

USING MODEL-INDEPENDENT LOWER BOUNDS TO IMPROVE PRICING OF ASIAN STYLE OPTIONS IN LÉVY MARKETS

BY

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ABSTRACT

Albrecher *et al.* (ALBRECHER, H., MAYER PH., SCHOUTENS, W. (2008) General lower bounds for arithmetic Asian option prices. *Applied Mathematical Finance*, **15**, 123–149) have proposed model-independent lower bounds for arithmetic Asian options. In this paper we provide an alternative and more elementary derivation of their results. We use the bounds as control variates to develop a simple Monte Carlo method for pricing contracts with Asian-style features. The conditioning idea that is inherent in our approach also inspires us to propose a new semi-analytic pricing approach. We compare both approaches and conclude that these both have their merits and are useful in practice. In particular, we point out that our newly proposed Monte Carlo method allows to deal with Asian-style products that appear in insurance (e.g., unit-linked contracts) in a very efficient way, and outperforms other known Monte Carlo methods that are based on control variates.

KEYWORDS

Asian-style options, conditional expectation, control variates, stochastic clock.

1. INTRODUCTION

An Asian option is a derivative whose payoff is based on the average of the trajectory of an underlying primitive asset. Such smoothing is often regarded as a desired protection against unexpected and brutal changes in the value of the underlying.¹ The same averaging feature also appears naturally when compounding or discounting cash flows, and hence Asian-style payoffs crop up in finance and actuarial science naturally. Among others we mention unit-linked products (see e.g., Schrager and Pelsser, 2004), ratchet equity indexed annuities (with Asian-end design) (see e.g., Lin and Tan 2003; Ballotta 2010) and variable annuities with flexible premiums (see e.g., Milevsky and Posner 2003).

Technically speaking, Asian options also appear in other disciplines such as physics (see e.g., Romeo *et al.* 2003) and engineering (see e.g., Fenton 1960). Unfortunately the averaging, and thus the path-dependence involved, makes options with Asian-style features much harder to value than their plain vanilla counterparts, and this even under the standard Black–Scholes model for the financial market. Hence, there has always been a great interest of the academic community in Asian options; see e.g., Boyle and Potapchik (2008) for an overview. They also exist in different flavors, but in this paper we focus on the discretely monitored fixed strike Asian option with arithmetic averaging.

We make several contributions in this paper. Our first contribution is that we derive model-independent lower bounds for Asian options valid in a rather general market setting. Model-independent bounds are important because they are solely implied by information available in the market and hence are neither prone to model error nor incompleteness of financial markets. The bounds we obtain are closely tied to the ones appearing in Albrecher *et al.* (2008), but are here obtained in a more straightforward way. Our approach does not make use of ad hoc probabilistic arguments but rather relies on simple conditioning techniques combined with well-established actuarial theory on stop-loss bounds. This result is also appealing as similar conditioning arguments can be useful to derive bounds for other options. In this paper we show that the idea of conditioning is also crucial in proposing a new semi-analytic approach.

Our second contribution is that, in the context of the subordinated Brownian motion for modeling the stock returns, we present a semi-analytic pricing method using a two-step approach. The idea of this method is that in the first step we condition on the stochastic clock which makes the nature of the problem multi-variate normal and allows us to make use of very accurate closed form approximations that we apply conditionally. Next, we use Monte Carlo simulation to account for the stochasticity of the clock itself. This method is astonishingly fast and provides tight lower bound approximations. The budget for carrying out simulations is important when economic (real-world) scenarios interplay with (risk-neutral) pricing, a situation that appears when assessing the risk of a portfolio of financial instruments over a given horizon (e.g., one year). In these instances this method appears to provide an attractive trade-off between accuracy and efficiency.

Monte Carlo simulation is a classical approach to value derivatives. It combines flexibility with accuracy, i.e., asymptotically the simulated value converges to the true value. The drawback is that the variance on the estimate decreases slowly, which creates room for potentially attractive improvements. Our third contribution is that we show how the bounds that we derive can also be used as the so-called control variates to increase the efficiency of traditional Monte Carlo schemes. We compare this method with other promising control variate approaches that were recently proposed by Dingec and Hormann (2012) and Fusai and Meucci (2008). Several numerical experiments allow us to conclude that our approach is outperforming when pricing insurance contracts with Asian-style features such as unit-linked contracts or variable annuities with

flexible premiums. In contrast, the method of Fusai and Meucci (2008) is better adapted when pricing contracts with a shorter maturity.

The structure of the paper is as follows. In Section 2 we derive model-independent lower bounds that were introduced in Albrecher *et al.* (2008) in a more straightforward way. In Section 3 we discuss the use of these bounds as control variates in Monte Carlo simulations. In Section 4 we introduce the market setting that we focus on, namely a subordinated Brownian motion framework with as particular examples the Variance Gamma (VG) and the Normal Inverse Gaussian (NIG) models. Section 5 is devoted to the derivation of analytical expressions for the lower bounds that we use as control variates in the setting of the VG and the NIG models. Section 6 presents the second approach for deriving precise and efficient estimations for the price of an Asian option. Several sets of numerical results are discussed in Section 7. Section 8 provides some extensions, in particular ratchet equity-indexed annuities and unit-linked insurances are discussed. Section 9 concludes the paper.

2. MODEL-INDEPENDENT BOUNDS FOR ASIAN OPTIONS

The bounds that we propose to price Asian options efficiently are rooted in actuarial theory on stop-loss order. We first define this ordering concept and provide a modest accompanying result that we further need in the paper. Next we derive three model-independent lower bounds for Asian options. These can also be found in Albrecher *et al.* (2008), but are here derived in a more straightforward way.

2.1. Stop-loss bounds

Stop-loss order is a well-established concept in the actuarial literature. It can be formally defined as follows.

Definition 1 (stop-loss order). *Let X and Y be two random variables. We say that the random variable² X is smaller than Y in stop-loss order, denoted as $X \leq_{sl} Y$, if for all $d \in \mathbb{R}$ it holds that $E[\text{Max}(X - d, 0)] \leq E[\text{Max}(Y - d, 0)]$.*

It is well known that $X \leq_{sl} Y \Leftrightarrow E[v(X)] \leq E[v(Y)]$ for all increasing convex functions v , which explains why stop-loss order also appears in the literature under the name increasing convex order.³ A well-known stop-loss lower bound for X is obtained by conditioning, i.e., for any random variable Z it holds that,

$$E[X | Z] \leq_{sl} X, \quad (1)$$

where Z can be any random variable. This result is also known as Jensen's inequality and is instrumental in this paper. A sufficient condition for $X \leq_{sl} Y$ is that $E[X] \leq E[Y]$ and their respective distribution functions cross only once (i.e., there exists a real c so that $F_X(x) \leq F_Y(x)$ for $x \leq c$ and $F_X(x) \geq F_Y(x)$

for $x \geq c$). See also Müller and Stoyan (2002) for example. It is then also clear that when g and h are non-decreasing,

$$g(X) + E[h(X)] \leq_{st} g(X) + h(X). \quad (2)$$

This last inequality will also be often used throughout the paper.

2.2. Asian options

In this paper we use the setup and the notation from Albrecher *et al.* (2008). Hence, we consider an arbitrage-free financial market containing a risky asset with price process $\{S_t, t \geq 0\}$ and a risk-free asset yielding the continuously compounded risk-free rate r . Let \mathbb{P} denote the physical probability measure. We consider the arithmetic Asian option with payoff at maturity $T > 0$ given by

$$\text{Max}(H_T - K, 0).$$

Here

$$H_T = \frac{1}{n} \sum_{i=1}^n S_{t_i},$$

K is the strike and the t_i ($i = 1, 2, \dots, n$) are the n monitoring times. Without loss of generality we assume that $t_n = T$. The absence of arbitrage opportunities implies there is an equivalent martingale (also called risk-neutral) measure \mathbb{Q} such that

$$E[S_s | S_t] = S_t e^{r(s-t)} \quad s \geq t \geq 0, \quad (3)$$

holds, where the expectation is taken with respect to \mathbb{Q} . Unless otherwise mentioned, all expectations appearing in this paper are \mathbb{Q} -expectations. The arbitrage-free value of the Asian option at $t = 0$ is denoted by $AC(K, n)$ and is determined by

$$AC(K, n) = e^{-rT} E[\text{Max}(H_T - K, 0)]. \quad (4)$$

It is well known that the absence of arbitrage is not sufficient to determine the risk-neutral measure \mathbb{Q} uniquely, hence the pricing of an Asian option (and other derivatives) might not be unambiguous. Moreover, for any possible choice for the pricing measure \mathbb{Q} the evaluation of (4) is by no means trivial and analytical solutions appear to be out of reach in general. Several methods have then been proposed in the literature, including (pseudo) Monte Carlo simulations, moment-matching techniques and Fast Fourier Transforms (FFTs). Unfortunately, the successful application of these methods require a further specification of the price process $\{S_t, t \geq 0\}$.

The uncertainty inherent with arbitrage-free pricing then gives an incentive to develop model-independent lower bounds, i.e., bounds for $AC(K, n)$ that are (as much as possible) based on available market information only and do not require modeling assumptions. Hence, in the next section we aim at finding lower bounds for $AC(K, n)$ that are “as close as possible” to $AC(K, n)$, while also “as

model free as possible” in the sense that they intend to hold irrespective of the choices one makes for the risk-neutral measure and the price process of the risky asset. Indeed, the bounds that we propose only require knowledge of prices of plain vanilla call prices essentially. Let us now introduce random variables $H_T^{l,t}$ defined as

$$H_T^{l,t} = E[H_T \mid S_t] \quad 0 \leq t \leq T.$$

Using the notation

$$j(t) = \text{Min}\{i \mid t_i \geq t\},$$

we find using condition (3) that $H_T^{l,t}$ can also be expressed as

$$H_T^{l,t} = A_T + \frac{1}{n} \sum_{i=j(t)}^n S_t e^{r(t_i-t)},$$

where A_T is given as

$$A_T = \sum_{i=1}^{j(t)-1} \frac{1}{n} E[S_{t_i} \mid S_t], \tag{5}$$

and where it is tacitly assumed that $\sum_{i=1}^{j(t)-1} = 0$ when $j(t) = 1$. Note that Jensen’s inequality (1) implies that

$$H_T^{l,t} \leq_{sl} H_T,$$

which shows, as per the definition of stop-loss order, that the $H_T^{l,t}$ might be useful to determine lower bounds for the Asian call price $AC(K, n)$, and this is the topic of the next section.

2.3. Albrecher *et al.*’s model-independent lower bounds

2.3.1. *First lower bound.* Let us consider the case $0 \leq t \leq t_1$. In this instance $H_T^{l,t}$ clearly writes as,

$$H_T^{l,t} = \frac{1}{n} \sum_{i=1}^n S_t e^{r(t_i-t)}.$$

Let us observe that

$$H_T^{l,t} = E[H_T^{l,t_1} \mid S_t].$$

In particular, this implies that for all $t \in [0, t_1]$, $H_T^{l,t} \leq_{sl} H_T^{l,t_1}$ so that the best lower bound for the Asian call price $AC(K, n)$, based on $H_T^{l,t}$, is reached when

$t = t_1$. For its value, denoted by LB_1 , we find that

$$\begin{aligned}
 LB_1 &= e^{-rT} \mathbb{E} \left[\text{Max}(H_T^{t_1} - K, 0) \right] \\
 &= \frac{e^{-rT}}{n} \mathbb{E} \left[\text{Max} \left(S_{t_1} - \frac{nK}{\sum_{i=1}^n e^{r(t_i-t_1)}}, 0 \right) \right] \sum_{i=1}^n e^{r(t_i-t_1)} \\
 &= \frac{1}{n} C \left(\frac{nK}{\sum_{i=1}^n e^{r(t_i-t_1)}}, t_1 \right) \sum_{i=1}^n e^{-r(T-t_i)}, \tag{6}
 \end{aligned}$$

where in the last step we have tacitly introduced the notation $C(K, t)$ to denote the value of a standard European Call with strike K and maturity $t > 0$. In summary, we have that

$$LB_1 \leq AC(K, n),$$

and only the non-arbitrage assumption was needed to derive the expression for LB_1 . Hence, the value for LB_1 can be observed in the market as soon as the European calls are quoted.

We remark that expression (6) was already derived in Albrecher *et al.* (2008, eq. (4) p. 127). They resorted to comonotonicity theory in order to do so, but this is unnecessary. In the next section we sharpen the bounds further. However, this goes at the cost of imposing some more structure on the market.

2.3.2. Second lower bound. The first lower bound might not be sharp because the variable $H_T^{t_1}$, obtained by taking the conditional expectation of H_T with respect to S_{t_1} , may not always depict the best possible approximation for H_T (which incorporates the randomness of S_t over the entire horizon $[0, T]$ and not only $[0, t_1]$). Hence, to improve the bounds we aim at considering choices for all $0 \leq t \leq T$. In order to do so successfully we need to impose the following condition on the market.

Condition 1. For all $0 \leq t \leq T$ and all $i = 1, \dots, j(t) - 1$ the random variables $E[S_i | S_t]$ are non-decreasing in S_t .

