(s,\chi_P)$ when $P\in\mathcal{P}_{2g+2}$. For simplicity, we only consider the odd degree case since the even degree case does not present any novelty.

For a more detailed discussion about Dirichlet L-functions and Dirichlet characters in function fields, we suggest the reader to consult [**11**], [**4**, § 2] and [**2**, **7**]. Throughout this paper, we will let $|f| = q^{\deg(f)}$ be the norm of f.

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The L–function associated with χ_P is the numerator of the zeta function associated with the hyperelliptic curve over $\mathbb{F}_q$ defined by the affine equation $C_P : y^2 = P(T)$ and, consequently, $L(s, \chi_P)$ is a polynomial of degree $2g$ in the variable $u = q^{-s}$ given by

$$\begin{aligned} L(s, \chi_P) &= \sum_{n=0}^{2g} A(n, \chi_P) q^{-ns} \\ &= \sum_{n=0}^{2g} \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_P(f) q^{-ns}. \end{aligned} \tag{2.5}$$

(see [**11**, propositions 14.6 and 17.7] and [**2**, § 3]).

It is well-known that this L–function satisfies a functional equation. Namely

$$L(s, \chi_P) = (q^{1-2s})^g L(1-s, \chi_P), \tag{2.6}$$

and the Riemann hypothesis for curves, proved by Weil [**13**], tells us that all the zeros of $L(s, \chi_P)$ have real part 1/2.

Before we proceed with the proof of the two main results of this paper, we would like to point out that there is a different method that would also allow us to obtain the same results. By differentiating on both sides of the functional equation of $L(s, \chi_P)$, it should be possible to obtain the asymptotic formula of $L'((1/2), \chi_P)$ using the results for the mean value of $L((1/2), \chi_P)$ from [**3**]. And with a little more work and by using the mean value of $L((1/2), \chi_P)$ and $L'((1/2), \chi_P)$, we could derive the mean value of $L''((1/2), \chi_P)$.

The approach described in the previous paragraph seems to work for arbitrary derivatives, although the full expansion as presented in theorem 2.1 and 2.3 are not easy to derive by the use of this method. Because of that, the proofs of the main results of this paper are done by taking the derivative of the functional equation and averaging the character sums over monic irreducibles. In this way, we can keep track and see more clearly all the lower order terms.

In a forthcoming paper, we consider the second moment of higher derivatives of this family of L-functions and we also present conjectures for all the integral moments of derivatives of L-functions in function fields.

3. The first moment of $L'(1/2, \chi_P)$

From now on $P \in \mathcal{P}_{2g+1,q}$. Changing D for P in the 'approximate' functional equation [**2**, lemma 3.3], we have that

$$L(s, \chi_P) = \sum_{\substack{f_1 \text{ monic} \\ \deg(f_1) \leqslant g}} ((\chi_P(f_1))/(|f_1|^s)) + (q^{1-2s})^g \sum_{\substack{f_2 \text{ monic} \\ \deg(f_2) \leqslant g-1}} ((\chi_P(f_2))/(|f_2|^{1-s})). \tag{3.1}$$

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The first derivative of the approximate functional equation (3.1) gives

$$
\begin{aligned}
L^{'}\left(\tfrac{1}{2}, \chi_P\right) &= -(\log(q)) \sum_{n=0}^{g} n q^{-n/2} A(n, \chi_P) \\
&\quad + (\log(q)) \sum_{m=0}^{g-1} A(m, \chi_P)(m-2g) q^{-m/2} \\
&= J_1 + J_2.
\end{aligned} \tag{3.2}
$$

To prove theorem 2.1, we need to average (3.2) over $\mathcal{P}_{2g+1,q}$. We will accomplish this task for each of the sums J_i, $i = 1, 2$ in (3.2). But before that, we need two results that will be used in the rest of the paper. The first one is the Polynomial Prime Theorem [**11**, theorem 2.2] which states that

$$
\#\mathcal{P}_{2g+1,q} = q^{2g+1}/2g + 1 + O\left(q^{g+(1/2)} 2g + 1\right). \tag{3.3}
$$

The second result we need is a bound for a non-trivial character sum over function fields. Assume that f is monic, $\deg(f) > 0$ and that f is not a perfect square. Rudnick has proved in [**12**] that

$$
\left| \sum_{\substack{P \text{ monic} \\ \text{irreducible} \\ \deg(P)=n}} (f/P) \right| \ll \deg(f)/n q^{n/2}. \tag{3.4}
$$

