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SPARSE SETS ON THE PLANE AND DENSITY POINTS DEFINED BY FAMILIES OF SEQUENCES

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Abstract

A condition equivalent to sparseness of a set on the plane is formulated and used as a motivation for a new concept of density point on the plane. This is investigated and compared with known previous versions.

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Sarkel and De in [7] introduced the notion of a set sparse at a point. Sparse subsets of the real line were also investigated by Filipczak in [1–3]. In [3] a condition equivalent to sparseness of a set on the real line was formulated and used as a motivation for a new concept of density point. This is a generalisation of a density point with respect to a fixed sequence [4].

In our paper similar results for subsets of the plane are stated and proved. A new concept of density point on the plane is considered and compared with others.

Let $\mathcal{L}_2$ stand for the family of all Lebesgue measurable sets on the plane and let λ_2 stand for two-dimensional Lebesgue measure. For $A, B \in \mathcal{L}_2$, the notation $A \sim B$ means that the symmetrical difference of A and B is a null set. For brevity, let $R((x, y), a, b)$ stand for the rectangle $(x - a, x + a) \times (y - b, y + b)$, where $x, y \in \mathbb{R}$, $a, b \in \mathbb{R}_+$ and $S((x, y), r) := R((x, y), r, r)$. Sequences $\{a_n\}_{n\in\mathbb{N}}$ are denoted by $\langle a\rangle$ for short. Let θ denote $(0, 0)$.

Definition 1 [8]. A measurable set $A \subset \mathbb{R}^2$ is said to be *sparse* at a point (x, y) if for any $\varepsilon > 0$ there exists $h > 0$ such that each interval (a, b) contained in $(0, h)$ with $a/b < h$ contains at least one point t such that

$$\frac{\lambda_2(A \cap S((x, y), t))}{4t^2} < \varepsilon.$$

The first theorem below is an analogue of the characterisation of sparseness in the one-dimensional case [3].

THEOREM 2. *A measurable set $A \subset \mathbb{R}^2$ is sparse at a point (x, y) if and only if for any $\varepsilon > 0$ there exists a decreasing sequence $\langle a \rangle$ satisfying $\lim_{n\to\infty} a_n = 0$ and $\liminf_{n\to\infty}(a_{n+1}/a_n) > 0$ and such that*

$$\frac{\lambda_2(A \cap S((x, y), a_n))}{4a_n^2} < \varepsilon \quad \textit{for each } n \in \mathbb{N}.$$

PROOF. Without loss of generality we can assume that $(x, y) = \theta$.

Suppose that A is sparse at θ and fix $\varepsilon > 0$. By the assumption there exists $h > 0$ such that, for each interval (a, b) contained in $(0, h)$ such that $a/b < h$, there exists a point $t \in (a, b)$ with the property that $\lambda_2(A \cap S(\theta, t))/4t^2 < \varepsilon$. We can assume that $h < 1$.

Let $c_n = (h/2)^n$ for $n \in \mathbb{N}$. Since $c_n < h$, which implies that $(c_{n+1}, c_n) \subset (0, h)$, and $c_{n+1}/c_n < h$ for $n \in \mathbb{N}$, we can find a sequence $\langle t \rangle$ such that $t_n \in (c_{n+1}, c_n)$ and $\lambda_2(A \cap S(\theta, t))/4t_n^2 < \varepsilon$ for every $n \in \mathbb{N}$. Of course, $\langle t \rangle$ is a decreasing sequence tending to zero, $t_{n+1}/t_n > c_{n+2}/c_n = (h/2)^2 > 0$, so $\liminf_{n\to\infty}(t_{n+1}/t_n) > 0$. This completes the proof of necessity.

