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ON ELLIPTIC PROBLEMS IN DOMAINS WITH UNBOUNDED BOUNDARY

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Abstract The paper deals with problems of the type $-\Delta u + a(x)u = |u|^{p-2}u$, $u > 0$, with zero Dirichlet boundary condition on unbounded domains in $\mathbb{R}^N$, $N \geqslant 2$, with $a(x) \geqslant c > 0$, $p > 2$ and $p < 2N/(N-2)$ if $N \geqslant 3$. The lack of compactness in the problem, related to the unboundedness of the domain, is analysed. Moreover, if the potential $a(x)$ has k suitable 'bumps' and the domain has h suitable 'holes', it is proved that the problem has at least $2(h+k)$ positive solutions (h or k can be zero). The multiplicity results are obtained under a geometric assumption on Ω at infinity which ensures the validity of a local Palais–Smale condition.

Keywords: unbounded boundary domains; Palais–Smale condition; multiplicity of solutions

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1. Introduction

This paper is concerned with problems of the form

$$\left.\begin{aligned} -\Delta u + (1+a(x))u &= u^{p-1} && \text{in } \mathcal{D},\\ u &> 0 && \text{in } \mathcal{D},\\ u &= 0 && \text{on } \partial\mathcal{D}, \end{aligned}\right\} \qquad P(a,\mathcal{D})$$

where $\mathcal{D}$ is an unbounded domain in $\mathbb{R}^N$, $p > 2$, $p < 2N/(N-2)$ if $N \geqslant 3$ and $a(x)$ is a non-negative function in $L^{N/2}(\mathcal{D})$.

Classical arguments show that, when $\mathcal{D}$ is bounded, problem $P(a,\mathcal{D})$ has a solution, whose existence does not depend either on the geometry or on the topology of $\mathcal{D}$. The geometrical-topological properties of $\mathcal{D}$ and the shape of a affect only the multiplicity of solutions; indeed, the number of solutions increases when the structure of $\mathcal{D}$ becomes more complex and when a has more peaks (see, for example, [**4**, **5**, **8**, **15**, **25**]).

The solutions of problem $P(a, \mathcal{D})$ correspond to the positive functions that are critical points of the energy functional

$$E_a(u) = \int_{\mathcal{D}} [|\nabla u|^2 + (1 + a(x))u^2] \, \mathrm{d}x$$

constrained on the manifold

$$M(\mathcal{D}) = \{u \in H_0^1(\mathcal{D}) : |u|_{L^p} = 1\}.$$

The unboundedness of the domain causes a lack of compactness for E_a constrained on $M(\mathcal{D})$, as a consequence of the non-compact embedding $H_0^1(\mathcal{D}) \hookrightarrow L^p(\mathcal{D})$; hence, the topological methods of the calculus of variations do not work directly. This is not only a technical problem. Indeed, there is a large class of domains $\mathcal{D}$ (that includes half-spaces, for instance) in which $P(0, \mathcal{D})$ has no solution (see [**13**]). On the other hand, it is well known (see [**6**, **14**, **16**, **22**]) that $P(0, \mathbb{R}^N)$ has a solution and that this solution, which is unique modulo translation, corresponds to the unique radially symmetric function v_p, $v_p > 0$, that realizes

$$m = \min_{u \in M(\mathbb{R}^N)} E_0(u). \tag{1.1}$$

When $\mathcal{D}$ is an exterior domain (that is $\mathbb{R}^N \setminus \mathcal{D}$ is bounded), E_a satisfies the well-known Palais–Smale compactness condition on $M(\mathcal{D})$ in the energy range $(m, 2^{1-2/p}m)$ (see [**2**, **3**]) and so it is possible to relate the number of solutions of $P(a, \mathcal{D})$ to the shape of $\mathcal{D}$ and a (see [**2**, **3**, **7**, **9**, **10**, **19**–**21**] and references therein).

If not only $\mathcal{D}$ but also $\partial\mathcal{D}$ is unbounded, different situations can occur, from the point of view of the Palais–Smale condition (we refer the reader to [**24**] for a detailed discussion on the Palais–Smale condition on unbounded domains). For example, in [**12**] it is proved that, in strip-like domains $\mathcal{D}$, problem $P(0, \mathcal{D})$ has a solution $u_{\mathcal{D}}$; consequently, for these domains the compactness condition cannot hold at level $E_0(u_{\mathcal{D}}/|u_{\mathcal{D}}|_{L^p})$, because $u_{\mathcal{D}}$ generates a non-compact family of solutions, by translation. Also, for the domains considered in [**11**], every solution causes a lack of compactness at the corresponding energy level. More precisely, in [**11**], domains with periodicity in some directions are studied and, after an analysis of the compactness failure, the multiplicity of solutions is proved. In [**17**], assumptions on the shape of $\mathcal{D}$ at infinity are stated which guarantee a local compactness condition for E_a constrained on $M(\mathcal{D})$. Let us remark that, in the examples mentioned above, a non-converging Palais–Smale sequence actually only exists at some energy levels. In § 2 we construct a domain (with unbounded boundary) such that for every energy level c there exists a non-relatively compact Palais–Smale sequence, at level c, and we remark that easy examples of domains, thin at infinity, can be given, in which the compactness condition globally holds (see also [**18**]). In this paper we also want to analyse the effect of the shape of $\mathcal{D}$ and of a on the number of solutions of $P(a, \mathcal{D})$, when $\mathcal{D}$ is a domain with unbounded boundary.

We make the following assumption on the shape of $\mathcal{D}$ at infinity.

Assumption (C).

$$\lim_{R\to+\infty} \inf\{r(x) : x \in \mathcal{D},\ |x| = R\} = +\infty, \tag{C$_1$}$$

$$\lim_{R\to+\infty} \sup\{h(y) : y \in \partial\mathcal{D},\ |y| = R\} = 0, \tag{C$_2$}$$

where, for $x \in \mathcal{D}$ and $y \in \partial\mathcal{D}$ we define

$$r(x) = \sup\{\rho > 0 : \exists \bar{x} \in \mathcal{D} \text{ such that } x \in B(\bar{x}, \rho) \text{ and } B(\bar{x}, \rho) \subset \mathcal{D}\},$$

$$h(y) = \sup\{\operatorname{dist}(z, T_{\partial\mathcal{D},y} \cap B(y, 1)) : z \text{ is in the connected component of } \partial\mathcal{D} \cap B(y, 1) \text{ containing } y\},$$

with $B(y, r)$, $r > 0$, the ball centred in y with radius r, and $T_{\partial\mathcal{D},y}$ the hyperplane tangent to $\partial\mathcal{D}$ in y.

Assumption (C) says that $\mathcal{D}$ enlarges at infinity and that its boundary flattens (or shrinks) at infinity.

The potential $a(x)$ is assumed to decay exponentially. Namely, we suppose that the following condition holds:

$$\int_{\mathbb{R}^N} a(x)\mathrm{e}^{2|x|}(1 + |x|^{(N-1)\sigma/2})\,\mathrm{d}x < +\infty \quad \text{for some } \sigma \in (1, 2]. \tag{1.2}$$

Now, our goal is to see in which way the presence of 'holes' in the domain and of 'bumps' in the potential affects the number of solutions of $P(a, \mathcal{D})$. To this aim, the effect on the functional E_a constrained on $M(\mathcal{D})$ of the holes and the bumps is analysed. Namely, one can see what happens when the holes enlarge, or narrow, and the bumps increase, or vanish. For example, every hole provides a kind of local maximum level in the constrained energy, whose value increases as the size of the hole increases, and the same holds for the bumps in the potential. Moreover, the interaction of two holes (or bumps) produces a saddle-type level in the constrained energy and an analogous effect is given by the interaction of the holes with the bumps and with the 'exterior' boundary of the domain. The saddle-type level related to the interaction of two holes (or bumps) goes down as the holes (bumps) move away from each other. Exploiting the effects described above, it is possible to state that if there are h suitably large holes in the domain and k suitably high bumps in the potential, with appropriate spacing in between them, then our problem has at least $2(h + k)$ solutions (see Theorem 4.1). Note that a key factor in the variational arguments used is a local compactness condition, which is related to Assumption (C) (see Proposition 2.2).

The method used in the proof of Theorem 4.1 can also be employed to cover the case of small holes and small bumps (see § 5).

The paper is organized as follows. Section 2 deals with Palais–Smale condition. In § 3 we introduce some tools and preliminary results, which are used in § 4 to prove Theorem 4.1. In § 5 we discuss the asymptotic behaviour of the solutions given by Theorem 4.1 and present other multiplicity results (see Theorem 5.1, for instance).

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2. The Palais–Smale condition

In order to find positive functions that are critical points for E_a on $M(\mathcal{D})$, we use minimax techniques of the calculus of variations. In general, a basic tool with which to apply these methods is the Palais–Smale compactness condition for E_a on $M(\mathcal{D})$. When $\mathcal{D}$ is bounded, it is well known that global compactness holds, as a consequence of the Rellich's compact embedding theorem, and variational methods work (see, for example, [**23**, §§ II-2, II-6]). Actually, to obtain the Palais–Smale condition it is sufficient to have

$$H_0^1(\mathcal{D}) \hookrightarrow L^p(\mathcal{D})$$

compact. Hence, taking into account [**1**, Theorem 6.16], for instance, if $\mathcal{D}$ becomes thin at infinity, then compactness holds. As an example, the following domain can be considered:

$$\mathcal{D}_0 = \left\{x = (x_1, \dots, x_N) \in \mathbb{R}^N : \sum_{i=1}^{N-1} |x_i|^2 < \frac{1}{1+|x_N|}\right\} \cup \bigcup_{n=2}^{+\infty} \left\{x \in \mathbb{R}^N : n < |x| < n + \frac{1}{n}\right\}. \tag{2.1}$$

While in $\mathcal{D}_0$ every Palais–Smale sequence at every energy level is relatively compact, the following example provides a domain in which, conversely, for every energy level there exist non-relatively compact Palais–Smale sequences.

Example 2.1. Let $(q_i)_i = \mathbb{Q}^+$ and define

$$S_i = \{(x_1, \dots, x_N) \in \mathbb{R}^N : -\tfrac{1}{2}q_i < x_1 < \tfrac{1}{2}q_i\}.$$

Then set

$$\hat{\mathcal{D}} = \left\{x = (x_1, \dots, x_N) \in \mathbb{R}^N : \sum_{j=2}^{N} x_j^2 < 1\right\} \cup S_1 \cup \left[\bigcup_{i=2}^{\infty} \left[S_i + \left(i + \sum_{j=1}^{i-1} q_j + \tfrac{1}{2}q_i, 0, \dots, 0\right)\right]\right].$$

For every $a \in L^{N/2}(\hat{\mathcal{D}})$, $E_a(M(\hat{\mathcal{D}})) = (m, +\infty)$ holds and, for every $c \in [m, +\infty)$, there exists a Palais–Smale sequence for E_a constrained on $M(\hat{\mathcal{D}})$, at level c, that is not relatively compact.

