Proceedings of the Edinburgh Mathematical Society (2010) **53**, 13–29 ©
DOI:10.1017/S001309150700017X Printed in the United Kingdom

NON-GENERICITY OF INFINITESIMAL VARIATIONS OF HODGE STRUCTURES ARISING IN SOME GEOMETRIC CONTEXTS

EMMANUEL ALLAUD[1] AND JAVIER FERNANDEZ[2]

[1] *Department of Mathematics, University of Utah, Salt Lake City, UT 84112-0090, USA* (allaud@math.utah.edu)
[2] *Instituto Balseiro, Universidad Nacional de Cuyo – CNEA, Bariloche R8402AGP, Argentina* (jfernand@ib.edu.ar)

(Received 30 January 2007)

Abstract We prove that the infinitesimal variations of Hodge structure arising in a number of geometric situations are non-generic. In particular, we consider the case of generic hypersurfaces in complete smooth projective toric varieties, generic hypersurfaces in weighted projective spaces and generic complete intersections in projective space and show that, for sufficiently high degrees, the corresponding infinitesimal variations are non-generic.

Keywords: Hodge theory; infinitesimal variations of Hodge structure; geometric variation

2000 *Mathematics subject classification:* Primary 14D07
Secondary 32G20

1. Introduction

A variation of Hodge structure can be described, using the language of exterior differential systems, as an integral manifold of Griffiths's differential system over the period domain. An important problem in Hodge theory is the study of the geometric locus: that is, the locus of those variations of Hodge structure that arise from the cohomology of a family of polarized projective varieties.

An infinitesimal version of this problem consists of describing the infinitesimal variations of Hodge structure—the integral elements of Griffiths's system—that arise from geometric variations. In [**1**] it was shown that the infinitesimal variations arising from deformations of hypersurfaces of sufficiently high degree in projective space are non-generic in the space of infinitesimal variations. The purpose of the present paper is to show that this property holds in a variety of geometric situations, namely, generic hypersurfaces of complete smooth toric varieties (Theorem 3.12), generic hypersurfaces in weighted projective space (Theorem 4.5) and generic complete intersections in projective space (Theorem 5.6). In all cases, requirements of sufficiently high degree apply.

By considering all these results we begin to see a general principle that infinitesimal variations of geometric origin (eventually satisfying some condition analogous to high degree) are non-generic.

The main tools used in this paper are the appropriate residue theories for simplicial toric varieties and for complete intersections in projective space, as well as infinitesimal Torelli theorems, dualities and Macaulay's Theorem.

The paper is organized as follows. In § 2 we review some results from [**1**] and describe an approach to proving the non-genericity of families of infinitesimal variations. In § 3 we study the infinitesimal variations associated to generic hypersurfaces in complete simplicial projective toric varieties and reduce the proof of non-genericity results to a numerical condition strongly related to infinitesimal Torelli theorems (Theorem 3.9). One such Torelli theorem, due to M. Green, allows us to conclude the non-genericity in the case of smooth ambient spaces. In § 4 we specialize the toric analysis to the case where the ambient space is a weighted projective space, in which case the numerical condition now follows from a result of L. Tu. Finally, in § 5, the general approach is specialized to the case of complete intersections.

2. The projection of the integral elements of the Griffiths system to a Grassmannian, and the symmetrizers' correspondence

Let us fix some notation (see [**12**,**14**] for more details). Recall that a *(real) Hodge structure of weight* k on the real vector space H with Hodge numbers $h^{k,0},\dots,h^{0,k}$, consists of a grading $H\otimes\mathbb{C} = H^{k,0}\oplus\cdots\oplus H^{0,k}$ such that $\dim H^{j,k-j} = h^{j,k-j}$ and $\overline{H^{j,k-j}} = H^{k-j,j}$. We will usually denote H and its complexification by H. We define $H^q := \hom(H^{k-q,q}, H^{k-q-1,q+1})$, and consider $\bigoplus_{0\leqslant q\leqslant k-1} H^q$ as a subset of $\hom(H,H)$. We also define the following maps for $0\leqslant q\leqslant k-1$:

$$p_q : \bigoplus_{0\leqslant r\leqslant k-1} H^r \to H^q$$
$$\alpha \mapsto \alpha|_{H^{k-q,q}}.$$

Remark 2.1. p_q is the natural projection of $\bigoplus_{0\leqslant a\leqslant k-1} H^a$ onto H^q.

The period space (i.e. the set of all *polarized* Hodge structures with fixed Hodge numbers and polarization Q [**12**]) is the homogeneous variety $\mathcal{D}\simeq G/P$ (with $G=\mathrm{SO}(H,Q)$ and P a parabolic subgroup). We denote by $\mathfrak{g}$ and $\mathfrak{p}$ the Lie algebras of G and P. Then $\mathfrak{g}$ is given by

$$\mathfrak{g} = \{X\in \mathrm{End}(H) \mid Q(Xv,w)+Q(v,Xw)=0 \text{ for all } v,w\in H\}. \tag{2.1}$$

Fixing a reference structure $H_0 := \{H_0^{k-q,q}\}\in\mathcal{D}$, we can also consider the following subspaces of $\mathrm{End}(H)$:

$$\mathrm{End}(H)^{p,-p} = \{X\in\mathrm{End}(H) \mid \forall r+s=k,\ X(H_0^{r,s})\subset H_0^{r+p,s-p}\} \tag{2.2}$$

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

and then define $\mathfrak{g}^{p,-p} := \mathfrak{g} \cap \mathrm{End}(H)^{p,-p}$. We also note that

$$\mathfrak{g}^0 := g^{0,0}, \quad \mathfrak{g}^- := \bigoplus_{p<0} \mathfrak{g}^{p,-p}, \quad \mathfrak{g}^+ := \bigoplus_{p>0} \mathfrak{g}^{p,-p},$$

so that we have $\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{g}^0 \oplus \mathfrak{g}^+$. Moreover, we have that $\mathfrak{p} = \mathfrak{g}^0 \oplus \mathfrak{g}^+$ and, as $\mathcal{D} \simeq G/P$, we conclude that

$$T_{H_0}\mathcal{D} \simeq \mathfrak{g}^-.$$

Let $V := \mathfrak{g}^{-1,1} \subset \bigoplus_{0 \leqslant a \leqslant k-1} H^a$, and let V_d denote the set of infinitesimal variations of Hodge structures of dimension d, then $V_d \subset \mathrm{Gr}(d, V)$, the *Grassmanian* of d dimensional subspaces of V. In fact, V_d is the algebraic subvariety of $\mathrm{Gr}(d, V)$ of all abelian d-subalgebras of V. Moreover, we have the following.

Lemma 2.2. *For all $i \in \{0, \dots, k\}$, $p_i : \mathrm{Gr}(d, V) \to \mathrm{Gr}(d, H^i)$ is a rational map.*

We now recall the definition of 'symmetrizer' due to Donagi [**8**].

Definition 2.3. Let $\psi : E \times F \to G$ be a bilinear map. We define

$$\mathrm{Symm}(\psi) := \{q \in \hom(E, F) \mid \forall \alpha, \alpha' \in E,\ \psi(\alpha, q(\alpha')) = \psi(\alpha', q(\alpha))\}.$$

The link between the infinitesimal variations of Hodge structures and the symmetrizers is given by the following proposition (see [**1**, Proposition 3.2]).

