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Semi-Strong Colouring of Intersecting Hypergraphs

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For any $c \geqslant 2$, a *c-strong colouring* of the hypergraph G is an assignment of colours to the vertices of G such that, for every edge e of G, the vertices of e are coloured by at least $\min\{c, |e|\}$ distinct colours. The hypergraph G is *t-intersecting* if every two edges of G have at least t vertices in common.

A natural variant of a question of Erdős and Lovász is: For fixed $c \geqslant 2$ and $t \geqslant 1$, what is the minimum number of colours that is sufficient to c-strong colour any t-intersecting hypergraphs? The purpose of this note is to describe some open problems related to this question.

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1. Introduction

The problem of colouring graphs and hypergraphs has a long and rich history (see, *e.g.*, [5, 7, 8, 9]). In the case of graphs, the notion of vertex colouring has a single natural definition: an assignment of labels to the vertices of a graph is a *proper colouring* if the

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endpoints of any edge in the graph are assigned distinct labels. For hypergraphs, however, there exist different natural definitions of vertex colouring. The most common definition, also called *weak* colouring, is an assignment of colours to the vertices such that no edge is monochromatic. Another common definition, called *strong* colouring, is an assignment of colours to the vertices such that all the vertices contained in an edge have distinct colours.

There is a more general notion of hypergraph vertex colouring that encompasses both the weak and strong colouring definitions. We call this notion *semi-strong* colouring.

Definition (semi-strong colouring). For a fixed $c \geqslant 2$, a *c-strong colouring* of the hypergraph G is an assignment of colours to its vertices such that each edge e of G covers vertices with at least $\min\{c, |e|\}$ distinct colours. The *c-strong chromatic number* of G, denoted $\chi(G, c)$, is the minimum number of colours required to c-strong colour G.

The definition of weak colouring corresponds to that of 2-strong colouring, and the definition of strong colouring is equivalent to ∞-strong colouring.[1]

The main focus of this note is the semi-strong colouring of intersecting hypergraphs. A hypergraph is *t-intersecting* if the intersection of any two of its edges contains at least t vertices. The set of edges of a t-intersecting hypergraph is often referred to as a t-intersecting family. Our goal is to determine the minimum number of colours that are sufficient to c-strong colour any t-intersecting hypergraph.

Definition (chromatic number of intersecting hypergraphs). Given two integers $c \geqslant 2$ and $t \geqslant 0$, the *c-strong chromatic number of t-intersecting hypergraphs*, denoted $\chi(t, c)$, is the minimum number of colours which suffices to c-strong colour *any* t-intersecting hypergraph.

With this notation, our goal can be restated as follows: determine $\chi(t, c)$ for every $t \geqslant 0$ and every $c \geqslant 2$. In their seminal paper, Erdős and Lovász [5] observed that the case where $c = 2$ is completely resolved by simple arguments. Specifically, $\chi(0, 2)$ is unbounded, $\chi(1, 2) = 3$, and $\chi(t, 2) = 2$ for every $t \geqslant 2$. (See also Exercise 13.33 in [8].) In the rest of this note, we focus on the case where $c > 2$.

A first step towards establishing the value of $\chi(t, c)$ for all $t \geqslant 0$ and $c > 2$ is to determine when this value is finite and when it is unbounded. As we show in the next sections, $\chi(t, c)$ is finite whenever $t \geqslant c$ and it is unbounded whenever $t \leqslant c - 2$. This leaves the case where $t = c - 1$.

Problem 1.1. *Determine whether or not $\chi(c - 1, c)$ is finite for every $c > 2$.*

[1] More generally, the notion of c-strong colouring of a hypergraph G is equivalent to the strong colouring of G whenever c is at least as large as the cardinality of the largest edge in G.

Following the online publication of this note, Chung [3] showed that $\chi(2,3) \leqslant 21$ and, independently, an anonymous referee showed that $\chi(2,3) \leqslant 7$. These results show that $\chi(2,3)$ is finite; Problem 1.1 currently remains open for all $c > 3$.

In Section 2, we show that for every $t \geqslant c \geqslant 2$ we have the lower bound $\chi(t,c) \geqslant 2(c-1)$. It seems reasonable to believe that this lower bound is tight. The best upper bound for the same chromatic numbers, however, is far from tight. We thus have the following open problem.