To prove the converse implication, for arbitrarily chosen $\varepsilon > 0$ find a decreasing sequence $\langle a \rangle$ tending to zero, with the property that $\liminf_{n\to\infty}(a_{n+1}/a_n) > 0$ and

$$\frac{\lambda_2(A \cap S(\theta, a_n))}{4a_n^2} < \varepsilon \quad \text{for } n \in \mathbb{N}. \tag{$*$}$$

Set $\delta \in (0, 1)$ such that $a_{n+1}/a_n > \delta$ for all $n \in \mathbb{N}$. Now let $h = \min\{\delta, a_1\}$ and consider any interval $(\alpha, \beta) \subset (0, h)$ with $\alpha/\beta < h$. Since $\beta \le a_1$, there exists $p \in \mathbb{N}$ such that $\beta \in (a_{p+1}, a_p]$. Thus the inequalities $\alpha/\beta < h \le \delta < a_{p+1}/a_p$ give $a_{p+1} > \alpha$, so $a_{p+1} \in (\alpha, \beta)$, and, using $(*)$ for $n = p + 1$, we conclude that A is sparse at θ. □

DEFINITION 3. We say that (x, y) is *a proximal density point* of a measurable set $A \subset \mathbb{R}^2$ if its complement $\mathbb{R}^2 \setminus A$ is sparse at (x, y).

We can also treat axes independently.

DEFINITION 4 [8]. A measurable set $A \subset \mathbb{R}^2$ is said to be *strongly sparse* at a point (x, y) if for any $\varepsilon > 0$ there exist $k_1 > 0$ and $k_2 > 0$ such that for each pair of intervals, (a_1, b_1) contained in $(0, k_1)$ with $a_1/b_1 < k_1$ and (a_2, b_2) contained in $(0, k_2)$ with $a_2/b_2 < k_2$, there exists a point $(u, v) \in (a_1, b_1) \times (a_2, b_2)$ such that

$$\frac{\lambda_2(A \cap R((x, y), u, v))}{4uv} < \varepsilon.$$

THEOREM 5. *A measurable set $A \subset \mathbb{R}^2$ is strongly sparse at a point (x, y) if and only if for any $\varepsilon > 0$ there exist two decreasing sequences $\langle a \rangle$, $\langle b \rangle$ tending to zero, satisfying $\liminf_{n\to\infty}(a_{n+1}/a_n) > 0$ and $\liminf_{n\to\infty}(b_{n+1}/b_n) > 0$ and such that*

$$\frac{\lambda_2(A \cap R((x, y), a_n, b_n))}{4a_nb_n} < \varepsilon \quad \textit{for each } n \in \mathbb{N}.$$

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The proof is a slight modification of that of Theorem 2.

Definition 6. We say that (x, y) is *a strong proximal density point* of a measurable set $A \subset \mathbb{R}^2$ if its complement $\mathbb{R}^2 \backslash A$ is strongly sparse at (x, y).

Theorems 2 and 5 motivate us to define a new kind of density point using a family of sequences. Following the ideas from [3], we call a family C of decreasing sequences convergent to 0 *acceptable* if $\inf\{a_1 : \langle a\rangle \in C\} = 0$.

Definition 7. We say that (x, y) is *a C-density point* of a measurable set $A \subset \mathbb{R}^2$ if and only if for every $\varepsilon > 0$ there exists $\langle a\rangle \in C$ (where C is assumed to be acceptable) such that

$$\frac{\lambda_2(A \cap S((x, y), a_n))}{4a_n^2} > 1 - \varepsilon \quad \text{for each } n \in \mathbb{N}.$$

For $A \in \mathcal{L}_2$ we define

$$\Phi_C(A) := \{(x, y) \in \mathbb{R}^2 : (x, y) \text{ is a } C\text{-density point of } A\}.$$

Let us denote by C_+ the family of all decreasing sequences convergent to 0 and satisfying $\liminf_{n\to\infty}(a_{n+1}/a_n) > 0$. Then Theorem 2 can be formulated in the following way.