Proposition 2.4. *For any $E^0 \in \mathrm{Gr}(d, H^0)$, we define the following bilinear map*

$$\begin{aligned} \phi_{E^0} : E^0 \times H^1 &\to \hom(H^{k,0}, H^{k-2,2}) \\ (\alpha, \beta) &\mapsto \beta \circ \alpha. \end{aligned}$$

Then we have, for $k \geqslant 3$,

$$p_1(p_0^{-1}(E^0) \cap V_d) \simeq \mathrm{Symm}\, \phi_{E^0}.$$

We need one more result to explain the principle of 'non-genericity' for infinitesimal variations of Hodge structures.

Proposition 2.5. *Let G^0, G^1, G^2 be three $\mathbb{C}$-vector spaces with $\dim G^0 > 1$. For all $E^0 \subset \hom(G^0, G^1)$, we consider the following bilinear map:*

$$\begin{aligned} \phi_{E^0} : E^0 \times \hom(G^1, G^2) &\to \hom(G^0, G^2) \\ (\alpha, \beta) &\mapsto \beta \circ \alpha. \end{aligned}$$

Let d be a positive integer, $d \leqslant \dim \hom(G^0, G^1)$. If we have the inequality

$$d \geqslant 3\left(\left[\frac{\dim G^1 - 1}{\dim G^0}\right] + 1\right), \tag{2.3}$$

then any generic $E^0 \in \mathrm{Gr}(d, \hom(G^0, G^1))$ satisfies

$$\mathrm{Symm}(\phi_{E^0}) = \{0\}.$$

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

Proof. See the proof of [**1**, Proposition 3.3]. The space of symmetrizers $\mathrm{Symm}(\phi_{E^0})$ is defined by a set of linear equations depending on $E^0 \in \mathrm{Gr}(d, \hom(G^0, G^1))$; then the condition $\mathrm{Symm}(\phi_{E^0}) = \{0\}$ is open in $\mathrm{Gr}(d, \hom(G^0, G^1))$ and the proof amounts to building a $\mathbb{C}$-vector space in $\hom(G^0, G^1)$ for which the rank of the linear equations defining the symmetrizers is maximal (the dimensions are such that maximal rank for those equations implies triviality of their zero set). □

In our context we will have $G^i := H^{k-i,i}$, $0 \leqslant i \leqslant 2$, and $E^0 := p_0(E)$, where E will be an infinitesimal variation of Hodge structure. In this setting the inequality (2.3) reads:

$$\dim p_0(E) \geqslant 3\left(\left[\frac{h^{k-1,1} - 1}{h^{k,0}}\right] + 1\right). \tag{2.4}$$

Now we can state the main theorem that we will use in our geometric applications.

Theorem 2.6. *Let E be an infinitesimal variations of Hodge structure such that $\dim p_0(E) = \dim E$, $p_1(E) \neq \{0\}$ and E satisfies inequality (2.4) then E is non-generic: that is, E must lie in a proper subvariety of $V_{\dim E}$.*

Proof. Define $E^0 := p_0(E)$ and $d := \dim E$. Using Proposition 2.4 we have that $p_1(E) \in p_1(p_0^{-1}(E^0) \cap V_d) \simeq \mathrm{Symm}(\phi_{E^0})$, and as we made the assumption that $p_1(E) \neq \{0\}$, this implies that $\mathrm{Symm}(\phi_{E^0}) \neq \{0\}$. But, as E satisfies (2.4), the generic element $X \in \mathrm{Gr}(d, H^0)$ satisfies $\mathrm{Symm}(\phi_X) = \{0\}$. Then $p_0(E)$ must lie in a proper subvariety S of $\mathrm{Gr}(\dim p_0(E), H^0)$; therefore, E lies in $p_0^{-1}(S) \cap V_{\dim E}$, which is a proper subvariety of V_d (because p_0 is regular and surjective). □

The following are applications of Theorem 2.6; they are the main results of this paper.

Theorem 2.7. *Let P be a complete smooth projective toric variety of dimension $n \geqslant 4$ and let $\beta \in A_{n-1}(P)$ be the degree of an ample Cartier divisor. Let E be the infinitesimal variation of Hodge structure associated to a generic $f \in S_{t\beta}$, where S is the homogeneous coordinate ring of P. Then there exists $t_0 \in \mathbb{N}$ such that, for all $t \geqslant t_0$, E is non-generic.*

Theorem 2.8. *Let $P = \mathbb{P}^n(q_0, \dots, q_n)$ be the weighted projective space with $n \geqslant 4$, $q_0 = \gcd(q_1, \dots, q_n) = 1$. Then, if $\mathrm{lcm}(q_1, \dots, q_n) \mid d$ and $\mathrm{lcm}(q_1, \dots, q_n) \mid \sum_{j=0}^{n} q_j$, there is a degree d_0 such that for all $d \geqslant d_0$ the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree d is non-generic.*

Theorem 2.9. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection defined by the intersection of c hypersurfaces of degrees $d_1, \dots, d_c$. If $\dim X \geqslant 3$, there exists $K \in \mathbb{N}$ such that if the degree of the canonical bundle $d(X) = \sum_{a=1}^{c} d_a - (n+1) \geqslant K$, then the infinitesimal variations of Hodge structure associated to the deformation of X, E, is non-generic in $V_{\dim E}$.*

Remark 2.10. These theorems state asymptotic results. Nevertheless, for concrete examples the conditions on the degrees of the varieties that ensure non-genericity can be stated and tested explicitly. For the last theorem, effective bounds are also included.

3. Hypersurfaces in toric varieties

In this section we describe the infinitesimal variation of Hodge structure associated to the family of hypersurfaces of fixed degree in a complete simplicial projective toric variety P. Using this description, the non-genericity of the infinitesimal variation for hypersurfaces of high degree can be established modulo some algebraic conditions. Finally, we show that the algebraic conditions hold, for instance, when the hypersurfaces and P are smooth and, also, when P is a weighted projective space.

Let P be the n-dimensional complete simplicial toric variety defined by the fan $\Sigma \subset N$, where N is a lattice of rank n in $\mathbb{R}^n$. P can be described as a geometric quotient as follows [**5**]. Let $S := \mathbb{C}[z_1, \dots, z_r]$ be the polynomial ring with variables corresponding to the integer generators n_j of the one-dimensional cones of Σ. Let $Z := \bigcap_{\sigma \subset \Sigma} \{z \in \mathbb{C}^r : \hat{z}_\sigma = 0\}$, where, for each cone $\sigma \subset \Sigma$, $\hat{z}_\sigma := \prod_{n_j \notin \sigma} z_j$. Then P is the geometric quotient of $U := \mathbb{C}^r \setminus Z$ by the algebraic group $G := \hom(A_{n-1}(P), \mathbb{C}^*)$, where $A_{n-1}(P)$ is the Chow group of P.

If M is the lattice dual to N, a grading on S can be induced from the exact sequence $0 \to M \to \mathbb{Z}^r \to A_{n-1}(P) \to 0$, where the second arrow is $m \mapsto (\langle m, n_1\rangle, \dots, \langle m, n_r\rangle)$ and the third is $(a_1, \dots, a_r) \mapsto a_1[D_1] + \cdots + a_r[D_r]$, where D_j is the divisor associated to the generator n_j. With this notation, $\deg(z_1^{a_1} \cdots z_r^{a_r}) := a_1[D_1] + \cdots + a_r[D_r] \in A_{n-1}(P)$. This grading coincides with the one induced by the action of G on $\mathbb{C}^r$. The graded piece of S of degree $\beta \in A_{n-1}(P)$ is denoted by S_β.