Assume that the stock price S_t at time $t > 0$ is known. Then the condition states that the expected stock prices at any intermediate time are increasing in S_t , which appears a natural and desirable property for the risky asset price process $\{S_t, t \geq 0\}$. From Condition 1 it follows that A_T as defined in (5) is increasing in S_t ($0 \leq t \leq T$) so that inequality (2) can be applied to find the stop-loss bound,

$$E[A_T] + \frac{1}{n} \sum_{i=j(t)}^n S_t e^{r(t_i-t)} \leq_{sl} H_T, \tag{7}$$

where $E[A_T]$ is explicitly given as

$$E[A_T] = \frac{1}{n} \sum_{i=1}^{j(t)-1} S_0 e^{rt_i}.$$

From (7) it follows that

$$\begin{aligned} & e^{-rT} \mathbb{E} \left[\text{Max} \left(\mathbb{E}[A_T] + \frac{1}{n} \sum_{i=j(t)}^n S_t e^{r(t_i-t)} - K, 0 \right) \right] \\ &= e^{-rT} \frac{\sum_{i=j(t)}^n e^{r(t_i-t)}}{n} \mathbb{E} \left[\text{Max} \left(S_t - \frac{n(K - \mathbb{E}[A_T])}{\sum_{i=j(t)}^n e^{r(t_i-t)}}, 0 \right) \right] \\ &= \frac{1}{n} \sum_{i=j(t)}^n e^{-r(T-t_i)} C(c_t^{(1)}, t) \\ &\leq AC(K, n), \end{aligned}$$

where $c_t^{(1)}$ is given as

$$c_t^{(1)} = \frac{(nK - \sum_{i=1}^{j(t)-1} S_0 e^{rt_i})}{\sum_{i=j(t)}^n e^{r(t_i-t)}}.$$

Since this holds for all $0 \leq t \leq T$, we find that

$$LB_t^{(1)} \leq AC(K, n),$$

with $LB_t^{(1)}$ given as

$$LB_t^{(1)} = \frac{e^{-rT}}{n} \text{Max}_{0 \leq t \leq T} \left(C(c_t^{(1)}, t) \sum_{i=j(t)}^n e^{rt_i} \right). \tag{8}$$

Note that expression (8) can also be found in Albrecher *et al.* (2008, eq. (10), p. 129), where more involved probabilistic arguments have been used to obtain it. Hence, the value for the $LB_t^{(1)}$ can be observed in the market as soon as the European calls are quoted. Furthermore, in order to derive the $LB_t^{(1)}$ we do not really need Condition 1. It is sufficient that (the less intuitive) condition (7) is fulfilled (see also Albrecher *et al.*, 2008). Finally, let us remark that $LB_1 \leq LB_t^{(1)}$.

2.3.3. *Third lower bound.* The following condition usually allows to further improve the bounds.

Condition 2. For all $i = 1, \dots, j(t) - 1$ we assume that $E[S_{t_i} | S_t] \geq S_0^{1 - \frac{t_i}{t}} S_t^{\frac{t_i}{t}}$ (almost surely).

Assume that the stock price S_t at time $t > 0$ is known. Then the expected stock price at intermediate times $0 \leq t_i \leq t$ is bounded below by the geometric average of the stock prices S_t and S_0 with weights equal to normalized distance between t_i and 0 respectively. It is not difficult to show that this assumption holds when log-returns are identically and independently distributed, a condition that holds true for exponential Lévy models (see also Proposition 2.1 in Albrecher *et al.*, 2008).

Condition 2 implies that

$$L_t := \frac{1}{n} \sum_{i=1}^{j(t)-1} S_0 \left(\frac{S_t}{S_0} \right)^{\frac{t_i}{t}} + \frac{1}{n} \sum_{i=j(t)}^n S_t e^{r(t_i-t)} \leq_{sl} \frac{1}{n} \sum_{i=1}^n S_{t_i}. \tag{9}$$

Taking into account that the left-hand side of the inequality is increasing in S_t , we find that

$$\begin{aligned} & e^{-rT} E[\text{Max}(L_t - K, 0)] \\ &= \frac{e^{-rT} S_0}{n} \sum_{i=1}^{j(t)-1} E \left[\text{Max} \left(\left(\frac{S_t}{S_0} \right)^{\frac{t_i}{t}} - \left(\frac{c_t^{(2)}}{S_0} \right)^{\frac{t_i}{t}}, 0 \right) \right] \\ &+ \frac{e^{-rT}}{n} \sum_{i=j(t)}^n e^{r(t_i-t)} E[\text{Max}(S_{t_i} - c_t^{(2)}, 0)] \\ &= \frac{e^{-rT} S_0}{n} \sum_{i=1}^{j(t)-1} E \left[\text{Max} \left(\left(\frac{S_t}{S_0} \right)^{\frac{t_i}{t}} - \left(\frac{c_t^{(2)}}{S_0} \right)^{\frac{t_i}{t}}, 0 \right) \right] \\ &+ \frac{e^{-rT}}{n} \sum_{i=j(t)}^n e^{rt_i} C(c_t^{(2)}, t), \end{aligned}$$

where $c_t^{(2)}$ is now the solution of

$$nK - \sum_{i=1}^{j(t)-1} S_0 \left(\frac{c_t^{(2)}}{S_0} \right)^{\frac{t_i}{t}} - c_t^{(2)} \sum_{i=j(t)}^n S_t e^{r(t_i-t)} = 0.$$

Hence, we find that

$$LB_t^{(2)} \leq AC(K, n),$$

where $LB_t^{(2)}$ is given by

$$LB_t^{(2)} = \frac{e^{-rT}}{n} \text{Max}_{0 \leq t \leq T} \left(S_0 \sum_{i=1}^{j(t)-1} E \left[\text{Max} \left(\left(\frac{S_t}{S_0} \right)^{\frac{t_i}{T}} - \left(\frac{c_t^{(2)}}{S_0} \right)^{\frac{t_i}{T}}, 0 \right) \right] + \sum_{i=j(t)}^n e^{rt_i} C(c_t^{(2)}, t) \right). \quad (10)$$

Note that in order to derive $LB_t^{(2)}$ we do not really need Condition 2. It is sufficient that (the less intuitive) condition (9) will hold. Also, this expression can be found in Albrecher *et al.* (2008, eq. (15), p. 132), but our approach is based on simpler considerations. Also note that $LB_t^{(2)}$ is maximum over different time instances of prices of a path-independent power call option. Carr and Chou (1997) explain that if an investor can trade in all European calls maturing at T , the path-independent option can be uniquely decomposed into a bond and a continuum of calls. Hence, when all call prices are available in the market, the value for $LB_t^{(2)}$ can be determined using their decomposition formula.

In Albrecher *et al.* (2008) the different lower bounds for Asian option prices have been numerically evaluated with varying success. Intuitively this is because the bounds used are essentially based on “approximating” the sum $\frac{1}{n} \sum_{i=1}^n S_{t_i}$ by a conditioned sum involving a single component S_{t_j} ($j = 1, 2, \dots, n$) only. While this procedure allows to obtain (almost) model-free lower bounds for Asian option prices, the reduction of dimensionality comes at some cost.

In this paper we aim at improving the accuracy. More precisely, we show the bounds can be used as control variates to improve efficiency of Monte Carlo approaches when estimating the price of Asian options. In doing so we focus on the third stop-loss bound. While there is no formal guarantee that it outperforms the second lower bound (and thus also the first lower bound), Albrecher *et al.* (2008) provides theoretical and empirical evidence that this is often the case. Recall also that this third lower bound exists in a Lévy setting and thus has practical⁴ appeal (see also proposition 2.1 in Albrecher *et al.*, 2008).

In the next Sections 3–5 we explain in detail the control variate approach that we propose. Numerical results can be found in Section 7.

3. BOUNDS AS CONTROL VARIATES FOR IMPROVED SIMULATION

In this section we show that the stop-loss bounds derived in Section 2 appear as a useful device to improve Monte Carlo simulation significantly, namely when using them as control variates.

3.1. Control variates

Clearly, the pricing of an Asian option amounts to determining the expected value of a random variable. A traditional technique to cope with such problem is Monte Carlo simulation which builds upon the law of large numbers and thereby upon large samples of random numbers. Hence, let Y_1, Y_2, \dots, Y_k be a series of k independently and identically distributed (i.i.d.) replications of the payoff $e^{-rT}(\frac{1}{n} \sum_{i=1}^n S_{t_i} - K)^+$. Then a classical estimator for $E[Y] := e^{-rT} E[(\frac{1}{n} \sum_{i=1}^n S_{t_i} - K)^+]$ is its sample mean \bar{Y} given as

$$\bar{Y} = \frac{1}{k} \sum_{i=1}^k Y_i. \quad (11)$$

The law of large numbers assures that a sample of independent, identically distributed random variables converges to the sample mean as the sample size k increases. More specifically, the statistic \bar{Y} is an unbiased and (strongly) consistent estimator for $E[Y]$, and

$$\frac{\sqrt{k}(\bar{Y} - E[Y])}{\sqrt{\text{Var}[Y]}} \Rightarrow N(0, 1),$$

where “ \Rightarrow ” denotes convergence in distribution, and $N(0, 1)$ denotes the standard normal distribution. To summarize: The Monte Carlo simulation consists in generating random draws Y_1, \dots, Y_k and next applying (11). We remark that the $O(k^{-\frac{1}{2}})$ convergence rate is independent of the number of dimensions n involved in the payoff sum, namely $\sum_{i=1}^n S_{t_i}$, making simulation for highly multi-dimensional problems often the preferred approach above other numerical techniques.

On the other hand, one of the main weaknesses with the Monte Carlo method is efficiency since standard simulation provides convergence rates of (only) $k^{-1/2}$. Using simulation, excessive run lengths or replications may thus be necessary to yield estimators with acceptable precision. In practice, simulation time can be of great importance, and consequently techniques have been developed to improve the performance of the Monte Carlo method, most of which are aimed at reducing the variance of the estimator. Variance reduction techniques thus allow to obtain greater precision for the same amount of simulation time, or to achieve a desired precision with less simulation time. One of the most effective variance reduction techniques is the method of control variates. The main idea is to use a highly correlated random variable with known mean to reduce the variance of the estimator. In our context, the geometric counterpart of the arithmetic Asian option has already been proposed in literature as a control variate by, e.g., Kemna and Vorst (1990) and Fu *et al.* (1999). Recently, this idea was further exploited and deepened by Fusai and Meucci (2008) and Dingec and Hormann (2012).

In this paper we show how the stop-loss bounds L_t that we discussed in Section 2 can appear as suitable control variates. More precisely, when Condition 2 is fulfilled the random variables L_t (see (9)) provide stop-loss bounds for $H_T = \frac{1}{n} \sum_{i=1}^n S_{t_i}$. We remark that the L_t are essentially obtained through conditioning and thus may preserve a significant amount of the information contained in H_T . In other words, one may intuitively expect the L_t to have good correlation with H_T making them possible candidates to improve convergence considerably.

First, let us consider only one time instance t_j and therefore only one random variable L_{t_j} . Suppose that for each replication Y_i for the payoff $e^{-rT}(\frac{1}{n} \sum_{i=k}^n S_{t_k} - K)^+$, we calculate another output X_i corresponding to the replication of $X := e^{-rT}(L_{t_j} - K)^+$ (X_i will be later in the text denoted by $X_i^{(t_j)}$, but as long as there is no confusion it is denoted just by X_i). Let us assume that $E[X]$ is known analytically (an analytical expression is given in Section 5 in the setting of VG and NIG models). We then consider the replications X_i ($i = 1, 2, \dots, k$) such that the subsequent pairs (X_i, Y_i) are i.i.d. The unbiased estimator of $E[X]$ is therefore given by

$$\bar{X} = \frac{1}{k} \sum_{i=1}^k X_i.$$

Then the control variate estimator \bar{Y}^c of $E[Y]$ given by

$$\begin{aligned} \bar{Y}^c &= \bar{Y} - b(\bar{X} - E[X]) \\ &= \frac{1}{k} \sum_{i=1}^k (Y_i - b(X_i - E[X])) \end{aligned}$$

is also unbiased and asymptotically normally distributed (see Glasserman, 2003).

The optimal coefficient b^* which minimizes the variance of the new control variate estimator, namely⁵

$$\text{Var}[\bar{Y}^c] = \text{Var}[\bar{Y}](1 - \rho_{XY}^2) \tag{12}$$

with ρ_{XY} the correlation between X and Y , is given by (see Glasserman, 2003)

$$b^* = \frac{\text{Cov}[X, Y]}{\text{Var}[X]}.$$

Quantities $\text{Cov}[X, Y]$ and $\text{Var}[X]$ are usually not known and in this case one has to use an estimate of b^* . Here we have chosen to use the estimate \hat{b} given by

$$\hat{b} = \frac{\sum_{i=1}^k (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_{i=1}^k (X_i - \bar{X})^2}.$$

In our setting we can choose any t_j to create a single control variate. It appears natural to use different L_{t_j} ($j = 1, 2, \dots, n$) together to create n control variates. Suppose that each replication of a simulation produces outputs Y_i and $\mathbf{X}_i = (X_i^{(t_1)}, \dots, X_i^{(t_n)})'$ (the exponent $'$ denotes that the vector is transposed) corresponding to a vector of payoff replications, namely $\mathbf{X} = e^{-rT}((L_{t_1} - K)^+, \dots, (L_{t_n} - K)^+)',$ and suppose that the vector of corresponding expectations $E[\mathbf{X}]$ is known. Then one can derive that the control variate estimator \bar{Y}^c based on the vector of sample means of the control variates $\bar{\mathbf{X}} := (\bar{X}^{(t_1)}, \bar{X}^{(t_2)}, \dots, \bar{X}^{(t_n)})'$ is given by

$$\bar{Y}^c = \bar{Y} - (B)'(\bar{\mathbf{X}} - E[\mathbf{X}]),$$

where $B^* = \Sigma_{\bar{\mathbf{X}}}^{-1}\Sigma_{\mathbf{X}Y}$ is optimal with minimized variance of \bar{Y}^c . $\Sigma_{\mathbf{X}}$ is a $n \times n$ covariance matrix and $\Sigma_{\mathbf{X}Y}$ is a $n \times 1$ covariance vector (see Glasserman, 2003). Since B^* is not known explicitly, it needs to be estimated. In order to avoid introducing bias we first generate pairs to estimate B^* and, in particular, estimates $\hat{\Sigma}_{\mathbf{X}}$ and $\hat{\Sigma}_{\mathbf{X}Y}$ for $\Sigma_{\mathbf{X}}$ and $\Sigma_{\mathbf{X}Y}$ respectively. Next, we use the remaining pairs to determine $\bar{Y}^c(B^*)$ independently, and then the minimized variance is

$$\text{Var}[\bar{Y}^c] = \text{Var}[\bar{Y}](1 - R^2), \tag{13}$$

where one estimates R^2 by $\hat{\Sigma}'_{\mathbf{X}Y}\hat{\Sigma}^{-1}_{\mathbf{X}}\hat{\Sigma}_{\mathbf{X}Y}/\text{Var}[\bar{Y}]$.

Whereas the random variables L_{t_j} are based on arithmetic averaging, we will propose in Section 3.3 control variates based on geometric averaging. These will be applied in the same way as above.

