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GRADIENT ESTIMATES VIA TWO-POINT FUNCTIONS FOR PARABOLIC EQUATIONS UNDER RICCI FLOW

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Abstract

We derive estimates relating the values of a solution at any two points to the distance between the points for quasilinear parabolic equations on compact Riemannian manifolds under Ricci flow.

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1. Introduction

Andrews and Clutterbuck [2, 3] and Andrews [1] studied two-point estimates and their applications in a variety of geometric contexts. Recently, Andrews and Xiong [4] used two-point estimates to deduce gradient estimates of the solutions of the quasilinear equations

$$\Big[\alpha(u,|Du|)\frac{D_iuD_ju}{|Du|^2}+\beta(u,|Du|)\Big(\delta_{ij}-\frac{D_iuD_ju}{|Du|^2}\Big)\Big]D_iD_ju+q(u,|Du|)=0, \qquad (1.1)$$

where the left-hand side of (1.1) is continuous on $\mathbb{R}\times TM_x^*\times\mathcal{L}_s^2(TM)$, α and β are nonnegative functions and $\beta(s,t)>0$ for $t>0$. They proved the following result.

Theorem 1.1 [4]. *Let M^n be a compact Riemannian manifold with* $\mathrm{Ric}\geq 0$ *and let u be a viscosity solution of (1.1). Suppose that the barrier* $\varphi:[a,b]\to[\inf u,\sup u]$ *satisfies*

$$\varphi'>0,$$
$$\frac{d}{dz}\Big(\frac{\varphi''\alpha(\varphi,\varphi')+q(\varphi,\varphi')}{\varphi'\beta(\varphi,\varphi')}\Big)<0.$$

If ψ is the inverse of φ (that is, $\psi(\varphi(z))=z$), then

$$\psi(u(y))-\psi(u(x))-d(x,y)\leq 0 \quad \textit{for all } x,y\in M.$$

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Allowing y to approach x leads to the following gradient estimate.

COROLLARY 1.2. *Under the conditions of Theorem 1.1, for every $u \in C^1(M)$ and $x \in M$,*

$$\nabla u(x) \leq \varphi'(\psi(u(x))).$$

The Modica-type gradient estimate obtained by using the P-function in [8] is a special case of this result with $\alpha = \beta = 1$ and a generalisation of Modica's result in [6] is also a special case with $\alpha = 2\Psi''(z)z + \Psi'(z)$ and $\beta = \Psi(z)$. The details of the two-point estimates are comparatively simple and geometric compared with the calculations involved in the P-function approach. Since this method does not involve differentiating the equation, it applies with minimal regularity requirements on the solution u, corresponding to the viscosity solution requirement.

We seek to apply this method to derive gradient estimates for parabolic equations. Azagra *et al.* [5] defined the viscosity solution and proved the parabolic maximum principle for semicontinuous functions on manifolds. Several authors considered gradient estimates under Ricci flow,

$$\frac{\partial}{\partial t} g(x,t) = -2\mathrm{Ric}(x,t).$$

For example, Liu [7] derived gradient estimates on a closed Riemannian manifold.

THEOREM 1.3 [7]. *Let $(M, g(t))$ be a closed Riemannian manifold, where $g(t)$ evolves by the Ricci flow in such a way that $-K_0 \leq \mathrm{Ric} \leq K_1$ for $t \in [0,T]$. If u is a positive solution to $(\Delta - \partial_t)u(x,t) = 0$, then, for $(x,t) \in M \times (0,T]$,*

$$\frac{|\nabla u(x,t)|^2}{u^2(x,t)} - \alpha\frac{u_t(x,t)}{u(x,t)} \leq \frac{n\alpha^2}{t} + \frac{n\alpha^3 K_0}{\alpha - 1} + n^{3/2}\alpha^2(K_0 + K_1)$$

for any $\alpha > 1$.

We find that the two-point function approach works when the metric evolves as a supersolution of the Ricci flow, that is, $\partial g/\partial t \geq -2\mathrm{Ric}$. We prove the following result.

THEOREM 1.4. *Let M^n be a compact Riemannian manifold with diameter $D(t)$ and let $g(t)$ be a time-dependent metric on M satisfying $\partial g/\partial t \geq -2\mathrm{Ric}$. Assume that the diameter $D(t)$ is bounded above by D, the Ricci curvature satisfies $\mathrm{Ric} \geq 0$ for $t \in [0,T)$ and $u : M \times [0,T) \to \mathbb{R}$ is a viscosity solution of the heat equation*

$$u_t = \left[\alpha(u,|Du|,t)\frac{D_iuD_ju}{|Du|^2} + \beta(t)\left(\delta_{ij} - \frac{D_iuD_ju}{|Du|^2}\right)\right]D_iD_ju + q(u,|Du|,t), \tag{1.2}$$

where $\beta(t) \geq 1$. Suppose that $\varphi : [0,D] \times [0,T] \to \mathbb{R}$ satisfies

$$\varphi' > 0$$
$$\frac{d}{ds}\left(\frac{\varphi_t - \varphi''\alpha(\varphi,\varphi',t) + q(\varphi,\varphi',t)}{\varphi'\beta(t)}\right) < 0.$$