Proposition 8. *A point (x, y) is a proximal density point of a measurable set $A \subset \mathbb{R}^2$ if and only if $(x, y) \in \Phi_{C_+}(A)$.*

Let C_1, C_2 be acceptable families of decreasing sequences convergent to 0.

Definition 9. We say that (x, y) is *a C_1, C_2-density point* of a measurable set $A \subset \mathbb{R}^2$ if and only if for every $\varepsilon > 0$ there exist two sequences $\langle a\rangle \in C_1$, $\langle b\rangle \in C_2$ such that

$$\frac{\lambda_2(A \cap R((x, y), a_n, b_n))}{4a_n b_n} > 1 - \varepsilon \quad \text{for each } n \in \mathbb{N}.$$

For $A \in \mathcal{L}_2$ we define

$$\Phi_{C_1,C_2}(A) := \{(x, y) \in \mathbb{R}^2 : (x, y) \text{ is a } C_1, C_2\text{-density point of } A\}.$$

Theorem 5 can thus be rephrased as follows.

Proposition 10. *A point (x, y) is a strong proximal density point of a measurable set $A \subset \mathbb{R}^2$ if and only if $(x, y) \in \Phi_{C_+,C_+}(A)$.*

It is obvious that for an acceptable family C and for any $A \in \mathcal{L}_2$, $\Phi_C(A) \subseteq \Phi_{C,C}(A)$ (as we can choose the same sequence twice), but we emphasise that for the same set A the values of Φ_C and $\Phi_{C,C}$ can be different.

Example 11. Let $a_1 = 1$, $a_{n+1} = (a_n/n)^2$ for $n \in \mathbb{N}$. It is clear that $\langle a\rangle$ is a decreasing sequence convergent to 0 and the family $C_{\langle a\rangle} := \{(a_k, a_{k+1}, \ldots) : k \in \mathbb{N}\}$ is acceptable.

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Let $A := \{(x, y) \in \mathbb{R}^2 : -x^2 \le y \le x^2\}$. Then

$$\frac{\lambda_2(A \cap S(\theta, a_n))}{4a_n^2} = \frac{a_n}{3} \quad \text{for each } n \in \mathbb{N},$$

so $\theta \notin \Phi_{C_{\langle a \rangle}}(A)$. However,

$$\frac{\lambda_2(A \cap R(\theta, a_n, a_{n+1}))}{4a_n a_{n+1}} = \frac{\lambda_2(A \cap R(\theta, a_n, (a_n/n)^2))}{\frac{4}{n^2}a_n^3} > 1 - \frac{1}{n} \quad \text{for each } n \in \mathbb{N},$$

so $\theta \in \Phi_{C_{\langle a \rangle}, C_{\langle a \rangle}}(A)$.

Now let us mention general properties of Φ_C and Φ_{C_1,C_2}.

Proposition 12. *For every $A, B \in \mathcal{L}_2$ and for any acceptable families C_1, C_2, C_1', C_2':*

(1) $\Phi_{C_1}(\emptyset) = \emptyset$, $\Phi_{C_1,C_2}(\emptyset) = \emptyset$, $\Phi_{C_1}(\mathbb{R}^2) = \mathbb{R}^2$, $\Phi_{C_1,C_2}(\mathbb{R}^2) = \mathbb{R}^2$;
(2) *if $A \sim B$ then $\Phi_{C_1}(A) = \Phi_{C_1}(B)$ and $\Phi_{C_1,C_2}(A) = \Phi_{C_1,C_2}(B)$;*
(3) *if $A \subset B$ then $\Phi_{C_1}(A) \subset \Phi_{C_1}(B)$ and $\Phi_{C_1,C_2}(A) \subset \Phi_{C_1,C_2}(B)$;*
(4) *if $C_1 \subset C_1'$ then $\Phi_{C_1}(A) \subset \Phi_{C_1'}(A)$, and if also $C_2 \subset C_2'$ then $\Phi_{C_1,C_2}(A) \subset \Phi_{C_1',C_2'}(A)$.*

One may ask whether this new concept of density is not just another way of describing proximal density, but leads also to some other density operators.