Given $f \in S_\beta$, let $V(f) := \{z \in \mathbb{C}^r : f(z) = 0\}$; $V(f) \cap U$ is G stable and hence descends to a hypersurface $X_f \subset P$. Since Σ is simplicial, P is a V-manifold. Moreover, by [**4**, Proposition 4.15], if β is Cartier and ample, for generic $f \in S_\beta$, X_f is a V-submanifold of P. Let $B_\beta \subset S_\beta$ be the Zariski open of those f for which X_f is a V-submanifold of P and let $\pi : \mathcal{X} \to B_\beta$ be the family of hypersurfaces $\mathcal{X}_f = X_f \subset P$. If we assume that P is a projective variety, classical results of Deligne show that $R^{n-1}\pi_*\mathbb{Q}$ defines a graded-polarized variation of mixed Hodge structure. On the other hand, by [**18**, § 1], it is known that, since X_f is a V-manifold, $H^{n-1}(X_f, \mathbb{Q})$ carries a pure Hodge structure of weight $n-1$. Furthermore, if $\omega := c_1(\mathcal{O}_P(1))$ and L_ω is the operator induced by left multiplication by ω,

$$H^{n-1}(X_f)_{\mathrm{p}} := \ker(L_\omega : H^{n-1}(X_f, \mathbb{Q}) \to H^{n+1}(X_f, \mathbb{Q})),$$

the *primitive cohomology*, is a sub-Hodge structure that is polarized. In what follows we will consider a sub-Hodge structure of $H^{n-1}(X_f)_{\mathrm{p}}$, namely, the *vanishing cohomology* (also known as *variable cohomology*) defined as

$$H^{n-1}(X_f)_{\mathrm{v}} := \ker(j_* : H^{n-1}(X_f, \mathbb{Q}) \to H^{n+1}(X_f, \mathbb{Q})),$$

where j_* is the Gysin morphism associated to the inclusion $j : X_f \to P$. Notice that in [**4**] this vanishing cohomology is called primitive cohomology. Since $L_\omega = j^* \circ j_*$ it is clear that $H^{n-1}(X_f)_{\mathrm{v}}$ is a sub-Hodge structure of $H^{n-1}(X_f)_{\mathrm{p}}$ that is polarized; hence, $H^{n-1}(X_f)_{\mathrm{v}}$ is a polarized sub-Hodge structure of $H^{n-1}(X_f)_{\mathrm{p}}$ of weight $n-1$. To sum up, we have the following proposition.

Proposition 3.1. *For $\beta \in A_{n-1}(P)$ Cartier and ample, the vanishing cohomology of the family of hypersurfaces $\pi : \mathcal{X} \to B_\beta$ defines the polarized variation of Hodge structure of weight $n-1$, whose fibres are $H^{n-1}(X_f, \mathbb{Q})_{\mathrm{v}}$.*

If $\mathcal{D}$ is the classifying space for the Hodge structures described in Proposition 3.1 and Γ is the monodromy group, the previous result defines a *period mapping* $\Phi : B_\beta \to \mathcal{D}/\Gamma$. In fact, upon restriction to a Zariski open $B'_\beta \subset B_\beta$, Φ factors through the *generic coarse moduli space for hypersurfaces of P with degree β*, $M_\beta := B'_\beta/\widetilde{\mathrm{Aut}}_\beta$, as defined in [**4**, § 13]:

$$\begin{array}{ccccc} B'_\beta & \hookrightarrow & B_\beta & \xrightarrow{\Phi} & \mathcal{D}/\Gamma \\ \downarrow{\scriptstyle\kappa} & & & \nearrow{\scriptstyle\Psi} & \\ M_\beta & & & & \end{array} \tag{3.1}$$

As noted in [**4**], for B'_β sufficiently small, κ is a smooth map. Taking differentials at the point $f \in B'_\beta$, we have $\mathrm{d}\Phi_f = \mathrm{d}\Psi_{[f]}\,\mathrm{d}\kappa_f$, with $\mathrm{d}\kappa_f$ onto. Notice that the horizontality of Φ implies that of Ψ. In what follows we will consider the infinitesimal variation of Hodge structure E_f associated to Ψ, $\mathrm{d}\Psi_{[f]} : T_{[f]}M_\beta \to T_{\Psi([f])}\mathcal{D}$. In fact, we will be only interested in the image of $\mathrm{d}\Phi_f$, which we continue to call E_f. Since $\mathrm{d}\kappa_f$ is onto, E_f is also the image of $\mathrm{d}\Psi_{[f]}$.

The following result shows some relations between the cohomology of a hypersurface X_f in a toric variety and graded parts of the Jacobian ring of f.

Theorem 3.2. *Let P be a complete simplicial projective toric variety of dimension $n \geqslant 4$, and let $X_f \subset P$ be a quasi-smooth ample hypersurface defined by $f \in S_\beta$ that defines a Cartier divisor. Let $R(f) := S/\langle \partial f/\partial z_1, \dots, \partial f/\partial z_r \rangle$ be the Jacobian ring of f, with the grading inherited from S. Then the following hold.*

(i) $H^0(P, \mathcal{O}_P(X_f)) \simeq S_\beta$. *The following diagram is commutative up to a multiplicative constant:*

$$\begin{array}{ccc} H^0(P, \mathcal{O}_P(X_f)) \otimes H^0(P, \Omega^n_P(pX_f)) & \longrightarrow & H^0(P, \Omega^n_P((p+1)X_f)) \\ \downarrow & & \downarrow \\ T_f B_\beta \otimes H^{n-p,p-1}(X_f)_{\mathrm{v}} & \longrightarrow & H^{n-p-1,p}(X)_{\mathrm{v}} \end{array} \tag{3.2}$$

where the top arrow is induced by the product of sections, the bottom one by the infinitesimal variation of Hodge structure and the vertical arrows are the surjective maps

$$\bar{\alpha}_p : H^0(P, \Omega^n_P(pX_f)) \to H^{n-p,p-1}(X_f)_{\mathrm{v}}$$

induced by the residue.

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

(ii) *The diagram*

$$\begin{array}{ccc} T_{[f]}M_\beta & \xrightarrow{\mathrm{d}\Psi_{[f]}} & \hom(H^{n-1,0}(X_f)_{\mathrm{v}}, H^{n-2,1}(X_f)_{\mathrm{v}}) \\ \simeq\downarrow & & \downarrow\simeq \\ R_\beta & \xrightarrow{\times} & \hom(R(f)_{\beta-\beta_0}, R(f)_{2\beta-\beta_0}) \end{array} \tag{3.3}$$

is commutative up to a multiplicative constant. Here $\beta_0 := \deg(z_1\cdots z_r) \in A_{n-1}(P)$.