3.1. Averaging J_1 and J_2

We now proceed to prove an asymptotic formula for the average of J_1 and J_2. From equation (3.2), we split the character sum $A(n, \chi_P)$ when f is a square of a polynomial and when f is not a square of a polynomial and this gives us that

$$
\begin{aligned}
\sum_{P \in \mathcal{P}_{2g+1,q}} J_1 &= -(\log q) \sum_{n=0}^{g} n q^{-n/2} \sum_{P \in \mathcal{P}_{2g+1,q}} \sum_{\substack{\deg(f)=n \\ f=\square}} \chi_P(f) \\
&\quad - (\log q) \sum_{n=0}^{g} n q^{-n/2} \sum_{P \in \mathcal{P}_{2g+1,q}} \sum_{\substack{\deg(f)=n \\ f\neq\square}} \chi_P(f).
\end{aligned} \tag{3.5}
$$

Using the bound for non-trivial character sums (3.4), we have that the above is

$$
\begin{aligned}
\sum_{P \in \mathcal{P}_{2g+1,q}} J_1 &= -(\log q) \sum_{n=0}^{g} n q^{-n/2} \sum_{P \in \mathcal{P}_{2g+1,q}} \sum_{\substack{\deg(f)=n \\ f=\square}} \chi_P(f) \\
&\quad + O\left(\sum_{n=0}^{g} n q^{-n/2} \sum_{\deg(f)=n} \frac{q^g n}{2g+1} \right).
\end{aligned} \tag{3.6}
$$

We have that the error term above is

$$\begin{aligned}\sum_{n=0}^{g} nq^{-n/2} \sum_{\deg(f)=n} \frac{q^g n}{2g+1} &= \sum_{n=0}^{g} n^2 q^{n/2} q^g \frac{1}{2g+1} \\ &\ll q^g g^2 q^{g/2} \frac{1}{2g+1} \\ &\ll |P|^{3/4} (\log_q |P|).\end{aligned} \tag{3.7}$$

For the sum when f is a perfect square, we have that

$$\begin{aligned}&-(\log q) \sum_{n=0}^{g} nq^{-n/2} \sum_{P \in \mathcal{P}_{2g+1,q}} \sum_{\substack{\deg(f)=n \\ f=\square}} \chi_P(f) \\ &= -(\log q) \sum_{n=0}^{g} nq^{-n/2} \sum_{\deg(l)=n/2} \sum_{P \in \mathcal{P}_{2g+1,q}} \chi_P(l^2) \\ &= -(\log q) \sum_{n=0}^{g} nq^{-n/2} \sum_{\deg(l)=n/2} \sum_{\substack{P \in \mathcal{P}_{2g+1,q} \\ P \nmid l}} 1.\end{aligned} \tag{3.8}$$

Using the Polynomial Prime theorem (3.3), the fact that

$$\sum_{\substack{P \in \mathcal{P}_{2g+1,q} \\ P \nmid l}} 1 = \sum_{P \in \mathcal{P}_{2g+1,q}} 1, \tag{3.9}$$

since $\deg(P) > \deg(l)$, and after a few arithmetic manipulations and performing the sum over n, we obtain

$$\begin{aligned}&-(\log q) \sum_{n=0}^{g} nq^{-n/2} \sum_{P \in \mathcal{P}_{2g+1,q}} \sum_{\substack{\deg(f)=n \\ f=\square}} \chi_P(f) \\ &= -(\log q) \frac{|P|}{\log_q |P|} \left[\frac{g}{2}\right] \left(\left[\frac{g}{2}\right] + 1\right) + O(|P|^{1/2} (\log_q |P|)).\end{aligned} \tag{3.10}$$

Combining (3.6) with the equation 3.10, we obtain

$$\sum_{P \in \mathcal{P}_{2g+1,q}} J_1 = -(\log q) \frac{|P|}{\log_q |P|} \left[\frac{g}{2}\right] \left(\left[\frac{g}{2}\right] + 1\right) + O(|P|^{3/4} (\log_q |P|)). \tag{3.11}$$

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Since the average over J_2 is similar, we shall not repeat the details. In the end, we obtain that

$$\sum_{P\in\mathcal{P}_{2g+1,q}} J_2 = -2(\log q)\frac{|P|}{\log_q|P|}g\left(\left[\frac{g-1}{2}\right]+1\right) + (\log q)\frac{|P|}{\log_q|P|}\left[\frac{g-1}{2}+1\right]\left(\left[\frac{g-1}{2}\right]+1\right) + O(|P|^{3/4}(\log_q|P|)). \tag{3.12}$$

Putting together the average of the quantities J_1 and J_2 over $P\in\mathcal{P}_{2g+1,q}$ allows us to deduce theorem 2.1. □