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Let Ψ be given by inverting φ for each t, so that $\varphi(\Psi(z,t),t) = z$ for each z and t. Assume that the range of $u(\cdot,0)$ is contained in the interval $[\varphi(0,0),\varphi(D,0)]$ and that, for all x and y in M,

$$\Psi(u(y,0),0) - \Psi(u(x,0),0) - d_0(x,y) \le 0.$$

Then

$$\Psi(u(y,t),t) - \Psi(u(x,t),t) - d_t(x,y) \le 0$$

for all $x, y \in M$ and $t \in [0,T)$.

The gradient estimate follows immediately.

COROLLARY 1.5. *Under the conditions of Theorem 1.4, for every $x \in M$ and $t \ge 0$,*

$$|\nabla u(x,t)| \le \varphi'(\Psi(u(x,t),t),t).$$

The graphical mean curvature flow and the Laplacian heat flow are two important examples of the heat equation of Theorem 1.4. We consider another type of heat equation which includes important examples such as p-Laplacian heat flow.

THEOREM 1.6. *Let M^n be a compact Riemannian manifold with diameter $D(t)$ and let $g(t)$ be a time-dependent metric on M satisfying $\partial g/\partial t \ge -2\mathrm{Ric}$. Assume that the diameter $D(t)$ is bounded above by D, the Ricci curvature satisfies $|\mathrm{Ric}| \le \kappa$ for $t \in [0,T)$ and $u : M \times [0,T) \to \mathbb{R}$ is a viscosity solution of the heat equation*

$$u_t = \Big[\alpha(|Du|,t)\frac{D_iuD_ju}{|Du|^2} + \beta(|Du|,t)\Big(\delta_{ij} - \frac{D_iuD_ju}{|Du|^2}\Big)\Big]D_iD_ju + q(|Du|,t), \qquad (1.3)$$

where $\alpha \ge 0$ and $\beta > 0$. Suppose that $\varphi : [0,D] \times [0,T) \to \mathbb{R}$ satisfies

$$\varphi_t \ge \varphi''\alpha(\varphi',t) + \kappa s|\varphi'(1-\beta(\varphi',t))|$$

and, that for all x and y in M,

$$u(y,0) - u(x,0) - 2\varphi\Big(\frac{d_0(x,y)}{2},0\Big) \le 0.$$

Then

$$u(y,t) - u(x,t) - 2\varphi\Big(\frac{d_t(x,y)}{2},t\Big) \le 0.$$

COROLLARY 1.7. *Under the conditions of Theorem 1.6, for every $x \in M$ and $t \ge 0$,*

$$|\nabla u(x,t)| \le \varphi'(0,t).$$

The paper is organised as follows. In Section 2 we recall background material including the definitions of $\bar{\mathcal{P}}^{2,-}f(t_0,x_0)$ and $\bar{\mathcal{P}}^{2,+}f(t_0,x_0)$ when f is a semicontinuous function and parabolic maximum principles for semicontinuous functions. In Section 3 we give the proof of Theorem 1.4 and Corollary 1.5 for one type of heat equation. Then Theorem 1.6 and Corollary 1.7 are proved in Section 4 for the other type of heat equation.

2. Preliminaries

First we present some definitions and results which will be used later.

Definition 2.1 [5]. Let $f : (0,T)\times M \to (-\infty,+\infty]$ be a lower semicontinuous (LSC) function. The parabolic second-order subjet of f at a point $(t_0,x_0)\in(0,T)\times M$ is defined by

$$\mathcal{P}^{2,-}f(t_0,x_0) := \{(D_t\varphi(t_0,x_0), D_x\varphi(t_0,x_0), D_x^2\varphi(t_0,x_0)) : \varphi \text{ is once continuously differentiable in } t\in(0,T), \text{twice continuously differentiable in } x\in M \text{ and } f-\varphi \text{ attains a local minimum at } (t_0,x_0)\}.$$

Similarly, for an upper semicontinuous (USC) function $f : (0,T)\times M\to[-\infty,+\infty)$, the parabolic second-order superjet of f at (t_0,x_0) is defined by

$$\mathcal{P}^{2,+}f(t_0,x_0) := \{(D_t\varphi(t_0,x_0), D_x\varphi(t_0,x_0), D_x^2\varphi(t_0,x_0)) : \varphi \text{ is once continuously differentiable in } t\in(0,T), \text{twice continuously differentiable in } x\in M \text{ and } f-\varphi \text{ attains a local maximum at } (t_0,x_0)\}.$$

Definition 2.2 [5]. Let $f : (0,T)\times M\to(-\infty,+\infty]$ be an LSC function and (t,x) be in $(0,T)\times M$. Then $\bar{\mathcal{P}}^{2,-}f(t_0,x_0)$ is the set of $(a,\zeta,A)\in\mathbb{R}\times TM_x^*\times\mathcal{L}_s^2(TM_x)$ such that there exists a sequence (x_k,a_k,ζ_k,A_k) in $M\times\mathbb{R}\times TM_{x_k}^*\times\mathcal{L}_s^2(TM_{x_k})$ satisfying:

(i) $(a_k,\zeta_k,A_k)\in\mathcal{P}^{2,-}f(t_k,x_k)$; and
(ii) $\lim_k\,(t_k,x_k,f(t_k,x_k),a_k,\zeta_k,A_k) = (t,x,f(t,x),a,\zeta,A)$.