While the technique of using control variates is rather straightforward, its successful application depends on a series of conditions that need to be met. First, using a control variate X for estimating $E[Y]$ more efficiently implies that $E[X]$ needs to be known. Second, as expressed by (12) and (13), the gain in efficiency depends on the strength of correlation. Finally, on a more practical note, it is important to observe that the computational cost to generate the replications (X_i, Y_i) should roughly be the same as generating the Y_i alone. We now discuss these conditions in some more detail for the specific context at hand.

3.2. Arithmetic lower bounds (ALBs)

We now show that the stop-loss bounds L_{t_j} given in (9) are usually useful as control variates in simulation schemes for pricing Asian options.

First, the intrinsic properties of the sum L_{t_j} make it often possible to compute $E[(L_{t_j} - K)^+]$ ($j = 1, \dots, n$) explicitly. Indeed, as shown in Section 2 and in Albrecher *et al.* (2008), every $E[(L_t - K)^+]$ can be expressed as

$$E[(L_t - K)^+] = \sum_{i=1}^{j(t)-1} \frac{S_0^{1-\frac{i}{t}}}{n} E[(S_t^{\frac{i}{t}} - k^{\frac{i}{t}})^+] + \sum_{i=j(t)}^n \frac{e^{r(t-i)}}{n} E[(S_t - k)^+], \tag{14}$$

where k is the solution of the equation:

$$\frac{1}{n} \sum_{i=1}^{j(t)-1} S_0 \left(\frac{k}{S_0} \right)^{\frac{t_i}{t}} + \frac{1}{n} \sum_{i=j(t)}^n k e^{r(t_i-t)} = K.$$

As can be easily noticed in (14), evaluating $E[(L_{t_j} - K)^+]$ essentially amounts to pricing a series of European power call payoffs of the type

$$(S_t^r - K)^+,$$

with r being a suitable real number and K a strike. We will see in Sections 4 and 5 that this is often in reach explicitly.

Second, the replications for both H_T and $\mathbf{L} := (L_{t_1}, L_{t_2}, \dots, L_{t_n})$ are fully determined by replications for the S_{t_i} ($i = 1, 2, \dots, n$) so that “almost no extra cost” is implied when generating the replications (H_T^i, \mathbf{L}^i) . Third, the random variable H_T is expected to be strongly related with the different L_{t_j} ($j = 1, 2, \dots, n$) so that significant efficiency gains are made.

We remark that (9) is in fact the arithmetic average of the vector

$$\left(S_0 \left(\frac{S_t}{S_0} \right)^{\frac{t_1}{t}}, \dots, S_0 \left(\frac{S_t}{S_0} \right)^{\frac{t_{j(t)-1}}{t}}, S_t e^{r(t_{j(t)}-t)}, \dots, S_t e^{r(t_n-t)} \right).$$

When using L_{t_j} ($j = 1, 2, \dots, n$) as control variates to speed up the Crude Monte Carlo (CMC), we label this in the following as the ALB approach, and we use the shorthand notation ALB_t to reflect $e^{-rT} E[(L_t - K)^+]$.

3.3. Geometric lower bounds (GLBs)

Using the lower bounds (10) at different spot times t_1, \dots, t_n to construct a multi-dimensional vector of control variates requires the evaluation of (14) for all these spot times. This needs some computational efforts which may slow down the pricing procedures. In order to derive in a more efficient way some control variates that are both highly correlated and fast to compute, we also propose to consider the geometric average of

$$\left(S_0 \left(\frac{S_t}{S_0} \right)^{\frac{t_1}{t}}, \dots, S_0 \left(\frac{S_t}{S_0} \right)^{\frac{t_{j(t)-1}}{t}}, S_t e^{r(t_{j(t)}-t)}, \dots, S_t e^{r(t_n-t)} \right),$$

denoted by \tilde{L}_t , i.e.,

$$\begin{aligned} \tilde{L}_t &:= \left(\prod_{i=1}^{j(t)-1} S_0 \left(\frac{S_t}{S_0} \right)^{\frac{t_i}{t}} \prod_{i=j(t)}^n S_t e^{r(t_i-t)} \right)^{1/n} \\ &= \alpha_t (S_t)^{\beta_t} \end{aligned} \tag{15}$$

with

$$\alpha_t = \left(\prod_{i=1}^{j(t)-1} (S_0)^{1-\frac{t_i}{t}} \prod_{i=j(t)}^n e^{r(t-t)} \right)^{1/n},$$

$$\beta_t = \frac{1}{n} \left(\sum_{i=1}^{j(t)-1} \frac{t_i}{t} + n - j(t) + 1 \right). \tag{16}$$

When using different \tilde{L}_{t_j} ($j = 1, 2, \dots, n$) to enhance simulation, we refer to GLB and denote $GLB_t := e^{-rT} E[(\tilde{L}_t - K)^+]$. Note that

$$GLB_t := e^{-rT} E[(\alpha_t(S_t)^{\beta_t} - K)^+].$$

Compared with the ALBs in (14), we now only need to price one European power call option for each spot time in (15). This saves a lot of computational efforts but using the geometric average is at the cost of the precision. Indeed, it is well known that the geometric averages are always smaller than the arithmetic ones in (almost surely) order, and therefore one also has

$$e^{-rT} E[(\tilde{L}_t - K)^+] \leq e^{-rT} E[(L_t - K)^+].$$

We now proceed by specifying an important class of market models for which our pricing method based on control variates appears useful.

4. MARKET SETTING: SUBORDINATED BROWNIAN MOTION

In this section we discuss subordinated Brownian motion as a suitable way to model asset returns. This provides a convenient framework for using the control variate approach. We will also use this setting in Section 6, where we develop our second method to price Asian options.

4.1. Subordination

It is convenient to express the risky asset price in terms of its log-returns, i.e., we write $S_t = S_0 e^{X_t}$, where X_t is to be interpreted as a cumulative log-return. The workhorse for modeling returns consists in assuming $\{X_t, t \geq 0\}$ is a Brownian motion with drift. This essentially amounts to assuming that the distribution of the increments $X_{t+s} - X_s$ over the time interval $[s, s + t]$, $s, t \geq 0$, is normally distributed. This setup can be traced back to Bachelier (1900), and is known in the literature as the famous Black–Scholes market. Numerous empirical studies have revealed the fact that asset price volatility tends to be time-varying and exhibits clustering effects. This in turn also implies that the assumption of normally distributed returns is not suitable to capture the long-tailed features of financial time series, especially when the problem at hand involves short-term returns; see e.g., Mandelbrot (1963), Eberlein *et al.* (1998) and Carr *et al.* (2002).

An idea to accommodate for this is to generalize the Brownian motion by making time itself stochastic, i.e., by considering time-changed Brownian motions. To this end, suppose that $\phi(u)$ is the characteristic function related to some distribution function (d.f.), and let $Z_t = \theta G_t + \sigma B_{G_t}$, where $\theta \in \mathbb{R}$, $\sigma \in \mathbb{R}_0^+$, B_t is the standard Brownian motion and where $\{G_t, t \geq 0\}$ is another positive stochastic process. The latter process is referred to as time-change, stochastic clock, chronometer or business time. We assume that it starts at zero, has independent and stationary increments and is such that the distribution of an increment over $[s, s+t]$, $s, t \geq 0$, i.e., $G_{t+s} - G_s$ has $(\phi(u))^t$ as its characteristic function. If $\{G_t, t \geq 0\}$ is a positive and strictly increasing Lévy process, it is called a subordinator. It can be shown that a Lévy process that is time-changed by a subordinator remains a Lévy process (see Sato, 1999).

The process $\{Z_t, t \geq 0\}$ constructed in the above way from a Brownian motion is called the subordinated Brownian motion. Such processes provide more flexibility and potential accuracy when modeling stochastic returns. For a full theoretical background, we refer to Cont and Tankov (2004). The focus on subordinated Brownian motions for modeling asset returns is no real restriction because Monroe (1978) essentially showed that arbitrage-free models for assets can be represented as time-changed Brownian motions. Note that many popular models in finance are based on time-changed Brownian motion where the time-change is chosen to be a subordinator. For example, the VG process or the Generalized Hyperbolic Model (including the NIG process) can be represented as Brownian motion time-changed by the respective Gamma process (Madan *et al.*, 1998) or the Generalized Inverse Gaussian process (Eberlein and Keller 1995). The Normal Tempered Stable process (also including the NIG process) can be written as Brownian motion time-changed by a tempered stable subordinator. We now formally present the first two models mentioned.

4.2. The variance Gamma model

This process was originally introduced by Madan and Seneta (1990) and further studied in Madan *et al.* (1998). It is a pure jump process that is obtained by *changing the clock* of a standard Brownian motion by a Gamma process. More precisely, let $\{B_t, t \geq 0\}$ denote a standard Brownian motion, $\sigma > 0$ and $\theta \in \mathbb{R}$; then the VG process $Z^{VG} = \{Z_t^{VG}, t \geq 0\}$, with parameters σ, ν and θ is defined as

$$Z_t^{VG} = \theta G_t + \sigma B_{G_t},$$

where G_t is a Gamma process with mean t and variance νt . The density of G_t can be written as

$$f_{G_t}(x; \frac{t}{\nu}, \nu) = x^{\frac{t}{\nu}-1} \frac{e^{-\frac{x}{\nu}}}{\Gamma(\frac{t}{\nu}) \nu^{\frac{t}{\nu}}}, \quad \nu > 0, \quad (17)$$

where $\Gamma(\cdot)$ stands for the Gamma function.

The density and the characteristic function of a Brownian motion time-changed by a subordinator, G_t , can be derived exploiting the fact that Z_t conditional on $G_t = g$ has a Gaussian distribution with mean θg and variance $\sigma^2 g$. For completeness we report the characteristic and the density function in case of univariate VG:

$$\begin{aligned} \varphi_{VG}(u; \sigma, \nu, \theta) &= \left(1 - iu\nu\theta + \frac{1}{2}\nu u^2\sigma^2 \right)^{-\frac{t}{\nu}}, \\ f_{VG}(x; \sigma, \nu, \theta) &= \frac{2e^{\frac{\theta x}{\sigma^2}}}{\nu^{\frac{t}{\nu}}\sqrt{2\pi\sigma}\Gamma(\frac{t}{\nu})} \left(\frac{x^2}{\frac{2\sigma^2}{\nu} + \theta^2} \right)^{\frac{t}{2\nu} - \frac{1}{4}} K_{\frac{t}{\nu} - \frac{1}{2}} \left(\frac{1}{\sigma^2} \sqrt{x^2(\frac{2\sigma^2}{\nu} + \theta^2)} \right), \end{aligned} \tag{18}$$

where $K_k(\cdot)$ denotes the modified Bessel function of the second kind of order k , see e.g., Madan *et al.* (1998).

In a Lévy market, there are many different equivalent martingale measures to choose. There exist two popular approaches to find an equivalent martingale measure, namely by using the so-called Esscher transform (see Gerber and Shiu, 1994, and the next paragraph) or by mean-correcting the exponential of a Lévy process (see Schoutens, 2003). This last method consists in changing the “drift” parameter in the Lévy process such that the discounted stock-price process becomes a martingale. For the VG model, we follow this last approach as in the paper by Albrecher *et al.* (2008) since we aim to use their parameter estimates in our numerical section.

More precisely, the risk-neutral process for the stock price dynamics in this section is given by the following VG process obtained by replacing the role of the Brownian motion in the original Black–Scholes model by the VG process and by taking into account the right drift such that the discounted stock-price process is a martingale:

$$\begin{aligned} S_t &= S_0 \exp((r + \omega)t + Z_t^{VG}) \\ &= S_0 \exp(\tilde{Z}_t^{VG}), \end{aligned} \tag{19}$$

where $\tilde{Z}_t^{VG} = (r + \omega)t + Z_t^{VG}$ with $\omega = \frac{1}{\nu} \ln(1 - \theta\nu - \frac{\sigma^2\nu}{2})$. Indeed, in this case one easily observes that the mean rate of return on the stock equals the risk-free interest rate r .

4.3. The normal inverse Gaussian model

The NIG process $Z^{NIG} = \{Z_t^{NIG}, t \geq 0\}$ has stationary and independent NIG-distributed increments with $Z_0^{NIG} = 0$. More precisely, Z_t^{NIG} has an NIG law with parameters α, η, δ and μ given by

$$f_{NIG}(x; \alpha, \eta, \delta, \mu) = \frac{\alpha\delta}{\pi} e^{\delta\sqrt{\alpha^2 - \eta^2} + \eta(x - \mu)} \frac{K_1(\alpha\sqrt{\delta^2 + (x - \mu)^2})}{\sqrt{\delta^2 + (x - \mu)^2}} \tag{20}$$

with $0 \leq |\eta| \leq \alpha$, $\delta \geq 0$ and $\mu \in \mathbb{R}$, where $K_1(\cdot)$ denotes the modified Bessel function of the second kind of order 1. For a survey on stochastic processes of the NIG-type, we refer to Barndorff-Nielsen (1998).

It is further well known (see e.g., Schoutens, 2003) that an NIG process Z_t^{NIG} with parameters α, η, δ and μ with density (20) can be rewritten as a drift term plus an Inverse Gaussian time-changed Brownian motion, namely

$$Z_t^{NIG} = \mu t + \eta \delta^2 I_t + \delta B_{I_t},$$

where $\{I_t, t \geq 0\}$ is an Inverse Gaussian process for which the density function is explicitly known, namely

$$f_{IG}(x; a, b) = \frac{a}{\sqrt{2\pi}} e^{ab} x^{-3/2} e^{-\frac{1}{2}(a^2 x^{-1} + b^2 x)}, \quad x > 0$$

with parameters $a = 1$ and $b = \delta \sqrt{\alpha^2 - \eta^2}$.

To apply the NIG model for market price modeling, we recall from Albrecher and Predota (2004) that log-returns of asset prices will be modeled by an NIG process Z_t^{NIG} , and therefore prices by $S_t = S_0 \exp(Z_t^{NIG})$. Since the market model is incomplete, there are many candidates of equivalent martingale measures for risk-neutral valuation and we use, as in the paper by Albrecher and Predota (2004), the method of Esscher transforms in order to be able to use their parameter estimates later on. We refer to Albrecher and Predota (2004) for details and further references.

