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CONTINUOUS REVIEW INVENTORY MODELS FOR PERISHABLE ITEMS WITH LEADTIMES

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We consider a continuous review (s, S) model of perishable items with lost sales. Once items are perished the entire inventory drops instantaneously to zero. The total cost includes the cost of: ordering, unsatisfied demand, units destroyed, holding, and fixed cost of perishability. Both the time to perishability and the lead times are assumed to be exponentially distributed while two cases of demand distribution are considered: Poisson and compound Poisson with general demand sizes. We study the average cost criterion and provide computational results on the problem of finding the optimal re-order level, s, and order up-to level, S. None of the known work on the subject is as general as the model presented here. Our analysis leads to several insights on the optimal (s, S) policies for perishable items in the presence of lead times. For example, we demonstrate that the effectiveness of a heuristic that ignores perishability (and is also analyzed here) decreases with the demand variability and that the cost may either increase or decrease with this variability.

Keywords: aging items, inventory, operations and supply chain, production: perishable, uncertainty: stochastic

1. INTRODUCTION AND LITERATURE REVIEW

Recently, there has been growing attention in the literature on the importance of planning a resilient supply chain. Sheffi [20] discusses how should businesses best prepare for confronting

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unexpected events. However, much of the recent discussion in this literature is on a high level and there is little guidance on how should the firm take its daily operational decisions, such as inventory and capacity decisions, in order to improve its resiliency. In contrast, there is much literature on stochastic inventory theory, starting with the seminal paper of Arrow, Harris, and Marschak [1], presenting a multi-period stochastic inventory problem. Since then, there have been several books on inventory management, including Whitin [24], Zipkin [26], and Porteus [18]. These books and many papers on inventory control, such as Shi et al. [21] that considers a continuous review of a single product with compound Poisson demand and lost sales, typically assume that items have an infinite lifetime. That is, a vast majority of inventory theory ignores unexpected events. In this paper, we aim at filling some of this gap. We consider a continuous review inventory model in the presence of unexpected events that cause the inventory level to instantaneously drop down to zero.

There is a considerable amount of literature on inventory items that are subject to perishability. This literature can be classified into four categories according to the causes of perishability: *decay, obsolescence, outdating, and disasters*. *Decay* means that as time progresses a fixed fraction of the inventory is vanished. Inventory is either lost continuously over time (under models of continuous review, see, e.g., Rajan, Steinberg, and Steinberg [19], which also consider pricing) or after every planning period (under models of periodic review). More recently, Li, Yu, and Wu [12] and Xue, Etti, and Yao [25] studied perishable items whose value deteriorates with time by increasing demand using promotions. *Obsolescence* means that an item is superseded by an improved version and is typical for high-tech products, software, and maps. In these examples the items themselves do not change, but the environment around them changes and, as a result, their utility declines. Usually the demand for the items is then gradually decreasing and their sale prices are changed accordingly (see, e.g., Song and Lau [22] and references therein). Both *Outdatings* and *Disasters* consider the perishability of individual items where a perishable unit has a binary (0-1) utility; full utility before perishability and zero utility after. Several surveys of inventory models with perishable items due to outdating and disasters are those of Nahmias [15], Karaesmen, Scheller-Wolf, and Deniz [10], and Baron [2]. Outdating means that every item has a, possibly random, expiration date and when time exceeds this expiration date the item's utility changed to 0. Most of the literature in the surveys above is devoted to models with outdatings. We note that such models are similar to models of single-server queue with abandonment (see, Boxma, Perry, and Stadje [5] and Perry and Asmussen [17]). In models with disasters all items stored are subject to external unexpected events (such as malfunctioning of storage equipment or spoilage because of extreme weather condition) that instantaneously bring the utility of all items on the shelf to 0.