To prove our assertions, let us recall that for every $i \in \mathbb{N}$ there exists a critical point u_i for E_0 on $M(S_i)$, corresponding to the 'minimal' solution of $P(0, S_i)$ (see [**12**]). Namely, u_i satisfies

$$\Theta(q_i) := \min_{M(S_i)} E_0(u) = E_0(u_i). \tag{2.2}$$

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From (2.2) it follows that the map $\Theta : \mathbb{Q} \to \mathbb{R}$ is a monotone decreasing continuous function and we claim that

$$\lim_{q_i \to +\infty} E_0(u_i) = m, \tag{2.3}$$

$$\lim_{q_i \to 0} E_0(u_i) = +\infty. \tag{2.4}$$

Taking into account (2.2), to prove (2.3) it is sufficient to construct, for every $i \in \mathbb{N}$, a function $w_i \in M(S_i)$ such that

$$\lim_{q_i \to +\infty} E_0(w_i) = m. \tag{2.5}$$

It is well known that the unique radially symmetric function v_p that realizes (1.1) satisfies

$$\lim_{|x| \to +\infty} |v_p(x)|\,|x|^{(N-1)/2} \mathrm{e}^{|x|} = d > 0, \tag{2.6}$$

$$\lim_{|x| \to +\infty} |\nabla v_p(x)|\,|x|^{(N-1)/2} \mathrm{e}^{|x|} = d, \tag{2.7}$$

for a suitable positive constant d (see [**6**]). Moreover, for every $i \in \mathbb{N}$ let us introduce the function $c_i : \mathbb{R}^N \to [0,1]$ defined by $c_i(x) = f(|x|/q_i)$, where $f \in C^\infty(\mathbb{R}^+, [0,1])$ is a non-increasing function such that

$$f \equiv \begin{cases} 1 & \text{in } [0, \frac{1}{4}], \\ 0 & \text{in } [\frac{1}{2}, +\infty). \end{cases}$$

From (2.6) and (2.7) it follows that

$$\lim_{q_i \to +\infty} c_i v_p = v_p \quad \text{in } L^p(\mathbb{R}^N) \text{ and in } H^1(\mathbb{R}^N). \tag{2.8}$$

So, (2.5) follows by setting $w_i = c_i v_p / |c_i v_p|_{L^p}$.

To prove (2.4) observe that, by the Poincaré inequality on a strip, there exists a constant $\bar{c} > 0$ such that

$$|u|_{L^p} \leqslant \bar{c} |\nabla u|_{L^2}, \quad \forall u \in H_0^1(S_1) \tag{2.9}$$

(see [**1**, Theorem 6.30], for example). To simplify the notation, let us assume $q_1 = 1$ and set $\hat{u}_i(x) = u_i(q_i x)$; applying (2.9) to $\hat{u}_i$ we get

$$q_i^{-[2N - p(N-2)]/(2p)} \leqslant \bar{c} |\nabla u_i|_{L^2(S_{q_i})},$$

because $|u_i|_{L^p} = 1 \ \forall i \in \mathbb{N}$, which implies our claim.

Now, if $c = \Theta(q_i)$ for some $i \in \mathbb{N}$, let us construct the sequence $(u_{i,j})_j$ in $M(\hat{\mathcal{D}})$ by

$$u_{i,j}(x_1, \dots, x_N) = u_i\left(x_1 - \left(i + \sum_{r=1}^{i-1} q_r + \tfrac{1}{2} q_i\right), x_2 + j, x_3, \dots, x_N\right)$$

(u_i is extended by 0 outside S_i). It is not then difficult to see that $(u_{i,j})_j$ is a Palais–Smale sequence for E_0 constrained on $M(\hat{\mathcal{D}})$, at level c, and it cannot have a converging subsequence.

If $c \in [m, +\infty) \setminus \Theta(\mathbb{Q})$, let $(q_{i_j})_j$ be a sequence in $\mathbb{Q}$ such that $c = \lim_{j\to+\infty} \Theta(q_{i_j})$. Then $(u_{i_j,j})_j$ is a Palais–Smale sequence for E_0 constrained on $M(\hat{\mathcal{D}})$ at level c that has no converging subsequence.

The equality $(m, +\infty) = E_0(M(\hat{\mathcal{D}}))$ is a direct consequence of (1.1), (2.3), (2.4) and of the continuity of E_0.

Now, to conclude, it is sufficient to observe that, for every fixed non-negative $a \in L^{N/2}(\hat{\mathcal{D}})$, $E_a(u) \geqslant E_0(u)$ $\forall u \in H_0^1(\hat{\mathcal{D}})$ and

$$\begin{aligned}
\lim_{j\to+\infty} |E_a(u_{i,j}) - E_0(u_{i,j})| = 0, \quad \forall i \in \mathbb{N}, \\
\lim_{j\to+\infty} |E_a(u_{i_j,j}) - E_0(u_{i_j,j})| = 0, \\
\lim_{j\to+\infty} \nabla E_a(u_{i,j}) - \nabla E_0(u_{i,j}) = 0, \quad \forall i \in \mathbb{N}, \\
\lim_{j\to+\infty} \nabla E_a(u_{i_j,j}) - \nabla E_0(u_{i_j,j}) = 0 \quad \text{in } H^{-1}(\hat{\mathcal{D}}).
\end{aligned}$$

In the domains we consider, we cannot expect a nice situation, as in (2.1), from the point of view of compactness, but, because of Assumption (C), the situation is also not as bad as in Example 2.1. In fact, we have the following local compactness result.

Proposition 2.2. *Assume that $\mathcal{D} \subset \mathbb{R}^N$ satisfies Assumption (C) and let a be a non-negative function in $L^{N/2}(\mathcal{D})$. If $(u_n)_n$ is a Palais–Smale sequence for E_a constrained on $M(\mathcal{D})$ at a level $c \in (m, 2^{1-2/p}m)$, then $(u_n)_n$ is relatively compact.*

Proof. The proof can be obtained arguing exactly as in the proof of [**17**, Lemma 3.1]. □

By using Proposition 2.2, in the proof of Theorem 4.1 we will find $2(h+k)$ distinct critical values for E_a on $M(\mathcal{D})$ in the energy range $(m, 2^{1-2/p}m)$. The following proposition states that the critical points that correspond to critical values in $(m, 2^{1-2/p}m)$ are functions that do not change sign in $\mathcal{D}$; actually, they have constant sign, by the maximum principle, and so solve $P(a, \mathcal{D})$.

Proposition 2.3. *Let $\mathcal{D}$ be an open set in $\mathbb{R}^N$ and let a be a non-negative function in $L^{N/2}(\mathcal{D})$. If $u \in H_0^1(\mathcal{D})$ is such that*

$$|u|_{L^p} = 1, \qquad E_a(u) = c, \qquad \nabla E_{a|M(\mathcal{D})}(u) = 0,$$

then $u^+ \not\equiv 0$ and $u^- \not\equiv 0$ implies $c > 2^{1-2/p}m$.

Proof. The proof is contained in the proof of [**9**, Theorem 1.1] and can be deduced by the minimality of m (see (1.1)). □

Finally, observe that, in our domains, the Palais–Smale condition cannot hold at level m and that $P(a, \mathcal{D})$ cannot be solved by a minimization argument, because of the following result.

Proposition 2.4. *Let $\mathcal{D}$ be an open set in $\mathbb{R}^N$ satisfying* (C_1) *and $a(x)$ be a non-negative function in $L^{N/2}(\mathbb{R}^N)$. Then*

$$\inf_{M(\mathcal{D})} E_a = m \tag{2.10}$$

and, if $\mathcal{D} \neq \mathbb{R}^N$ or $a \not\equiv 0$, the infimum in (2.10) is not achieved.

Proof. Observe that, by (C_1), for every $n \in \mathbb{N}$ there exists $z_n \in \mathcal{D}$ such that $B(z_n, n) \subset \mathcal{D}$. Then, arguing as for (2.5), we have

$$\inf_{u \in M(\mathcal{D})} E_a(u) = m. \tag{2.11}$$

If $u^* \in M(\mathcal{D})$ realizes (2.11), then u^* also realizes (1.1), because

$$m \leqslant E_0(u^*) \leqslant E_a(u^*) = m. \tag{2.12}$$

Then, by the uniqueness of the minimizers of (1.1), there exists $y^* \in \mathbb{R}^N$ such that $u^*(x) = v_p(x - y^*)$. Since $v_p > 0$ in $\mathbb{R}^N$, we can conclude that $\mathcal{D} = \mathbb{R}^N$. Moreover,

$$\begin{aligned} m = E_a(u^*) &= \int_{\mathbb{R}^N} [|\nabla v_p(x - y^*)|^2 + (1 + a(x)) v_p^2(x - y^*)] \,\mathrm{d}x \\ &= \int_{\mathbb{R}^N} [|\nabla v_p(x - y^*)|^2 + v_p^2(x - y^*)] \,\mathrm{d}x + \int_{\mathbb{R}^N} a(x) v_p^2(x - y^*) \,\mathrm{d}x \\ &= m + \int_{\mathbb{R}^N} a(x) v_p^2(x - y^*) \,\mathrm{d}x, \end{aligned} \tag{2.13}$$

which implies that $a \equiv 0$ because a is non-negative. □

3. Tools, preliminary results and known facts

For every smooth domain $\mathcal{D} \subset \mathbb{R}^N$, we define a cut-off function $c_{\mathcal{D}}$, which is a function in $C^\infty(\mathbb{R}^N, [0, 1])$ such that $c_{\mathcal{D}} = 0$ on $\mathbb{R}^N \setminus \mathcal{D}$, $c_{\mathcal{D}}(x) = 1$ if $x \in \mathcal{D}$ and $\operatorname{dist}(x, \partial\mathcal{D}) \geqslant 1$. If $\mathcal{D} = \mathbb{R}^N$, set $c_{\mathcal{D}} \equiv 1$.