The corresponding definition of $\bar{\mathcal{P}}^{2,+}f(t_0,x_0)$ when f is an upper semicontinuous function is then clear.

Theorem 2.3 [5]. *Let $M_1,\dots,M_k$ be Riemannian manifolds and $\Omega_i\in M_i$ open subsets. Define $\Omega=(0,T)\times\Omega_1\times\cdots\times\Omega_k$. Let u_i be upper semicontinuous functions on $(0,T)\times\Omega_i$ for $i=1,\dots,k$. Let φ be a function defined on Ω which is once continuously differentiable in $t\in(0,T)$ and twice continuously differentiable in $x:=(x_1,\dots,x_k)\in\Omega_1\times\cdots\times\Omega_k$ and set*

$$w(t,x_1,\dots,x_k)=u_1(t,x_1)+\cdots+u_k(t,x_k)\quad \text{for } (t,x_1,\dots,x_k)\in\Omega.$$

Assume that $(\hat{t},\hat{x}_1,\dots,\hat{x}_k)$ is a maximum of $\omega-\varphi$ in Ω. Further, assume that there is a $\tau>0$ such that for every $M>0$ there is $C>0$ such that for $i=1,\dots,k$,

$$\begin{cases} a_i\le C \text{ whenever } (a_i,\zeta_i,A_i)\in\bar{\mathcal{P}}^{2,+}_{M_i}u_i(t,x_i),\\ d(x_i,\hat{x}_i)+|t-\hat{t}|\le\tau \text{ and } |u_i(t,x_i)|+|\zeta_i|+\|A_i\|\le M.\end{cases}$$

Then, for each $\epsilon>0$, there exist real numbers b_i and bilinear forms $B_i\in\mathcal{L}_s^2(TM_i)_{\hat{x}_i}$ for $i=1,\dots,k$ such that

$$(b_i, D_{x_i}\varphi(\hat{t},\hat{x}_1,\dots,\hat{x}_k), B_i)\in(a_i,\zeta_i,A_i)\in\bar{\mathcal{P}}^{2,+}_{M_i}u_i(\hat{t},\hat{x}_i)$$

for $i = 1, \ldots, k$ and the block diagonal matrix with entries B_i satisfies

$$-\Big(\frac{1}{\epsilon} + \|A\|\Big)I \leq \begin{pmatrix} B_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & B_k \end{pmatrix} \leq A + \epsilon A^2,$$

where $A = D_x^2\varphi(\hat{t}, \hat{x}_1, \ldots, \hat{x}_k)$ and $b_1 + \cdots + b_k = \partial\varphi/\partial t(\hat{t}, \hat{x}_1, \ldots, \hat{x}_k)$.

We prove the next lemma by the same method as [3, Lemma 8].

Lemma 2.4. *Let u be a continuous function and $\varphi : \mathbb{R} \times [0, T) \to \mathbb{R}$ be a $C^{2,1}$ function with $\varphi \geq 0$. Let $\Psi : \mathbb{R} \times [0, T) \to \mathbb{R}$ be the inverse of φ, so that*

$$\Psi(\varphi(u(y, t), t), t) = u(y, t).$$

(i) *If $(a, \zeta, A) \in \mathcal{P}^{2,+}(\Psi \circ u)(\hat{x}, \hat{t}))$, then*

$$(\varphi_t + \varphi', \varphi'\zeta, \varphi''\zeta \otimes \zeta + \varphi' A) \in \mathcal{P}^{2,+}u(\hat{x}, \hat{t}),$$

where all derivatives of φ are evaluated at $\Psi(u(\hat{x}, \hat{t}), \hat{t})$.

(ii) *If $(a, \zeta, A) \in \mathcal{P}^{2,-}(\Psi \circ u)(\hat{x}, \hat{t})$, then*

$$(\varphi_t + \varphi', \varphi'\zeta, \varphi''\zeta \otimes \zeta + \varphi' A) \in \mathcal{P}^{2,-}u(\hat{x}, \hat{t}),$$

where all derivatives of φ are evaluated at $\Psi(u(\hat{x}, \hat{t}), \hat{t})$.