In particular, Albrecher and Predota (2004) show that for a Lévy process $\{Z_t^{NIG}, t \geq 0\}$ it is possible to define a locally equivalent probability measure P^θ , called the Esscher equivalent measure, through

$$dP^\theta = \exp(\theta Z_t^{NIG} - t \log M_{NIG}(\theta)) dP,$$

with $M_{NIG}(\theta)$, the moment generating function of (20)⁶. This probability measure P^θ is a risk-neutral probability measure if θ is the solution of

$$r = \mu + \delta(\sqrt{\alpha^2 - (\eta + \theta)^2} - \sqrt{\alpha^2 - (\eta + \theta + 1)^2}).$$

The stock prices under this probability measure P^θ also follow an NIG process with new parameter $\eta' = \eta + \theta$. Therefore, under this risk-neutral measure, we consider $S_t = S_0 \exp(\tilde{Z}_t^{NIG})$, where \tilde{Z}_t^{NIG} is the NIG($\alpha, \eta + \theta, \delta, \mu$) process.

Note that the NIG distribution is a special case $\lambda = -\frac{1}{2}$ of the generalized hyperbolic distribution given by the density

$$f_{GH}(x; \alpha, \eta, \delta, \lambda, \mu) = a(\alpha, \eta, \delta, \lambda) \frac{K_{\lambda - \frac{1}{2}}(\alpha \sqrt{\delta^2 + (x - \mu)^2})}{(\delta^2 + (x - \mu)^2)^{\frac{1}{4} - \frac{\lambda}{2}}} e^{\eta(x - \mu)}, \tag{21}$$

$$a(\alpha, \eta, \delta, \lambda) = \frac{(\alpha^2 - \eta^2)^{\frac{\lambda}{2}}}{\sqrt{2\pi} \alpha^{\lambda - \frac{1}{2}} \delta^\lambda K_\lambda(\delta \sqrt{\alpha^2 - \eta^2})},$$

where $K_\nu(x)$ denotes as above the modified Bessel function of the second kind of order ν . Derivation of the characteristic function of the generalized hyperbolic distribution (as well as its Lévy measure) can be found in Prause (1999), namely

$$\varphi_{\text{GH}}(u; \alpha, \eta, \delta, \lambda, \mu) = e^{iu\mu} \left(\frac{\alpha^2 - \eta^2}{\alpha^2 - (\eta + iu)^2} \right)^{\frac{\lambda}{2}} \frac{K_\lambda(\delta\sqrt{\alpha^2 - (\eta + iu)^2})}{K_\lambda(\delta\sqrt{\alpha^2 - \eta^2})}. \tag{22}$$

The generalized hyperbolic process is a pure jump process that has been introduced in finance by the work of Eberlein and Keller (1995). Its construction is similar to the one of the VG and NIG processes. However, this time the stochastic clock follows a generalized inverse Gaussian distribution.

The VG process can also be seen as a special case of the generalized hyperbolic process. The density and characteristic function (18) can be obtained from the generalized hyperbolic density and characteristic functions, namely (21) and (22), by taking $\alpha = \sqrt{(2/\nu) + (\theta^2/\nu^4)}$, $\eta = \theta/\sigma^2$, $\lambda = \sigma^2/\nu$, $\mu = 0$ and $\delta \rightarrow 0$, see e.g., Schoutens (2003).

5. ALB AND GLB AS AN EXPRESSION OF POWER CALLS

We remark that both ALB and GLB are based on the evaluation of power call options with price

$$e^{-rT} E[(\alpha_t (S_t)^{\beta_t} - K)^+]. \tag{23}$$

Indeed, the value of GLB_t is then immediately obtained by substituting the expressions of α_t and β_t stated in (16). The value of ALB_t follows from rewriting (14) as follows:

$$\begin{aligned} e^{-rT} E[(L_t - K)^+] &= e^{-rT} \sum_{i=1}^{j(t)-1} \frac{S_0^{1-\frac{t_i}{T}}}{n} E[(\alpha_{t_i} (S_t)^{\beta_{t_i}} - k^{\frac{t_i}{T}})^+] \\ &\quad + e^{-rT} \sum_{i=j(t)}^n \frac{e^{r(t_i-t)}}{n} E[(\alpha_{t_i} (S_t)^{\beta_{t_i}} - k)^+], \end{aligned}$$

with $\alpha_{t_i} = 1$ for all i , $\beta_{t_i} = \frac{t_i}{T}$ for all $1 \leq i \leq j(t) - 1$, $\beta_{t_i} = 1$ for all $j(t) \leq i$; and with the appropriate strikes (see Sections 3.2 and 3.3).

We now will derive expressions for (23) in VG and NIG models.

5.1. Calculation of power call options in the variance Gamma model

We recall from Section 4 that under the chosen risk neutral measure the risk neutral process is given by (19), namely

$$S(t) = S_0 \exp(\tilde{Z}_t^{VG}),$$

where $\tilde{Z}_t^{VG} = (r + \omega)t + Z_t^{VG}$, $\omega = \frac{1}{\nu} \ln(1 - \theta\nu - \frac{\sigma^2\nu}{2})$, and where $\{Z_t^{VG}, t \geq 0\}$ is a VG process with parameters σ, ν and θ .

As mentioned in the beginning of this section, we study essentially the price of a general power call option, namely

$$\begin{aligned} e^{-rT} E[(\alpha_t(S_t)^{\beta_t} - K)^+] &= e^{-rT} E[(\alpha_t(S_0)^{\beta_t} e^{\beta_t \tilde{Z}_t^{VG}} - K)^+] \\ &= e^{-rT} E[E[(\alpha_t(S_0)^{\beta_t} e^{\beta_t \tilde{Z}_t^{VG}} - K)^+ | G(t) = g]] \\ &= e^{-rT} E \left[\int_{k^*}^{+\infty} (\alpha_t(\xi_t)^{\beta_t} e^{\beta_t x} - K) f(x) dx \right], \end{aligned}$$

where $k^* = \frac{1}{\beta_t} \ln(\frac{K}{\alpha_t(\xi_t)^{\beta_t}})$, $\xi_t = S_0 \exp(rt + \omega t)$ is a deterministic factor, and where $f(x)$ is the Gaussian probability density function with mean θg and variance $\sigma^2 g$.

Hence, the inner expectation is given by the following Black–Scholes-type formula, denoted by $\Phi(g)$:

$$\begin{aligned} \Phi(g) &:= \int_{k^*}^{+\infty} (\alpha_t(\xi_t)^{\beta_t} e^{\beta_t x} - K) f(x) dx \\ &= \alpha_t(\xi_t)^{\beta_t} \exp\left(\beta_t \theta g + \frac{\sigma^2 g \beta_t^2}{2}\right) (1 - \mathcal{N}(k^*; \theta g + \sigma^2 g, \sigma \sqrt{g})) \\ &\quad - K(1 - \mathcal{N}(k^*; \theta g, \sigma \sqrt{g})), \end{aligned}$$

where $\mathcal{N}(x; \mu, \sigma)$ is the Gaussian cumulative distribution function with mean μ and variance σ^2 .

With this notation, the price of a general power call option equals

$$e^{-rT} E[(\alpha_t(S_t)^{\beta_t} - K)^+] = e^{-rT} \int_0^\infty \Phi(g) f_{G(t)}(g) dg.$$

Using the density function $f_{G(t)}(g)$ of (17) and by substituting $y = \frac{g}{\nu}$, one easily finds the following expression:

$$\begin{aligned} e^{-rT} E[(\alpha_t(S_t)^{\beta_t} - K)^+] &= \int_0^\infty \Phi(y\nu) y^{\frac{\nu}{\nu}-1} \nu^{\frac{\nu}{\nu}-1} \frac{e^{-y}}{\Gamma(\frac{\nu}{\nu}) \nu^{\frac{\nu}{\nu}}} \nu dy \\ &= \int_0^\infty y^{\frac{\nu}{\nu}-1} e^{-y} \frac{\Phi(y\nu)}{\Gamma(\frac{\nu}{\nu})} dy. \end{aligned} \tag{24}$$

This integral can be computed numerically by using the generalized Gauss Laguerre integration method. Indeed, the generalized Gauss Laguerre quadrature (GLQ) is a numerical method to evaluate integrals of the form $\int_0^{+\infty} w(y; \alpha) f(y) dy$, where $w(y; \alpha) = y^\alpha e^{-y}$ with $\alpha > -1$ (so in our case

$f(y) = \frac{\Phi(yv)}{\Gamma(\frac{t}{v})}$ and $\alpha = \frac{t}{v} - 1$). More precisely, the GLQ proposes the following approximation:

$$\int_0^{+\infty} y^\alpha e^{-y} f(y) dy \approx \sum_i^n w_i f(y_i),$$

where y_i are the zeros of the generalized Laguerre polynomial $L_n^{(\alpha)}(x)$ (see Abramowitz and Stegun, 1964). These polynomials can be evaluated by using the following recurrence relation:

$$(n + 1)L_{n+1}^{(\alpha)}(y) = (2n + 1 + s - y)L_n^{(\alpha)}(y) - (n + s)L_{n-1}^{(\alpha)}(y)$$

with initial conditions $L_0^{(\alpha)}(y) = 1$ and $L_{-1}^{(\alpha)}(y) = 0$. The weights w_i are given by the formula

$$w_i = \frac{-\Gamma(n + \alpha)}{\Gamma(n + 1)L_{n-1}^{(\alpha)}(x_i) \frac{dL_n^{(\alpha)}}{dx}(x_i)},$$

where the derivatives can be obtained by using the following relation:

$$y \frac{d}{dx} L_n^{(\alpha)}(y) = nL_n^{(\alpha)}(y) - (n + s)L_{n-1}^{(\alpha)}(y).$$

For more details about this method, we refer to Press *et al.* (1992) and Stroud and Secrest (1966). There are already a few papers that successfully applied GLQ for pricing various derivatives under the VG model in an accurate and fast way (see for example, Madan *et al.*, 2011 and Garcia *et al.*, 2007). Browne *et al.* (2003) used generalized GLQ for liquidity premium calculation. In our work, as we have explained above, the generalized GLQ is a good method to compute (24) showing up in the ALB and GLB values. The generalized GLQ leads to accurate and fast approximations.

We remark that another famous and fast method to compute call options is FFT, see e.g., Carr and Madan (1998). Indeed, one could use FFT to compute expressions of the form (23). However, the FFT is adequate if one wants to price options for multiple strikes because it returns a vector of option prices simultaneously for different strikes. This makes it a suitable approach in the context of model calibration, for instance. But in our case we need only the price of one option for a fixed strike for different α_t and β_t . Hence, an integration routine which concentrates on single option prices saves significant computation time. Therefore, we prefer the generalized GLQ.

5.2. Calculation of power call options in the normal inverse Gaussian model

To apply the NIG model, we recall from Section 4.3 that we model stock prices by $S_t = S_0 \exp(\tilde{Z}_t^{NIG})$, where \tilde{Z}_t^{NIG} is a NIG($\alpha, \eta + \theta, \delta, \mu$) process under a well-chosen risk-neutral measure related to the Esscher transform.

Along the same lines as before, we find that

$$\begin{aligned}
 & e^{-rT} E[(\alpha_t(S_t)^{\beta_t} - K)^+] \\
 &= e^{-rT} \int_{\frac{1}{\beta_t} \ln(\frac{K}{\alpha_t S_0^{\beta_t}})}^{\infty} (\alpha_t S_0^{\beta_t} e^{\beta_t x} - K) f_{\text{NIG}}(x; \alpha, \eta + \theta, \delta, \mu) dx.
 \end{aligned}$$

Note that $\exp(\beta_t x) f_{\text{NIG}}(x; \alpha, \eta + \theta, \delta, \mu)$ can be rewritten as

$$f_{\text{NIG}}(x; \alpha, \eta + \theta + \beta_t, \delta, \mu) c'_t,$$

where

$$c'_t = e^{\delta_t(\sqrt{\alpha^2 - (\eta + \theta)^2} - \sqrt{\alpha^2 - (\eta + \theta + \beta_t)^2}) + \beta_t \mu}.$$

Therefore, one can easily derive an expression based on the cumulative distribution functions $F_{\text{NIG}}(\cdot)$:

$$\begin{aligned}
 & e^{-rT} E[(\alpha_t(S_t)^{\beta_t} - K)^+] \\
 &= e^{-rT} c'_t \alpha_t S_0^{\beta_t} \left(1 - F_{\text{NIG}} \left(\frac{1}{\beta_t} \ln \left(\frac{K}{\alpha_t S_0^{\beta_t}} \right); \alpha, \eta + \theta + \beta_t, \delta, \mu \right) \right) \\
 &\quad - K \left(1 - F_{\text{NIG}} \left(\frac{1}{\beta_t} \ln \left(\frac{K}{\alpha_t S_0^{\beta_t}} \right); \alpha, \eta + \theta, \delta, \mu \right) \right).
 \end{aligned}$$

6. LOWER BOUND DERIVED USING THE STOCHASTIC CLOCK

Inspired by the idea of conditioning used to derive the model-independent bounds in Section 2, we now present another method. Indeed, we will condition on different values in the path of the subordinator, also called the stochastic clock. Assuming that $S_t = S_0 \exp(\mu t + \theta G_t + \sigma B_{G_t})$, where B_t is a Brownian motion and $\{G_t, t \geq 0\}$ is the stochastic clock, we can write the average $\frac{1}{n} \sum_{i=1}^n S_{t_i}$ as $\sum_{i=1}^n \alpha(t_i, G_{t_i}) \exp(\sigma B_{G_{t_i}})$, with $\alpha(t_i, G_{t_i}) = \frac{S_0 e^{\mu t_i + \theta G_{t_i}}}{n}$. As before, we assume that $t_n = T$.