Then we introduce the map $v_{p,\mathcal{D}} : \mathbb{R}^N \to M(\mathcal{D})$ as

$$v_{p,\mathcal{D}}[y](x) = \frac{c_{\mathcal{D}}(x) v_p(x - y)}{|c_{\mathcal{D}} v_p(\cdot - y)|_{L^p}}.$$

Fixing $\zeta \in \partial B(0, 1)$ and $z \in \mathbb{R}^N$, we define

$$\Sigma_z = \partial B(z + \zeta, 2) = \{y \in \mathbb{R}^N : |y - (z + \zeta)| = 2\}$$

and, for $\rho > 0$, we define

$$\Psi_{\mathcal{D},\rho,z} : \Sigma_z \times [0, 1] \to M(\mathcal{D})$$

as

$$\Psi_{\mathcal{D},\rho,z}[y, t](x) = \frac{c_{\mathcal{D}}(x)[(1 - t) v_p(x - [z + \rho(y - z)]) + t v_p(x - (z + \rho\zeta))]}{|c_{\mathcal{D}}[(1 - t) v_p(\cdot - [z + \rho(y - z)]) + t v_p(\cdot - (z + \rho\zeta))]|_{L^p}}.$$

Notice that

$$\Psi_{\mathcal{D},\rho,z}[y,0] = v_{p,\mathcal{D}}[z+\rho(y-z)], \quad \Psi_{\mathcal{D},\rho,z}[y,1] = v_{p,\mathcal{D}}[z+\rho\zeta], \quad \forall z \in \mathbb{R}^N,\ \forall y \in \Sigma_z.$$

For $z \in \mathbb{R}^N$, let $\beta_z : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$ be the function defined by

$$\beta_z(u) = z + \frac{1}{|u|^p_{L^p}} \int_{\mathbb{R}^N} \chi(x-z)|u(x)|^p \,\mathrm{d}x, \tag{3.1}$$

where

$$\chi(x) = \frac{x}{1+|x|}. \tag{3.2}$$

For $u \in L^p(\mathbb{R}^N)$ we set

$$\tilde{u}(x) = \frac{1}{\omega_N} \int_{B(x,1)} |u(y)| \,\mathrm{d}y,$$

where ω_N is the measure of the unit ball in $\mathbb{R}^N$, and

$$\hat{u}(x) = \Big[\tilde{u}(x) - \tfrac{1}{2} \max_{\mathbb{R}^N} \tilde{u}(x)\Big]^+.$$

We then define the map

$$\beta : L^p(\mathbb{R}^N) \setminus \{0\} \to \mathbb{R}^N$$

by

$$\beta(u) = \frac{1}{|\hat{u}|^p_{L^p}} \int_{\mathbb{R}^N} [\hat{u}(x)]^p x \,\mathrm{d}x. \tag{3.3}$$

The 'barycentre-type' functions β_z and β are well defined and continuous in the L^p-norm. Moreover,

$$\beta(u(x-\bar{x})) = \beta(u) + \bar{x}, \quad \forall u \in L^p(\mathbb{R}^N) \setminus \{0\},\ \forall \bar{x} \in \mathbb{R}^N, \tag{3.4}$$

and, by the radial symmetry of v_p,

$$\beta(v_p(\cdot - y)) = y, \quad \forall y \in \mathbb{R}^N. \tag{3.5}$$

Finally, for $x, y \in \mathbb{R}^N$, we denote the segment joining x and y by

$$[x,y] = \{x + t(y-x) : t \in [0,1]\}.$$

Proposition 3.1. *Let $a(x)$ be a non-negative function in $L^{N/2}(\mathbb{R}^N)$, and Ω and ω be open domains in $\mathbb{R}^N$, $\bar{\Omega}, \omega \neq \mathbb{R}^N$. If $a(x) \not\equiv 0$ or $\omega \neq \emptyset$, then there exists $\mu > m$ such that*

$$\inf\{E_{a(\cdot-\bar{x})}(u) : \bar{x} \in \mathbb{R}^N,\ u \in M(\Omega \setminus (\bar{\omega}+\bar{x})),\ \beta(u) = 0\} > \mu, \tag{3.6}$$

$$\inf\{E_{a(\cdot-\bar{x})}(u) : \bar{x} \in \mathbb{R}^N,\ u \in M(\Omega \setminus (\bar{\omega}+\bar{x})),\ \beta(u) = \bar{x}\} > \mu \tag{3.7}$$

(*with the notation* $\inf \emptyset = +\infty$).

Proof. Since $M(\Omega \setminus (\bar{\omega}+\bar{x})) \subseteq M(\Omega)$ and $E_{a(\cdot-\bar{x})}(u) \geqslant E_0(u)$, for all $u \in H^1(\mathbb{R}^N)$ and all $\bar{x} \in \mathbb{R}^N$, in order to prove (3.6) it is sufficient to verify that

$$\inf\{E_0(u) : u \in M(\Omega),\ \beta(u) = 0\} > m. \tag{3.8}$$

Assume, by contradiction, that (3.8) does not hold, i.e. there exists a sequence $(u_n)_n$ in $M(\Omega)$ such that $\beta(u_n) = 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} E_0(u_n) = m$. Since $(u_n)_n$ is a minimizing sequence for (1.1), there exists, as proved in [**3**], a sequence of points $(y_n)_n$ in $\mathbb{R}^N$ and a sequence of functions $(w_n)_n$ in $H^1(\mathbb{R}^N)$ such that

$$u_n(x) = v_p(x-y_n) + w_n(x) \quad \text{with} \lim_{n\to+\infty} \|w_n\|_{H^1} = 0. \tag{3.9}$$

The sequence $(y_n)_n$ has to be unbounded. Indeed, if $y_n \xrightarrow[n\to+\infty]{} \bar{y}$, up to a subsequence, then from (3.9) it follows that

$$\lim_{n\to+\infty} E_0(u_n) = \int_\Omega [|\nabla v_p(x-\bar{y})|^2 + (v_p(x-\bar{y}))^2]\,\mathrm{d}x < m, \tag{3.10}$$

because the function v_p that realizes (1.1) is strictly positive on $\mathbb{R}^N$, while $\bar{\Omega} \neq \mathbb{R}^N$; hence, a contradiction arises.

Observe that, from (3.4), (3.5) and the continuity of β, it follows that

$$\begin{aligned}\lim_{n\to+\infty} |\beta(u_n) - y_n| &= \lim_{n\to+\infty} |\beta(v_p(x-y_n) + w_n(x)) - y_n| \\ &= \lim_{n\to+\infty} |\beta(v_p(x) + w_n(x+y_n))| = 0, \end{aligned} \tag{3.11}$$

and this contradicts the fact that $(y_n)_n$ is unbounded, because $\beta(u_n) \equiv 0$. So (3.8), and thus (3.6), is proved.

Taking into account (3.4) and $M((\Omega - \bar{x}) \setminus \bar{\omega}) \subset M(\mathbb{R}^N \setminus \bar{\omega})$, in order to prove (3.7) it is sufficient to verify that

$$\inf\{E_a(u) : u \in M(\mathbb{R}^N \setminus \bar{\omega}),\ \beta(u) = 0\} > m. \tag{3.12}$$

If $\omega \neq \emptyset$, then (3.12) can be proved by arguing as for the proof of (3.8). Therefore, assume that $\omega = \emptyset$ and suppose, by contradiction, that (3.12) does not hold, i.e. there exists a sequence $(v_n)_n$ in $H^1(\mathbb{R}^N)$ such that

$$E_a(v_n) \xrightarrow[n\to+\infty]{} m$$

and $\beta(v_n) = 0$ for all $n \in \mathbb{N}$. Since $m \leqslant E_0(v_n) \leqslant E_a(v_n)$, because $m = \min_{M(\mathbb{R}^N)} E_0$ and $a(x) \geqslant 0$ for all $x \in \mathbb{R}^N$, we find that $(v_n)_n$ is a minimizing sequence for (1.1). There then exists a sequence of points (y_n) in $\mathbb{R}^N$ and a sequence of functions $(w_n)_n$ in $H^1(\mathbb{R}^N)$ such that (3.9) holds with v_n in place of u_n. The sequence $(y_n)_n$ is unbounded, otherwise $y_n \xrightarrow[n\to+\infty]{} \bar{y}$, up to a subsequence, and

$$\lim_{n\to+\infty} E_a(v_n) = \int_{\mathbb{R}^N} [|\nabla v_p(x-\bar{y})|^2 + (1+a(x))(v_p(x-\bar{y}))^2]\,\mathrm{d}x > m,$$

because $v_p > 0$ in $\mathbb{R}^N$ and $a \not\equiv 0$. But $(y_n)_n$ cannot be unbounded; indeed, arguing as in (3.11), we find that

$$|\beta(v_n) - y_n| \xrightarrow[n\to+\infty]{} 0$$

and that $\beta(v_n) \equiv 0$ holds. So a contradiction arises and (3.12) is proved. □

Proposition 3.2. *Let Ω and ω be open domains in $\mathbb{R}^N$ and let $a \in L^q(\mathbb{R}^N)$ for some $q \in [1, +\infty)$; moreover, assume that ω is bounded and that Ω verifies* (C$_1$). *If $(z_n)_n$ is such that $B(z_n, n) \subset \Omega$, then*

(a) $\displaystyle\lim_{n\to+\infty} \sup_{y\in\partial B(z_n,n/2)} E_{a(\cdot-z_n)}(v_{p,\Omega\setminus(\bar\omega+z_n)}[y]) = m,$

(b) $\displaystyle\lim_{n\to+\infty} \sup_{y\in\partial B(z_n,n/2)} |\beta(v_{p,\Omega\setminus(\bar\omega+z_n)}[y]) - y| = 0.$

Proof. From (2.6) and (2.7) we may infer that

$$\lim_{n\to+\infty} \sup_{y\in\partial B(z_n,n/2)} |v_p(\cdot - y) - c_{\Omega\setminus(\bar\omega+z_n)} v_p(\cdot - y)|_{L^p} = 0, \tag{3.13}$$

$$\lim_{n\to+\infty} \sup_{y\in\partial B(z_n,n/2)} \|v_p(\cdot - y) - c_{\Omega\setminus(\bar\omega+z_n)} v_p(\cdot - y)\|_{H^1} = 0; \tag{3.14}$$

therefore

$$\lim_{n\to+\infty} \sup_{y\in\partial B(z_n,n/2)} \|v_p(\cdot - y) - v_{p,\Omega\setminus(\bar\omega+z_n)}[y]\|_{H^1} = 0. \tag{3.15}$$

Moreover, we claim that

$$\int_{\mathbb{R}^N} a(x-\bar z) v_p^2(x-\bar y)\,\mathrm{d}x = \int_{\mathbb{R}^N} a(x) v_p^2(x-(\bar y-\bar z))\,\mathrm{d}x \to 0 \quad \text{as} \quad |\bar y - \bar z| \to +\infty. \tag{3.16}$$

Indeed, assume that $q > 1$ and call $q' = q/(q-1)$ the conjugate exponent. Let us fix $\eta > 0$ and choose R to be large enough to have

$$|a|_{L^q(\mathbb{R}^N\setminus B(0,R))} < (\eta/2)|v_p^2|^{-1}_{L^{q'}(\mathbb{R}^N)}$$

and let $|\bar y - \bar z|$ be large enough to have

$$|v_p^2|_{L^{q'}(B(\bar z-\bar y,R))} < (\eta/2)|a|^{-1}_{L^q(\mathbb{R}^N)}$$

(from (2.6), it follows that $v_p^2 \in L^r(\mathbb{R}^N)$ for all $r \in [1, +\infty)$). Then

$$\begin{aligned}
\int_{\mathbb{R}^N} a(x) v_p^2(x-(\bar y-\bar z))\,\mathrm{d}x &= \int_{B(0,R)} a(x) v_p^2(x-(\bar y-\bar z))\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N\setminus B(0,R)} a(x) v_p^2(x-(\bar y-\bar z))\,\mathrm{d}x \\
&\leqslant |a|_{L^q(\mathbb{R}^N)} |v_p^2|_{L^{q'}(B(\bar z-\bar y,R))} + |a|_{L^q(\mathbb{R}^N\setminus B(0,R))} |v_p^2|_{L^{q'}(\mathbb{R}^N)} \\
&\leqslant \eta
\end{aligned}$$

and (3.16) follows on letting $\eta \to 0$.