In practice, instances of perishability due to outdating and disasters are followed by a random lead time. That is after a random lead time a new batch of items arrives at the shelf. We consider a continuous review (s, S) (with variable lot size) inventory model of perishable items with lost sales and lead time and aim at cost minimizing. We note that our model is general and the assumption of a variable order size after a lead time in particular is practical in many settings such as when the total shelf space is limited or in the case of vendor managed inventory. Under vendor managed inventory, the vendor of the product is responsible for maintaining the inventory in the retail outlets. Then, the vendor may be waiting for several outlets to require new inventory, contributing to the stochasticity of the lead time, and fills the orders from sufficiently large trucks, allowing to bring the inventory at each retail outlet back to S. Other common controls such as (r, Q) (with a fixed lot size) and models with backlogging are also possible, but we leave their investigation for future work. The total cost includes the costs of: ordering, unsatisfied demand, units destroyed, holding, and fixed cost of perishability. The stochasticity of the system is due to four factors:

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time to perishability, lead time, demand arrival time, and demand size (at each arrival). Both the lead time and the time to perishability are exponentially distributed and the demand follows a compound Poisson with generally distributed demand sizes. We focus on the average cost criterion where we obtain exact closed-form expressions for the cost. Berk and Gürler [4] analyze a (r, Q) model with Poisson (unit) demand, and deterministic lead time and time to perishability. They use an embedded Markov chain to characterize the effective shelf life distribution (whenever the inventory level is exactly Q) because "an analysis based on the regenerative cycles ... is quite intractable." We pursue the analysis using regenerative cycles for our model, establishing that while this analysis is challenging, it is still tractable.

We note that the assumptions of exponential lifetime and lead time are common in the literature not only because they are more tractable but also because they are quite reasonable whenever a combination of different factors affect these time intervals. For example, the exponential time to perishability is well defended when perishability is caused by disasters. Disasters can occur due to unexpected events, which may include external disasters such as thefts, earthquakes, weather related (e.g., a hurricane, a tsunami or a super storm) and terrorism; internal disasters such as fires and strikes; and emergency orders from other vendors that are willing to generously compensate the supplier for the order. We conjecture that even though any individual such unexpected event is not likely to occur frequently, the occurrence of any unexpected event may not be that rare and if the variability in demand size is high, disasters better be taken into consideration when making inventory control decisions.

Several of the recent continuous (s, S) models with perishability are: (Liu and Shi [13]) who considered the average cost case with exponential lifetime of inventory and general renewal demand through a Markov process with zero lead time. A continuous (s, S) model with random shelf life with a general distribution, zero lead time and renewal arrivals is discussed by Gürler and Özkaya in [7]. A continuous (s, S) model with exponential lifetime, exponential lead time, Poisson demand, and with backlogging is presented by Liu and Yang in [14] for the average cost case. A similar model, but with lost sales assuming that the number of outstanding replenishment orders is at most one and that items perished independently of each other, is shown by Kalpakam and Sapna in [9]. Varghese and Krishnamoorthy [23] assumed that disasters hit only a fraction of the existing items in a discrete demand model. Recently, Baron et al. [3] developed a continuous review $(0, S)$ model of perishable items under compound Poisson demand with either unit or exponentially distributed demand sizes, exponential time to perishability, and zero lead times. We note that with zero lead time there is no reason to order when inventory is positive. Thus, the inclusion of lead time in the model substantially changes the derivation due to that $s > 0$. This inclusion implies not only that a different methodology is required for the analysis, but also leads to different results and insights. More recently, Chen, Pang, and Pan [6] considered a joint pricing and inventory control of perishable items over a finite horizon.

Our inventory model, which considers the four uncertainties mentioned above, is comprehensive and, to the best of our knowledge, no other relevant article is that general. As a matter of fact *even the special case of our model with general demand size without perishability* (which is easily derived by taking the time to perishability to infinity in our analysis), as far as we know, has never been investigated in the literature. For example, approximated cost formulation (albeit in the (r, Q) settings) is used by Nagarajan and Rajagopalan in [16] for vendor-managed non-perishable inventory (when the vendor uses continuous review and the lead time is positive). This suggests that the analysis of this case is challenging. Moreover, Karaesmen et al. [10, p. 27] write "within the fixed ordering cost model with variable lot sizes, for the zero lead time case research is quite mature, but for positive lead

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times and/or batch demands there are still opportunities for research into both the optimal policy structure and effective heuristics. These models are both complex and practically applicable, making this yet another problem that is both challenging and important." Our paper derives the optimal control of the (s, S) policy for these settings (with both positive lead time and possible batch demand-in the compound Poisson case).