(iii) *The same statements hold if we replace the semijets by their closures.*

Proof. (i) Assume that $(a, \zeta, A) \in \mathcal{P}^{2,+}(\Psi \circ u)(\hat{x}, \hat{t})$. By Definition 2.1, there is a $C^{2,1}$ function h such that $\Psi(u(x, t), t) - h(x, t)$ has a local maximum at $(\hat{x}, \hat{t})$ and $(h_t, Dh, D^2h)(\hat{x}, \hat{t}) = (a, \zeta, A)$. Since φ is increasing,

$$u(x, t) - \varphi(h(x, t), t) = \varphi(\Psi(u(x, t), t), t) - \varphi(h(x, t), t)$$

has a local maximum at $(\hat{x}, \hat{t})$. It follows that

$$(\varphi_t + \varphi' a, \varphi'\zeta, \varphi''\zeta \otimes \zeta + \varphi' A) \in \mathcal{P}^{2,+}u(\hat{x}, \hat{t}).$$

Part (ii) can be proved by a similar argument. Part (iii) follows by an approximation. □

3. Proof of Theorem 1.4

Let $\epsilon > 0$ be arbitrary and consider the first time $t_0 > 0$ and points x_0 and y_0 in M at which the inequality

$$\Psi(u(y, t), t) - \Psi(u(x, t), t) - d_t(x, y) - \epsilon(1 + t) \leq 0$$

reaches equality. Note that if $\epsilon > 0$, then we necessarily have $y_0 \neq x_0$. Even though the length of the curve depends explicitly on t through the time dependence of the metric g, we can still replace $d_t(x, y)$ by a smooth function $\tilde{d}_t(x, y)$ as in [4, proof of Theorem 6]

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within a neighbourhood of (x_0, y_0) at any fixed time t. Let $\gamma_0(s)$ be a minimising geodesic joining x_0 and y_0 parametrised by arc length at time t_0, that is, $|\gamma_0'(s)|_{g(t_0)} = 1$ with length $l = \mathscr{L}_{g(t_0)}(\gamma_0) = d_{t_0}(x_0, y_0)$. Let $\{e_i(s)\}_{i=1}^n$ be parallel orthonormal vector fields along $\gamma_0(s)$ with $e_n(s) = \gamma_0'(s)$. Then in small neighbourhoods U_{x_0} of x_0 and U_{y_0} of y_0, there are mappings $x \mapsto (a_1(x), \dots, a_n(x))$ and $y \mapsto (b_1(y), \dots, b_n(y))$ such that

$$x = \exp_{x_0}\Big(\sum_{i=1}^n a_i(x) e_i(0)\Big), \quad y = \exp_{y_0}\Big(\sum_{i=1}^n b_i(y) e_i(l)\Big).$$

Then $\tilde{d}_t(x, y)$ can be defined by

$$\tilde{d}_t(x, y) = \mathscr{L}_{g(t)}\Big(\exp_{\gamma_0(s)}\Big(\frac{l-s}{l}\sum a_i(x)e_i(s) + \frac{s}{l}\sum b_i(y)e_i(s)\Big)\Big) \quad \text{for } s \in [0, l].$$

Therefore,

$$\Psi(u(y,t),t) - \Psi(u(x,t),t) - \tilde{d}_t(x,y) \le \epsilon(1+t)$$

for any $(x, y, t) \in U_{x_0} \times U_{y_0} \times [0, T]$ and with equality at (x_0, y_0, t_0). Thus, we can apply the parabolic maximum principle to conclude that for each $\lambda > 0$ there exist $X \in \mathcal{L}_s^2(TM_{x_0})$, $Y \in \mathcal{L}_s^2(TM_{y_0})$ such that

$$(b_1, D_y\tilde{d}_t(x,y))\big|_{(t_0,x_0,y_0)}, Y) \in \mathcal{P}^{2,+}(\Psi \circ u)(y_0, t_0),$$
$$(-b_2, -D_x\tilde{d}_t(x,y))\big|_{(t_0,x_0,y_0)}, X) \in \mathcal{P}^{2,-}(\Psi \circ u)(x_0, t_0),$$
$$\epsilon + \frac{d}{dt}(\tilde{d}_t(x,y))\big|_{(t_0,x_0,y_0)} = b_1 + b_2$$

and

$$\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \le H + \lambda H^2,$$

where $H = D^2\tilde{d}_t(x,y)|_{(t_0,x_0,y_0)}$. We compute

$$\begin{aligned}\frac{d}{dt}(\tilde{d}_t(x,y))\big|_{(t_0,x_0,y_0)} &= \int_0^l \frac{d}{dt}(\langle\gamma_0'(s), \gamma_0'(s)\rangle_{g(t)}^{1/2})\Big|_{t=t_0} ds \\ &= \frac{1}{2}\int_0^l \frac{dg}{dt}\langle\gamma_0'(s), \gamma_0'(s)\rangle\Big|_{t=t_0} ds \ge -\int_0^l \mathrm{Ric}_{t_0}(e_n(s), e_n(s))\, ds.\end{aligned}$$