We now recall two classical kinds of density on the plane.

Definition 13 [9]. We say that a point $(x, y) \in \mathbb{R}^2$ is *an ordinary density point* of the set $A \in \mathcal{L}_2$ if

$$\lim_{h \to 0^+} \frac{\lambda_2(A \cap S((x, y), h))}{4h^2} = 1.$$

Definition 14 [9]. We say that a point $(x, y) \in \mathbb{R}^2$ is *a strong density point* of the set $A \in \mathcal{L}_2$ if

$$\lim_{h \to 0^+, k \to 0^+} \frac{\lambda_2(A \cap R((x, y), h, k))}{4hk} = 1.$$

If we require here that only upper limits be equal to 1 we say respectively that a point (x, y) is an *upper ordinary density point of A* or an *upper strong density point of A*.

As usual, let $\Phi_0(A)$ denote the set of all ordinary density points of a set $A \in \mathcal{L}_2$, $\Phi_s(A)$ the set of all strong density points of $A \in \mathcal{L}_2$, $\overline{\Phi}_0(A)$ the set of all upper ordinary density points of A, and $\overline{\Phi}_s(A)$ the set of all upper strong density points of A.

Let $\widetilde{C}$ denote the family of all decreasing sequences convergent to 0. Obviously $\widetilde{C}$ is acceptable.

Proposition 15. *We have*

$$\Phi_{\widetilde{C}} = \overline{\Phi}_0 \quad \textit{and} \quad \Phi_{\widetilde{C},\widetilde{C}} = \overline{\Phi}_s.$$

Proposition 16. *For any acceptable families C_1, C_2 and for every $A \in \mathcal{L}_2$:*

(1) $\Phi_0(A) \subseteq \Phi_{C_1}(A) \subseteq \overline{\Phi}_0(A)$, $\Phi_s(A) \subseteq \Phi_{C_1,C_2}(A) \subseteq \overline{\Phi}_s(A)$;

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(2) $\Phi_{C_1}(A) \in \mathcal{L}_2$, $\Phi_{C_1,C_2}(A) \in \mathcal{L}_2$;
(3) $\Phi_{C_1}(A) \sim A$, $\Phi_{C_1,C_2}(A) \sim A$.

PROOF. Property (1) is obvious. Properties (2) and (3) come from (1) and the Lebesgue density theorem which says that $\Phi_0(A) \sim A$ and $\Phi_s(A) \sim A$. □

By Proposition 12 for every $A, B \in \mathcal{L}_2$ and for any acceptable families C_1, C_2:

$$\Phi_{C_1}(A \cap B) \subseteq \Phi_{C_1}(A) \cap \Phi_{C_1}(B)$$

and

$$\Phi_{C_1,C_2}(A \cap B) \subseteq \Phi_{C_1,C_2}(A) \cap \Phi_{C_1,C_2}(B),$$

but equality need not hold. For example, we can find disjoint sets A and B such that $\overline{\Phi}_0(A) \cap \overline{\Phi}_0(B)$ is nonempty, so $\Phi_{\widetilde{C}}(A \cap B) \neq \Phi_{\widetilde{C}}(A) \cap \Phi_{\widetilde{C}}(B)$, and since $\overline{\Phi}_0(C) \subset \overline{\Phi}_s(C)$ for any $C \in \mathcal{L}_2$, then also $\Phi_{\widetilde{C},\widetilde{C}}(A \cap B) \neq \Phi_{\widetilde{C},\widetilde{C}}(A) \cap \Phi_{\widetilde{C},\widetilde{C}}(B)$.