Using this notation, we can express the price of the Asian call option at time $t = 0$ as

$$\begin{aligned}
 AC(K, n) &= e^{-rT} E \left[\left(\frac{1}{n} \sum_{i=1}^n S_{t_i} - K \right)^+ \right] \\
 &= e^{-rT} E \left[E \left[\left(\sum_{i=1}^n \alpha(t_i, G_{t_i}) \exp(\sigma B_{G_{t_i}}) - K \right)^+ \mid G_{t_1}, \dots, G_{t_n} \right] \right] \\
 &= e^{-rT} E \left[E_G \left[\left(\sum_{i=1}^n \alpha(t_i, G_{t_i}) \exp(\sigma B_{G_{t_i}}) - K \right)^+ \right] \right],
 \end{aligned}$$

where $E_G[\cdot]$ is a short-hand notation to reflect $E[\cdot \mid G_{t_1}, \dots, G_{t_n}]$.

Note that by conditioning, the inner expectation is essentially an option on an average of log-normally distributed random variables. Indeed, after conditioning we have that $\sigma B_{G_{t_i}}$ are normally distributed random variables with means and variances that only depend on the values of G_{t_i} .

Vanduffel *et al.* (2008a) proposed for $E[(\sum_{i=1}^n \alpha_i e^{X_i} - K)^+]$ with X_i (multivariate) normal random variables, a lower bound based on the so-called maximal variance approximation, namely

$$E \left[\left(E \left[\sum_{i=1}^n \alpha_i e^{X_i} \mid \Lambda = \sum_{i=1}^n \lambda_i X_i \right] - K \right)^+ \right],$$

with an appropriate choice for the coefficients of the conditioning variable $\Lambda = \sum_{i=1}^n \lambda_i X_i$. Indeed, λ_i are chosen in order to maximize the first-order approximation of the variance of $E[\sum_{i=1}^n \alpha_i e^{X_i} \mid \Lambda = \sum_{i=1}^n \lambda_i X_i]$. This method leads to lower very accurate analytical lower bound approximations of the true prices (see e.g., Kaas *et al.*, 2000; Vanduffel *et al.*, 2008a,b; Valdez *et al.*, 2009).

Applying the appropriate formulae of Vanduffel *et al.* (2008a), namely their equations (10), (17), (37) and (40), we derive the lower bound

$$\begin{aligned} AC(K, n) &= e^{-rT} E \left[E_G \left[\left(\sum_{i=1}^n \alpha(t_i, G_{t_i}) \exp(\sigma B_{G_{t_i}}) - K \right)^+ \right] \right] \\ &\geq e^{-rT} E \left[E_G \left[\left(E \left[\sum_{i=1}^n \alpha(t_i, G_{t_i}) \exp(\sigma B_{G_{t_i}}) \mid \Lambda = \sum_{i=1}^n \lambda_i \sigma B_{G_{t_i}} \right] - K \right)^+ \right] \right] \\ &= e^{-rT} E \left[E_G \left[\sum_{i=1}^n \alpha(t_i, G_{t_i}) E[e^{\sigma B_{G_{t_i}}}] \mathcal{N}(R_{G_{t_i}} - \mathcal{N}^{-1}(p)) - K(1 - p) \right] \right] \end{aligned} \tag{25}$$

with

$$\begin{aligned} \lambda_i &= \alpha(t_i, G_{t_i}) E[e^{\sigma B_{G_{t_i}}}] = \alpha(t_i, G_{t_i}) \exp \left(E[\sigma B_{G_{t_i}}] + \frac{1}{2} \text{Var}[\sigma B_{G_{t_i}}] \right), \\ R_{G_{t_i}} &= \frac{\sum_{j=1}^n \alpha(t_j, G_{t_j}) E[e^{\sigma B_{G_{t_j}}}] \text{cov}[B_{G_{t_i}}, B_{G_{t_j}}] \sigma^2}{\sigma_\Lambda} \\ \sigma_\Lambda^2 &= \sum_{i=1}^n \sum_{j=1}^n \alpha(t_i, G_{t_i}) \alpha(t_j, G_{t_j}) E[e^{\sigma B_{G_{t_i}}}] E[e^{\sigma B_{G_{t_j}}}] \text{cov}[B_{G_{t_i}}, B_{G_{t_j}}] \sigma^2, \end{aligned}$$

and with p the solution of the equation

$$Q_p \left[E_G \left[\sum_{i=1}^n \alpha(t_i, G_{t_i}) \exp(\sigma B_{G_{t_i}}) \mid \Lambda = \sum_{i=1}^n \lambda_i \sigma B_{G_{t_i}} \right] \right] = K,$$

where $Q_p(X)$ is the p -quantile risk measure for a random variable X defined as

$$Q_p(X) = \inf\{x \in \mathbb{R} \mid F_X(x) \geq p\}, \quad p \in (0, 1),$$

where $F_X(x) = P[X \leq x]$, and $\mathcal{N}(\cdot)$ is the cumulative distribution function of a standard normal distribution.

Since the expressions in (25) only depend on the values G_{t_i} of the path of the subordinator, we can denote (25) as $e^{-rT} E[H(G_{t_1}, \dots, G_{t_n})]$ for a certain function H given by

$$H(G_{t_1}, \dots, G_{t_n}) := E_G \left[\sum_{i=1}^n \alpha(t_i, G_{t_i}) E[e^{\sigma B_{G_{t_i}}}] \mathcal{N}(R_{G_{t_i}} - \mathcal{N}^{-1}(p)) - K(1 - p) \right]. \quad (26)$$

Therefore, the lower bound (25) could be calculated by a numerical multi-dimensional integration method. However, since there is usually a large number of spot times to take into account, one would need a high-dimensional numerical integration method which usually turns out to be slow and moreover not always precise due to the accumulated errors. Therefore, in this paper we use a Monte Carlo simulation with GLB as control variates to compute the values in (25). This method will be referred to as the lower bound derived by using the Stochastic Clock (i.e., the LBSC approach). As explained before, such a Monte Carlo simulation is easy to implement, and as our numerical results in Section 7 will show, this method leads to very precise approximations. This method has the advantage that one has not to simulate Brownian motions. Indeed, for this approach one only needs to generate the values of the clock and the analytical formula (26). Therefore, one only needs to generate half the numbers in comparison with the case of time-changed Brownian motion. Furthermore, the variance of the conditional expectation $H(G_{t_1}, \dots, G_{t_n})$ is smaller than the variance of the Asian payoff so that convergence is expected to be much faster than the CMC (see also Section 7.2.3).

7. NUMERICAL ILLUSTRATIONS

In this section we present different sets of numerical results in order to illustrate the methods introduced in this paper. We focus on the two subordinated Brownian motion Lévy models that were described in Section 4, namely the VG and NIG processes. For deriving the numerical results, we made use of the CPU Intel Core i7-3630 QM 2.40 GHz and RAM 8.00 GB under the Windows 8 OS 64 bit.

7.1. General setup

We present our numerical results for Asian options in the context of a VG economy (see Section 7.2.1) and an NIG economy (see Section 7.2.2). We use the

CMC) approach as the benchmark to compare the results of the Monte Carlo approaches that use as control variates the ALB, resp. the GLB as presented in Section 3, with the results of the semi-analytic LBSC approach explained in Section 6. In all instances, we use the same 50,000 simulations to perform the calculations. Specifically, for the different methods mentioned we provide the estimated mean, the standard error (SE), the computation time (C), the variance reduction factor (Var-RF) and the efficiency reduction factor (Eff-RF). These parameters are well known and do not require further explanation except the last two.

The Var-RF is defined as the ratio between the variance of the estimator for the method at hand and the variance of the estimator under the CMC approach. Note indeed that variance reduction techniques precisely allow to obtain greater precision for the same budget of computer time, or, equivalently, to obtain a given degree of precision with less simulation time. Therefore, it is natural to consider the Var-RF as a first performance measure. However, to measure the efficiency of approaches, we should not only consider the variance reduction but also the computation time and the possibility of bias. Let μ be the unknown quantity that we estimate (i.e., the mean of the control variate at hand), and let \bar{Y} be the corresponding estimator. The bias,⁷ variance and mean square error of \bar{Y} are defined as,

$$\begin{aligned}\text{Bias}(\bar{Y}) &= E[\bar{Y}] - \mu, \\ \text{Var}(\bar{Y}) &= E[(\bar{Y} - E[\bar{Y}])^2], \\ \text{MSE}(\bar{Y}) &= E[(\bar{Y} - \mu)^2] = (\text{Bias}(\bar{Y}))^2 + \text{Var}(\bar{Y}),\end{aligned}$$

where the mean square error reflects both bias and variance. A second possible measure (see Glasserman, 2003; L'Ecuyer, 1994) to reflect the efficiency (Eff) of an estimator \bar{Y} is then to use the inverse of the average computation time (C) multiplied by the mean square error:

$$\text{Eff}(\bar{Y}) = \frac{1}{C(\bar{Y})\text{MSE}(\bar{Y})}.$$

This efficiency measure balances the issues of computation time, bias and variance reduction and has a strong intuitive appeal. Note, however, that computation time also depends on the efficiency of the particular code used and computer specifications. Furthermore, the penalty for bias in this measure appears in an ad hoc way and lacks a strong theoretical foundation. In some situations (e.g., in the context of risk management) a biased approach may be perfectly acceptable to some degree whereas in other cases (e.g., a trading context) this feature may be considered as completely unacceptable. In other words, the precise interpretation of Eff-RF also depends on the context at hand. This remark also explains why we in the numerical illustrations also provide the computation time for all methods used, and also why we separately show the (estimated) relative bias for the LBSC approach. Formally, the relative bias

(Rel. Bias) is defined as

$$\text{Rel. Bias} = \left| \frac{E[\bar{Y}_{\text{LBSC}}] - \mu}{\mu} \right|,$$

where \bar{Y}_{LBSC} is the statistical estimator under the LBSC approach. Note that μ is not known and needs to be estimated. To this end, we use an unbiased estimator based on 10,000,000 simulations, namely the CMC with the ALBs as control variates (CMC with ALBs).

We also compare the results with two other methods that were recently proposed in the literature. In Dingec and Hormann (2012), the authors propose a Monte Carlo method with control variate for the pricing of arithmetic Asian options in a Lévy setting. The idea of this method is to use as a control variate the geometric average Asian option payoff as calculated under the hypothesis of a geometric Brownian motion. The method is further based on the numerical inversion of the cumulative distribution function of the increments and is therefore applicable to Lévy processes for which the transition law is known. In Fusai and Meucci (2008), the authors propose a Monte Carlo method in which the control variate is the corresponding geometric Asian option but now computed in the (true) Lévy setting. The theoretical value of the geometric Asian option is here obtained using the FFT method. We denote the methods by Fusai and Meucci (2008) and Dingec and Hormann (2012) in this paper further as CMC-GA_Lévy resp. CMC-GA_BS.

Finally, we make some remarks regarding the reported computation time. All methods require some pre-processing. In particular, the regression coefficients that are inherent in the approaches that use control variates need to be computed separately (to avoid bias). All these coefficients are computed from the same sample using 100,000 replications. The computation time for estimating these coefficients is not included in the reported computation time results. Furthermore, the method by Fusai and Meucci (2008) uses the FFT approach to price the value of the geometric Asian option that is used as control variate. Setting up this framework is not completely straightforward and the method needs to be adapted (i.e., the level of discretization and/or truncation used) to the specificities (e.g., the strike) of the payoff at hand. We do not consider these efforts when reporting computation time. Finally, the computational effort that is needed to estimate the bias of the LBSC approach is not included in the reported computation time for this method. Indeed, this computational effort only arises because it is not possible to estimate bias with a closed-form formula and does not reveal information on the intrinsic quality (speed) of the method itself.

7.2. Numerical results for a variance Gamma economy

We first calculate Asian option prices in a VG model. We refer to (19) for the risk-neutral price process and we take the parameters as in Albrecher *et al.* (2008, Table 8), namely $\sigma = 0.2684$, $\nu = 1.1737$ and $\theta = -0.1280$. This setting

leads to an annual volatility for the log-returns of 30 (211%) (see also Dingec and Hormann, 2012 for more details). In our numerical example we take as starting value $S_0 = 100$, risk-free interest rate $r = 0.03$ and consider that the options are averaged monthly with maturity $T = 10$ years, namely the number of averaging $n = 120$ (see Table 1). We also give results for higher and lower volatility cases, namely $\sigma = 0.5884$ and 0.058 , corresponding to annual volatilities of 60 (452%) and 15 (031%) respectively (see Tables 2 and 3 respectively).

In Tables 1–3, the mean, relative bias, SE, C, Eff-RF and Var-RF are presented in these VG settings for different approaches and different strikes. We also present the “True Value”, which is estimated by using CMC with the ALB method and an extremely large number of simulations (10 millions).

7.3. Numerical results for a normal inverse Gaussian economy

In this section we calculate Asian option prices in the settings of an NIG model. In Table 4, results are presented for Asian option prices in the settings of an NIG process, with the log-return following the density (20) and the parameter values taken from Albrecher and Predota (2004), namely $\alpha = 136.29$, $\eta = -8.95$, $\delta = 0.0059$, $\mu = 0.00079$. The initial value S_0 is chosen to be 50, the maturity is 20 days and the number of averaging days n equals the number of days until maturity, namely $n = 20$. The yearly risk-free interest rate equals $r = 0.1$. We also give results for a higher volatility, namely $\delta = 0.1059$, keeping the short maturity of 20 days (see Table 5). Finally, we present results for a high volatility, namely $\delta = 0.5$, and a long maturity of five years with monthly averaging, $n = 60$ (see Table 6). The performances of different methods are summarized for different strikes.

7.4. Discussion

From an analysis of Tables 1 to 6, we observe that the LBSC approach is performing very well in terms of the variance reduction. It is in many cases the approach with the highest values for Var-RF (Tables 1, 2, 4–6). In particular, the approach is performing outstanding when the volatility of the underlying risky asset reaches average to high levels. Recall that all other things being equal, the higher the Var-RF, the smaller the number of simulations that are needed to obtain a given accuracy. Therefore, for a given acceptable bound on the SE, the LBSC approach often needs the smallest calculation time (even if we consider the fact that for each replication the method requires more running resources). We explain the feature of the high variance reduction ability for the LBSC as follows: The approach is based on conditioning, which allows to avoid simulation of the Brownian motion. More precisely, for each realization of the stochastic clock one uses an analytical formula for approximating the (conditional) option price. Next, by generating different scenarios for the clock one obtains an estimated price for the (unconditional) option price. It is then clear that the variance of the estimator using the LBSC approach is lower than the estimator

TABLE 1

SUMMARY OF THE SIMULATIONS FOR VG MODELS WITH $\sigma = 0.2684$, $\nu = 1.1737$, $\theta = -0.1280$, $S_0 = 100$, $r = 0.03$, THE MATURITY IS 10 YEARS ($T = 10$), AND THERE IS MONTHLY AVERAGING ($n = 120$).