Therefore,

$$b_1 + b_2 \ge \epsilon - \int_0^l \mathrm{Ric}_{t_0}(e_n(s), e_n(s))\, ds. \tag{3.1}$$

Note that $D_y\tilde{d}_t(x,y)|_{(t_0,x_0,y_0)} = e_n(l)$ and $D_x\tilde{d}_t(x,y)|_{(t_0,x_0,y_0)} = -e_n(0)$. By Lemma 2.4,

$$(b_1\varphi'(z_{y_0}, t_0) + \varphi_t(z_{y_0}, t_0), \varphi'(z_{y_0}, t_0)e_n(l), \varphi''(z_{y_0}, t_0)Y + \varphi'(z_{y_0}, t_0)e_n(l) \otimes e_n(l))$$

is in $\mathcal{P}^{2,+}(u)(y_0, t_0)$ and

$$(-b_2\varphi'(z_{x_0}, t_0) + \varphi_t(z_{x_0}, t_0), \varphi'(z_{x_0}, t_0)e_n(0), \varphi''(z_{x_0}, t_0)X + \varphi'(z_{x_0}, t_0)e_n(0) \otimes e_n(0))$$

is in $\mathcal{P}^{2,-}(u)(x_0,t_0)$, where $z_{x_0} = \Psi(u(x_0,t_0),t_0)$ and $z_{y_0} = \Psi(u(y_0,t_0),t_0)$. On the other hand, since u is both a subsolution and a supersolution of (1.2),

$$\begin{aligned}\varphi_t(z_{y_0},t_0) + \varphi'(z_{y_0},t_0)b_1 + q(\varphi(z_{y_0},t_0),\varphi'(z_{y_0},t_0),t_0)\\ - \operatorname{tr}(\varphi'(z_{y_0},t_0)A_2Y + \varphi''(z_{y_0},t_0)A_2e_n(l)\otimes e_n(l)) \le 0\end{aligned}$$

and

$$\begin{aligned}\varphi_t(z_{x_0},t_0) - \varphi'(z_{x_0},t_0)b_2 + q(\varphi(z_{x_0},t_0),\varphi'(z_{x_0},t_0),t_0)\\ - \operatorname{tr}(\varphi'(z_{x_0},t_0)A_1Y + \varphi''(z_{x_0},t_0)A_1e_n(0)\otimes e_n(0)) \le 0,\end{aligned}$$

where

$$A_1 = \begin{pmatrix}\beta(t_0) & & & \\ & \ddots & & \\ & & \beta(t_0) & \\ & & & \alpha(\varphi(z_{x_0},t_0),\varphi'(z_{x_0},t_0),t_0)\end{pmatrix},$$

$$A_2 = \begin{pmatrix}\beta(t_0) & & & \\ & \ddots & & \\ & & \beta(t_0) & \\ & & & \alpha(\varphi(z_{y_0},t_0),\varphi'(z_{y_0},t_0),t_0)\end{pmatrix}.$$

For the inequality at y_0,

$$\begin{aligned}\varphi_t(z_{y_0},t_0) - (\alpha(\varphi(z_{y_0},t_0),\varphi'(z_{y_0},t_0),t_0))\varphi''(z_{y_0},t_0) + q(\varphi(z_{y_0},t_0),\varphi'(z_{y_0},t_0),t_0))\\ + \varphi'(z_{y_0},t_0)\left(b_1 - \operatorname{tr}\begin{pmatrix}0 & C\\ C & A_2\end{pmatrix}\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\right) \le 0,\end{aligned}$$

where C is an $n\times n$ matrix to be determined. Dividing by $\varphi'(z_{y_0},t_0)\beta(t_0)$ gives

$$\left(\frac{\varphi_t - \varphi''\alpha(\varphi,\varphi',t) + q(\varphi,\varphi',t)}{\varphi'\beta(t)}\right)\bigg|_{(z_{y_0},t_0)} + \frac{1}{\beta(t_0)}\left(b_1 - \operatorname{tr}\begin{pmatrix}0 & C\\ C & A_2\end{pmatrix}\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\right) \le 0.$$

Similarly, for the inequality at x_0,

$$\left(\frac{\varphi_t - \varphi''\alpha(\varphi,\varphi',t) + q(\varphi,\varphi',t)}{\varphi'\beta(t)}\right)\bigg|_{(z_{x_0},t_0)} - \frac{1}{\beta(t_0)}\left(b_2 - \operatorname{tr}\begin{pmatrix}A_1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\right) \ge 0.$$

Let

$$C = \begin{pmatrix}\beta(t_0) & & & \\ & \ddots & & \\ & & \beta(t_0) & \\ & & & 0\end{pmatrix}.$$

Then

$$\left(\frac{\varphi_t - \varphi''\alpha(\varphi,\varphi',t) + q(\varphi,\varphi',t)}{\varphi'\beta(t)}\right)\bigg|_{(z_{x_0},t_0)}^{(z_{y_0},t_0)} + \frac{1}{\beta(t_0)}(b_1+b_2) - \operatorname{tr}\left(W\begin{pmatrix}-X & 0\\ 0 & Y\end{pmatrix}\right) \le 0,$$