If $q = 1$, then (3.16) follows from the Lebesgue theorem and (2.6).

By using (3.16) and (3.15) we obtain

$$\lim_{n\to+\infty} \sup_{y\in\partial B(z_n,n/2)} \int_{\mathbb{R}^N} a(x-z_n)(v_{p,\Omega\setminus(\bar{\omega}+z_n)}(x-y))^2 \,\mathrm{d}x = 0. \tag{3.17}$$

Assertion (a) follows from (3.15) and (3.17).

By (3.15) we have

$$\lim_{n\to+\infty} \sup_{y\in\partial B(z_n,n/2)} |v_{p,\Omega\setminus(\bar{\omega}+z_n)}[y] - v_p(\cdot - y)|_{L^p} = 0,$$

and hence (b) follows from the continuity of β and from (3.5). □

Remark 3.3. By Proposition 3.2 (b) we find that, for n large enough, there exists $y \in \partial B(z_n, \frac{1}{2}n)$ such that $\beta(v_{p,\Omega\setminus(\bar{\omega}+z_n)}[y]) \in [0, z_n]$.

The same arguments used to prove Propositions 2.4, 3.1 and 3.2 can be used to state the following results, where the effects on the energy functional of the 'holes' in the domain and of the bumps in the potential are described. In particular, these results show that the holes and the bumps play the same role.

Proposition 3.4. *For $x_1, x_2 \in \mathbb{R}^N$ and D_1, D_2 bounded non-empty open sets in $\mathbb{R}^N$, define $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^2 (\bar{D}_i + x_i)$. There then exists $\mu_{D_1,D_2} > m$ such that*

$$\inf_{x_1,x_2\in\mathbb{R}^N} \inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_1, x_2\}\} > \mu_{D_1,D_2} > m.$$

Moreover,

(a) $\inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_1, x_2]\} > m$,

(b) $\displaystyle\lim_{|x_1-x_2|\to+\infty} \sup\{E_0(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|)\} = m$,

(c) $\displaystyle\lim_{|x_1-x_2|\to+\infty} \sup_{y\in\partial B(x_1,|x_1-x_2|/2)} |\beta(v_{p,\mathcal{D}}[y]) - y| = 0$.

Proposition 3.5. *Let a_1 and a_2 be non-negative functions in $L^{N/2}(\mathbb{R}^N)$, with $a_2 \not\equiv 0$, and let ω be a bounded open set in $\mathbb{R}^N$. Define $\mathcal{D} = \mathbb{R}^N \setminus (\bar{\omega} + x_1)$ and $a(x) = a_1(x - x_1) + a_2(x - x_2)$, $x_1, x_2 \in \mathbb{R}^N$; if $a_1 \not\equiv 0$ or $\omega \neq \emptyset$, then there exists $\mu_{\omega,a_1,a_2} > m$ such that*

$$\inf_{x_1,x_2\in\mathbb{R}^N} \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_1, x_2\}\} > \mu_{\omega,a_1,a_2} > m.$$

Moreover,

(a) $\inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_1, x_2]\} > m$,

(b) $\displaystyle\lim_{|x_1-x_2|\to+\infty} \sup\{E_a(v_{p,\mathcal{D}}(\cdot - y)) : y \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|)\} = m$,

(c) $\displaystyle\lim_{|x_1-x_2|\to+\infty} \sup_{y\in\partial B(x_1,|x_2-x_1|/2)} |\beta(v_{p,\mathcal{D}}[y]) - y| = 0$.

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Remark 3.6. From Proposition 3.4 (c) it follows that, for sufficiently large $|x_2 - x_1|$, there exists $y \in \partial B(x_1, \frac{1}{2}|x_2 - x_1|)$ such that $\beta(v_{p,\mathcal{D}}[y]) \in [x_1, x_2]$, where $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^2 (\bar{D}_i + x_i)$. An analogous assertion follows from Proposition 3.5 (c).

Proposition 3.7. *If $a \in L^{N/2}(\mathbb{R}^N)$ is a non-negative function, $a \not\equiv 0$, and $\bar{x} \in \mathbb{R}^N$, then*

(a) $I_a := \inf\{E_{a(\cdot - \bar{x})}(u) : u \in M(\mathbb{R}^N),\ \beta_{\bar{x}}(u) = \bar{x}\} > m$,

(b) $\lim_{\rho \to +\infty} \sup\{E_{a(\cdot - \bar{x})}(\Psi_{\mathbb{R}^N,\rho,\bar{x}}[y, 0]) : y \in \Sigma_{\bar{x}}\} = m$.

Remark 3.8. From the symmetry of v_p, it follows that, for all $\rho > 0$ and all $y \in \Sigma_{\bar{x}}$,

$$\beta_{\bar{x}}(\Psi_{\mathbb{R}^N,\rho,\bar{x}}[y, 0]) = \bar{x} + \theta(\rho|y - \bar{x}|)\frac{y - \bar{x}}{|y - \bar{x}|}, \quad \text{with } \theta(\tau) > 0,\ \forall \tau > 0.$$

Then, as a consequence of the continuity of $\beta_{\bar{x}}$, for every $\rho > 0$ there exists $(\hat{y}, \hat{t}) \in \Sigma_{\bar{x}} \times [0, 1]$ such that $\beta_{\bar{x}}(\Psi_{\mathbb{R}^N,\rho,\bar{x}}[\hat{y}, \hat{t}]) = \bar{x}$.

Proposition 3.9. *Let $\omega \subset \mathbb{R}^N$ be a bounded non-empty open set and $\bar{x} \in \mathbb{R}^N$. If $\mathcal{D} = \mathbb{R}^N \setminus (\bar{\omega} + \bar{x})$, then*

(a) $J_\omega := \inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta_{\bar{x}}(u) = \bar{x}\} > m$,

(b) $\lim_{\rho \to +\infty} \sup\{E_0(\Psi_{\mathcal{D},\rho,\bar{x}}[y, 0]) : y \in \Sigma_{\bar{x}}\} = m$,

(c) $\lim_{\rho \to +\infty} \sup_{y \in \Sigma_{\bar{x}}} |\beta_{\bar{x}}(\Psi_{\mathcal{D},\rho,\bar{x}}[y, 0]) - (\bar{x} + \chi(\rho(y - \bar{x})))| = 0$
(see (3.2) for the definition of χ).

Remark 3.10. From Proposition 3.9 (c) it follows that, for large ρ, there exists $(\hat{y}, \hat{t}) \in \Sigma_{\bar{x}} \times [0, 1]$ such that $\beta_{\bar{x}}(\Psi_{\mathcal{D},\rho,\bar{x}}[\hat{y}, \hat{t}]) = \bar{x}$.

Now, let us establish what happens when ω enlarges and a increases. For $\omega \subset \mathbb{R}^N$ let us set

$$D(\omega) = \sup\{r \in \mathbb{R}^+ : B(y, r) \subset \omega \text{ for some } y \in \mathbb{R}^N\}. \tag{3.18}$$

Proposition 3.11. *Let $\omega \subset \mathbb{R}^N$ be a bounded open set, $\bar{x} \in \mathbb{R}^N$ and define $\mathcal{D} = \mathbb{R}^N \setminus (\bar{\omega} + \bar{x})$. There then exists $\bar{\rho} = \bar{\rho}(\omega)$ such that*

$$\max_{\Sigma_{\bar{x}} \times [0,1]} E_0(\Psi_{\mathcal{D},\rho,\bar{x}}[y, t]) < 2^{1-2/p} m, \quad \forall \rho > \bar{\rho}. \tag{3.19}$$

Moreover, if $B(0, D(\omega)) \subset \bar{\omega}$, then

$$\lim_{D(\omega) \to +\infty} J_\omega = 2^{1-2/p} m. \tag{3.20}$$

(J_ω was introduced in Proposition 3.9 (a).)

Inequality (3.19) is [**10**, Lemma 3.5]; (3.20) is [**19**, Lemma 3.3].

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Proposition 3.12. *Let $(a_n)_n$ be a sequence of non-negative functions in $L^{N/2}(\mathbb{R}^N)$ that verify condition (1.2), and let $\bar{x} \in \mathbb{R}^N$. For every $n \in \mathbb{N}$ there exists $\bar{\rho} = \bar{\rho}(a_n)$ such that*

$$\max_{\Sigma_{\bar{x}} \times [0,1]} E_{a_n(\cdot - \bar{x})}(\Psi_{\mathbb{R}^N,\rho,\bar{x}}[y,t]) < 2^{1-2/p} m, \quad \forall \rho > \bar{\rho}. \tag{3.21}$$

If

$$\lim_{n \to +\infty} a_n(x) = +\infty \quad \text{for a.a. } x \in \mathbb{R}^N,$$

then

$$\lim_{n \to +\infty} I_{a_n} = 2^{1-2/p} m. \tag{3.22}$$

(I_a was introduced in Proposition 3.7 (a).)

We can prove (3.21) and (3.22) by arguing as in the proof of [**21**, Lemmas 2.7 and 3.2], where condition (1.2) has been considered with $\sigma = 2$.

4. A multiplicity result

Theorem 4.1. *Let $h, k \in \mathbb{N}$, $h + k \neq 0$. Assume that Ω is a smooth open set in $\mathbb{R}^N$, $\Omega \neq \mathbb{R}^N$, that satisfies Assumption (C) and that $(a_n^1)_n, \dots, (a_n^k)_n$ are k sequences of non-negative functions in $L^{N/2}(\mathbb{R}^N)$ verifying condition (1.2) and such that*

$$\lim_{n \to +\infty} a_n^j(x) = +\infty \quad \text{for a.a. } x \in \mathbb{R}^N, \ \text{for } j = 1, \dots, k.$$

There then exist

$$\begin{aligned}
D_2 &= D_2(\omega_1), \\
D_3 &= D_3(\omega_2), \\
&\ \vdots \\
D_h &= D_h(\omega_{h-1}), \\
\bar{n}_1 &= \bar{n}_1(\omega_h), \\
\bar{n}_2 &= \bar{n}_2(a_{n_1}^1), \\
&\ \vdots \\
\bar{n}_k &= \bar{n}_k(a_{n_{k-1}}^{k-1}), \\
\bar{x}_1 &= \bar{x}_1(\Omega, \omega_1), \\
&\ \vdots \\
\bar{x}_h &= \bar{x}_h(\Omega, \omega_1, \bar{x}_1, \dots, \omega_{h-1}, \bar{x}_{h-1}, \omega_h), \\
\bar{x}_{h+1} &= \bar{x}_{h+1}(\mathcal{D}, a_{n_1}^1), \\
&\ \vdots \\
\bar{x}_{h+k} &= \bar{x}_{h+k}(\mathcal{D}, a_{n_1}^1, \bar{x}_{h+1}, \dots, a_{n_{k-1}}^{k-1}, \bar{x}_{h+k-1}, a_{n_k}^k),
\end{aligned}$$

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such that problem $P(a,\mathcal{D})$, where $\mathcal{D}=\Omega\setminus\bigcup_{i=1}^h(\bar\omega_i+\bar x_i)$, has at least $2(k+h)$ distinct solutions whenever ω_i are open bounded sets in $\mathbb{R}^N$ such that $D(\omega_i)>D_i$, for $i=2,\dots,h$, and $a(x)$ has the form

$$a(x)=\sum_{j=1}^k a^j_{n_j}(x-\bar x_{h+j}),\tag{4.1}$$

with $n_j>\bar n_j$, for $j=1,\dots,k$.