VG Model with an Annual Volatility of 30.211% and a Maturity of 10 Years								
Strike	True Value	Approach	Mean	Rel. Bias	SE	C	Eff-RF	Var-RF
K = 60	43.6440	CMC	43.7069	–	0.2043	3.09	1.0	1.0
		CMC-GA_BS	43.7912	–	0.1547	8.99	0.6	1.7
		CMC-GA_Lévy	43.6111	–	0.0263	4.10	45.5	60.3
		CMC with ALBs	43.6194	–	0.0184	3.80	100.2	123.3
		CMC with GLBs	43.5956	–	0.0318	3.74	34.1	41.3
		LBSC	43.6163	0.0633%	0.0019	50.30	3.3	11 561.9
K = 80	32.1815	CMC	32.2381	–	0.1918	3.07	1.0	1.0
		CMC-GA_BS	32.3366	–	0.1470	9.04	0.6	1.7
		CMC-GA_Lévy	32.1553	–	0.0243	4.04	47.3	62.3
		CMC with ALBs	32.1567	–	0.0179	3.77	93.5	114.8
		CMC with GLBs	32.1318	–	0.0290	3.71	36.2	43.7
		LBSC	32.1815	0.0357%	0.0018	50.40	16.6	11 354.1
K = 100	23.1446	CMC	23.1672	–	0.1747	3.09	1.0	1.0
		CMC-GA_BS	23.2488	–	0.1372	9.21	0.5	1.6
		CMC-GA_Lévy	23.1437	–	0.0226	4.16	44.4	59.8
		CMC with ALBs	23.1186	–	0.0170	3.70	88.2	105.6
		CMC with GLBs	23.0971	–	0.0253	3.62	40.7	47.7
		LBSC	23.1136	0.1339%	0.0015	50.62	1.9	13 564.5
K = 120	16.4151	CMC	16.3973	–	0.1558	3.08	1.0	1.0
		CMC-GA_BS	16.4650	–	0.1257	9.21	0.5	1.5
		CMC-GA_Lévy	16.4400	–	0.0213	4.16	39.6	53.5
		CMC with ALBs	16.3942	–	0.0161	3.72	77.5	93.6
		CMC with GLBs	16.3790	–	0.0212	3.63	45.8	54.0
		LBSC	16.3870	0.1710%	0.0010	50.20	1.9	24 273.6
K = 150	9.7445	CMC	9.6845	–	0.1282	3.08	1.0	1.0
		CMC-GA_BS	9.7259	–	0.1077	9.01	0.5	1.4
		CMC-GA_Lévy	9.7753	–	0.0203	4.13	29.7	39.9
		CMC with ALBs	9.7265	–	0.0154	3.47	61.5	69.3
		CMC with GLBs	9.7214	–	0.0169	3.43	51.7	57.5
		LBSC	9.7152	0.1084%	0.0007	50.60	8.9	33 541.3

when using the traditional Monte Carlo simulation. Intuitively, this difference in variance is expected to be quite significant because the stochastic clock is only one source of randomness for the Asian option price, and is independent of the second source, which is the Brownian motion. Unfortunately, and mainly driven by the presence of bias, this efficiency in terms of variance reduction is not reflected in the other efficiency measure, Eff-Rf. One also observes that the bias is increasing in maturity and volatility, but remains rather small. We believe that the LBSC approach is particularly useful in a risk management context because

TABLE 2

SUMMARY OF THE SIMULATIONS FOR VG MODELS WITH $\sigma = 0.5884$, $\nu = 1.1737$, $\theta = -0.1280$, $S_0 = 100$, $r = 0.03$, THE MATURITY IS 10 YEARS ($T = 10$), AND THERE IS MONTHLY AVERAGING ($n = 120$).

VG Model with an Annual Volatility of 60.452% and a Maturity of 10 Years								
Strike	True Value	Approach	Mean	Rel. Bias	SE	C	Eff-RF	Var-RF
K = 60	50.5757	CMC	49.4934	–	0.8367	3.24	1.0	1.0
		CMC-GA_BS	49.3145	–	0.7660	9.18	0.4	1.2
		CMC-GA_Lévy	50.1774	–	0.3715	4.09	4.0	5.1
		CMC with ALBs	50.3151	–	0.1160	3.96	42.6	52.0
		CMC with GLBs	49.8362	–	0.4730	3.85	2.6	3.1
		LBSC	50.1856	0.7712%	0.0056	51.07	0.3	22323.6
K = 80	43.8048	CMC	42.6894	–	0.8279	3.26	1.0	1.0
		CMC-GA_BS	42.5279	–	0.7628	9.30	0.4	1.2
		CMC-GA_Lévy	43.4574	–	0.3698	4.19	3.9	5.0
		CMC with ALBs	43.5767	–	0.1077	3.77	51.1	59.1
		CMC with GLBs	43.1395	–	0.4590	3.67	2.9	3.3
		LBSC	43.4394	0.8341%	0.0107	50.64	0.3	5 986.7
K = 100	38.7141	CMC	37.5352	–	0.8185	3.24	1.0	1.0
		CMC-GA_BS	37.4047	–	0.7589	9.13	0.4	1.2
		CMC-GA_Lévy	38.3246	–	0.3688	4.19	3.8	4.9
		CMC with ALBs	38.4763	–	0.1001	3.93	55.1	66.9
		CMC with GLBs	38.0813	–	0.4444	3.86	2.8	3.4
		LBSC	38.3370	0.9741%	0.0110	50.15	0.3	5 536.7
K = 120	34.6617	CMC	33.5189	–	0.8091	3.26	1.0	1.0
		CMC-GA_BS	33.4197	–	0.7547	9.26	0.4	1.1
		CMC-GA_Lévy	34.3530	–	0.3685	4.22	3.7	4.8
		CMC with ALBs	34.4984	–	0.0933	3.74	65.6	75.2
		CMC with GLBs	34.0747	–	0.4297	4.09	2.8	3.5
		LBSC	34.3474	0.9067%	0.0112	50.80	0.4	5 218.8
K = 150	30.0585	CMC	28.9148	–	0.7954	3.27	1.0	1.0
		CMC-GA_BS	28.8438	–	0.7479	9.17	0.4	1.1
		CMC-GA_Lévy	29.8106	–	0.3693	4.22	3.6	4.6
		CMC with ALBs	29.9484	–	0.0848	3.68	78.2	88.0
		CMC with GLBs	29.5825	–	0.4081	3.38	3.7	3.8
		LBSC	29.7612	0.9892%	0.0103	51.00	0.5	5 963.4

the feature of variance reduction may be considered as more important than the fact that the approach provides estimates that are biased. Note indeed that especially in risk management computation time is still an issue because simulations are typically carried out jointly in the real world and the risk-neutral world (e.g., when calculating the risk for a portfolio of options over a one-year horizon, one needs to compute prices in the risk-neutral world conditional on the occurrence of real-world scenarios). In this instance, small pricing errors can

TABLE 3

SUMMARY OF THE SIMULATIONS FOR VG MODELS WITH $\sigma = 0.058$, $\nu = 1.1737$, $\theta = -0.1280$, $S_0 = 100$, $r = 0.03$, THE MATURITY IS 10 YEARS ($T = 10$), AND THERE IS MONTHLY AVERAGING ($n = 120$).

VG Model with an Annual Volatility of 15.031% and a Maturity of 10 Years								
Strike	True Value	Approach	Mean	Rel. Bias	S.E.	C.	Eff-RF	Var-RF
K = 60	42.1930	CMC	42.2141	-	0.0964	3.05	1.0	1.0
		CMC-GA_BS	42.2457	-	0.0794	9.06	0.5	1.5
		CMC-GA_Lévy	42.1844	-	0.0055	4.02	233.1	307.2
		CMC with ALBs	42.1899	-	0.0053	3.85	262.1	330.8
		CMC with GLBs	42.1869	-	0.0077	3.55	134.7	156.7
		LBSC	42.1876	0.0127%	0.0064	51.15	8.0	226.9
K = 80	28.2478	CMC	28.2458	-	0.0903	3.03	1.0	1.0
		CMC-GA_BS	28.2773	-	0.0750	9.18	0.5	1.4
		CMC-GA_Lévy	28.2397	-	0.0047	4.01	278.9	369.1
		CMC with ALBs	28.2340	-	0.0068	3.74	142.9	176.3
		CMC with GLBs	28.2321	-	0.0082	3.54	103.8	121.3
		LBSC	28.2377	0.0357%	0.0066	50.65	3.4	187.2
K = 100	16.3258	CMC	16.3192	-	0.0757	3.04	1.0	1.0
		CMC-GA_BS	16.3471	-	0.0642	9.34	0.5	1.4
		CMC-GA_Lévy	16.3300	-	0.0038	4.10	294.2	396.8
		CMC with ALBs	16.3292	-	0.0076	3.92	76.9	99.2
		CMC with GLBs	16.3285	-	0.0082	3.87	66.9	85.2
		LBSC	16.3204	0.0333%	0.0061	50.33	5.2	153.5
K = 120	7.7350	CMC	7.7266	-	0.0540	3.05	1.0	1.0
		CMC-GA_BS	7.7447	-	0.0473	9.61	0.4	1.3
		CMC-GA_Lévy	7.7528	-	0.0033	4.12	198.2	267.8
		CMC with ALBs	7.7311	-	0.0073	3.86	43.2	54.7
		CMC with GLBs	7.7301	-	0.0072	3.60	47.7	56.3
		LBSC	7.7306	0.0568%	0.0050	51.10	3.9	116.6
K = 150	1.4991	CMC	1.4876	-	0.0221	3.04	1.0	1.0
		CMC-GA_BS	1.4937	-	0.0204	9.39	0.4	1.2
		CMC-GA_Lévy	1.5112	-	0.0032	4.13	35.1	47.7
		CMC with ALBs	1.5056	-	0.0049	3.92	15.8	20.3
		CMC with GLBs	1.5088	-	0.0057	3.62	12.6	15.0
		LBSC	1.5007	0.1084%	0.0034	50.40	2.1	42.3

surely be neglected but speed is important. Therefore, for asset-liability management purposes, the LBSC approach is a very suitable candidate.

As for the unbiased Monte Carlo approaches that are based on control variates, the analysis of different tables shows that there are two approaches that depart from the others and show nice performance statistics. These are the crude MonteCarlo with the ALBs as control variates (CMC with ALBs approach) and the control variate approach (CMC-GA_Lévy) of Fusai and Meucci (2008).

TABLE 4

SUMMARY OF THE SIMULATIONS FOR NIG MODELS WITH $\alpha = 136.29$, $\eta = -8.95$, $\delta = 0.0059$, $\mu = 0.00079$, $S_0 = 50$, $r = 0.1$, THE MATURITY IS 20 DAYS ($T = 20/365$), AND THERE IS DAILY AVERAGING ($n = 20$).

NIG Model with an Annual Volatility of 12.611% and a Maturity of 20 Days								
Strike	True Value	Approach	Mean	Rel. Bias	SE	C	Eff-RF	Var-RF
K = 47	3.1271	CMC	3.1248		0.003959	0.48	1.0	1.0
		CMC-GA_BS	3.1271		0.000788	1.07	11.3	25.2
		CMC-GA_Lévy	3.1270		0.000017	1.00	27 357.7	56 995.3
		CMC with ALBs	3.1271		0.000034	0.68	9 776.0	13 849.3
		CMC with GLBs	3.1271		0.000038	0.70	7 255.4	10 580.8
		LBSC	3.1271	0.0001%	0.000002	14.80	34 667.5	3 207 703.5
K = 48.5	1.6484	CMC	1.6466		0.003821	0.50	1.0	1.0
		CMC-GA_BS	1.6485		0.000738	1.08	12.4	26.8
		CMC-GA_Lévy	1.6492		0.000015	1.02	33 277.2	67 885.5
		CMC with ALBs	1.6487		0.000194	0.69	280.1	386.6
		CMC with GLBs	1.6487		0.000196	0.72	264.3	380.6
		LBSC	1.6484	0.0006%	0.000010	15.20	2 548.1	161 081.9
K = 50	0.4249	CMC	0.4241		0.002524	0.49	1.0	1.0
		CMC-GA_BS	0.4247		0.000526	1.08	10.5	23.1
		CMC-GA_Lévy	0.4250		0.000011	1.05	25 742.6	55 162.6
		CMC with ALBs	0.4246		0.000306	0.71	47.0	68.2
		CMC with GLBs	0.4246		0.000306	0.69	48.3	68.0
		LBSC	0.4250	0.0017%	0.000009	15.10	1 494.2	73 918.4
K = 51.5	0.0258	CMC	0.0256		0.000632	0.53	1.0	1.0
		CMC-GA_BS	0.0253		0.000231	1.11	3.6	7.4
		CMC-GA_Lévy	0.0280		0.000008	1.09	3 206.2	6 593.8
		CMC with ALBs	0.0257		0.000176	0.75	9.1	12.8
		CMC with GLBs	0.0257		0.000177	0.76	8.9	12.7
		LBSC	0.0258	0.0021%	0.000009	15.00	180.4	5 125.9

Tables 1, 2 and 6 show that the CMC with ALBs method is outperforming for the different criteria considered, whereas Tables 3–5 support the use of CMC-GA_Lévy. A closer inspection of the specifications of different numerical illustrations, in particular considering the aspects of maturity and volatility, suggests that Fusai and Meucci’s method (2008) is better adapted to deal with products that have a rather short maturity (Tables 4 and 5) or when the volatility of the underlying is rather low (Table 3). In contrast, the CMC with ALBs approach is outperforming when the maturities involved are longer (like in insurance products) and in these instances its performance increases with increasing volatility; see Tables 1, 2 and 6. We further investigate these observations by performing a further sensitivity analysis of maturity and volatility factors using an NIG economy. The results of this analysis are shown in Tables 7 and 8, and confirm the above findings. In summary, the numerical examples provide strong evidence

TABLE 5

SUMMARY OF THE SIMULATIONS FOR NIG MODELS WITH $\alpha = 136.29$, $\eta = -8.95$, $\delta = 0.1059$, $\mu = 0.00079$, $S_0 = 50$, $r = 0.1$, THE MATURITY IS 20 DAYS ($T = 20/365$), AND THERE IS DAILY AVERAGING ($n = 20$).