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where the matrix

$$W = \begin{pmatrix} I_{n-1} & 0 & I_{n-1} & 0 \\ 0 & \dfrac{\alpha(\varphi(z_{x_0}, t_0), \varphi'(z_{x_0}, t_0), t_0)}{\beta(t_0)} & 0 & 0 \\ I_{n-1} & 0 & I_{n-1} & 0 \\ 0 & 0 & 0 & \dfrac{\alpha(\varphi(z_{y_0}, t_0), \varphi'(z_{y_0}, t_0), t_0)}{\beta(t_0)} \end{pmatrix}$$

is positive semidefinite. Since

$$\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \le H + \lambda H^2,$$

it follows that

$$\operatorname{tr}\left(W\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right) \le \operatorname{tr}(WH) + \lambda \operatorname{tr}(WH^2).$$

Letting $\lambda \to 0$,

$$\left(\frac{\varphi_t - \varphi''\alpha(\varphi, \varphi', t) + q(\varphi, \varphi', t)}{\varphi'\beta(t)}\right)\Bigg|_{(z_{x_0}, t_0)}^{(z_{y_0}, t_0)} + \frac{1}{\beta(t_0)}(b_1 + b_2) \le \operatorname{tr}(WH).$$

Now we compute $\operatorname{tr}(WH)$:

$$\begin{aligned} \operatorname{tr}(WH) = &\sum_{i=1}^{n-1} (D_{x_i} D_{x_i} \tilde{d}_t + 2D_{x_i} D_{y_i} \tilde{d}_t + D_{y_i} D_{y_i} \tilde{d}_t)\big|_{(t_0, x_0, y_0)} \\ &+ \frac{\alpha(\varphi(z_{x_0}, t_0), \varphi'(z_{x_0}, t_0), t_0)}{\beta(t_0)} D_{x_n} D_{x_n} \tilde{d}_t\big|_{(t_0, x_0, y_0)} \\ &+ \frac{\alpha(\varphi(z_{y_0}, t_0), \varphi'(z_{y_0}, t_0), t_0)}{\beta(t_0)} D_{y_n} D_{y_n} \tilde{d}_t\big|_{(t_0, x_0, y_0)}. \end{aligned}$$

The sum in the first line is

$$\frac{d^2}{d\mu^2}\bigg|_{\mu=0} \tilde{d}_{t_0}(\exp_{x_0}(\mu e_i(0)), \exp_{y_0}(\mu e_i(l))) = \frac{d^2}{d\mu^2}\bigg|_{\mu=0} \mathscr{L}_{g(t_0)}(\exp_{\gamma_0(s)}(\mu e_i(s))_{s\in[0,l]}).$$

By the second variation formulae,

$$\frac{\partial^2}{\partial\mu^2}\bigg|_{\mu=0} \mathscr{L}(\gamma(\mu, \cdot)) = \int_0^l \left(|\nabla_{\gamma_s}(\gamma_\mu^\perp)|^2 - R(\gamma_s, \gamma_\mu, \gamma_\mu, \gamma_s)\right) ds + \langle \gamma_s, \nabla_{\gamma_\mu}\gamma_\mu \rangle\big|_0^l,$$

where $\gamma_\mu^\perp$ means the normal part of the variational vector. Since $\gamma_\mu = e_i(s)$, we have $\nabla_{\gamma_s}\gamma_\mu^\perp = 0$ and $\nabla_{\gamma_\mu}\gamma_\mu = 0$ and so

$$\sum_{i=1}^{n-1} (D_{x_i} D_{x_i} \tilde{d}_t + 2D_{x_i} D_{y_i} \tilde{d}_t + D_{y_i} D_{y_i} \tilde{d}_t)\big|_{(t_0, x_0, y_0)} = -\int_0^l \operatorname{Ric}_{t_0}(e_n(s), e_n(s))\, ds.$$

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Similarly,

$$D_{x_n} D_{x_n} \tilde{d}_t \big|_{(t_0,x_0,y_0)} = 0, \quad D_{y_n} D_{y_n} \tilde{d}_t \big|_{(t_0,x_0,y_0)} = 0.$$

In summary,

$$\operatorname{tr}(WH) = -\int_0^l \operatorname{Ric}_{t_0}(e_n(s), e_n(s))\, ds,$$

which implies that

$$\left(\frac{\varphi_t - \varphi''\alpha(\varphi, \varphi', t) + q(\varphi, \varphi', t)}{\varphi'\beta(t)}\right)\Bigg|_{(z_{x_0},t_0)}^{(z_{y_0},t_0)} + \frac{1}{\beta(t_0)}(b_1 + b_2) \le -\int_0^l \operatorname{Ric}_{t_0}(e_n(s), e_n(s))\, ds.$$