4. The first moment of $L''(1/2,\chi_P)$

In this section, we prove theorem 2.3. The second derivative of the approximate functional equation (3.1) at $s=1/2$ gives

$$\begin{aligned} L''(\tfrac{1}{2},\chi_P) &= (\log(q))^2\sum_{n=0}^{g} n^2 q^{-n/2}A(n,\chi_P) \\ &\quad + (\log(q))^2\sum_{m=0}^{g-1} A(m,\chi_P)(m-2g)^2 q^{-m/2} \\ &= S_1+S_2, \end{aligned} \tag{4.1}$$

where

$$A(n,\chi_P) = \sum_{\substack{f \text{ monic} \\ \deg(f)=n}} \chi_P(f). \tag{4.2}$$

4.1. Averaging S_1 and S_2

From equation (4.1), we have that

$$\begin{aligned} \sum_{P\in\mathcal{P}_{2g+1,q}} S_1 &= \sum_{P\in\mathcal{P}_{2g+1,q}} (\log(q))^2\sum_{n=0}^{g} n^2 q^{-n/2}A(n,\chi_P) \\ &= \sum_{P\in\mathcal{P}_{2g+1,q}} (\log(q))^2\sum_{n=0}^{g} n^2 q^{-n/2}\sum_{\substack{\deg(f)=n \\ f=\square}}\chi_P(f) \\ &\quad + \sum_{P\in\mathcal{P}_{2g+1,q}} (\log(q))^2\sum_{n=0}^{g} n^2 q^{-n/2}\sum_{\substack{\deg(f)=n \\ f\neq\square}}\chi_P(f). \end{aligned} \tag{4.3}$$

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For f not a perfect square, we use the bound given in equation (3.4) and the same reasoning used in 3.7 to write

$$\begin{aligned}\sum_{P\in\mathcal{P}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{\deg(f)=n\\ f\neq\square}} \chi_P(f) \\ \ll \sum_{n=0}^{g}\sum_{\deg(f)=n} n^2 q^{-n/2}\frac{q^g n}{2g+1} \qquad (4.4)\\ \ll |P|^{3/4}(\log_q|P|)^2.\end{aligned}$$

When f is a perfect square, we use the Prime Polynomial theorem (3.3) to write

$$\begin{aligned}&\sum_{P\in\mathcal{P}_{2g+1,q}} (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{\deg(f)=n\\ f=\square}} \chi_P(f)\\ &= (\log(q))^2 \sum_{n=0}^{g} n^2 q^{-n/2} \sum_{\substack{\deg(f)=n\\ f=l^2}} \sum_{P\in\mathcal{P}_{2g+1,q}} \chi_P(l^2)\\ &= (\log(q))^2 \sum_{n=0}^{[g/2]} 4m^2 q^{-m} \sum_{\deg(l)=m} \left(\frac{q^{2g+1}}{2g+1} + O\left(\frac{q^g}{g}\right)\right) \qquad (4.5)\\ &= \frac{2}{3}(\log q)^2 \frac{|P|}{\log_q|P|}\left[\frac{g}{2}\right]\left(\left[\frac{g}{2}\right]+1\right)\left(2\left[\frac{g}{2}\right]+1\right) + O(q^g g^2).\end{aligned}$$

Invoking equations (4.4) and (4.5), we proved that

$$\begin{aligned}\sum_{P\in\mathcal{P}_{2g+1,q}} S_1 = \frac{2}{3}(\log q)^2 \frac{|P|}{\log_q|P|}\left[\frac{g}{2}\right]\left(\left[\frac{g}{2}\right]+1\right)\left(2\left[\frac{g}{2}\right]+1\right)\\ + O(|P|^{3/4}(\log_q|P|)^2). \qquad (4.6)\end{aligned}$$

Using the Prime Polynomial theorem (3.3), the bound for non-trivial character sums (3.4) and equation (4.1) a similar argument can be used to establish

$$\begin{aligned}\sum_{P\in\mathcal{P}_{2g+1,q}} S_2 = \frac{2}{3}(\log q)^2 \frac{|P|}{\log_q|P|}\left(\left(\left[\frac{g-1}{2}\right]+1\right)\right.\\ \left.\times\left(6g^2 + \left[\frac{g-1}{2}\right] - 6g\left[\frac{g-1}{2}\right] + 2\left[\frac{g-1}{2}\right]^2\right)\right) \qquad (4.7)\\ + O(|P|^{3/4}(\log_q|P|)^2).\end{aligned}$$

Combining equations (4.6) and (4.7), we establish theorem 2.3. □

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