NIG Model with an Annual Volatility of 53.428% and a Maturity of 20 Days							
Strike	True Value	Approach	Mean	Rel. Bias	SE	C	Eff-RF Var-RF
K = 47	3.5060	CMC	3.5012		0.014344	0.48	1.0
		CMC-GA_BS	3.5064		0.000365	1.07	691.1
		CMC-GA_Lévy	3.5069		0.000204	1.00	2 378.0
		CMC with ALBs	3.5073		0.001231	0.68	95.8
		CMC with GLBs	3.5073		0.001258	0.70	89.1
		LBSC	3.5059	0.0030%	0.000006	14.80	601.0
K = 48.5	2.4177	CMC	2.4169		0.012600	0.50	1.0
		CMC-GA_BS	2.4180		0.000339	1.08	638.3
		CMC-GA_Lévy	2.4188		0.000187	1.02	2 221.7
		CMC with ALBs	2.4197		0.001291	0.69	69.0
		CMC with GLBs	2.4198		0.001311	0.72	64.2
		LBSC	2.4176	0.0045%	0.000001	14.90	450.0
K = 50	1.5549	CMC	1.5542		0.010476	0.49	1.0
		CMC-GA_BS	1.5554		0.000311	1.08	514.6
		CMC-GA_Lévy	1.5553		0.000174	1.05	1 689.1
		CMC with ALBs	1.5551		0.001272	0.71	46.8
		CMC with GLBs	1.5552		0.001281	0.69	47.5
		LBSC	1.5550	0.0060%	0.000002	15.00	411.7
K = 51.5	0.9288	CMC	0.9267		0.008248	0.53	1.0
		CMC-GA_BS	0.9292		0.000278	1.11	420.2
		CMC-GA_Lévy	0.9311		0.000163	1.09	1 244.4
		CMC with ALBs	0.9276		0.001197	0.75	33.6
		CMC with GLBs	0.9276		0.001197	0.76	33.1
		LBSC	0.9289	0.0113%	0.000001	15.10	216.7

that the CMC with ALBs approach is suitable in an insurance context that involves longer maturities, whereas Fusai and Meucci's method (2008) is more suitable in a trading context involving shorter maturities. On a more qualitative note, we point out that the CMC with ALBs approach and the CMC with GLBs approach computes the expectations of control variates using Gauss Laguerre integration whereas Fusai and Meucci's method (2008) requires the FFT approach. We found the former approach rather easy to implement and did not encounter major difficulties. The implementation of the FFT requires a careful handling of discretization and truncation (see also Carr and Madan, 1998), which makes it sometimes more difficult to use.

8. APPLICATIONS TO OTHER PRODUCTS

8.1. Unit-linked insurance

Unit-linked insurance products are life-type insurance products where policyholder's premiums are used to purchase units in investment funds generally comprising different assets such as, for example, bonds and stocks, usually selected

TABLE 6

SUMMARY OF THE SIMULATIONS FOR NIG MODELS WITH $\alpha = 136.29$, $\eta = -8.95$, $\delta = 0.5$, $\mu = 0.00079 \times 30$, $S_0 = 50$, $r = 0.1$, THE MATURITY IS FIVE YEARS ($T = 5$), AND THERE IS MONTHLY AVERAGING ($n = 60$).

NIG Model with an Annual Volatility of 21.05% and a Maturity of Five Years								
Strike	True Value	Approach	Mean	Rel. Bias	SE	C	Eff-RF	Var-RF
K = 45	12.6220	CMC	12.6258		0.050729	1.12	1.0	1.0
		CMC-GA_BS	12.6403		0.004346	1.82	83.8	136.2
		CMC-GA_Lévy	12.6238		0.004364	2.00	75.7	135.1
		CMC with ALBs	12.6234		0.003498	1.54	153.0	210.3
		CMC with GLBs	12.6238		0.005214	1.88	56.4	94.7
		LBSC	12.6191	0.0232%	0.000004	31.20	10.8	130 013 302.8
K = 50	10.1101	CMC	10.1218		0.048370	1.14	1.0	1.0
		CMC-GA_BS	10.1292		0.004246	1.85	80.0	129.8
		CMC-GA_Lévy	10.1141		0.004254	2.20	67.0	129.3
		CMC with ALBs	10.1110		0.003777	1.54	121.4	164.0
		CMC with GLBs	10.1113		0.005139	1.82	55.5	88.6
		LBSC	10.1075	0.0255%	0.000004	32.20	12.5	142 851 816.1
K = 55	7.9216	CMC	7.9205		0.045000	1.12	1.0	1.0
		CMC-GA_BS	7.9329		0.004065	1.82	75.4	122.5
		CMC-GA_Lévy	7.9192		0.004067	2.33	58.8	122.4
		CMC with ALBs	7.9207		0.003865	1.55	98.0	135.6
		CMC with GLBs	7.9201		0.004743	1.89	53.3	90.0
		LBSC	7.9193	0.0288%	0.000003	31.50	13.8	177 883 174.4
K = 60	6.0872	CMC	6.1020		0.041040	1.13	1.0	1.0
		CMC-GA_BS	6.0955		0.003918	1.84	67.4	109.7
		CMC-GA_Lévy	6.0835		0.003910	2.29	54.4	110.2
		CMC with ALBs	6.0835		0.003822	1.56	83.5	115.3
		CMC with GLBs	6.0815		0.004264	1.91	54.8	92.6
		LBSC	6.0852	0.0323%	0.000003	31.30	15.7	220 145 773.8

by the policyholder.⁸ Since the fund comprises risky assets, the insurance company generally proposes a minimum guarantee at maturity in order to protect the policyholder against an economy downfall. Let us assume that the policyholder invests only in a single investment fund or stock and that investment premiums P_i are paid at time t_i , $i = 0, 1, 2, \dots, n - 1$, which allow the policyholder to purchase P_i/S_i units and each unit has a value S_T at expiry $T = t_n$. The payoff of the contract at maturity T (conditional upon survival of the insured until time T) is therefore given by the maximum of the fund value (FV_n) and a certain minimum payment K ,

$$\max(FV_n, K) = \max\left(\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}}, K\right) = K + \max\left(\sum_{i=0}^{n-1} P_i \frac{S_T}{S_{t_i}} - K, 0\right).$$

For example, in the case of a guaranteed rate of return R , insurance companies often determine K to be equal to $K = \sum_{i=0}^{n-1} P_i e^{R(T-t_i)}$ (see Schrager and Pelsser, 2004), where the investment premiums P_i can be a function of the guaranteed rate of return R .

TABLE 7

SUMMARY OF THE SIMULATIONS FOR NIG MODELS WITH $\alpha = 136.29$, $\eta = -8.95$, $\mu = 0.00079$, $\delta = 1.02$, $S_0 = 50$, $r = 0.1$, THE MATURITY PERIODS ARE 1, 5 AND 10 YEARS, AND THERE IS MONTHLY AVERAGING.

NIG Model with an Annual Volatility of 30.07% and a Maturity of One Year						
Strike	Approach	Mean	SE	C	Eff-RF	Var-RF
K = 45	CMC	7.8910	0.0354	0.23	1.0	1.0
	CMC-GA_Lévy	7.9164	0.0014	0.37	399.6	642.8
	CMC with ALBs	7.9119	0.0030	0.22	149.4	142.9
K = 60	CMC	1.3985	0.0172	0.25	1.0	1.0
	CMC-GA_Lévy	1.4057	0.0012	0.40	137.0	219.2
	CMC with ALBs	1.4054	0.0027	0.26	38.8	40.3
NIG Model with an Annual Volatility of 30.07% and a Maturity of Five Years						
Strike	Approach	Mean	SE	C	Eff-RF	Var-RF
K = 45	CMC	13.4823	0.0720	1.00	1.0	1.0
	CMC-GA_Lévy	13.4578	0.0081	1.70	46.8	79.5
	CMC with ALBs	13.4667	0.0059	1.15	129.3	148.7
K=60	CMC	7.8520	0.0615	1.10	1.0	1.0
	CMC-GA_Lévy	7.8560	0.0075	1.83	40.3	67.0
	CMC with ALBs	7.8518	0.0057	1.32	98.5	118.1
NIG Model with an Annual Volatility of 30.07% and a Maturity of 10 Years						
Strike	Approach	Mean	SE	C	Eff-RF	Var-RF
K = 45	CMC	16.0506	0.0972	5.60	1.0	1.0
	CMC-GA_Lévy	16.0415	0.0200	8.90	14.8	23.5
	CMC with ALBs	16.0302	0.0088	6.72	102.3	122.7
K=60	CMC	12.2261	0.0923	5.60	1.0	1.0
	CMC-GA_Lévy	12.2298	0.0193	9.05	14.2	22.9
	CMC with ALBs	12.2151	0.0083	6.65	104.2	123.8

Following Schrage and Pelsser (2004), we assume that at time $i = 0, \dots, n - 1$ the premium P_i is given by $P_i = GP_i - FC_i - c_i FV_i$, where GP_i is the gross premium at time i which is paid at regular intervals until the expiry of the insurance contract; FC_i is a fixed cost at time i , including investment costs, administration costs; c_i is the fund value-related cost deduction (including mortality charges) and FV_i is the fund value at time i . In Schrage and Pelsser (2004), it is shown that

$$\sum_{i=0}^{n-1} P_i \frac{S_T}{S_i} = \sum_{i=0}^{n-1} \tilde{P}_i^n \frac{S_T}{S_i},$$

where $\tilde{P}_i^n = (GP_i - FC_i) \prod_{j=1}^{n-i-1} (1 - c_{n-j})$, which is deterministic. In particular, if c_i is constant over time and denoted by $c_i \equiv c$, we have

TABLE 8

SUMMARY OF THE SIMULATIONS FOR NIG MODELS WITH $\alpha = 136.29$, $\eta = -8.95$, $\mu = 0.00079$, $\delta = 1.81$, $S_0 = 50$, $r = 0.1$, THE MATURITY PERIODS ARE 1, 5 AND 10 YEARS, AND THERE IS MONTHLY AVERAGING.

NIG Model with an Annual Volatility of 40.05% and a Maturity of One Year						
Strike	Approach	Mean	SE	C	Eff-RF	Var-RF
K = 45	CMC	8.6853	0.0461	0.25	1.0	1.0
	CMC-GA_Lévy	8.7163	0.0025	0.40	218.8	350.1
	CMC with ALBs	8.7129	0.0042	0.27	113.8	122.9
K = 60	CMC	2.4488	0.0277	0.27	1.0	1.0
	CMC-GA_Lévy	2.4598	0.0021	0.44	102.4	166.9
	CMC with ALBs	2.4586	0.0038	0.30	47.2	52.5
NIG Model with an Annual Volatility of 40.05% and a Maturity of Five Years						
Strike	Approach	Mean	SE	C	Eff-RF	Var-RF
K = 45	CMC	14.6706	0.0996	1.10	1.0	1.0
	CMC-GA_Lévy	14.6663	0.0147	1.80	28.2	46.2
	CMC with ALBs	14.6692	0.0089	1.30	105.4	124.6
K = 60	CMC	9.7760	0.0890	1.15	1.0	1.0
	CMC-GA_Lévy	9.7992	0.0138	1.88	25.4	41.5
	CMC with ALBs	9.7969	0.0081	1.35	103.7	121.7
NIG Model with an Annual Volatility of 40.05 % and a Maturity of 10 Years						
Strike	Approach	Mean	SE	C	Eff-RF	Var-RF
K = 45	CMC	17.0460	0.1473	6.00	1.0	1.0
	CMC-GA_Lévy	17.0598	0.0448	9.20	7.0	10.8
	CMC with ALBs	17.0398	0.0172	7.00	62.6	73.0
K = 60	CMC	13.8717	0.1423	5.88	1.0	1.0
	CMC-GA_Lévy	13.8974	0.0437	9.14	6.8	10.6
	CMC with ALBs	13.8764	0.0160	6.86	67.9	79.2

$\tilde{P}_i^n = (G P_i - F C_i)(1 - c)^{n-i-1}$. In that case, the no-arbitrage price of the unit-linked contract can be rewritten as

$$P = K + E \left[\max \left(\sum_{i=0}^{n-1} \tilde{P}_i^n \frac{S_T}{S_i} - K, 0 \right) \right].$$

Following our approach of modeling the spot price by an exponential VG or NIG process, namely $S_t = S_0 e^{Z_t}$, where Z_t is a VG or NIG process, and using the independent and stationary increment properties of Lévy processes, we obtain

$$\sum_{i=0}^{n-1} \tilde{P}_i^n \frac{S_T}{S_i} = \sum_{i=0}^{n-1} \tilde{P}_i^n e^{Z_T - Z_i} \stackrel{D}{=} \sum_{i=0}^{n-1} \tilde{P}_i^n e^{Z'_{T-i}},$$

where Z' is an independent copy of the Lévy process Z , and $\stackrel{D}{=}$ reflects equality in law.

The no-arbitrage price of the unit-linked contract is now given by

$$P = K + E \left[\max \left(\sum_{i=0}^{n-1} \tilde{P}_i^n e^{Z'_{T-t_i}} - K, 0 \right) \right].$$

It is easy to observe the relation between unit-linked insurance products and Asian options, and therefore the approaches developed in this paper are suitable to price such insurance contracts.