Combining this with (3.1),

$$\epsilon \le (1 - \beta(t_0)) \int_0^l \operatorname{Ric}_{t_0}(e_n(s), e_n(s))\, ds,$$

which gives a contradiction. Therefore,

$$\Psi(u(y,t), t) - \Psi(u(x,t), t) - d_t(x,y) \le 0.$$

4. Proof of Theorem 1.6

Let $\epsilon > 0$ be arbitrary and consider the first time $t_0 > 0$ and points x_0 and y_0 in M at which the inequality

$$u(y,t) - u(x,t) - 2\varphi\left(\frac{d_t(x,y)}{2}, t\right) - \epsilon(1+t) \le 0$$

reaches equality. If $\epsilon > 0$, then we necessarily have $y_0 \ne x_0$. We replace $d_t(x,y)$ by a smooth function $\tilde{d}_t(x,y)$ as in the proof of Theorem 1.4 within a neighbourhood of (x_0, y_0). Then

$$u(y,t) - u(x,t) - 2\varphi\left(\frac{\tilde{d}_t(x,y)}{2}, t\right) - \epsilon(1+t) \le 0$$

for any $(x,y,t) \in U_{x_0} \times U_{y_0} \times [0,T]$ and with equality at (x_0, y_0, t_0). Assume that $l = d_{t_0}(x_0, y_0) = 2s_0$. We apply the parabolic maximum principle to conclude that for each $\lambda > 0$, there exist $X \in \mathcal{L}_s^2(TM_{x_0})$, $Y \in \mathcal{L}_s^2(TM_{y_0})$ such that

$$(b_1, \varphi'(s_0,t_0) D_y \tilde{d}_t(x,y))\big|_{(t_0,x_0,y_0)}, Y) \in \mathcal{P}^{2,+}(u)(y_0,t_0),$$

$$(-b_2, -\varphi'(s_0,t_0) D_x \tilde{d}_t(x,y))\big|_{(t_0,x_0,y_0)}, X) \in \mathcal{P}^{2,-}(u)(x_0,t_0),$$

$$\epsilon + \frac{d}{dt}\left(2\varphi\left(\frac{d_t(x,y)}{2}, t\right)\right) = b_1 + b_2$$

and

$$\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix} \le H + \lambda H^2,$$

where $H = D^2\psi$ and $\psi = 2\varphi(\tilde{d}_t(x,y)/2, t)$.

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We compute

$$b_1 + b_2 \geq \epsilon - \varphi'(s_0, t_0) \int_0^l \mathrm{Ric}_{t_0}(e_n(s), e_n(s))\, ds + 2\varphi_t(s_0, t_0). \tag{4.1}$$

Since u is both a subsolution and a supersolution of (1.3),

$$b_1 \leq \mathrm{tr}\,(A_2 Y) + q(\varphi'(s_0, t_0), t_0), \quad -b_2 \geq \mathrm{tr}\,(A_1 X) + q(\varphi'(s_0, t_0), t_0),$$

where

$$A_1 = A_2 = \begin{pmatrix} \beta(\varphi'(s_0,t_0),t_0) & & & \\ & \ddots & & \\ & & \beta(\varphi'(s_0,t_0),t_0) & \\ & & & \alpha(\varphi'(s_0,t_0),t_0) \end{pmatrix}.$$

Therefore,

$$b_1 \leq \mathrm{tr}\left(\begin{pmatrix} 0 & C \\ C & A_2 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right) + q(\varphi'(s_0, t_0), t_0), \tag{4.2}$$

$$-b_2 \geq -\mathrm{tr}\left(\begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right) + q(\varphi'(s_0, t_0), t_0). \tag{4.3}$$

Set $A = A_1 = A_2$ and

$$C = \begin{pmatrix} \beta(\varphi'(s_0,t_0),t_0) & & & \\ & \ddots & & \\ & & \beta(\varphi'(s_0,t_0),t_0) & \\ & & & 0 \end{pmatrix}.$$

Combining (4.2) with (4.3),

$$b_1 + b_2 \leq \mathrm{tr}\left(\begin{pmatrix} A & C \\ C & A \end{pmatrix}\begin{pmatrix} -X & 0 \\ 0 & Y \end{pmatrix}\right) \leq \mathrm{tr}\left(\begin{pmatrix} A & C \\ C & A \end{pmatrix}H + \lambda\begin{pmatrix} A & C \\ C & A \end{pmatrix}H^2\right).$$

Dividing by $\beta(\varphi'(s_0, t_0), t_0)$ gives

$$\frac{b_1 + b_2}{\beta(\varphi'(s_0, t_0), t_0)} \leq \mathrm{tr}\,(WH) + \lambda \mathrm{tr}\,(WH^2), \tag{4.4}$$

where

$$W = \begin{pmatrix} I_{n-1} & 0 & I_{n-1} & 0 \\ 0 & \frac{\alpha(\varphi'(s_0,t_0),t_0)}{\beta(\varphi'(s_0,t_0),t_0)} & 0 & 0 \\ I_{n-1} & 0 & I_{n-1} & 0 \\ 0 & 0 & 0 & \frac{\alpha(\varphi'(s_0,t_0),t_0)}{\beta(\varphi'(s_0,t_0),t_0)} \end{pmatrix}$$

is positive semidefinite.