Proof. The first assertion of (i) is [**4**, Lemma 4.11]. Theorem 6.13 from [**21**] (modified for toric varieties using the residue theory developed in [**4**]) gives that the diagram

$$\begin{array}{ccc} H^0(P,\Omega^n_P(pX_f)) & \longrightarrow & \hom(H^0(P,\mathcal{O}_P(X_f)), H^0(P,\Omega^n_P((p+1)X_f))) \\ \downarrow & & \downarrow \\ H^{n-p,p-1}(X_f)_{\mathrm{v}} & \longrightarrow & \hom(T_fB_\beta, H^{n-p-1,p}(X_f)_{\mathrm{v}}) \end{array} \tag{3.4}$$

is commutative up to a multiplicative constant. The vertical arrows are induced by the surjective residue maps $\bar{\alpha}_p$ and the isomorphism $H^0(P,\mathcal{O}_P(X_f)) \simeq S_\beta \simeq T_fB_\beta$. The rest of part (i) follows from (3.4).

To prove part (ii), we start by showing that the vertical arrows are isomorphisms. Proposition 13.7 in [**4**] does this for the left arrow, while Theorem 10.13 in [**4**] implies the result for the right arrow.

Next we establish the commutativity of diagram (3.3). Starting from (3.4) and using [**4**, Lemma 4.11 and Theorem 9.7], we have that

$$\begin{array}{ccc} S_{p\beta-\beta_0} & \longrightarrow & \hom(S_\beta, S_{(p+1)\beta-\beta_0}) \\ \simeq\downarrow & & \downarrow\simeq \\ H^0(P,\Omega^n_P(pX_f)) & \longrightarrow & \hom(H^0(P,\mathcal{O}_P(X_f)), H^0(P,\Omega^n_P((p+1)X_f))) \\ \downarrow & & \downarrow \\ H^{n-p,p-1}(X_f)_{\mathrm{v}} & \longrightarrow & \hom(T_fB_\beta, H^{n-p-1,p}(X_f)_{\mathrm{v}}) \end{array}$$

commutes up to a multiplicative constant. From here it is easy to see that the same holds for

$$\begin{array}{ccc} R_{p\beta-\beta_0} & \longrightarrow & \hom(S_\beta, R_{(p+1)\beta-\beta_0}) \\ \simeq\downarrow & & \downarrow\simeq \\ H^{n-p,p-1}(X_f)_{\mathrm{v}} & \longrightarrow & \hom(T_fB_\beta, H^{n-p-1,p}(X_f)_{\mathrm{v}}) \end{array}$$

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

and, eventually, for

$$\begin{array}{ccc} S_\beta & \longrightarrow & \hom(R_{p\beta-\beta_0}, R_{(p+1)\beta-\beta_0}) \\ \simeq\downarrow & & \downarrow\simeq \\ T_f B_\beta & \longrightarrow & \hom(H^{n-p,p-1}(X_f)_{\mathrm{v}}, H^{n-p-1,p}(X_f)_{\mathrm{v}}) \end{array}$$

and then (3.3) follows by noticing that the horizontal arrows factor through R_β and $T_{[f]}M$, respectively. □

Batyrev gives a similar description of the differential of the period mapping in his work on affine hypersurfaces [**3**, Proposition 11.8].

To state the following result we will write $O(t^k)$ to denote a function (a polynomial in this case) with $O(t^k)/t^k$ bounded as $t \to \infty$. Also, if $M \simeq \mathbb{Z}^n \subset \mathbb{R}^n$ is a lattice, $\mathrm{vol}(\pi)$ is the volume of the lattice polytope $\pi \subset \mathbb{R}^n$, normalized so that the unit n-cube of the lattice M is 1.

Lemma 3.3. *Let D be an ample Cartier divisor in the n-dimensional complete simplicial toric variety P and $\Delta \subset \mathbb{R}^n$ its associated polytope. If X is a generic ample hypersurface of degree $[tD] \in A_{n-1}(P)$ for $t \in \mathbb{N}$, then $\mathrm{vol}(\Delta) > 0$ and*

(i) $h_t^{n-1,0} := \dim H^{n-1,0}(X) = \mathrm{vol}(\Delta)t^n + O(t^{n-1})$,

(ii) $h_t^{n-2,1} := \dim H^{n-2,1}(X) = (2^n - (n+1))\,\mathrm{vol}(\Delta)t^n + O(t^{n-1})$ *and*

(iii) $\mu_t := \dim M_{[tD]} \geqslant \mathrm{vol}(\Delta)t^n + O(t^{n-1})$.

Proof. Let $\Delta_t := t\Delta$ be the polytope in $M \otimes \mathbb{R} \simeq \mathbb{R}^n$ defined by tD. By [**6**, (5.5)] we have

$$\left.\begin{aligned} h_t^{n-1,0} &:= \dim H^{n-1,0}(X) = l^*(\Delta_t), \\ h_t^{n-2,1} &:= \dim H^{n-2,1}(X) = l^*(2\Delta_t) - (n+1)l^*(\Delta_t) - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} l^*(\Gamma), \end{aligned}\right\} \tag{3.5}$$

where $\mathcal{F}_{n-1}(\Delta_t)$ is the set of $(n-1)$-dimensional faces of Δ_t and $l^*(\pi)$ denotes the number of lattice points that lie in the relative interior of the polytope π.

The number of lattice points in a lattice polytope $\pi_t := t\pi \subset \mathbb{R}^n$ is $E_\pi(t) := l(\pi_t)$, where $l(\pi_t)$ is the number of lattice points in π_t and E_π is the Ehrhart polynomial of π [**9**]. E_π is a polynomial of degree at most n, where the coefficient of t^n is $\mathrm{vol}(\pi)$. Furthermore, the reciprocity law says that $E_\pi(-t) = (-1)^n l^*(\pi_t)$.

Since D is an ample Cartier divisor, Δ is an n-dimensional lattice polytope [**16**, § 2.2] and we have $l^*(\Delta_t) = (-1)^n E_\Delta(-t) = \mathrm{vol}(\Delta)t^n + O(t^{n-1})$ with $\mathrm{vol}(\Delta) > 0$. Hence,

$$h_t^{n-1,0} = \mathrm{vol}(\Delta)t^n + O(t^{n-1}).$$

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

Similarly,

$$\begin{aligned} h_t^{n-2,1} &= l^*(2\Delta_t) - (n+1)l^*(\Delta_t) - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} l^*(\Gamma) \\ &= (-1)^n E_{2\Delta}(-t) - (n+1)(-1)^n E_\Delta(-t) - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} (-1)^{n-1} E_\Gamma(-t) \\ &= \operatorname{vol}(2\Delta)t^n - (n+1)\operatorname{vol}(\Delta)t^n - \sum_{\Gamma \in \mathcal{F}_{n-1}(\Delta_t)} \operatorname{vol}(\Gamma)t^{n-1} + O(t^{n-1}) \\ &= (2^n - (n+1))\operatorname{vol}(\Delta)t^n + O(t^{n-1}). \end{aligned}$$

Now we turn to μ_t. By Theorem 3.2 we have $\mu_t = \dim R(f)_{[tD]}$ for any generic polynomial $f \in S_{[tD]}$. Since $\dim R(f)_{[tD]} = \dim S_{[tD]} - \dim J(f)_{[tD]}$, we study each term separately. Writing $D = \sum_{j=1}^r b_j D_j$, where the D_j are the torus-invariant divisors associated to the one-dimensional cones in the fan of P, we have

$$\begin{aligned} \dim S_{[tD]} &= \#\{(a_1, \dots, a_r) \in \mathbb{Z}^r_{\geqslant 0} : \deg(z_1^{a_1} \cdots z_r^{a_r}) = [tD]\} \\ &= \#\{m \in M : \langle m, n_j \rangle + tb_j \geqslant 0 \text{ for } j = 1, \dots, r\} \\ &= \#(\Delta_t \cap M) \\ &= l(\Delta_t). \end{aligned}$$