Proof. We assume, in the first steps, that $\Omega=\mathbb{R}^N$. Moreover, we can assume that $B(0,D(\omega_i))\subset\bar\omega_i$, $i=1,\dots,h$.

Step 1. Let us fix $i\in\{1,\dots,h\}$, $x_i\in\mathbb{R}^N$ and set $\mathcal{D}_i=\mathbb{R}^N\setminus(\bar\omega_i+x_i)$.

From Propositions 2.4, 3.9 and 3.11, and taking into account Remark 3.10, it follows that there exists $\bar\rho_i>0$ such that

$$\beta_{x_i}(\Psi_{\mathcal{D}_i,\bar\rho_i,x_i}[y,0])\text{ is homotopically equivalent in }\mathbb{R}^N\setminus\{x_i\}\text{ to the identity map on }\Sigma_{x_i}\tag{4.2}$$

and

$$\begin{aligned}m&<\sup\{E_0(\Psi_{\mathcal{D}_i,\bar\rho_i,x_i}[y,0]):y\in\Sigma_{x_i}\}\\&<\inf\{E_0(u):u\in M(\mathcal{D}_i),\ \beta_{x_i}(u)=x_i\}\\&\leqslant\sup\{E_0(\Psi_{\mathcal{D}_i,\bar\rho_i,x_i}[y,t]):(y,t)\in\Sigma_{x_i}\times[0,1]\}\\&<2^{1-2/p}m.\end{aligned}\tag{4.3}$$

Moreover, again by Proposition 3.11, we find that for every $i\in\{1,\dots,h-1\}$ there exists $D_{i+1}=D_{i+1}(\omega_i)>0$ such that

$$\begin{aligned}\sup\{E_0(\Psi_{\mathcal{D}_i,\bar\rho_i,x_i}[y,t]):(y,t)\in\Sigma_{x_i}\times[0,1]\}&\\<\inf\{E_0(u):u\in M(\mathcal{D}_{i+1}),\ \beta_{x_{i+1}}(u)=x_{i+1}\}&\end{aligned}\tag{4.4}$$

whenever $D(\omega_{i+1})>D_{i+1}$. In the following, ω_i will be considered fixed, for $i=1,\dots,h$, as chosen in this step.

Step 2. Let us fix $j\in\{1,\dots,k\}$, $x_{h+j}\in\mathbb{R}^N$ and $n\in\mathbb{N}$. By Propositions 2.4, 3.7, 3.12 and Remark 3.8, we can find $\bar\rho^n_j>0$ such that

$$\begin{aligned}\beta_{x_{h+j}}(\Psi_{\mathbb{R}^N,\bar\rho^n_j,x_{h+j}}[y,0])\text{ is homotopically equivalent in }\mathbb{R}^N\setminus\{x_{h+j}\}&\\\text{to the identity map on }\Sigma_{x_{h+j}}&\end{aligned}\tag{4.5}$$

and

$$\begin{aligned}m&<\sup\{E_{a^j_n(\cdot-x_{h+j})}(\Psi_{\mathbb{R}^N,\bar\rho^n_j,x_{h+j}}[y,0]):y\in\Sigma_{x_{h+j}}\}\\&<\inf\{E_{a^j_n(\cdot-x_{h+j})}(u):u\in M(\mathbb{R}^N),\ \beta_{x_{h+j}}(u)=x_{h+j}\}\\&\leqslant\sup\{E_{a^j_n(\cdot-x_{h+j})}(\Psi_{\mathbb{R}^N,\bar\rho^n_j,x_{h+j}}[y,t]):(y,t)\in\Sigma_{x_{h+j}}\times[0,1]\}\\&<2^{1-2/p}m.\end{aligned}\tag{4.6}$$

Moreover, by Proposition 3.12, we can choose $\bar{n}_1 \in \mathbb{N}$ such that

$$\begin{aligned}\sup\{E_0(\Psi_{\mathcal{D}_h,\bar{\rho}_h,x_h}[y,t]) : (y,t) \in \Sigma_{x_h} \times [0,1]\} \\ < \inf\{E_{a_n^1(\cdot - x_{h+1})}(u) : u \in M(\mathbb{R}^N),\ \beta_{x_{h+1}}(u) = x_{h+1}\} \quad (4.7)\end{aligned}$$

for every $n > \bar{n}_1$ and, fixing $n_{j-1} \in \mathbb{N}$ for $j \in \{2, \dots, k\}$, there exists $\bar{n}_j = \bar{n}_j(a^{j-1}_{n_{j-i}}) \in \mathbb{N}$ such that

$$\begin{aligned}\sup\{E_{a^{j-1}_{n_{j-1}}(\cdot - x_{h+j-1})}(\Psi_{\mathbb{R}^N,\bar{\rho}^{n_{j-1}}_{j-1},x_{h+j-1}}[y,t]) : (y,t) \in \Sigma_{x_{h+j-1}} \times [0,1]\} \\ < \inf\{E_{a^j_n(\cdot - x_{h+j})}(u) : u \in M(\mathbb{R}^N),\ \beta_{x_{h+j}}(u) = x_{h+j}\}, \quad (4.8)\end{aligned}$$

for every $n > \bar{n}_j$. In the following, n_j will be considered fixed, for $j = 1, \dots, k$, as chosen in this step. Moreover, we will define $\bar{\rho}^{n_j}_j = \bar{\rho}_{h+j}$.

Notice that in Steps 1 and 2 $\bar{\rho}_i$, D_i and $\bar{n}_i$ are independent of the translation points x_i.

Step 3. From Proposition 3.4, with ω_i in place of D_i (hence $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^2 (\bar{\omega}_i + x_i)$), and from Remark 3.6, it follows that there exists $R_2 = R_2(\omega_1, \omega_2)$ such that, if $|x_2 - x_1| > R_2$, then we have

$$\begin{aligned} m &< \inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_1, x_2]\} \\ &\leqslant \sup\{E_0(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|)\} \\ &< \inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_1, x_2\}\} \end{aligned} \quad (4.9)$$

and

$$\sup\{E_0(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|)\} < \inf\{E_0(u) : u \in M(\mathbb{R}^N \setminus \bar{\omega}_1),\ \beta_0(u) = 0\}. \quad (4.10)$$

Moreover, we can choose R_2 large enough that we also have the following:

$$\begin{aligned}&\text{the homotopy } \mathcal{H}_1 : \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|) \times [0,1] \to \mathbb{R}^N \setminus \{x_1, x_2\} \\ &\qquad \text{given by } \mathcal{H}_1(y,t) = (1-t)y + t\beta(v_{p,\mathcal{D}}[y]) \text{ is well defined.} \quad (4.11)\end{aligned}$$

We can apply this procedure recursively for $j = 3, \dots, h$, fixing x_i, $i = 1, \dots, j-1$. Proposition 3.4, with $D_1 = \bigcup_{i=1}^{j-1} (\omega_i + x_i)$ and $D_2 = \omega_j$ (hence $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^{j} (\bar{\omega}_i + x_i)$), and Remark 3.6 guarantee the existence of $R_j = R_j(\omega_1, \omega_2, x_2, \dots, x_{j-1}, \omega_j)$ such that, for $|x_j - x_{j-1}| > R_j$,

$$\begin{aligned} m &< \inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_{j-1}, x_j]\} \\ &\leqslant \sup\{E_0(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_{j-1}, \tfrac{1}{2}|x_j - x_{j-1}|)\} \\ &< \inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_{j-1}, x_j\}\} \end{aligned} \quad (4.12)$$

and

$$\begin{aligned}\sup\{E_0(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_{j-1}, \tfrac{1}{2}|x_j - x_{j-1}|)\} \\ < \inf\left\{E_0(u) : u \in M\left(\mathbb{R}^N \setminus \bigcup_{i=1}^{j-1} (\bar{\omega}_i + x_i)\right),\ \beta(u) \in [x_{j-2}, x_{j-1}]\right\}. \quad (4.13)\end{aligned}$$

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Moreover, R_j can be chosen large enough that we also have the following:

$$\text{the homotopy } \mathcal{H}_{j-1} : \partial B(x_{j-1}, \tfrac{1}{2}|x_j - x_{j-1}|) \times [0,1] \to \mathbb{R}^N \setminus \{x_{j-1}, x_j\} \\ \text{given by } \mathcal{H}_{j-1}(y,t) = (1-t)y + t\beta(v_{p,\mathcal{D}}[y]) \text{ is well defined.} \tag{4.14}$$

Step 4. Fix $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar{\omega}_i + x_i)$ and, for $n \in \mathbb{N}$ and $x_{h+1} \in \mathbb{R}^N$, set $a(x) = a_n^1(x - x_{h+1})$.

From Proposition 3.5, with $a_1 \equiv 0$ and $a_2 = a_n^1$, and Remark 3.6, it follows that there exists $R_{h+1} = R_{h+1}(\mathcal{D}, n) > 0$ such that, if $|x_{h+1} - x_h| > R_{h+1}$,

$$\begin{aligned} m &< \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_h, x_{h+1}]\} \\ &\leqslant \sup\{E_a(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_h, \tfrac{1}{2}|x_{h+1} - x_h|)\} \\ &< \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_h, x_{h+1}\}\} \end{aligned} \tag{4.15}$$

and

$$\sup\{E_a(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_h, \tfrac{1}{2}|x_{h+1} - x_h|)\} \\ < \inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_{h-1}, x_h]\}. \tag{4.16}$$

Moreover, we can choose R_{h+1} large enough that we also have the following:

$$\text{the homotopy } \mathcal{H}_h : \partial B(x_h, \tfrac{1}{2}|x_{h+1} - x_h|) \times [0,1] \to \mathbb{R}^N \setminus \{x_h, x_{h+1}\} \\ \text{given by } \mathcal{H}_h(y,t) = (1-t)y + t\beta(v_{p,\mathcal{D}}[y]) \text{ is well defined.} \tag{4.17}$$

Step 5. Fix $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar{\omega}_i + x_i)$ and, for $j = 2, \dots, k$, define

$$a(x) = \sum_{i=1}^{j} a_{n_i}^i (x - x_{h+i}),$$

$n_i \in \mathbb{N}$ and $x_{h+i} \in \mathbb{R}^N$. Taking into account Proposition 3.5 and Remark 3.6, we see that there exists $R_{h+j} = R_{h+j}(\mathcal{D}, n_1, x_{h+1}, \dots, n_{j-1}, x_{h+j-1}, n_j)$ such that, for $|x_{h+j} - x_{h+j-1}| > R_{h+j}$,

$$\begin{aligned} m &< \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_{h+j-1}, x_{h+j}]\} \\ &\leqslant \sup\{E_a(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_{h+j-1}, \tfrac{1}{2}|x_{h+j} - x_{h+j-1}|)\} \\ &< \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_{h+j-1}, x_{h+j}\}\} \end{aligned} \tag{4.18}$$

and

$$\sup\{E_a(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_{h+j-1}, \tfrac{1}{2}|x_{h+j} - x_{h+j-1}|)\} \\ < \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_{h+j-2}, x_{h+j-1}]\}. \tag{4.19}$$