In view of Propositions 15 and 16(1) we can ask whether it is possible to find a family of sequences C, $C \neq C_+$ for which there exists a set $A \in \mathcal{L}_2$ such that $\Phi_0(A) \subset \Phi_C(A) \subset \overline{\Phi}_0(A)$; whether there exists C such that $\Phi_0 = \Phi_C$; and whether there exists $C \subset \widetilde{C}$ for which $\Phi_C = \overline{\Phi}_0$ or $\Phi_C = \overline{\Phi}_s$. We obtain some answers here using a special kind of operator $\Phi_{\langle s\rangle\langle t\rangle} : \mathcal{L}_2 \to \mathcal{L}_2$ which was defined in [5] for fixed sequences $\langle s\rangle, \langle t\rangle$ from the family $\mathcal{S}$ of all unbounded and nondecreasing sequences of positive reals. For convenience we reformulate this definition for decreasing sequences convergent to 0.

DEFINITION 17. Let $\langle a\rangle, \langle b\rangle \in \widetilde{C}$. For $A \in \mathcal{L}_2$ we define an operator

$$\Phi_{\langle a\rangle\langle b\rangle}(A) = \left\{(x, y) \in \mathbb{R}^2 : \lim_{n\to+\infty} \frac{\lambda_2(A \cap R((x, y), a_n, b_n))}{4a_n b_n} = 1\right\}.$$

Let $\langle a\rangle \in \widetilde{C}$. As in Example 11, we see that the family

$$C\langle a\rangle := \{(a_k, a_{k+1}, \ldots) : k \in \mathbb{N}\}$$

is acceptable.

The following results are straightforward consequences of the definitions.

PROPOSITION 18. *For each $\langle a\rangle \in \widetilde{C}$,*

$$\Phi_{C\langle a\rangle} = \Phi_{\langle a\rangle\langle a\rangle}.$$

PROPOSITION 19. *For each $\langle a\rangle, \langle b\rangle \in \widetilde{C}$,*

$$\Phi_{C\langle a\rangle,C\langle b\rangle} = \Phi_{\langle a\rangle\langle b\rangle}.$$

If, instead of the family $C\langle a\rangle$, we consider the family of all subsequences of a fixed sequence $\langle a\rangle$, we can obtain a different set of density points, as explained in the following example.

EXAMPLE 20. Let $a_n = 1/n!$ for $n \in \mathbb{N}$ and define $A := \bigcup_{n\in\mathbb{N}}(a_{2n+1}, a_{2n})$ and $B := (A \cup (-A)) \times \mathbb{R}$.

Then in the family of all subsequences of $\langle a\rangle$ we can find a sequence $\{b_n\}_{n\in\mathbb{N}}$, $b_n = a_{2n}$, such that $\lim_{n\longrightarrow\infty}(\lambda_2(B\cap S(\theta, b_n))/4b_n^2) = 1$, so for this family θ is a C-density point of B. However, since in each sequence from $C\langle a\rangle$ there is a subsequence $\{a_{2k+1}\}_{k\geq k_0}$ for a certain k_0 and $\lim_{k\longrightarrow\infty}(\lambda_2(B\cap S(\theta, a_{2k+1}))/4a_{2k+1}^2) = 0$, we have that $\theta\notin\Phi_{C\langle a\rangle}(B)$.

The family of density operators that can be described via the notion of C-density is strictly bigger than that defined by $\Phi_{\langle a\rangle\langle a\rangle}$, $\langle a\rangle\in\widetilde{C}$. Indeed, for example, $\Phi_{\widetilde{C}}$ cannot be generated by any sequence since it is not distributive under intersection.

The next theorem states that proximal density cannot be defined by means of $\Phi_{\langle a\rangle\langle b\rangle}$ either.