8.2. Ratchet equity-indexed annuities (EIAs)

In this section, we focus on pricing ratchet equity-indexed annuities (EIAs) with Asian-end design. An equity-indexed annuity contract is a fixed annuity earning a minimum rate of interest and offering a potential gain which is tied to the performance of an underlying equity index S and is, in general, measured on the basis of the returns generated by the index over the lifetime of the contract. There exist different types of EIAs and the most popular ones are the annual ratchet EIAs. In the case of annual ratchet EIAs, the index return level R_t is measured each year based on the equity index performances. In the literature you can find different designs for R_t , and according to Lin and Tan (2003) the most common type of design is based on averaging (which according to Marrion, 2000, 2001 represents approximately 60% of Annual Reset EIAs), for example, the index return is often determined by $R_t = \frac{1}{N} \frac{\sum_{i=0}^{N-1} S_{t-i/N}}{S_{t-1}} - 1$, where N is the number of reset dates in one year to calculate the index return measure.

According to Lin and Tan (2003) and Marrion (2000, 2001), ratchet EIAs are among the most popular contracts in the North America market in terms of their sales volume. According to Marrion (2000, 2001), 70% of EIAs sold in the market are of this type. There exist two types of ratchet EIAs, namely the simple ratchet EIAs and the compounded ratchet EIAs. Following Ballotta (2010), simple ratchet EIAs payoffs $B_T^{(s)}$ and compounded ratchet EIAs payoffs $B_T^{(c)}$ at maturity T are respectively given by

$$B_T^{(s)} = 1 + T F + \sum_{t=1}^T [(\alpha R_t - F)^+ - (\alpha R_t - C)^+],$$

$$B_T^{(c)} = \prod_{t=1}^T [1 + F + (\alpha R_t - F)^+ - (\alpha R_t - C)^+],$$

where C and $F \in (0, 1)$ denote fixed cap and floor rates providing respectively an upper bound to the rate of return and a minimum guarantee, and where R_t reflects the index return and α a fixed percentage, called the participation rate.

Therefore, ratchet EIAs can be seen as a portfolio comprising a risk-free component and a sequence of bull spreads composed of call options of types depending on the design of R_t . Using the moment-matching method and assuming a VG economy for the equity index S_t , Ballotta (2010) has derived an approximation for a ratchet EIA where the index return is measured by $R_t = \frac{1}{N} \frac{\sum_{i=0}^{N-1} S_{t-i/N}}{S_{t-1}} - 1$, where N is the number of reset dates in one year to calculate the index-level measure.

Modeling the equity index S by an exponential VG or NIG process, namely $S_t = S_0 e^{Z_t}$, where Z_t is a VG or NIG process (see Section 4), and because of independent and stationary increment properties of Lévy processes, the no-arbitrage price of the ratchet EIAs is in the settings of Ballotta (2010) in the simple case given by

$$B_0^{(s)} = e^{-rT} (1 + T F) + e^{-rT} E \left[T \left(\frac{\alpha}{N} \sum_{i=0}^{N-1} e^{Z'_{1-i/N}} - K_F \right)^+ \right] - e^{-rT} E \left[T \left(\frac{\alpha}{N} \sum_{i=0}^{N-1} e^{Z'_{1-i/N}} - K_C \right)^+ \right],$$

and in the compounded case by

$$B_0^{(c)} = e^{-rT} E \left[1 + F + \left(\frac{\alpha}{N} \sum_{i=0}^{N-1} e^{Z'_{1-i/N}} - K_F \right)^+ - \left(\frac{\alpha}{N} \sum_{i=0}^{N-1} e^{Z'_{1-i/N}} - K_C \right)^+ \right]^T,$$

where $K_F = \alpha + F$ and $K_C = \alpha + C$; and, as before, Z' is an independent copy of the Lévy process Z . Hence, the pricing problem of such ratchet EIAs with Asian-end design reduces to the problem of pricing two Asian options with a maturity of one year and a strike of K_F and K_C . It is now easy to understand that the approximating price approaches developed in this paper to price Asian options can be useful for insurance companies that deal with ratchet EIAs.

9. CONCLUSIONS

In this paper we propose two methods for pricing Asian-style payoffs in a Lévy market setting. The first approach consists in using model-independent lower bounds for Asian options as control variates in a traditional Monte Carlo scheme (CMC with ALBs approach). Numerical results show that this approach outperforms other (unbiased) methods when the contracts involve longer maturities. In contrast, the Monte Carlo method with geometric control variate of Fusai and Meucci (2008) seems to be more efficient in a rather short maturity context. Note that the CMC with ALBs approach avoids the use of FFT

(needed in the well-known Fusai and Meucci (2008) method), which requires more programming efforts and needs to be adjusted to the specificities of the payoff at hand (issues of discretization, truncation and singularities). In this respect Monte Carlo methods are more general approaches that often require less programming efforts than deterministic numerical techniques. We conclude that the (unbiased) CMC with ALBs approach is suitable in a pricing context, especially for the pricing of insurance Asian-type derivatives as these involve longer maturities. Note that we also briefly discussed the application of the methods presented in this paper to other insurance products such as ratchet EIAs and unit-linked insurance agreements.

The second approach (LBSC) is based on the conditioning on the stochastic clock, which makes the nature of the problem multi-variate normal and allows us to make use of very accurate closed-form approximations that we apply conditionally. Next, Monte Carlo simulations with control variates are used to account for the stochasticity of the clock itself. This approach is biased, but the bias appears to be small in practical examples. Moreover, the method is simple to implement and its variance-reduction ability usually surpasses all the other methods developed in this paper as well as the popular control variate Monte Carlo methods of Fusai and Meucci (2008) and Dingec and Hormann (2012). Therefore, the last method offers an attractive trade-off between accuracy and efficiency (speed), which is particularly useful in a risk management context.

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NOTES

1. In contrast with standard calls they are less sensitive to speculators who could drive up the gains from the option by manipulating the price of the underlying asset near maturity.
2. In the remainder of the paper all expectations are tacitly assumed to exist.
3. For more details on the properties of the stop-loss order and its applications in actuarial science, we refer to Shaked and Shanthikumar (1994) or Kaas *et al.* (2008), among others.
4. It is well known that Lévy processes are suitable candidates to model log-returns of risky assets; see e.g., Schoutens (2003).
5. We denote by (X, Y) a generic pair of random variables with the same distribution as each (X_i, Y_i) .
6. As mentioned in footnote 2, we assume tacitly the existence of all expectations, and moreover of all moment generating functions.

7. In this paper we do not consider the bias that arises because of the error caused by the numerical method at hand, but only the bias that appears if one uses a biased estimator. In other words, we only consider the bias when using the LBSC approach to approximate the true value (see Section 6). However, note that all other control variate-based methods that we discuss are also biased to some extent even when the estimator used is unbiased. Indeed, numerical integration methods, such as the GLQ or FFTs, also have an error term and are therefore biased.

8. Some insurers even give the possibility to invest in individual stocks (see Schrager and Pelsler, 2004)

REFERENCES

- ABRAMOWITZ, M. and STEGUN, C.A. (1964) *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*. Dover.
- ALBRECHER, H. and PREDOTA, M. (2004) On asian option pricing for NIG lévy processes. *Journal of Computational and Applied Mathematics*, **172**, 153–168.
- ALBRECHER, H., MAYER, PH. and SCHOUTENS, W. (2008) General lower bounds for arithmetic Asian option prices. *Applied Mathematical Finance*, **15**, 123–149.
- BACHELIER, L. (1900) *Théorie de la spéculation*. PhD thesis, Gauthier-Villars, Paris, France.
- BALLOTTA, L. (2010) Efficient pricing of ratchet equity-indexed annuities in a variance-gamma economy. *North American Actuarial Journal*, **14**, 355–368.
- BARNDORFF-NIELSEN, O.E. (1998) Processes of normal inverse gaussian type. *Finance and Stochastics*, **2**, 41–68.
- BOYLE, P. and POTAPCHIK, A. (2008) Prices and sensitivities of Asian options: A survey. *Insurance: Mathematics and Economics*, **42**, 189–211.
- BROWNE, S., MILEVSKY, M.A. and SALISBURY, T.S. (2003) Asset allocation and the liquidity premium for illiquid annuities. *Journal of Risk and Insurance*, **70**, 509–226.
- CARR, P. and CHOU, A. (1997) Breaking barriers. *Risk*, **10**, 139–145.
- CARR, P. and MADAN, D. (1998) Option valuation using the fast fourier transform. *Journal of Computational Finance*, **2**, 61–73.
- CARR, P., GEMAN, H., MADAN, D. and YOR, M. (2002) The fine structure of asset returns: An empirical investigation. *Journal of Business*, **75**, 305–332.
- CONT, R. and TANKOV, P. (2004) *Financial Modeling with Jump Processes*, CRC Financial Mathematics Series. London: Chapman & Hall.
- DINGEC, K.D. and HORMANN, W. (2012) A general control variate method for option pricing under lévy process. *European Journal of Operational Research*, **221**, 368–377.
- EBERLEIN, E. and KELLER, U. (1995) Hyperbolic distributions in finance. *Bernoulli*, **1**, 281–299.
- EBERLEIN, E., KELLER, U. and PRAUSE, K. (1998) New insights into smile, mis-pricing and value at risk: The hyperbolic model. *Journal of Business*, **71**, 371–405.
- FENTON, L.F. (1960) The sum of log-normal probability distributions in scatter transmission systems. *IEEE Transactions on Communication Systems*, **8**, 57–67.
- FU, M., MADAN, D. and WANG, T. (1999) Pricing continuous Asian options: A comparison of Monte Carlo and Laplace transform inversion methods. *Journal of Computational Finance*, **2**, 49–74.
- FUSAI, G. and MEUCCI, A. (2008) Pricing discretely monitored Asian option under lévy processes. *Journal of Banking and Finance*, **32**, 2076–2088.
- GARCIA, J., GOSENS, S., VIKTORIYA, M. and SCHOUTENS, W. (2007) *Lévy Base Correlation*. Technical report 07-04, Section of Statistics, K.U. Leuven, TU/e, the Netherlands.
- GERBER, H. and S.W. SHIU (1994) Option pricing by Esscher transforms. *Transactions of the Society of Actuaries*, **46**, 99–191.
- GLASSERMAN, P. (2003) *Monte Carlo Methods in Financial Engineering*. New York, NY: Springer.
- KAAS, R., DHAENE, J. and GOOVAERTS, M. (2000) Upper and lower bounds for sums of random variables. *Insurance: Mathematics and Economics*, **27**, 151–168.
- KAAS, R., GOOVAERTS, M., DHAENE, J. and DENUIT, M. (2008) *Modern Actuarial Risk Theory*. New York, NY: Springer.
- KEMNA, A. and VORST, A. (1990) A pricing method for options based on average asset values. *Journal of Banking and Finance*, **14**, 113–130.

- L'ECUYER, P. (1994) Efficiency improvement and variance reduction. In *Proceedings of the 1994 Winter Simulation Conference*, Dec. 1994, 122–132.
- LIN, X. and TAN, K. (2003) Valuation of equity-indexed annuities under stochastic interest rates. *North American Actuarial Journal*, **7**, 72–93.
- MADAN, D. and SENATA, E. (1990) The variance gamma model for share market returns. *Journal of Business*, **63**, 511–524.
- MADAN, D., CARR, P. and CHANG, E. (1998) The variance gamma process and option pricing. *European Finance Review*, **2**, 79–105.
- MADAN, D.B., PISTORIUS, M.R. and SCHOUTENS, W. (2011) The valuation of structured products using Markov chain models. *Quantitative Finance*, **13**, 125–136.
- MANDELBROT, B. (1963) The variation of certain speculative prices. *Journal of Business*, **36**, 394–419.
- MARRION, J. (2000) Advantage 2000 equity index report (November). Available online at <http://www.indexannuity.org>.
- MARRION, J. (2001) Advantage 2001 equity index report (March). Available online at <http://www.indexannuity.org>.
- MILEVSKY, M. and POSNER, S. (2003) The titanic option: Valuation of the guaranteed minimum death benefit in variable annuities and mutual funds. *Journal of Risk and Insurance*, **68**, 93–128.
- MONROE, I. (1978) Processes that can be embedded in Brownian motion. *Annals of Probability*, **6**, 42–56.
- MÜLLER, A. and STOYAN, D. (2002) *Comparison Methods for Stochastic Models and Risks*. Hoboken, NJ: Wiley.
- PRAUSE, K. (1999) *The Generalized Hyperbolic Model: Estimation, Financial Derivatives, and Risk Measures*. PhD thesis, University of Freiburg, Germany.
- PRESS, W.H., TEUKOLSKY, S.A., VETTERLING, W.T. and FLANNERY, B.P. (1992) *Numerical Recipes in C*. Cambridge, UK: Cambridge University Press.
- ROMEO, M., DA COSTA, V. and BARDOU, F. (2003) Broad distribution effects in sums of lognormal random variables. *European Physical Journal B*, **32**, 513–525.
- SATO, K. (1999) *Lévy Processes and Infinitely Divisible Distributions*. Cambridge, UK: Cambridge University Press.
- SCHOUTENS, W. (2003) *Lévy Processes in Finance: Pricing Financial Derivatives*. Hoboken, NJ: Wiley.
- SCHRAGER, D. and PELSSER, A. (2004) Pricing rate of return guarantees in regular premium unit linked insurance. *Insurance: Mathematics and Economics*, **35**, 369–398.
- SHAKED, M. and SHANTHIKUMAR, J.G. (1994) *Stochastic Orders and Their Applications*. New York, NY: Academic Press.
- STROUD, A.H. and SECREST, D. (1966) *Gaussian Quadrature Formulas*. Upper Saddle River, NJ: Prentice-Hall.
- VALDEZ, E.A., DHAENE, J., MAJ, M. and VANDUFFEL, S. (2009) Bounds and approximations for sums of dependent log-elliptical random variables. *Insurance: Mathematics and Economics*, **44**(3), 385–397.
- VANDUFFEL, S., CHEN, X., DHAENE, J., GOOVAERTS, M., HENRARD, L. and KAAS, R. (2008a) Optimal approximations for risk measure of sums of log normals based on conditional expectations. *Journal of Computational and Applied Mathematics*, **221**, 202–218.
- VANDUFFEL, S., SHANG, Z., HENRARD, L., DHAENE, J. and E.A. VALDEZ (2008b) Analytic bounds and approximations for annuities and asian options. *Insurance: Mathematics and Economics*, **42**(3), 1109–1117.

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