Problem 1.2. *Determine whether* $\chi(c,c) = 2(c-1)$ *for every* $c > 2$.

For any $t' > t$, the inequality $\chi(t',c) \leqslant \chi(t,c)$ follows immediately from the observation that t'-intersecting hypergraphs are also t-intersecting. A positive answer to Problem 1.2 would therefore immediately imply that $\chi(t,c) = 2(c-1)$ for every $t \geqslant c$. It might be easier to first determine whether $\chi(t,c) = 2(c-1)$ for values of t that are much greater than c. But even the problem of determining whether the limit of $\chi(t,c)$ as $t \to \infty$ equals $2(c-1)$ is open.

Problem 1.3. *For every* $c > 2$, *determine whether* $\lim_{t\to\infty} \chi(t,c) = 2(c-1)$.

Following the presentation of this problem, Alon [1] showed that when $t \geqslant 2c^2$, we have $\chi(t,c) \leqslant 2c-1$. This bound is obtained by showing that, for any t-intersecting hypergraph, a random $(2c-1)$-colouring of the hypergraph is c-strong with positive probability.

For the last problem we return to the chromatic number $\chi(c-1,c)$. If it is finite, can we determine its exact value? In Section 2, we show that $\chi(c-1,c) \geqslant 2c-1$. The final problem asks whether this bound is tight.

Problem 1.4. *For every* $c > 2$, *determine whether* $\chi(c-1,c) = 2c-1$.

In the rest of this note, we present some results on the chromatic numbers of intersecting hypergraphs. Section 2 establishes lower bounds on the values of $\chi(t,c)$ for every $t \geqslant 0$. Section 3 introduces the probabilistic argument for obtaining upper bounds on $\chi(t,c)$ when $t \geqslant c-1$.

2. General lower bounds

As mentioned in the Introduction, the trivial observation that $(t+1)$-intersecting hypergraphs are also t-intersecting implies that the c-strong chromatic number of t-intersecting hypergraphs is non-increasing in t. In other words, for any $c \geqslant 2$ and any $t \geqslant 0$, we have $\chi(t+1,c) \leqslant \chi(t,c)$. The following proposition shows that the semi-strong chromatic number of intersecting hypergraphs satisfies a different monotonicity property when we increase both t and c.

Proposition 2.1. *For any* $c \geqslant 2$ *and any* $t \geqslant 0$, *we have* $\chi(t+1,c+1) \geqslant \chi(t,c)+1$.

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Proof. Let G be a t-intersecting hypergraph with c-strong chromatic number $\chi(G,c) = \chi(t,c)$. Define G' to be the $(t+1)$-intersecting hypergraph obtained by adding a new vertex v and including it in each of the edges of G. Since $\chi(t+1,c+1) \geqslant \chi(G',c+1)$, it suffices to show that $\chi(G',c+1) \geqslant \chi(G,c)+1 = \chi(t,c)+1$.

Consider any $(c+1)$-strong colouring of G' that uses ℓ colours. For each edge $e \cup \{v\}$ of G', this colouring must assign at least $\min\{c+1, |e|+1\}$ distinct colours to the vertices covered by this edge. This implies that the vertices in the edge e (without the new vertex v) must be coloured by $\min\{c,|e|\}$ distinct colours that are all different from the colour assigned to v. Since this is true for any edge e of G, we obtain a c-strong colouring of G with $\ell - 1$ colours by arbitrarily recolouring any vertex of G that received the same colour as v. Therefore, $\chi(G',c+1) \geqslant \chi(G,c)+1$, as we wanted to show. □

Proposition 2.1 immediately implies that $\chi(t,c)$ is unbounded whenever $t \leqslant c-2$.

Corollary 2.2. *For any $c \geqslant 2$ and any $t \leqslant c-2$, we have $\chi(t,c) = \infty$.*

Proof. Applying Proposition 2.1 a total of t times, we obtain

$$\chi(t,c) \geqslant \chi(t-1,c-1) \geqslant \chi(t-2,c-2) \geqslant \cdots \geqslant \chi(0,c-t).$$

But when $c-t \geqslant 2$, no finite number of colours is sufficient to $(c-t)$-strong colour all 0-intersecting hypergraphs since this class includes all hypergraphs. □

The following two propositions give the lower bounds on $\chi(t,c)$ when $t \geqslant c-1$.