Intricate derivations, using the methodologies of renewal theory, level crossing theory, and Markov chains, enables us to provide exact, closed form, analysis for this comprehensive model. The closed-form results allow an immediate optimization of the values of the reorder level, s, and order up-to level, S. We provide numerical results for the optimal cost and controls of systems with perishability. We further compared these results to the ones of a heuristic that ignores perishability (an independent and not previously studied model). We note that a heuristic that considers perishability but ignores lead time will choose the reorder level $s = 0$, and is, thus, less practical in the presence of lead time.

Based on our numerical results, we derive several managerial guidelines on inventory management for perishable items. The generality of our model allows us to consider the effects of the four uncertainties on the optimal (s, S) inventory control and to compare the resulting cost to the cost of the optimal (s, S) control for a model that ignores perishability. To clarify the discussion, we slightly abuse the definitions of cycle and safety stock as follows: because the cycle stock is controlled by both S, and s, we call $S - s$ the "cycle stock" and, similarly, because the safety stock is dictated by s, we call s the "safety stock" (note that in most of our numerical examples, the safety stock: s-demand over lead time, is negative). We note that such insights cannot be obtained based upon an analysis that ignores the lead time and thus has $s = 0$. The insights are:

Efficiency of the heuristic: We found that considering perishability in inventory control becomes more important as the variability of the demand size increases from a unit to an exponential. Specifically, for the Poisson demand case, the heuristic is quite effective, with a typical relative error (compared with the optimal (s, S) cost) of less than 1%, if perishability is not too frequent; in contrast, in the compound Poisson demand case, the heuristic performed quite poorly (relative errors between 7% and 113%). As expected, the effectiveness of the heuristic deteriorates as perishability becomes more common. We also observe that the heuristic's effectiveness increases with the lead time, and this effectiveness is almost independent of the rate of demand.

Safety stock (s): We found that, in contrast to the intuition in the non perishable case, as the variability of demand size increases the safety stock decreases. This decrease reflects a higher need for managers to hedge against perishability with the increase variability. As expected, the safety stock increases with lead time and with time to perishability. Moreover, we observed that the safety stock grows linearly with the demand rate. While in the non perishable case a linear growth of the safety stock is expected, this is a somewhat surprising in the presence of perishability.

Cycle stock ($S - s$): We found that as the time to perishability increases the cycle stock is reduced slower than the safety stock. The intuition behind this observation is as follows: Without perishability safety stock is carried most of the time, whereas only half of the cycle stock is carried on average, thus in the events of perishability the relative cost savings from decreasing the safety stock are higher than this savings when decreasing the cycle stock. We also found that the cycle stock remains fixed with changes in the lead time. This is somewhat surprising because our initial intuition suggested that in the presence of perishability the cycle stock will decrease with the lead time in order to reduce the inventory that can be lost (during overall longer inventory cycles). Also, surprisingly, it appears that demand size variability has almost no effect on the cycle stock. As expected, the cycle stock increases with the demand rate.

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Cost: As expected the cost is increasing with the perishability, lead time, and the demand rate. Surprisingly, the cost may either increase or decrease with the demand size variability. This occurs, because in some cases the system with higher demand size variability carries much less inventory in order to better hedge for the increase uncertainty. So while the cost may decrease with the demand size variability, so will the profit. The implications are that for a model with disasters (i.e., if even though any individual unexpected event is not likely to occur frequently, the occurrence of any unexpected event may not be rare) if the variability in demand size is high, disasters better be considered when making inventory control decisions.

Our plan for the paper is as follows. We present the model in Section 2. The cost functionals are derived in Section 3. Examples of Poisson and compound Poisson with exponential demand sizes are discussed in Section 4. In Section 5, we focus on optimization and computational results. We conclude the paper by discussing the resulting intuitions in Section 6. All proofs are in the Appendix.