Theorem 21. *For every $\langle a\rangle, \langle b\rangle\in\widetilde{C}$ there exists a set $A\in\mathcal{L}_2$ such that the set of all its proximal density points differs from $\Phi_{\langle a\rangle\langle b\rangle}(A)$.*

Proof. Let $\langle a\rangle, \langle b\rangle\in\widetilde{C}$ and let $\{a_{n_k}\}_{k\in\mathbb{N}}$ be a subsequence of $\langle a\rangle$ belonging to C_0. For simplicity denote $\langle s\rangle := \{a_{n_k}\}_{k\in\mathbb{N}}$ and $\langle u\rangle := \{b_{n_k}\}_{k\in\mathbb{N}}$.

Define $B := \bigcup_{n\in\mathbb{N}}(s_n/2, s_n)$ and $D := (B\cup(-B))\times\mathbb{R}$.

Since

$$\frac{\lambda_2(D\cap R(\theta, s_n, u_n))}{4s_nu_n} \geq \frac{2(s_n - \frac{s_n}{2})2u_n}{4s_nu_n} = \frac{1}{2},$$

we have $\theta\notin\Phi_{\langle a\rangle\langle b\rangle}(A)$ where $A := \mathbb{R}^2\backslash D$.

Our next claim is to show that θ is a proximal density point of A, that is, D is sparse at θ.

Let $\varepsilon > 0$. We may assume that $\varepsilon < 1$. From the fact that $\langle s\rangle\in C_0$ it follows that there exists $n_0\in\mathbb{N}$ such that, for each $n\geq n_0$, $s_{n+1}/s_n < \varepsilon/2$. Let $h := \min\{s_{n_0}/2, \varepsilon/4\}$ and let $(\alpha,\beta)\subset(0, h)$ be an interval such that $\alpha/\beta < h$. We want to find a point $t\in(\alpha,\beta)$ satisfying $\lambda_2(D\cap S(\theta, t))/4t^2 < \varepsilon$.

If there exists $k\in\mathbb{N}$ such that $s_k/2\in(\alpha,\beta)$, then $s_k\leq 2\beta\leq 2h\leq s_{n_0}$ and $k > n_0$. Put $t := s_k/2$. Therefore,

$$\frac{\lambda_2(D\cap S(\theta, t))}{4t^2} = \frac{\lambda_2(D\cap S(\theta, \frac{s_k}{2}))}{4(\frac{s_k}{2})^2} \leq \frac{4s_{k+1}\frac{s_k}{2}}{4(\frac{s_k}{2})^2} = 2\frac{s_{k+1}}{s_k} < \varepsilon.$$

If $s_n/2\notin(\alpha,\beta)$ for each $n\in\mathbb{N}$, then there exists $k\in\mathbb{N}$ such that $s_k/2\leq\alpha<\beta\leq s_{k-1}/2$. Thus, for $t := (\alpha+\beta)/2$, we have $t\in(\alpha,\beta)$ and

$$\frac{\lambda_2(D\cap S(\theta, t))}{4t^2} \leq \frac{4s_kt}{4\frac{\beta}{2}t} = 4\frac{\alpha}{\beta} < 4h\leq\varepsilon.$$

This completes the proof. □

Remark. We can also prove without difficulty that the set D appearing in the proof above is strongly sparse at θ, so strong proximal density cannot be defined by $\Phi_{\langle a\rangle\langle b\rangle}$ either.

Proposition 22. *There exists an acceptable family C such that*

$$\Phi_C = \Phi_0.$$

Proof. Since, for any $\langle a\rangle \in \widetilde{C}$ with the property that $\liminf_{n\to\infty}(a_{n+1}/a_n) > 0$, $\Phi_{\langle a\rangle\langle a\rangle} = \Phi_0$, as was shown in [5], we can choose as C a family $C\langle a\rangle$ and then $C\langle a\rangle$-density coincides with ordinary density on the plane. □

According to [6, Corollary 1.5 and Theorem 1.11] there is continuum of different operators of the type $\Phi_{\langle a\rangle\langle b\rangle}$, so we have at least continuum many operators of the type Φ_C.

We can also consider more general families of sequences, not only the families consisting of subsequences of a fixed sequence.