Proposition 2.3. *For any $c \geqslant 2$, we have $\chi(c-1,c) \geqslant 2c-1$.*

Proof. Fix $c \geqslant 2$ and consider the hypergraph

$$G = \left([3c-3], \binom{[3c-3]}{2c-2}\right).$$

This hypergraph is $(c-1)$-intersecting and all its edges have size $2c-2$. Consider any colouring of the vertices in G that uses at most $2c-2$ colours. The most common $c-1$ colours in such a colouring must cover at least

$$(c-1)\left\lceil \frac{3c-3}{2c-2} \right\rceil = (c-1)\cdot 2 = 2c-2$$

vertices. So one of the edges of G covers vertices with at most $c-1$ distinct colours and the colouring of G is not c-strong. Thus, $\chi(c-1,c) \geqslant \chi(G,c) \geqslant 2c-1$. □

Proposition 2.4. *For any $t \geqslant c \geqslant 2$, we have $\chi(t,c) \geqslant 2(c-1)$.*

Proof. Fix $t \geqslant c \geqslant 2$ and consider the hypergraph

$$G = \left([(2c-1)t], \binom{[(2c-1)t]}{ct}\right).$$

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The hypergraph G is t-intersecting and all its edges have size ct. Consider any colouring of the vertices in G that uses at most $2c-3$ colours. The most common $c-1$ colours in such a colouring must cover at least

$$\left\lceil \frac{c-1}{2c-3}(2c-1)t \right\rceil = \left\lceil \frac{c(2c-3)+1}{2c-3} t \right\rceil = \left\lceil ct + \frac{t}{2c-3} \right\rceil > ct$$

vertices. So one of the edges of G covers vertices with at most $c-1$ distinct colours and the colouring cannot be c-strong. Thus, $\chi(t,c) \geqslant \chi(G,c) \geqslant 2(c-1)$. □

3. Probabilistic upper bound

For a fixed $0 < p < 1$, the *p-biased measure* of a family $\mathcal{F}$ over $[n]$ is $\mu_p(\mathcal{F}) := \Pr_S[S \in \mathcal{F}]$, where the probability over S is obtained by including each element $i \in [n]$ in S independently with probability p. Such a set S is called a *p-biased subset* of $[n]$. Dinur and Safra [4] showed that when p is small enough, 2-intersecting families have small p-biased measure. Friedgut [6] showed how the same result also extends to t-intersecting families for every $t > 2$.

Theorem 3.1 (Dinur and Safra [4]; Friedgut [6]). *Fix $t \geqslant 1$. Let $\mathcal{F}$ be a t-intersecting family. For any $p < \frac{1}{t+1}$, the p-biased measure of $\mathcal{F}$ is bounded by $\mu_p(\mathcal{F}) \leqslant p^t$.*

We obtain upper bounds on the chromatic number of intersecting hypergraphs by applying an immediate corollary of Theorem 3.1.

Corollary 3.2. *Fix $t \geqslant 1$. Let $\mathcal{F}$ be a t-intersecting family. For any $p < \frac{1}{t+1}$, the probability that a p-biased subset of $[n]$ contains a set $S \in \mathcal{F}$ is at most p^t.*

Proof. Fix $\mathcal{F}$ to be some t-intersecting family and define $\mathcal{F}'$ to be the t-intersecting family obtained from $\mathcal{F}$ by adding any set which contains a member of $\mathcal{F}$. That is, $\mathcal{F}' = \{T' \subseteq [n] \mid \exists T \in \mathcal{F} \text{ s.t. } T \subseteq T'\}$. Fix $p < \frac{1}{t+1}$ and let $S \subseteq [n]$ be a random p-biased subset of $[n]$. The set S contains some set of $\mathcal{F}$ if and only if $S \in \mathcal{F}'$. By Theorem 3.1, the probability that this event occurs is at most p^t. □

We use the corollary to argue that when ℓ is large enough, a random ℓ-colouring of a t-intersecting hypergraph is c-strong with positive probability.

Theorem 3.3. *For every $t \geqslant c \geqslant 2$, let ℓ be an integer that satisfies $\ell > (c-1)(t+1)$ and*

$$\binom{\ell}{c-1}\left(\frac{c-1}{\ell}\right)^t < 1.$$

Then $\chi(t,c) \leqslant \ell$. In particular, since $\ell = t^t$ satisfies both conditions, $\chi(t,c)$ is finite.