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Next,

$$
\begin{aligned}
\sum_{i=1}^{n-1}&(D_{x_i}D_{x_i}\psi + 2D_{x_i}D_{y_i}\psi + D_{y_i}D_{y_i}\psi)\big|_{(t_0,x_0,y_0)}\\
&= \sum_{i=1}^{n-1}\bigg(\varphi'(s_0,t_0)\frac{d^2}{d\mu^2}\bigg|_{\mu=0}\tilde{d}_{t_0}(\exp_{x_0}(\mu e_i(0)),\exp_{y_0}(\mu e_i(l)))\\
&\quad + \varphi''(s_0,t_0)\frac{d}{d\mu}\bigg|_{\mu=0}\tilde{d}_{t_0}(\exp_{x_0}(\mu e_i(0)),\exp_{y_0}(\mu e_i(l)))\bigg)\\
&= \sum_{i=1}^{n-1}\bigg(\varphi'(s_0,t_0)\frac{d^2}{d\mu^2}\bigg|_{\mu=0}\mathscr{L}_{g(t_0)}(\exp_{\gamma_0(s)}(\mu e_i(s))_{s\in[0,l]})\\
&\quad + \varphi''(s_0,t_0)\frac{d}{d\mu}\bigg|_{\mu=0}\mathscr{L}_{g(t_0)}(\exp_{\gamma_0(s)}(\mu e_i(s))_{s\in[0,l]})\bigg)\\
&= -\varphi'(s_0,t_0)\int_0^l \mathrm{Ric}_{t_0}(e_n(s),e_n(s))\,ds.
\end{aligned}
$$

The summand in the first line is

$$
\begin{aligned}
(D_{x_n}&D_{x_n}\psi + 2D_{x_n}D_{y_n}\psi + D_{y_n}D_{y_n}\psi)\big|_{(t_0,x_0,y_0)}\\
&= \varphi'(s_0,t_0)\frac{d^2}{d\mu^2}\bigg|_{\mu=0}\tilde{d}_{t_0}(\exp_{x_0}(\mu e_n(0)),\exp_{y_0}(\mu e_n(l)))\\
&\quad + \frac{1}{2}\varphi''(s_0,t_0)\bigg(\frac{d}{d\mu}\bigg|_{\mu=0}\tilde{d}_{t_0}(\exp_{x_0}(\mu e_n(0)),\exp_{y_0}(\mu e_n(l)))\bigg)^2\\
&= \varphi'(s_0,t_0)\frac{d^2}{d\mu^2}\bigg|_{\mu=0}\mathscr{L}_{g(t_0)}(\exp_{\gamma_0(s)}(\mu e_n(s))_{s\in[0,l]})\\
&\quad + \frac{1}{2}\varphi''(s_0,t_0)\bigg(\frac{d}{d\mu}\bigg|_{\mu=0}\mathscr{L}_{g(t_0)}(\exp_{\gamma_0(s)}(\mu e_n(s))_{s\in[0,l]})\bigg)^2\\
&= 2\varphi''(s_0,t_0).
\end{aligned}
$$

In summary,

$$
\mathrm{tr}\,(WH) = 2\varphi''(s_0,t_0)\frac{\alpha(\varphi'(s_0,t_0),t_0)}{\beta(\varphi'(s_0,t_0),t_0)} - \varphi'(s_0,t_0)\int_0^l \mathrm{Ric}_{t_0}(e_n(s),e_n(s))\,ds.
$$

Substituting this into (4.4) and letting $\lambda \to 0$,

$$
\frac{b_1+b_2}{\beta(\varphi'(s_0,t_0),t_0)} \le 2\varphi''(s_0,t_0)\frac{\alpha(\varphi'(s_0,t_0),t_0)}{\beta(\varphi'(s_0,t_0),t_0)} - \varphi'(s_0,t_0)\int_0^l \mathrm{Ric}_{t_0}(e_n(s),e_n(s))\,ds.
$$

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Combining this equation with (4.1),

$$\begin{aligned}\epsilon \le -2\varphi_t(s_0, t_0) + 2\alpha(\varphi'(s_0, t_0), t_0)\varphi''(s_0, t_0) \\ + \varphi'(s_0, t_0)(1 - \beta(\varphi'(s_0, t_0), t_0)) \int_0^{2s_0} \mathrm{Ric}_{t_0}(e_n(s), e_n(s))\, ds,\end{aligned}$$

which gives a contradiction. Consequently,

$$u(y, t) - u(x, t) - 2\varphi\Big(\frac{d_t(x, y)}{2}, t\Big) - \epsilon(1 + t) \le 0.$$

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