Also, since

$$J(f)_{[tD]} = \biggl\{ \sum_{j=1}^r g_j(z) \frac{\partial f}{\partial z_j} : g_j \in S_{[D_j]} \text{ for all } j \biggr\},$$

we have $\dim J(f)_{[tD]} \leqslant K$, for $K := \sum_{j=1}^r \dim S_{[D_j]}$. Notice that K is independent of t (in fact, it is determined by the one-dimensional cones of the fan of P). We then conclude that

$$\begin{aligned} \mu_t &= \dim R(f)_{[tD]} \\ &= \dim S_{[tD]} - \dim J(f)_{[tD]} \geqslant l(\Delta_t) - K \\ &= \operatorname{vol}(\Delta)t^n + O(t^{n-1}). \end{aligned}$$

□

Proposition 3.4. *Let $\beta \in A_{n-1}(P)$ be the class of an ample Cartier divisor in the complete simplicial toric variety P of dimension $n \geqslant 4$. Then there exists $t_0 \in \mathbb{N}$ such that the cohomology of the generic ample hypersurface X of degree $t\beta$ for $t \geqslant t_0$ satisfies*

$$\mu_t \geqslant 3\biggl(\biggl[\frac{h_{\mathrm{v}}^{n-2,1}(X) - 1}{h_{\mathrm{v}}^{n-1,0}(X)}\biggr] + 1\biggr), \tag{3.6}$$

where $\mu_t := \dim M_{t\beta}$.

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

Proof. Let D be an ample Cartier divisor with $[D]=\beta$. Then by Lemma 3.3 we have

$$\left[\frac{h_t^{n-2,1}-1}{h_t^{n-1,0}}\right] \leqslant \frac{h_t^{n-2,1}-1}{h_t^{n-1,0}} \leqslant \frac{(2^n-(n+1))\operatorname{vol}(\Delta)t^n+O(t^{n-1})}{\operatorname{vol}(\Delta)t^n+O(t^{n-1})}$$
$$= \frac{(2^n-(n+1))\operatorname{vol}(\Delta)+O(t^{n-1})/t^n}{\operatorname{vol}(\Delta)+O(t^{n-1})/t^n}$$

and, since the last expression converges to $2^n-(n+1)$ as $t\to\infty$, we see that the whole expression is bounded from above.

From

$$H^{n-1}(X_f,\mathbb{Q})_{\mathrm{p}} = H^{n-1}(X_f,\mathbb{Q})_{\mathrm{v}} \oplus j^*H^{n-1}(P,\mathbb{Q})$$

(see [**21**, Proposition 2.27]), Lefschetz's Theorem and Bott's Formula (see [**13**, Theorem 2.14]), it is easy to conclude that, for $n\geqslant 4$, $H^{n-1,0}(X)=H^{n-1,0}(X)_{\mathrm{v}}$ and $H^{n-2,1}(X)=H^{n-2,1}(X)_{\mathrm{v}}$. Hence, the Hodge numbers $h_t^{n-1,0}$ and $h_t^{n-2,1}$ compute the corresponding dimensions of the vanishing cohomology of X.

Thus, the right-hand side of (3.6) is bounded from above. On the other hand, also from Lemma 3.3, we have $\mu_t\geqslant \operatorname{vol}(\Delta)t^n+O(t^{n-1})$ with $\operatorname{vol}(\Delta)>0$, so that the left-hand side of (3.6) grows like t^n, and the inequality holds for t sufficiently large. □

Remark 3.5. In the proof of Proposition 3.4 a choice of divisor D is used. This choice determines how large t should be so that (3.6) holds. But any other divisor D' with the same degree would have an associated polytope Δ' that is a lattice translate of that of D; in particular, this implies that they have the same number of lattice points and hence the left-hand and right-hand sides of (3.6) would be the same for D and D'. Hence, a value of t that makes (3.6) true for D works for any other D' with degree β.

Next we study the p_1 projection of the infinitesimal variation of Hodge structure associated to sufficiently ample hypersurfaces in toric varieties. We first recall, in a slightly adapted form, Lemma 1.28 of [**11**].

Lemma 3.6. *Let P be a projective variety and let $\mathcal{E}_1$, $\mathcal{E}_2$ be two coherent sheaves over P. For $\mathcal{L}$ a sufficiently ample invertible sheaf, the multiplication map*

$$H^0(P,\mathcal{E}_1\otimes\mathcal{L}^a)\otimes H^0(P,\mathcal{E}_2\otimes\mathcal{L}^b)\to H^0(P,\mathcal{E}_1\otimes\mathcal{E}_2\otimes\mathcal{L}^{a+b})$$

is surjective for $a,b\geqslant 1$.

Remark 3.7. The ampleness condition in Lemma 3.6 is that

$$H^1(P\times P,\mathcal{I}_\Delta\otimes\mathcal{E}_1\otimes\mathcal{E}_2\otimes\pi_1^*\mathcal{L}^a\otimes\pi_2^*\mathcal{L}^b)=\{0\} \tag{3.7}$$

for all $a,b\geqslant 1$, where $\mathcal{I}_\Delta$ is the ideal sheaf of the diagonal in $P\times P$.

Proposition 3.8. *Let P be a complete simplicial projective toric variety of dimension n and $\beta\in A_{n-1}(P)$ an ample Cartier degree. Then there is a $t_0\in\mathbb{N}$ with the property that if E is the infinitesimal variation of Hodge structure associated to a generic $f\in S_{t\beta}$ for $t\geqslant t_0$, then $p_1(E)\neq\{0\}$.*

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

Proof. Let $\mathcal{E}_1 := \mathcal{O}_P$, $\mathcal{E}_2 := \Omega^n_P$, $a := 1$ and $1 \leqslant b \leqslant n-1$. If $[D] = \beta$ for an ample Cartier divisor D, then $\mathcal{L} := \mathcal{O}_P(tD)$ will satisfy condition (3.7) for all $t \geqslant t_0$ for some $t_0 \in \mathbb{N}$. Using Lemma 3.6 with $\mathcal{E}_1$, $\mathcal{E}_2$, a and b as above and $\mathcal{L} := \mathcal{O}_P(tD)$ for $t \geqslant t_0$ we see that, for a generic $f \in S_{t\beta}$,

$$H^0(P, \mathcal{O}_P(X_f)) \otimes H^0(P, \Omega^n_P(bX_f)) \to H^0(P, \Omega^n_P((b+1)X_f))$$

is surjective. Then, by (3.2),

$$T_f B_\beta \otimes H^{n-p,p-1}(X_f)_{\mathrm{v}} \to H^{n-p-1,p}(X_f)_{\mathrm{v}} \tag{3.8}$$

is surjective. Therefore, the map obtained by iterating (3.8) $n-1$ times,

$$(T_f B_\beta)^{\otimes(n-1)} \otimes H^{n-1,0}(X_f)_{\mathrm{v}} \to H^{0,n-1}(X_f)_{\mathrm{v}},$$

is surjective. On the other hand, making t_0 larger if necessary, $h^{n-1,0} \neq 0$ by Lemma 3.3; hence, we conclude that $p_1(E) \neq 0$. □

Theorem 3.9. *Let P be a complete simplicial projective toric variety of dimension $n \geqslant 4$ and let $\beta \in A_{n-1}(P)$ be the degree of an ample Cartier divisor. For $t \in \mathbb{N}$ let $E_{t\beta}$ be the infinitesimal variation of Hodge structure on $M_{t\beta}$ (see (3.1)) associated to a generic $f \in S_{t\beta}$. Then there exists a $t_0 \in \mathbb{N}$ such that, if $E_{t\beta}$ satisfies*

$$\dim(p_0(E_{t\beta})) = \dim E_{t\beta} \tag{3.9}$$

for $t \geqslant t_0$, $E_{t\beta}$ is non-generic in $V_{\dim E_{t\beta}}$.