2. PRELIMINARIES AND PROBLEM FORMULATION

The inventory model we consider assumes an (s, S) control without backlogging. Specifically, in this continuous inventory control policy an order is triggered when the inventory level decreases to or below s. The order is received after a lead time and it brings the inventory level to S. Note that the order size for (s, S) models with lead time is determined at the order fulfillment time. This model is very reasonable under the practice of vendor managed inventory. With this practice, the product's vendor maintains the inventory in the retail outlets. Then, after a random lead time (possibly due to waiting for several outlets to require inventory) the vendor fills the orders from sufficiently large trucks, bringing the inventory at each retail outlet back to S. (For a review of vendor managed inventory, see, Kleywegt, Nori, and Savelsbergh [11] and references therein.) We note that we focus on the analysis of the inventory system, rather on the coordination issues faced in cases of vendor managed inventory (see, e.g., Nagarajan and Rajagopalan [16] who consider this issues in a (r, Q) settings.) Because we assume the time to perishability is exponential, it is immaterial if items from the previous cycle remain at the beginning of the next cycle (the memoryless property implies that items' time to perishability is independent of their age). Whenever the inventory available is not sufficient to meet the entire customers' demand size, the demand is filled as much as possible and the unsatisfied demand is lost. While in models with demand backlog or long lead times there are situations where allowing several outstanding orders is optimal, we only allow a single outstanding order. This assumption is common in the literature, see, e.g., the discussion of the papers in Table 3 of Karaesmen et al. [10].

We let $\mathbf{W} = \{W(t) : t \geq 0\}$ denote the inventory level process. This process, $\mathbf{W}$, is a regenerative process with a step function where the negative jump sizes represent the satisfied demands and the positive jumps are order arrivals. Let T be the generic cycle that is the time between two order arrivals into the warehouse. We assume that $0 \leq \mathbf{W} \leq S$ with $W(0) = S$.

Two important features of our model are the random lead time L_ξ that is exponentially distributed with a rate ξ and the random time to perishability D_η that is exponentially distributed with a rate η. The lead time L_ξ is the time it takes from the instant that an order is placed until it arrives at the system and the end of the lead time is also the end of the cycle for W (i.e., at the end of the lead time $W = S$). At an instant of perishability all items (if any) are removed from the shelf. However, some perishability events are not

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effective (if $W(t) = 0$). As a result, the times between *effective perishability events* (times at which items are removed from the shelf) are not exponentially distributed, but as will be seen in the sequel, are independent and identically distributed (iid) random variables (RVs), so that the arrival of effective perishability events forms a renewal process.

We consider a compound Poisson demand process, with inter-arrival times $V_i \sim \exp(\lambda)$, and demand sizes that are iid RVs denoted by Y_i that follow a general distribution, $F_Y(\cdot)$. We demonstrate our derivation and provide numerical results on the two most common (and easy) demand size examples – the Poisson demand, where demand sizes $Y_i = 1$, and – the compound Poisson demand with demand sizes $Y_i \sim \exp(\mu)$ (with Y being the generic RV). These demand examples are identical to the ones considered by Graves in [8].

To describe the dynamic of $\mathbf{W}$ we define first the compound Poisson process (with arrival rate λ)$\bar{\mathbf{W}} = \{\bar{W}(t) : t \geq 0\}$ where

$$\bar{W}(t) = S - [Y_1 + \cdots + Y_{N(t)}] \tag{1}$$

and $\mathbf{N} = \{N(t) : t \geq 0\}$ is the Poisson arrival process with rate λ. We assume that the three random quantities, time to perishability, D_η, lead time, L_ξ, and the compound Poisson process, $\bar{\mathbf{W}}$, are independent of each other.