Moreover, R_{h+j} can be chosen large enough that we also have:

$$\text{the homotopy } \mathcal{H}_{h+j-1} : \partial B(x_{h+j-1}, \tfrac{1}{2}|x_{h+j} - x_{h+j-1}|) \times [0,1] \to \mathbb{R}^N \setminus \{x_{h+j-1}, x_{h+j}\} \\ \text{given by } \mathcal{H}_{h+j-1}(y,t) = (1-t)y + t\beta(v_{p,\mathcal{D}}[y]) \text{ is well defined.} \tag{4.20}$$

Step 6. First, set $\mathcal{D}_{1,2} = \mathbb{R}^N \setminus \bigcup_{i=1}^{2}(\bar{\omega}_i + x_i)$. From (2.6), (2.7) it follows that, for $i = 1, 2$,

$$\lim_{|x_2-x_1|\to+\infty} \max\{E_0(\Psi_{\mathcal{D}_{1,2},\bar{\rho}_i,x_i}[y,t]) : (y,t) \in K\} = \max\{E_0(\Psi_{\mathcal{D}_i,\bar{\rho}_i,x_i}[y,t]) : (y,t) \in K\}, \tag{4.21}$$

for every compact subset K in $\Sigma_{x_i} \times [0,1]$. Moreover,

$$\lim_{|x_2-x_1|\to+\infty} \beta_{x_i}(\Psi_{\mathcal{D}_{1,2},\bar{\rho}_i,x_i}[y,0]) = \beta_{x_i}(\Psi_{\mathcal{D}_i,\bar{\rho}_i,x_i}[y,0]) \quad \text{uniformly in } \Sigma_{x_i},\ i = 1, 2. \tag{4.22}$$

Hence, in particular, for $|x_2 - x_1|$ sufficiently large, (4.2) holds with $\mathcal{D}_{1,2}$ in place of $\mathcal{D}_i$, so

$$\exists(\bar{z}_i, \bar{t}_i) \in \Sigma_{x_i} \times [0,1] \qquad \text{such that } \beta_{x_i}(\Psi_{\mathcal{D}_{1,2},\bar{\rho}_i,x_i}[\bar{z}_i, \bar{t}_i]) = x_i. \tag{4.23}$$

Now observe that

$$\inf\{E_0(u) : u \in M(\mathcal{D}_i),\ \beta_{x_i}(u) = x_i\} \leqslant \inf\{E_0(u) : u \in M(\mathcal{D}_{1,2}),\ \beta_{x_i}(u) = x_i\}, \quad i = 1, 2. \tag{4.24}$$

From (4.21), (4.23) and (4.24) it follows that, if $|x_2 - x_1|$ is sufficiently large, (4.3) holds with $\mathcal{D}_{1,2}$ in place of $\mathcal{D}_i$, $i = 1, 2$, and (4.4) holds with $i = 1$ and $\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_{1,2}$.

Furthermore, we can also take $|x_2 - x_1|$ to be sufficiently large to get (4.9)–(4.11).

Now, set $\mathcal{D}_{1,2,3} = \mathbb{R}^N \setminus \bigcup_{i=1}^{3}(\bar{\omega}_i + x_i)$, with x_1 and x_2 fixed as in the previous claim. We find that

$$\lim_{|x_3-x_2|\to+\infty} \max\{E_0(\Psi_{\mathcal{D}_{1,2,3},\bar{\rho}_3,x_3}[y,t]) : (y,t) \in K\} = \max\{E_0(\Psi_{\mathcal{D}_3,\bar{\rho}_3,x_3}[y,t]) : (y,t) \in K\}, \tag{4.25}$$

for every compact subset K in $\Sigma_{x_3} \times [0,1]$. Moreover,

$$\lim_{|x_3|\to+\infty} \beta_{x_3}(\Psi_{\mathcal{D}_{1,2,3},\bar{\rho}_3,x_3}[y,0]) = \beta_{x_3}(\Psi_{\mathcal{D}_3,\bar{\rho}_3,x_3}[y,0]) \quad \text{uniformly in } \Sigma_{x_3}, \tag{4.26}$$

so, for $|x_3|$ sufficiently large, (4.2) holds for $i = 3$, with $\mathcal{D}_{1,2,3}$ in place of $\mathcal{D}_3$ and

$$\exists(\bar{z}_3, \bar{t}_3) \in \Sigma_{x_3} \times [0,1] \quad \text{such that } \beta_{x_3}(\Psi_{\mathcal{D}_{1,2,3},\bar{\rho}_3,x_3}[\bar{z}_3, \bar{t}_3]) = x_3. \tag{4.27}$$

Observe also that

$$\inf\{E_0(u) : u \in M(\mathcal{D}_3),\ \beta_{x_3}(u) = x_3\} \leqslant \inf\{E_0(u) : u \in M(\mathcal{D}_{1,2,3}),\ \beta_{x_3}(u) = x_3\}. \tag{4.28}$$

From (4.25), (4.27) and (4.28) it follows that, for $|x_3|$ large enough, (4.3) holds, for $i = 3$, with $\mathcal{D}_{1,2,3}$ in place of $\mathcal{D}_3$.

We have, for $i = 1, 2$,

$$\lim_{|x_3|\to+\infty} \max\{E_0(\Psi_{\mathcal{D}_{1,2,3},\bar{\rho}_i,x_i}[y,t]) : (y,t) \in K\} = \max\{E_0(\Psi_{\mathcal{D}_{1,2},\bar{\rho}_i,x_i}[y,t]) : (y,t) \in K\}, \tag{4.29}$$

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for every compact subset K in $\Sigma_{x_i} \times [0,1]$. Furthermore,

$$\lim_{|x_3|\to+\infty} \beta_{x_i}(\Psi_{\mathcal{D}_{1,2,3},\bar\rho_i,x_i}[y,0]) = \beta_{x_i}(\Psi_{\mathcal{D}_{1,2},\bar\rho_i,x_i}[y,0]) \quad \text{uniformly in } \Sigma_{x_i}. \tag{4.30}$$

Hence, for $|x_3|$ sufficiently large, (4.2) holds with $\mathcal{D}_{1,2,3}$ in place of $\mathcal{D}_i$, for $i = 1, 2$, and

$$\exists (\bar z_i, \bar t_i) \in \Sigma_{x_i} \times [0,1] \quad \text{such that } \beta_{x_i}(\Psi_{\mathcal{D}_{1,2,3},\bar\rho_i,x_i}[\bar z_i, \bar t_i]) = x_i. \tag{4.31}$$

Now, observe that

$$\begin{aligned} &\inf\{E_0(u) : u \in M(\mathcal{D}_{1,2}),\ \beta_{x_i}(u) = x_i\} \\ &\qquad \leqslant \inf\{E_0(u) : u \in M(\mathcal{D}_{1,2,3}),\ \beta_{x_i}(u) = x_i\} \quad \text{for } i = 1, 2. \end{aligned} \tag{4.32}$$

So (4.29), (4.31) and (4.32) imply that, for sufficiently large $|x_3|$, (4.3) also holds, for $i = 1, 2$, with $\mathcal{D}_{1,2,3}$ in place of $\mathcal{D}_i$.

Moreover, taking into account (4.25), (4.29) and (4.28), (4.32), we get (4.4) with $i = 1, 2$ and $\mathcal{D}_i = \mathcal{D}_{i+1} = \mathcal{D}_{1,2,3}$.

We have

$$\begin{aligned} &\lim_{|x_3|\to+\infty} \max\{E_0(v_{p,\mathcal{D}_{1,2,3}}[y]) : y \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|)\} \\ &\qquad = \max\{E_0(v_{p,\mathcal{D}_{1,2}}[y]) : y \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|)\} \end{aligned} \tag{4.33}$$

and

$$\begin{aligned} &\inf\{E_0(u) : u \in M(\mathcal{D}_{1,2}),\ \beta(u) \in \{x_1, x_2\}\} \\ &\qquad \leqslant \inf\{E_0(u) : u \in M(\mathcal{D}_{1,2,3}),\ \beta(u) \in \{x_1, x_2\}\}, \quad \forall x_3 \in \mathbb{R}^N. \end{aligned} \tag{4.34}$$

Furthermore,

$$\lim_{|x_3|\to+\infty} \beta(v_{p,\mathcal{D}_{1,2,3}}[y]) = \beta(v_{p,\mathcal{D}_{1,2}}[y]) \quad \text{uniformly in } \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|), \tag{4.35}$$

so, for $|x_3|$ sufficiently large, (4.11) holds with $\mathcal{D}_{1,2,3}$ in place of $\mathcal{D}$ and, in particular,

$$\exists y_{1,2} \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|) \quad \text{such that } \beta(v_{p,\mathcal{D}_{1,2,3}}[y_{1,2}]) \in [x_1, x_2]. \tag{4.36}$$

Then, from (4.33)–(4.36) it follows that, for sufficiently large $|x_3|$, (4.9)–(4.11) hold with $\mathcal{D}_{1,2,3}$ in place of $\mathcal{D}$.

We can also take $|x_3|$ sufficiently large to verify (4.12)–(4.14) with $j = 3$.

Iterating these arguments, we get (4.2), (4.3) with $\mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar\omega_i + x_i)$ in place of $\mathcal{D}_i$, $i = 1, \dots, h$, (4.4) with

$$\mathcal{D}_i = \mathcal{D}_{i+1} = \mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar\omega_i + x_i), \quad i = 1, \dots, h-1,$$

and (4.9)–(4.14) with $\mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar\omega_i + x_i)$ in place of $\mathcal{D}$.

Now, fix $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar{\omega}_i + x_i)$.

Notice that, arguing as for (3.16), for every $a \in L^{N/2}(\mathbb{R}^N)$ we obtain

$$\lim_{|z|\to+\infty} E_{a(\cdot - z)}(w) = E_0(w) \quad \text{uniformly in } w \in K, K \subset H_0^1(\mathcal{D}) \text{ a compact set.} \tag{4.37}$$

Moreover, for $a(x) = a_{n_1}^1(x - x_{h+1})$, $x_{h+1} \in \mathbb{R}^N$,

$$\begin{aligned} &\inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta_{x_i}(u) = x_i\} \\ &\qquad \leqslant \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta_{x_i}(u) = x_i\}, \qquad i = 1, \dots, h, \end{aligned} \tag{4.38}$$

and

$$\begin{aligned} &\inf\{E_0(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_i, x_{i+1}\}\} \\ &\qquad \leqslant \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_i, x_{i+1}\}\}, \quad i = 1, \dots, h-1. \end{aligned} \tag{4.39}$$

Hence, we can choose $|x_{h+1}|$ sufficiently large in such a way that not only do (4.15)–(4.17) hold, but also (4.3), (4.4), (4.9), (4.10) and (4.12), (4.13) hold, on the domain $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar{\omega}_i + x_i)$, with the functional E_a in place of E_0.