Proof. This is immediate from Propositions 3.4 and 3.8, and Theorem 2.6. □

Example 3.10. If $P := \mathbb{P}^n$, the infinitesimal Torelli and Macaulay Theorems imply that condition (3.9) holds. Fixing D as the hyperplane divisor, we conclude that the infinitesimal variation associated to hypersurfaces of sufficiently high degree are non-generic: that is, we recover the result of [**1**].

We can now extend the result of the previous example to the case of variations of generic smooth hypersurfaces in smooth toric varieties. To show that in this case (3.9) holds, we quote Theorem 0.1 from [**11**].

Theorem 3.11. *Let P be a smooth complete algebraic variety of dimension $n \geqslant 2$ and let X be a smooth member of the linear system determined by a sufficiently ample line bundle on P. Then the map*

$$H^1(X, \mathcal{T}_X) \to \mathrm{hom}(H^0(X, \Omega^{n-1}_X), H^1(X, \Omega^{n-2}_X))$$

is injective.

Then, in the smooth case, we conclude the following result from Theorems 3.9 and 3.11.

Theorem 3.12. *Let P be a complete smooth projective toric variety of dimension $n \geqslant 4$ and $\beta \in A_{n-1}(P)$ the degree of an ample Cartier divisor. Let E be the infinitesimal variation of Hodge structure associated to a generic $f \in S_{t\beta}$. Then there exists $t_0 \in \mathbb{N}$ such that, for all $t \geqslant t_0$, E is non-generic.*

Proof. Let t_0 be the value produced by Theorem 3.9, and let X_f be the hypersurface associated to a generic $f \in S_{t\beta}$ for $t \geqslant t_0$. Generically, X_f is smooth. Since, by increasing t_0 if needed, $T_f M_{t\beta} \hookrightarrow H^1(X_f, \mathcal{T}_{X_f})$, by Theorem 3.11, we conclude that $E \simeq p_0(E)$ so that (3.9) holds and so does this result. □

Remark 3.13. Assuming an infinitesimal Torelli theorem valid for (generic) hypersurfaces of sufficiently high degree in toric varieties, it is possible to remove the smoothness requirement from Theorem 3.12. In the next section we show that smoothness is not an essential ingredient for the non-genericity result.

4. Hypersurfaces in weighted projective spaces

Let $(q_0, \dots, q_n) \in \mathbb{N}^{n+1}$ be such that $q_0 = \gcd(q_1, \dots, q_n) = 1$ and $m \mid s$, where $m := \operatorname{lcm}(q_1, \dots, q_n)$ and $s := \sum_{j=0}^n q_j$.

Let $P := \mathbb{P}^n(q_0, \dots, q_n)$ be the weighted projective space with weights $(q_0, \dots, q_n)$ (so that P is 'well formed' by the conditions on the weights). P is a complete simplicial toric variety of dimension n with fan $\Sigma := \{\sigma_I : I \subset \{0, \dots, n\}\}$, where $\sigma_{\{i_1,\dots,i_k\}}$ is the cone generated by $e_{i_1}, \dots, e_{i_k}$, with $\{e_1, \dots, e_n\}$ the canonical basis of $\mathbb{R}^n$ and $e_0 := -\sum_{j=1}^n e_j$. In particular, the one-dimensional cones are $\Sigma^{(1)} := \{\sigma_j : j = 0, \dots, n\}$, where $\sigma_j := \mathbb{R}_{\geqslant 0} e_j$ and the n-dimensional cones are $\Sigma^{(n)} := \{\hat{\sigma}_j : j = 0, \dots, n\}$, where $\hat{\sigma}_j$ is the cone generated by all the e_l except for e_j. The lattice $N \subset \mathbb{R}^n$ is generated by

$$\left\{ f_1 := \frac{1}{q_1} e_1, \ \dots, \ f_n := \frac{1}{q_n} e_n \right\},$$

and we define $f_0 := e_0 = -\sum_{j=1}^n q_j f_j$. Notice that f_j is the integral generator of σ_j for $j = 0, \dots, n$.

$A_{n-1}(P) \simeq \mathbb{Z}$ and if D_j is the torus-invariant divisor associated to the cone σ_j, then $\deg(D_j) = q_j$. Hence, the grading on $S := \mathbb{C}[z_0, \dots, z_n]$ is given by

$$\deg(z_0^{a_0} \cdots z_n^{a_n}) := \sum_{j=0}^n q_j a_j$$

Proposition 4.1. *Let $D := \sum_{j=0}^n a_j D_j$ be a Weil divisor in P with $d := \deg(D)$. Then D is a Cartier divisor if and only if $m \mid d$. Also, D is ample if and only if $d > 0$.*

Proof. It is known (see [**10**, Exercise, p. 62]) that $\sum_{j=0}^n a_j D_j$ is Cartier if and only if for each maximal cone σ there is $u(\sigma) \in M$ such that

$$\langle u(\sigma), f_j \rangle = -a_j \tag{4.1}$$

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

holds for all lattice generators f_j of the one-dimensional cones contained in σ. If D is Cartier, there are $u(\hat{\sigma}_k) \in M$ for each n-dimensional cone $\hat{\sigma}_k$, $k = 0, \ldots, n$, that satisfy (4.1). Evaluating each $u(\hat{\sigma}_k)$ on the lattice generators f_j and using (4.1), it is easy to see that

$$u(\hat{\sigma}_k) = -\left(\sum_{j=1}^{n} a_j f_j^*\right) + \alpha_k \quad \text{with } \alpha_k := \begin{cases} 0 & \text{if } k = 0, \\ \dfrac{d}{q_k} f_k^* & \text{otherwise.} \end{cases} \tag{4.2}$$

Hence, since $u(\hat{\sigma}_k) \in M$, $q_k \mid d$ for all k, and so $m \mid d$.

Conversely, if $m \mid d$, for each maximal cone $\hat{\sigma}_k$, (4.2) defines an element of M that satisfies (4.1). Therefore, D is Cartier.