Proposition 23. *Let $\delta \in (0, 1]$ and $C_\delta := \{\langle a\rangle \in \widetilde{C} : \liminf_{n\to\infty}(a_{n+1}/a_n) \geq \delta\}$. Then*

$$\Phi_{C_\delta} = \Phi_0.$$

Proof. By Proposition 16(1) it is enough to show, for any $A \in \mathcal{L}_2$, that $\Phi_{C_\delta}(A) \subset \Phi_0(A)$. To derive a contradiction, suppose that $\Phi_{C_\delta}(A) \setminus \Phi_0(A) \neq \emptyset$ for some measurable A. We can assume that $\theta \in \Phi_{C_\delta}(A)\setminus\Phi_0(A)$. Thus there exists a positive ε_0 and $\langle h\rangle \in \widetilde{C}$ such that, for all $n \in \mathbb{N}$,

$$\frac{\lambda_2(A' \cap S(\theta, h_n))}{4h_n^2} > \varepsilon_0.$$

Since $\theta \in \Phi_{C_\delta}(A)$, there exists a sequence $\langle a\rangle \in \widetilde{C}_\delta$ such that

$$\frac{\lambda_2(A' \cap S(\theta, a_n))}{4a_n^2} < \frac{\varepsilon_0\delta^2}{4}.$$

Choose two positive integers n_0 and p such that, for $n \geq n_0$, we have $a_{n+1}/a_n > \delta/2$ and $h_p \leq a_{n_0}$. Thus for each $n \geq p$ there exists $k_n \geq n_0$ such that $a_{k_n+1} \leq h_n \leq a_{k_n}$, and

$$\begin{aligned}\frac{\lambda_2(A' \cap S(\theta, h_n))}{4h_n^2} &\leq \frac{\lambda_2(A' \cap S(\theta, a_n))}{4a_{k_n+1}^2}\\ &= \frac{\lambda_2(A' \cap S(\theta, a_{k_n}))}{4a_{k_n}^2} \cdot \frac{a_{k_n}^2}{a_{k_n+1}^2}\\ &< \frac{\varepsilon_0\delta^2}{4} \cdot \frac{4}{\delta^2} = \varepsilon_0.\end{aligned}$$

This contradiction completes the proof. □

Proposition 24. *Let $C_0 := \{\langle a\rangle \in \widetilde{C} : \lim_{n\to\infty}(a_{n+1}/a_n) = 0\}$. Then*

$$\Phi_{C_0} = \overline{\Phi}_0.$$

Proof. By Propositions 15 and 16(1) it is enough to show, for any $A \in \mathcal{L}_2$, that $\Phi_{\widetilde{C}}(A) \subset \Phi_{C_0}(A)$. That is the case since from every sequence belonging to $\widetilde{C}$ we can choose a subsequence belonging to C_0. □

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PROPOSITION 25. *Let*

$$C_0^1 := \Big\{ \langle a \rangle \in \widetilde{C} : \liminf_{n\to\infty} (a_{n+1}/a_n) = 0 \textit{ and } \limsup_{n\to\infty} (a_{n+1}/a_n) = 1 \Big\}.$$

Then

$$\Phi_{C_0^1} = \overline{\Phi}_0.$$

PROOF. Let

$$C_{\underline{0}} := \Big\{ \langle a \rangle \in \widetilde{C} : \liminf_{n\to\infty} (a_{n+1}/a_n) = 0 \Big\}.$$