Notice that from (2.6) and (2.7) it follows, for every $a \in L^{N/2}(\mathbb{R}^N)$, that, for every compact set $K \subset \Sigma_0 \times [0, 1]$ and for every $\rho > 0$,

$$\lim_{|z|\to+\infty} E_{a(\cdot - z)}(\Psi_{\mathcal{D},\rho,z}[y + z, t]) = E_a(\Psi_{\mathbb{R}^N,\rho,0}[y, t]) \quad \text{uniformly in } K. \tag{4.40}$$

Moreover,

$$\lim_{|z|\to+\infty} \sup_{y\in\Sigma_0} |(\beta_z(\Psi_{\mathcal{D},\rho,z}[y + z, 0]) - z) - \beta_0(\Psi_{\mathbb{R}^N,\rho,0}[y, 0])| = 0. \tag{4.41}$$

So, also taking into account relations (4.37), (4.40), (4.41) and arguing for the bumps $a_{n_i}^i(x - x_{h+i})$, $i = 1, \dots, k$, as we have done for the holes $(\bar{\omega}_i + x_i)$, $i = 1, \dots, h$, we can choose recursively the 'centre' of the bumps x_{h+i} in such a way that the inequalities stated in Steps 1–5 hold with $\mathcal{D} = \mathbb{R}^N \setminus \bigcup_{i=1}^h (\bar{\omega}_i + x_i)$ and E_a in place of E_0, where $a(x) = \sum_{i=1}^k a_{n_i}^i(x - x_{h+i})$. Namely, we obtain, for $i = 1, \dots, h + k$,

$$\begin{aligned} m &< \sup\{E_a(\Psi_{\mathcal{D},\bar{\rho}_i,x_i}[y, 0]) : y \in \Sigma_{x_i}\} \\ &\quad < b_{1,i} := \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta_{x_i}(u) = x_i\} \\ &\quad \leqslant b_{2,i} := \sup\{E_a(\Psi_{\mathcal{D},\bar{\rho}_i,x_i}[y, t]) : (y, t) \in \Sigma_{x_i} \times [0, 1]\} < 2^{1-2/p} m, \end{aligned} \tag{4.42}$$

$$\beta_{x_i}(\Psi_{\mathcal{D},\bar{\rho}_i,x_i}[y, 0]) \text{ is homotopically equivalent in } \mathbb{R}^N \setminus \{x_i\} \text{ to the identity map on } \Sigma_{x_i} \tag{4.43}$$

and, for $i = 1, \dots, h + k - 1$,

$$\sup\{E_a(\Psi_{\mathcal{D},\bar{\rho}_i,x_i}[y, t]) : (y, t) \in \Sigma_{x_i} \times [0, 1]\} < \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta_{x_{i+1}}(u) = x_{i+1}\}. \tag{4.44}$$

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Moreover, for $i = 1, \dots, h+k-1$,

$$\begin{aligned} m < d_{1,i} &:= \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_i, x_{i+1}]\} \\ &\leqslant d_{2,i} := \sup\{E_a(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_i, \tfrac{1}{2}|x_{i+1} - x_i|)\} \\ &< \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{x_i, x_{i+1}\}\}, \end{aligned} \tag{4.45}$$

$$\text{the homotopy } \mathcal{H}_i : \partial B(x_i, \tfrac{1}{2}|x_{i+1} - x_i|) \times [0,1] \to \mathbb{R}^N \setminus \{x_i, x_{i+1}\} \\ \text{given by } \mathcal{H}_i(y,t) = (1-t)y + t\beta(v_{p,\mathcal{D}}[y]) \text{ is well defined.} \tag{4.46}$$

Furthermore,

$$\sup\{E_a(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_1, \tfrac{1}{2}|x_2 - x_1|)\} \\ < \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta_{x_1}(u) = x_1\} < 2^{1-2/p} m \tag{4.47}$$

and, for $i = 2, \dots, h+k-1$,

$$\sup\{E_a(v_{p,\mathcal{D}}[y]) : y \in \partial B(x_i, \tfrac{1}{2}|x_{i+1} - x_i|)\} \\ < \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [x_{i-1}, x_i]\} < 2^{1-2/p} m. \tag{4.48}$$

Step 7. By Assumption (C), we can now consider a sequence of points $z_n \in \Omega$ such that $B(z_n, n) \subset \Omega$. Fix

$$D = \bigcup_{j=1}^{h} (\bar{\omega}_j + x_j) \quad \text{and} \quad a(x) = \sum_{j=1}^{k} a^j_{n_j}(x - x_{h+j}).$$

For $i = 1, \dots, h+k$ we have

$$\lim_{n\to+\infty} \sup\{|E_{a(\cdot - z_n)}(\Psi_{\mathbb{R}^N \setminus (D+z_n), \bar{\rho}_i, x_i + z_n}[y + z_n, t]) \\ - E_{a(\cdot - z_n)}(\Psi_{\Omega \setminus (D+z_n), \bar{\rho}_i, x_i + z_n}[y + z_n, t])| : (y,t) \in K\} = 0, \\ K \text{ compact in } \Sigma_{x_i} \times [0,1], \tag{4.49}$$

$$\lim_{n\to+\infty} \sup_{y \in \Sigma_{x_i + z_n}} |\beta_{x_i + z_n}(\Psi_{\mathbb{R}^N \setminus (D+z_n), \bar{\rho}_i, x_i + z_n}[y, 0]) - \beta_{x_i + z_n}(\Psi_{\Omega \setminus (D+z_n), \bar{\rho}_i, x_i + z_n}[y, 0])| = 0, \tag{4.50}$$

$$\inf\{E_{a(\cdot - z_n)}(u) : u \in M(\mathbb{R}^N \setminus (D + z_n)),\ \beta_{x_i + z_n}(u) = x_i + z_n\} \\ \leqslant \inf\{E_{a(\cdot - z_n)}(u) : u \in M(\Omega \setminus (D + z_n)),\ \beta_{x_i + z_n}(u) = x_i + z_n\}, \quad \forall n \in \mathbb{N}. \tag{4.51}$$

Moreover, for every compact K in $\mathbb{R}^N$,

$$\lim_{n\to+\infty} \sup_{y \in K} |E_{a(\cdot - z_n)}(v_{p,\mathbb{R}^N \setminus (D+z_n)}[y + z_n]) - E_{a(\cdot - z_n)}(v_{p,\Omega \setminus (D+z_n)}[y + z_n])| = 0, \tag{4.52}$$

$$\lim_{n\to+\infty} \sup_{y \in K} |\beta(v_{p,\mathbb{R}^N \setminus (D+z_n)}[y + z_n]) - \beta(v_{p,\Omega \setminus (D+z_n)}[y + z_n])| = 0, \tag{4.53}$$

$$\inf\{E_{a(\cdot - z_n)}(u) : u \in M(\mathbb{R}^N \setminus (D + z_n)),\ \beta(u) \in K + z_n\} \\ \leqslant \inf\{E_{a(\cdot - z_n)}(u) : u \in M(\Omega \setminus (D + z_n)),\ \beta(u) \in K + z_n\}, \quad \forall n \in \mathbb{N}. \tag{4.54}$$

From (4.49)–(4.54) it follows that (4.42)–(4.48) hold, for n large, with

$$\mathcal{D} = \Omega \setminus \bigcup_{i=1}^{h} (\bar{\omega}_i + (x_i + z_n)), \qquad a(x) = \sum_{i=1}^{k} a^i_{n_i}(x - (x_{h+i} + z_n))$$

and setting

$$\bar{x}_i := x_i + z_n, \quad i = 1, \dots, h+k, \tag{4.55}$$

in place of x_i. Furthermore, by Propositions 3.1, 3.2 and Remark 3.3, n can be chosen to be large enough to also have

$$\begin{aligned} m < d_{1,h+k} &:= \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [0, z_n]\} \\ \leqslant d_{2,h+k} &:= \sup\{E_a(v_{p,\mathcal{D}}) : y \in \partial B(z_n, \tfrac{1}{2}n)\} \\ &< \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in \{0, z_n\}\}, \end{aligned} \tag{4.56}$$

$$\begin{aligned} &\text{the homotopy } \mathcal{H}_{h+k} : \partial B(z_n, \tfrac{1}{2}n) \times [0,1] \to \mathbb{R}^N \setminus \{0, z_n\} \\ &\qquad \text{given by } \mathcal{H}_{h+k}(y,t) = (1-t)y + t\beta(v_{p,\mathcal{D}}[y]) \text{ is well defined} \end{aligned} \tag{4.57}$$

and, moreover,

$$\begin{aligned} m &< \sup\{E_a(v_{p,\mathcal{D}}) : y \in \partial B(z_n, \tfrac{1}{2}n)\} \\ &< \inf\{E_a(u) : u \in M(\mathcal{D}),\ \beta(u) \in [\bar{x}_{h+k-1}, \bar{x}_{h+k}]\} < 2^{1-2/p}m. \end{aligned} \tag{4.58}$$

Step 8. We now prove Theorem 4.1 with $D_2, \dots, D_h$ as in Step 1, $\bar{n}_1, \dots, \bar{n}_k$ as in Step 2 and $\bar{x}_1, \dots, \bar{x}_{h+k}$ as in Step 7. Henceforth,

$$\mathcal{D} = \Omega \setminus \bigcup_{i=1}^{h} (\bar{\omega}_i + \bar{x}_i) \quad \text{and} \quad a(x) = \sum_{i=1}^{k} a^i_{n_i}(x - \bar{x}_{h+i}). \tag{4.59}$$

We denote the sublevels of E_a in $M(\mathcal{D})$ by

$$E^c_a = \{u \in M(\mathcal{D}) : E_a(u) \leqslant c\}, \quad c \in \mathbb{R}.$$

For $i = 1, \dots, h+k$, we claim that there exists a critical value $c_i \in [b_{1,i}, b_{2,i}]$ (see (4.42), with a and $\mathcal{D}$ as in (4.59) and $\bar{x}_i$ in place of x_i). Assume, by contradiction, that such a critical value does not exist. Then, taking into account Proposition 2.2, standard arguments (see [**23**], for example) show that there exist $\eta_i > 0$ and a continuous map

$$\mathcal{G}_i : E^{b_{2,i}}_a \to E^{b_{1,i}-\eta_i}_a$$

such that

$$\mathcal{G}_i(u) = u, \quad \forall u \in E^{b_{1,i}-\eta_i}_a. \tag{4.60}$$

Moreover, by (4.42), η_i can be chosen small enough to have

$$b_{1,i} - \eta_i > \sup\{E_a(\Psi_{\mathcal{D},\bar{\rho}_i,\bar{x}_i}[y, 0]) : y \in \Sigma_{x_i}\}. \tag{4.61}$$

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If we identify $B(\bar{x}_i, 1)$ with $\Sigma_{\bar{x}_i} \times [0,1]$, by polar coordinates (with the points of the form $(y,1)$, $y \in \Sigma_{\bar{x}_i}$, corresponding to $\bar{x}_i$ in $B(\bar{x}_i, 1)$), then, by (4.42), (4.60), (4.61) and (4.43), the map $g_i : B(\bar{x}_i, 1) \to \mathbb{R}^N$ given by

$$g_i(y,t) = \beta_{x_i}(\mathcal{G}_i(\Psi_{\mathcal{D},\bar{\rho}_i,\bar{x}_i}[y,t]))$$

is well defined, continuous and is such that $g_{i|_{\partial B(\bar{x}_i,1)}}$ is homotopically equivalent to the identity in $\mathbb{R}^N \setminus \{\bar{x}_i\}$. Hence, there exists $(\bar{y}_i, \bar{t}_i) \in \Sigma_{\bar{x}_i} \times [0,1]$ such that $g_i(\bar{y}_i, \bar{t}_i) = \bar{x}_i$, i.e. $\beta_{x_i}(\mathcal{G}_i(\Psi_{\mathcal{D},\bar{\rho}_i,\bar{x}_i}[y,t])) = \bar{x}_i$, that is in contradiction to (4.60) and proves our claim.