By [**2**, Theorem 4.6], D is ample if and only if its piecewise linear support function ψ_D satisfies

$$\psi_D(f_{i_1} + \cdots + f_{i_k}) - (\psi_D(f_{i_1}) + \cdots + \psi_D(f_{i_k})) > 0 \tag{4.3}$$

for all primitive collections $\{f_{i_1}, \ldots, f_{i_k}\}$. In our case, the only primitive collection is $\{f_0, \ldots, f_n\}$, for which the left-hand side of (4.3) evaluates to $d/\max\{q_0, \ldots, q_n\}$ and the second assertion follows. □

Corollary 4.2. *mD_0 is an ample Cartier divisor. In particular, P is a projective variety.*

Let E be the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree $d = tm$ for some $t \in \mathbb{N}$. We are interested in the dimension of $p_0(E)$, which will be computed using the Weighted Macaulay Theorem. Let us state it here, in a slightly simplified form. Let $\rho := (n+1)d - 2s$.

Theorem 4.3 (Weighted Macaulay Theorem [20]). *If c is a multiple of m and $e \in \mathbb{N}$ such that $\rho - (c+e) > -s + mn$, then the natural map $R_e \to \hom(R_c, R_{c+e})$ induced by the ring multiplication is injective.*

We can then derive the following.

Corollary 4.4. *If $d \geqslant 2m$ we have $\dim p_0(E) = \dim E$.*

Proof. To determine $\dim p_0(E)$ we apply Theorem 4.3 for $e = d$ and $c = d - s$. Then our assumptions ensure that m divides c. Moreover,

$$\rho - (c+e) = (n+1)d - 2s - d - d + s = (n-1)d - s.$$

Then as $d \geqslant 2m$ we have that $\rho - (c+e) \geqslant 2m(n-1) - s \geqslant mn - s$. The Weighted Macaulay Theorem then gives us that $R_d \hookrightarrow \hom(R_{d-s}, R_{2d-s})$. Using the diagram (3.3), this gives us the following injection:

$$E \hookrightarrow \hom(H^{n-1,0}, H^{n-2,2}),$$

which means exactly that $p_0(E) \simeq E$. □

Theorem 4.5. *Let $P = \mathbb{P}^n(q_0, \ldots, q_n)$ with $n \geqslant 4$, $q_0 = \gcd(q_1, \ldots, q_n) = 1$. Then, if $m \mid d$ and $m \mid s$, there is a degree d_0 such that for all $d \geqslant d_0$ the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree d is non-generic.*

Proof. Recall that E is the infinitesimal variation of Hodge structure associated to a generic hypersurface of degree tm. By Corollary 4.4, if $t \geqslant 2$, condition (3.9) holds. On the other hand, since m is the degree of an ample Cartier divisor, the result follows by Theorem 3.9. □

5. Complete intersections

In this section we prove that the infinitesimal variations of Hodge structure associated to generic complete intersections of sufficiently high degree in $\mathbb{P}^n$ are non-generic. The context is the following: we have $X \subset \mathbb{P}^n$ given as the complete intersection of c hypersurfaces of degrees $d_1 \geqslant \cdots \geqslant d_c$, each one determined by a homogeneous polynomial F_a for $a = 1, \ldots, c$. We will assume that X is smooth. The degree of the canonical bundle of X, K_X, will be denoted by $d(X)$, so that $d(X) = \sum_{a=1}^{c} d_a - (n+1)$. In this section we use the *primitive* Hodge numbers: that is, $h^{p,q} := \dim H^{p,q}(X)_{\mathrm{p}}$.

The following theorem states the relevant known results on the cohomology of a complete intersection.

Theorem 5.1. *With the notation as above, let $S := \mathbb{C}[x_0, \ldots, x_n, \mu_1, \ldots, \mu_c]$, $F := \sum_{a=1}^{c} \mu_a F_a \in S$,*

$$J := \left\langle \frac{\partial F}{\partial \mu_1}, \ldots, \frac{\partial F}{\partial \mu_c}, \frac{\partial F}{\partial x_0}, \ldots, \frac{\partial F}{\partial x_n} \right\rangle$$

be the Jacobian ideal of F and let $R := S/J$ be seen as a bigraded ring with the grading induced by $\deg(x_j) := (0,1)$ and $\deg(\mu_a) := (1, -d_a)$. Then we have the following.

(i) *$H^{n-c-p,p}(X)_{\mathrm{p}} \simeq R_{(p,d(X))}$, where H_{p} is the primitive cohomology. In particular, $H^{n-c,0}(X)_{\mathrm{p}} \simeq R_{(0,d(X))}$ and $H^{n-c-1,1}(X)_{\mathrm{p}} \simeq R_{(1,d(X))}$.*

(ii) *$H^1(X, \mathcal{T}_X) \simeq R_{(1,0)}$, where $\mathcal{T}_X$ is the tangent sheaf of X. Also, if $d(X) \geqslant 0$, the tangent space to the moduli of X is isomorphic to $H^1(X, \mathcal{T}_X)$.*

(iii) *The diagram*

$$\begin{array}{ccc} H^1(X, \mathcal{T}_X) & \longrightarrow & \hom(H^{n-c-p,p}(X)_{\mathrm{p}}, H^{n-c-p-1,p+1}(X)_{\mathrm{p}}) \\ \simeq \downarrow & & \downarrow \simeq \\ R_{(1,0)} & \longrightarrow & \hom(R_{(p,d(X))}, R_{(p+1,d(X))}) \end{array} \tag{5.1}$$

(where the top horizontal arrow is the action of the infinitesimal variation and the bottom one is induced by multiplication) commutes up to a multiplicative constant.

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

Part (i) is due to Terasoma *et al*. [**7**, § 2]. The first part of (ii) and part (iii) are obtained by adapting Terasoma's proofs of [**19**, Propositions 2.5 and 2.6] to the bigraded context. These results are proved using a *Cayley trick*: that is, by associating to X a hypersurface $\mathcal{X} := (F = 0)$ in a larger ambient space and studying the relation between the cohomologies of X and $\mathcal{X}$. The second assertion of (ii) is [**17**, Theorem 3.5].

Remark 5.2. The complete intersection assumption ensures that the sequence $(F_1, \dots, F_c)$ is regular.

In order to apply Theorem 2.6, we need to meet condition (2.4) and check that the conditions on the projections of the infinitesimal variation hold. The purpose of the following result is to solve the first problem.

Proposition 5.3. *Assume that* $\dim X = n - c \geqslant 2$. *Then there is* $K_1 \in \mathbb{N}$ *such that, for* $d(X) \geqslant K_1$,

$$\mu \geqslant 3\left(\left[\frac{h^{n-c-1,1} - 1}{h^{n-c,0}}\right] + 1\right), \tag{5.2}$$

where μ *is the dimension of the infinitesimal variation of Hodge structure associated to the family of deformations of* X.

Proof. This proposition follows readily from polynomial estimates of the different dimensions that appear in (5.2).