Then $C_0 \subset C_{\underline{0}}$ and, by Proposition 12(4) and Proposition 24,

$$\Phi_{C_{\underline{0}}} = \overline{\Phi}_0.$$

Since $C_0^1 \subset C_{\underline{0}}$ it is sufficient to show that $\Phi_{C_{\underline{0}}}(A) \subset \Phi_{C_0^1}(A)$ for every $A \in \mathcal{L}_2$. Moreover, we need only show that if $\theta \in \Phi_{C_{\underline{0}}}(A)$ then $\theta \in \Phi_{C_0^1}(A)$. Let $A \in \mathcal{L}_2$ and $\theta \in \Phi_{C_{\underline{0}}}(A)$. Fix $\varepsilon > 0$. We may assume that $\varepsilon < 1$. There exists $\langle a \rangle \in C_{\underline{0}}$ such that

$$\frac{\lambda_2(A \cap S(\theta, a_n))}{4a_n^2} > 1 - \frac{\varepsilon}{2} \quad \text{for every } n \in \mathbb{N}.$$

Without loss of generality we can assume that $\lim_{n\to\infty}(a_{n+1}/a_n) = 0$. For every $n \in \mathbb{N}$ we fix $\alpha_n > 0$ such that

$$\alpha_n \le \frac{\varepsilon}{8} a_{n+1} \quad \text{and} \quad \alpha_n < \frac{a_n}{n}.$$

We now define inductively a decreasing sequence $\langle b \rangle$ which consists of all terms of the sequence $\langle a \rangle$, and if $a_{2k+1} - a_{2k+2} > \alpha_{2k+1}$ we put between them a new term equal to $a_{2k+1} - \alpha_{2k+1}$. Of course $\langle b \rangle \in \widetilde{C}$. Since $\{a_{2k}\}_{k\in\mathbb{N}}$ is a subsequence of $\langle b \rangle$ and in the sequence $\langle b \rangle$ the next term after a_{2k} is a_{2k+1}, we get $\liminf_{n\to\infty}(b_{n+1}/b_n) = 0$.

We now consider the subsequence $\{a_{2k+1}\}_{k\in\mathbb{N}}$ of the sequence $\langle b \rangle$. In the sequence $\langle b \rangle$ the next term after a_{2k+1} is a_{2k+2} when $a_{2k+1} - a_{2k+2} \le \alpha_{2k+1}$, and $a_{2k+1} - \alpha_{2k+1}$ otherwise. In both cases the relevant ratio is not smaller then $1 - (\alpha_{2k+1}/a_{2k+1}) > 1 - (1/n)$, so $\limsup_{n\to\infty}(b_{n+1}/b_n) = 1$.

It remains to prove that

$$\frac{\lambda_2(A \cap S(\theta, b_n))}{4b_n^2} \ge 1 - \varepsilon \quad \text{for every } n \in \mathbb{N}.$$

If there exists $k \in \mathbb{N}$ such that $b_n = a_k$ then the required inequality holds. Otherwise there exists $k \in \mathbb{N}$ such that $b_n = a_k - \alpha_k$. Then

$$\begin{aligned}
\frac{\lambda_2(A \cap S(\theta, b_n))}{4b_n^2} &= \frac{\lambda_2(A \cap S(\theta, a_k - \alpha_k))}{4(a_k - \alpha_k)^2} \\
&= \frac{\lambda_2(A \cap S(\theta, a_k)) - (\lambda_2(A \cap S(\theta, a_k)) - \lambda_2(A \cap S(\theta, a_k - \alpha_k)))}{4(a_k - \alpha_k)^2}
\end{aligned}$$

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$$
\begin{aligned}
&\geq \frac{\lambda_2(A \cap S(\theta, a_k))}{4a_k^2} - \frac{8\alpha_k(a_k - \alpha_k) + 4\alpha_k^2}{4(a_k - \alpha_k)^2} \\
&\geq 1 - \frac{\varepsilon}{2} - \frac{2\alpha_k}{a_{k+1}} - \left(\frac{\alpha_k}{a_{k+1}}\right)^2 \\
&\geq 1 - \frac{\varepsilon}{2} - 2\frac{\varepsilon}{8} - \left(\frac{\varepsilon}{8}\right)^2 \geq 1 - \varepsilon.
\end{aligned}
$$

This concludes the proof. □

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