The inventory level $\mathbf{W}$ is regulated by the process $\bar{\mathbf{W}}$ and four stopping times. Before presenting the stopping times for $\mathbf{W}$, we introduce several relevant stopping times for $\bar{\mathbf{W}}$. Recall that $\bar{W}(0) = S$ and define the stopping times $\tilde{\tau}_{S-s} = \inf\{t > 0 : \bar{W}(t) \leq s\}$, denoting the first time when $\bar{\mathbf{W}}$ drops to or below level s, and $\tilde{\tau}_S = \inf\{t > 0 : \bar{W}(t) \leq 0\}$ denoting the first time when $\bar{\mathbf{W}}$ drops to or below level 0. Also, let $\tilde{\tau}_{s-Y} = \tilde{\tau}_S - \tilde{\tau}_{S-s} \geq 0$, denote the duration of time from the instant when $\bar{\mathbf{W}}$ drops to or below level s and until $\bar{\mathbf{W}}$ drops to or below level 0.

Note that in the Poisson demand case $\bar{W}(\tilde{\tau}_{S-s}) = s$, and when demand sizes are exponential, it follows by the memoryless property that $\bar{W}(\tilde{\tau}_{S-s}) = s - Y$. Moreover, for any demand size, it follows, by the strong Markov property, that $\tilde{\tau}_{S-s}$ and $\tilde{\tau}_{s-Y}$ are conditionally independent. Also, the distribution of $\tilde{\tau}_{s-Y}$ may have an atom at 0. The Laplace Stieltjes transform (LST) of the stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$ (where for the Poisson demand case $s - Y = s$), and $\tilde{\tau}_S$ are known before for several demand sizes such as a unit demand or phase type, see, e.g., Baron, Berman and Perry [3].

We next define the four stopping times that are associated with $\mathbf{W}$: The first stopping time is the time to perishability D_η. The second stopping time is

$$\tau_{S-s} = \min\{\tilde{\tau}_{S-s}, D_\eta\}. \tag{2}$$

The stopping time τ_{S-s} can be interpreted as the first time the inventory level drops to or below level s, either due to demand, then $\tau_{S-s} = \tilde{\tau}_{S-s}$ or due to perishability, then $\tau_{S-s} = D_\eta$. We assume that orders are initiated whenever $W(t) \leq s$, whether the cause for the decrease is a demand or a perishability, so that after a lead time L_ξ, an order arrives, bringing the inventory level back to S; then, the cycle ends. We define the third stopping time as the end of a cycle

$$T = \tau_{S-s} + L_\xi. \tag{3}$$

Finally, the fourth stopping time is

$$\tau_S = \min\{\tilde{\tau}_S, D_\eta\}. \tag{4}$$

The stopping time τ_S can be interpreted as the first time, after τ_{S-s}, when the inventory level drops to or below level 0. Note that in the compound Poisson case $\tau_S = \tilde{\tau}_{S-s}$ with a positive probability. This occurs whenever $\tilde{\tau}_{s-Y} = 0$, then τ_S may be longer than T.

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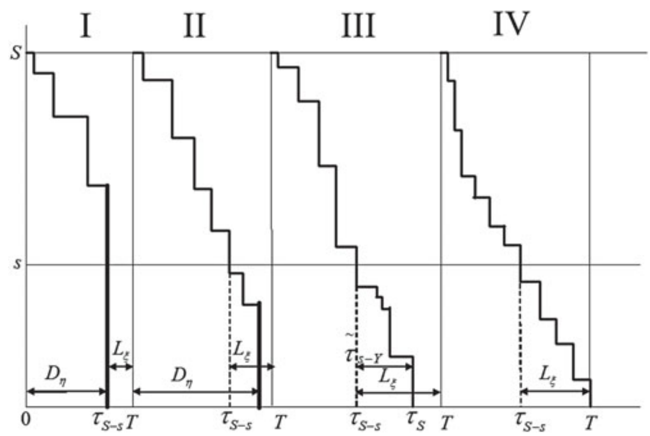

FIGURE 1. Sample paths of the inventory process for the four different possible cycles. The bold line of downwards jumps denote perishability events. (This example shows a compound Poisson demand, as can be seen from the different sizes of the orthogonal jumps.)