Using combinatorics, it is not difficult to show the following bounds:

$$h^{n-c-1,1} \leqslant \frac{c}{n!} d(X)^n + O(d(X)^{n-1}),$$
$$\dim H^1(X, \mathcal{T}_X) \geqslant \frac{1}{n!c^n} d(X)^n + O(1)$$

and, for $c \geqslant 2$,

$$h^{n-c,0} \geqslant \frac{1}{n^n} d(X)^{n-c},$$

while, for $c = 1$,

$$h^{n-1,0} \geqslant \frac{1}{n!} d(X)^n.$$

For $c \geqslant 2$, the right-hand side of (5.2) is bounded from above by

$$3\frac{n^n c 2^n}{n!} d(X)^c + O(d(x)^{c-1}),$$

and

$$\mu = \dim H^1(X, \mathcal{T}_X) \geqslant \frac{1}{n!c^n} d(X)^n + O(1).$$

Then, since $c < n$, the statement follows. The case for $c = 1$ is similar. □

Remark 5.4. It is possible to give an effective version of Proposition 5.3. Indeed, if

$$d(x) \geqslant \begin{cases} \max\{n,\ \sqrt[n-c]{3n^n c^{n+1} 2^n + 1},\ \sqrt[c]{n!c^n(3 + c^2 + (n+1)^2)}\} & \text{if } c \geqslant 2, \\ \max\{n,\ \sqrt[n]{n!(2^n 3 + 4 + (n+1)^2)}\} & \text{if } c = 1, \end{cases}$$

https://doi.org/10.1017/S001309150700017X Published online by Cambridge University Press

then (5.2) holds. Since we do not find this explicit bound very illuminating, or useful, we omit the details.

Next we prove that the conditions on the projections of the infinitesimal variation required by Theorem 2.6 hold.

Proposition 5.5. *There is a $K_2 \in \mathbb{N}$ such that if E is a complete intersection infinitesimal variation of Hodge structure with $d(X) \geqslant K_2$, then we have*

$$p_0(E) \simeq E \quad \text{and} \quad p_1(E) \neq \{0\}$$

Proof. First we refer to the local Torelli theorem proved in [**17**]. Therein Peters proved that the following map is injective:

$$\begin{aligned} E &\hookrightarrow \hom(H^{n-c,0}(X)_{\mathrm{p}}, H^{n-c-1,1}(X)_{\mathrm{p}}), \\ \alpha &\mapsto \alpha|_{H^{n-c,0}(X)_{\mathrm{p}}}. \end{aligned}$$

This means exactly that $E \simeq p_0(E)$. Next, using [**15**, Lemma 3.4], we see that the map

$$R_{(1,0)} \otimes R_{(1,d(X))} \to R_{(2,d(X))}$$

induced by multiplication is surjective, so using diagram (5.1) we find that the corresponding map in cohomology $E \otimes H^{n-c-1,1}(X)_{\mathrm{p}} \to H^{n-c-2,2}(X)_{\mathrm{p}}$ is surjective; therefore, $p_1(E) \neq \{0\}$ as soon as $h^{n-c-2,2}$ is non-zero. But, using the surjectivity of the multiplication in R and diagram (5.1) once more, the following map is also surjective:

$$E \otimes H^{2,n-c-2}(X)_{\mathrm{p}} \to H^{1,n-c-1}(X)_{\mathrm{p}}.$$

Now, as seen in the proof of Proposition 5.3,

$$\dim H^1(X, \mathcal{T}_X) \geqslant \frac{1}{n!c^n} d(X)^n + O(1),$$

so that there exists $K_2 \in \mathbb{N}$ with the property that $H^1(X, \mathcal{T}_X) \neq \{0\}$ if $d(X) \geqslant K_1$. If this last condition holds, then $E \neq \{0\}$. Therefore, $h^{1,n-c-1} = h^{n-c-1,1}$ is non-zero because E injects into $\hom(H^{n-c,0}(X)_{\mathrm{p}}, H^{n-c-1,1}(X)_{\mathrm{p}})$; this then implies that $h^{n-c-2,2} = h^{2,n-c-2} \neq 0$. □

From Theorem 2.6, whose hypotheses are satisfied because of Propositions 5.3 and 5.5, we derive the final result, as follows.

Theorem 5.6. *Let $X \subset \mathbb{P}^n$ be a smooth complete intersection defined by the intersection of c hypersurfaces of degrees $d_1, \dots, d_c$. If $\dim X \geqslant 3$, there exists $K \in \mathbb{N}$ such that if the degree of the canonical bundle $d(X) = \sum_{a=1}^{c} d_a - (n+1) \geqslant K$, then the infinitesimal variation of Hodge structure associated to the deformation of X, E, is non-generic in $V_{\dim E}$.*

Acknowledgements. We thank D. Cox, E. Cattani, A. Dickenstein and J. M. Landsberg for many helpful discussions.

References

1. E. Allaud, Nongenericity of variations of Hodge structure for hypersurfaces of high degree, *Duke Math. J.* **129** (2005), 201–217.
2. V. V. Batyrev, Quantum cohomology rings of toric manifolds, *Astérisque* **218** (1993), 9–34.
3. V. V. Batyrev, Variations of the mixed Hodge structure of affine hypersurfaces in algebraic tori, *Duke Math. J.* **69** (1993), 349–409.
4. V. V. Batyrev and D. A. Cox, On the Hodge structure of projective hypersurfaces in toric varieties, *Duke Math. J.* **75** (1994), 293–338.
5. D. A. Cox, The homogeneous coordinate ring of a toric variety, *J. Alg. Geom.* **4** (1995), 17–50.
6. V. I. Danilov and A. G. Khovanskiĭ, Newton polyhedra and an algorithm for calculating Hodge–Deligne numbers, *Izv. Akad. Nauk SSSR Ser. Mat.* **50** (1986), 925–945.
7. A. Dimca, Residues and cohomology of complete intersections, *Duke Math. J.* **78** (1995), 89–100.
8. R. Donagi, Generic Torelli for projective hypersurfaces, *Compositio Math.* **50** (1983) 325–353.
9. E. Ehrhart, *Polynômes arithmétiques et méthode des polyèdres en combinatoire*, International Series of Numerical Mathematics, Volume 35 (Birkhäuser, 1977).
10. W. Fulton, *Introduction to toric varieties*, Annals of Mathematics Studies, Volume 131 (Princeton University Press, 1993).
11. M. L. Green, The period map for hypersurface sections of high degree of an arbitrary variety, *Compositio Math.* **55** (1985), 135–156.
12. P. Griffiths, *Topics in transcendental algebraic geometry*, Annals of Mathematics Studies, Volume 106 (Princeton University Press, 1984).
13. E. N. Materov, The Bott formula for toric varieties, *Moscow Math. J.* **2** (2002), 161–182, 200.
14. R. Mayer, Coupled contact systems and rigidity of maximal dimensional variations of Hodge structure, *Trans. Am. Math. Soc.* **352** (1999), 2121–2144.
15. J. Nagel, The Abel–Jacobi map for complete intersections, *Indagationes Math.* **8** (1997), 95–113.
16. T. Oda, *Convex bodies and algebraic geometry*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Volume 15 (Springer, 1988).
17. C. Peters, The local Torelli theorem, I, Complete intersections, *Math. Annalen* **217** (1975), 1–16.
18. J. H. M. Steenbrink, Mixed Hodge structure on the vanishing cohomology, in *Real and complex singularities*, pp. 525–563 (Sijthoff and Noordhoff, Alphen aan den Rijn, 1977).
19. T. Terasoma, Infinitesimal variation of Hodge structures and the weak global Torelli theorem for complete intersections, *Annals Math. (2)* **132** (1990), 213–235.
20. L. Tu, Macaulay's Theorem and local Torelli for weighted hypersurfaces, *Compositio Math.* **60** (1986), 33–44.
21. C. Voisin, *Hodge theory and complex algebraic geometry, II*, Cambridge Studies in Advanced Mathematics, Volume 77 (Cambridge University Press, 2003).