Our next goal is to show that for every $i = 1, \dots, h+k-1$ there exists a critical value $c_{i,i+1} \in [d_{1,i}, d_{2,i}]$ (see (4.45) with a and $\mathcal{D}$ as in (4.59) and $\bar{x}_i$ in place of x_i). Observe that $[d_{1,i}, d_{2,i}] \subset (m, 2^{1-2/p}m)$ by (4.45), (4.47) and (4.48), and assume, by contradiction, that such a critical point does not exist. Then, taking into account Proposition 2.2, we see that there exist $\eta_{i,i+1} > 0$ and a continuous deformation $\mathcal{F}_i : E_a^{d_{2,i}} \times [0,1] \to E_a^{d_{2,i}}$ such that

$$\mathcal{F}_i(u,0) = u, \ \forall u \in E_a^{d_{2,i}}, \quad \text{and} \quad \mathcal{F}_i(u,1) \subset E_a^{d_{1,i}-\eta_{i,i+1}}, \ \forall u \in E_a^{d_{2,i}}. \tag{4.62}$$

Now, let us define $f_i : \partial B(\bar{x}_i, \frac{1}{2}|\bar{x}_{i+1} - \bar{x}_i|) \times [0,1] \to \mathbb{R}^N \setminus \{\bar{x}_i, \bar{x}_{i+1}\}$ by

$$f_i(y,t) = \begin{cases} \mathcal{H}_i(y, 2t) & \text{if } t \in [0, \frac{1}{2}], \\ \beta(\mathcal{F}_i(v_{p,\mathcal{D}}[y], 2t-1)) & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \tag{4.63}$$

By (4.46), (4.62) and (4.45), f_i is a well-defined continuous deformation that verifies

$$\begin{aligned} &f_i(y,0) = y, && \forall y \in \partial B(\bar{x}_i, \tfrac{1}{2}|\bar{x}_{i+1} - \bar{x}_i|), \\ &f_i(y,1) \cap [\bar{x}_i, \bar{x}_{i+1}] = \emptyset, && \forall y \in \partial B(\bar{x}_i, \tfrac{1}{2}|\bar{x}_{i+1} - \bar{x}_i|), \\ &f_i(y,t) \notin \{\bar{x}_i, \bar{x}_{i+1}\}, && \forall (y,t) \in \partial B(\bar{x}_i, \tfrac{1}{2}|\bar{x}_{i+1} - \bar{x}_i|) \times [0,1], \end{aligned}$$

which is impossible. So the existence of the critical value $c_{i,i+1}$ is proved.

By using relations (4.56)–(4.58) and exactly the same argument developed to prove the existence of the critical values $c_{i,i+1}$, $i = 1, \dots, h+k-1$, we can find a critical value $c_{h+k,h+k+1} \in [d_{1,h+k}, d_{2,h+k}]$.

To summarize: from (4.44), (4.47), (4.48) and (4.58) it follows that the critical levels c_i, $c_{i,i+1}$, we have found for $i = 1, \dots, h+k$, verify

$$m < c_{h+k,h+k+1} < c_{h+k-1,h+k} < \cdots < c_{1,2} < c_1 < \cdots < c_{h+k} < 2^{1-2/p}m. \tag{4.64}$$

Hence, they actually give rise to distinct critical points for the functional E_a constrained on $M(\mathcal{D})$. These critical points are positive functions, by (4.64), Proposition 2.3 and the maximum principle, so they provide $2(h+k)$ distinct solutions to problem $P(a, \mathcal{D})$.

□

5. Final remarks

In the proof of Theorem 4.1, h critical levels are related to the ω_i, k to the a_i, $h+k-1$ to the interactions between the holes in the domain and the bump in the potential, and another critical level comes from the action of the boundary of Ω. In order to distinguish the critical levels we have found (hence the solutions), a crucial role has been played by the asymptotic behaviour stated in Proposition 3.2 (a), in Propositions 3.4 (b) and 3.5, and in (3.20), (3.22). The application of these results in the proof of Theorem 4.1 shows that if u_{ω_i} are the solutions of $P(a, \mathcal{D})$ related to ω_i, $u_{a^i_{n_i}}$ are the solutions related to $a^i_{n_i}$, and $u_{i,i+1}$ are the solutions given by the interactions, then the following energy estimates hold:

$$\lim_{D(\omega_i)\to+\infty} E_a\left(\frac{u_{\omega_i}}{|u_{\omega_i}|_{L^p}}\right) = 2^{1-2/p}m, \quad i=1,\dots,h, \tag{5.1}$$

$$\lim_{\bar{n}_i\to+\infty} E_a\left(\frac{u_{a^i_{n_i}}}{|u_{a^i_{n_i}}|_{L^p}}\right) = 2^{1-2/p}m, \quad i=1,\dots,k, \tag{5.2}$$

$$\lim_{|\bar{x}_{i+1}-\bar{x}_i|\to+\infty} E_a\left(\frac{u_{i,i+1}}{|u_{i,i+1}|_{L^p}}\right) = m, \qquad i=1,\dots,h+k-1, \tag{5.3}$$

$$\lim_{n\to+\infty} E_a\left(\frac{u_{h+k,h+k+1}}{|u_{h+k,h+k+1}|_{L^p}}\right) = m \tag{5.4}$$

(see (4.56) in Step 7 of the proof of Theorem 4.1 for the dependence on n of the solution $u_{h+k,h+k+1}$). In fact, (5.1) and (5.2) follow from (4.42), (5.3) follows from (4.45), and (5.4) follows from (4.56).

A natural question is what happens if the 'holes' ω_i shrink, instead of enlarging, and the bumps a^i vanish, instead of increasing. To analyse this situation, arguing as in the proof of [**5**, Theorem A.1] we can obtain the following asymptotic estimate:

$$\lim_{\operatorname{cap}\bar{\omega}\to 0} \sup\{E_0(v_{p,\mathbb{R}^N\setminus\bar{\omega}}[y]) : y\in\mathbb{R}^N\} = m, \tag{5.5}$$

where, for a closed bounded set $G\subset\mathbb{R}^N$, the capacity of G is defined by

$$\operatorname{cap} G = \inf\{\|u\|_{H^1} : u\in H^1(\mathbb{R}^N),\ u\geqslant 1 \text{ on } G \text{ in the } H^1\text{-sense}\}.$$

Moreover, it is easily seen that

$$\lim_{|a|_{L^{N/2}}\to 0} \sup\{E_a(v_p(\cdot - y)) : y\in\mathbb{R}^N\} = m. \tag{5.6}$$

By the asymptotic behaviour stated in (5.5) and (5.6), the following result can be proved by working as in Theorem 4.1.

Theorem 5.1. *Let $h,k\in\mathbb{N}$, $h+k\neq 0$, and assume that Ω is an open set in $\mathbb{R}^N$ that satisfies Assumption (C). If*

$$\mathcal{D} = \Omega\setminus\bigcup_{i=1}^{h}(\bar{\omega}_i+\bar{x}_i), \quad \bar{x}_i\in\mathbb{R}^N,$$

with ω_i bounded open sets in $\mathbb{R}^N$, and a has the form

$$a(x) = \sum_{j=1}^{k} a_j(x - \bar{x}_{h+j}), \quad x_{h+j} \in \mathbb{R}^N,$$

with a_j non-negative functions in $L^{N/2}(\mathbb{R}^N)$ that satisfy assumption (1.2), then there exist

$$\begin{aligned}
C_2 &= C_2(\omega_1),\\
C_3 &= C_3(\omega_2),\\
&\vdots\\
C_h &= C_h(\omega_{h-1}),\\
L_1 &= L_1(\omega_h),\\
L_2 &= L_2(a_1),\\
&\vdots\\
L_k &= L_k(a_{k-1}),\\
\bar{x}_1 &= \bar{x}_1(\Omega, \omega_1),\\
&\vdots\\
\bar{x}_h &= \bar{x}_h(\Omega, \omega_1, \bar{x}_1, \dots, \omega_{h-1}, \bar{x}_{h-1}, \omega_h),\\
\bar{x}_{h+1} &= \bar{x}_{h+1}(\mathcal{D}, a_1),\\
&\vdots\\
\bar{x}_{h+k} &= \bar{x}_{h+k}(\mathcal{D}, a_1, \bar{x}_{h+1}, \dots, a_{k-1}, \bar{x}_{h+k-1}, a_k)
\end{aligned}$$

such that problem $P(a, \mathcal{D})$ has at least $2(k+h)$ distinct solutions $u_{\omega_1}, \dots, u_{\omega_h}$, $u_{a_1}, \dots, u_{a_k}$, $u_{i,i+1}$, $i = 1, \dots, h+k$, whenever $\operatorname{cap}\bar{\omega}_i < C_i$, $i = 1, \dots, h$, and $|a_i|_{L^{N/2}} < L_i$, $i = i, \dots, k$.

In the case of Theorem 5.1, as a consequence of (5.5) and (5.6), the following estimates can be obtained, besides (5.3) and (5.4):

$$\lim_{\operatorname{cap}\bar{\omega}_i \to 0} E_a\left(\frac{u_{\omega_i}}{|u_{\omega_i}|_{L^p}}\right) = m, \quad i = 1, \dots, h, \tag{5.7}$$

$$\lim_{|a_i|_{L^{N/2}} \to 0} E_a\left(\frac{u_{a_i}}{|u_{a_i}|_{L^p}}\right) = m, \quad i = 1, \dots, k. \tag{5.8}$$

Observe also that results similar to those stated in Theorems 4.1 and 5.1 can be proved when there are holes that expand and holes that shrink and bumps that increase and bumps that vanish simultaneously.

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Remark 5.2. If in Theorems 4.1 and 5.1 we consider $\Omega = \mathbb{R}^N$, then we find that problem $P(a, \mathcal{D})$ has at least $2(h+k)-1$ solutions.

In fact, if we consider Theorem 4.1, for example, we can repeat the proof developed in § 4, with the exception of Step 7, so the desired number of solutions can be found.

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