Proof. Let $G = ([n], E)$ be a t-intersecting hypergraph and let ℓ be an integer that satisfies both conditions of the theorem. Consider a random colouring of G where each vertex is

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assigned a colour that is chosen independently and uniformly at random from $[\ell]$. Fix C to be a set of $c-1$ colours. The set S of vertices that receive one of the colours in C is a random subset of $[n]$ where each element is included in S independently with probability $p = \frac{c-1}{\ell} < \frac{1}{t+1}$. By Corollary 3.2, the probability that S contains any edge in E is at most $(\frac{c-1}{\ell})^t$. Applying the union bound over all possible choices of $c-1$ colours, the probability that some edge in G contains vertices that have at most $c-1$ colours is at most $\binom{\ell}{c-1}(\frac{c-1}{\ell})^t < 1$. Therefore, there exists a c-strong colouring of G that requires only ℓ colours. □

Remark. The proof of Theorem 3.3 does more than is required for establishing the value of $\chi(t,c)$. It shows that when ℓ is large enough, a *random* colouring of a t-intersecting hypergraph with ℓ colours is c-strong with high probability.

Theorem 3.3 yields different upper bounds for different values of t with respect to a given c. When $t = c$, the best bound obtained by the theorem is exponential in c.

Corollary 3.4. *For every $c \geqslant 2$, $\chi(c,c) < \sqrt{c} \cdot e^c$.*

When $t = 2c$, the bound is already much stronger and shows that the chromatic number $\chi(t,c)$ is polynomial in c.

Corollary 3.5. *For every $c \geqslant 2$ and $t \geqslant 2c$, $\chi(t,c) < 2c^2$.*

As t grows beyond $2c+1$, the bound obtained by Theorem 3.3 does not continue to improve. In fact, it gets much worse. Note also that because of the condition $\ell > (c-1)(t+1)$, the theorem does not yield a sub-quadratic upper bound on $\chi(t,c)$ for any $t \geqslant c$.

Remark. The topic of semi-strong colouring of intersecting hypergraphs came up in the authors' study of property testing of boolean functions [2]. A common approach in such testing algorithms is that of implicit learning, where we randomly partition some domain and identify a small subset of special parts in the partition. Often the main obstacle is proving that when the function is far from satisfying the questioned property, no choice of a small number of special parts would fool the tester. Theorem 3.3, and particularly Corollary 3.5, guarantees that when we randomly partition the domain into a polynomial number of parts (which are analogous to colours), with high probability the union of any small number of parts will satisfy some criteria (such as not completely containing any member of some *bad* intersecting family). See [2] for more details.

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References

[1] Alon, N. (2013) Paul Erdős and probabilistic reasoning. In *Erdős Centennial*, Vol. 25 of *Bolyai Society Mathematical Studies* (L. Lovász, I. Ruzsa and V. Sós, eds), Springer, pp. 11–33.

[2] Blais, E., Weinstein, A. and Yoshida, Y. (2012) Partially symmetric functions are efficiently isomorphism-testable. In *Proc. 53rd Annual IEEE Symposium on Foundations of Computer Science: FOCS*, IEEE Computer Society, pp. 551–560.

[3] Chung, P. N. (2013) On the c-strong chromatic number of t-intersecting hypergraphs. *Discrete Math.* **313** 1063–1069.

[4] Dinur, I. and Safra, S. (2005) On the hardness of approximating minimum vertex cover. *Ann. of Math.* **162** 439–485.

[5] Erdős, P. and Lovász, L. (1973) Problems and results on 3-chromatic hypergraphs and some related questions. *Coll. Math. Soc. J. Bolyai* **10** 609–627.

[6] Friedgut, E. (2008) On the measure of intersecting families, uniqueness and stability. *Combinatorica* **28** 503–528.

[7] Jensen, T. R. and Toft, B. (1994) *Graph Coloring Problems*, Wiley.

[8] Lovász, L. (1993) *Combinatorial Problems and Exercises*, second edition, AMS.

[9] Molloy, M. and Reed, B. (2001) *Graph Colouring and the Probabilistic Method*, Springer.