Formally, we define the inventory level process $\mathbf{W}$ during any one cycle by the compound Poisson process $\bar{\mathbf{W}}$ and the four stopping times as follows (see Figure 1):

$$W(t) = \begin{cases} \bar{W}(t) & 0 \le t \le D_\eta \quad \text{if } \tau_{S-s} = D_\eta \\ 0 & D_\eta \le t < T, \quad \text{(cycle 1, in Figure 1)}, \\ \bar{W}(t) & 0 \le t \le D_\eta \quad \text{if } \tau_{S-s} < D_\eta \text{ and } D_\eta \le \min(T, \tilde{\tau}_S) \\ 0 & D_\eta \le t < T, \quad \text{(cycle 2, in Figure 1)}, \\ \bar{W}(t) & 0 \le t \le \tau_S \quad \text{if } \tau_S < D_\eta \text{ and } \tilde{\tau}_{s-Y} < L_\xi \\ 0 & \tau_S \le t < T, \quad \text{(cycle 3, in Figure 1)}, \\ \bar{W}(t) & 0 \le t \le T, \quad \text{if } \tau_S < D_\eta \text{ and } \tilde{\tau}_{s-Y} > L_\xi \\ & \quad \text{(cycle 4, in Figure 1)}. \end{cases}$$

2.1. Objective Function

We focus on deriving the average cost per time unit. The total cost is composed of four components: ordering cost (both variable and fixed), holding costs, cost of unsatisfied demand, and cost of perishability. We denote by K_o the order set-up cost, c is the cost per item, K_u is the cost per unit of an unsatisfied demand, K_d is the fixed penalty for (effective) perishability events that occur when the shelf is not empty, and h is the holding cost for one unit of inventory per time unit. We further denote the time between unsatisfied demands by U and the time between effective perishability events by Z.

With these definitions and using the key renewal theorem the average cost can be expressed as follows: First, in the beginning of each cycle the controller pays the set-up ordering cost K_o; resulting in an average cost rate of $K_0/E(T)$. Second, because each demand is for $1/\mu$ units on average, demand arrives at a rate λ and is unsatisfied at a rate $1/E(U)$, the variable cost rate of ordering is $(c/\mu)(\lambda - (1/E(U)))$. We note that in general there are two types of unsatisfied demands: arriving at an empty system, i.e., when $W(t) = 0$, with an average size of $1/\mu$ units, and arriving when there is some but not enough inventory in the system, i.e., $0 < W(t) < Y$, with an average size of $E(Y|Y \ge W(t))$ units.

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In the Poisson demand case, where $Y = 1$, every unsatisfied demand arrives when $W(t) = 0$ and is for 1 unit. In the compound Poisson case with $Y \sim \exp(\mu)$ demands, due to the memoryless property, we have $E(Y) = E(Y - W(t) | Y \geq W(t)) = 1/\mu$. Third, by Poisson arrivals see time averages, each perishability event destroys $E(W)$ items on average and these are destroyed at a rate η and a cost c_1. For simplicity, we do not consider any additional cost per items destroyed as this cost is captured via their replenishment cost. Thus, we use $c_1 = c$. Fourth, the holding cost rate is $hE(W)$ and, as in the calculation of the variable ordering cost rate, the expected cost of unsatisfied demand is $K_u/(\mu E(U))$. Finally, the average fixed cost of perishability is $K_d/E(Z)$. Thus, letting R denote the average cost we have

$$\begin{aligned} R &= \frac{K_o}{E(T)} + c\left(\frac{1}{\mu}\left(\lambda - \frac{1}{E(U)}\right) + \eta E(W)\right) + hE(W) + \frac{K_u}{\mu E(U)} + \frac{K_d}{E(Z)} \\ &= \frac{K_o}{E(T)} + c\frac{\lambda}{\mu} + c\eta E(W) + hE(W) + \frac{K_u - c}{\mu E(U)} + \frac{K_d}{E(Z)}. \end{aligned} \quad \textbf{(5)}$$

Note that the cost of unsatisfied demand (lost sales) is offset by the cost savings from not having to meet this demand, so that the actual cost of a unit of un-met demand is $K_u - c$. From a modeling perspective, it is reasonable that $K_u - c > 0$. That is, a unit of lost demand is more costly than a unit of a satisfied one.

In the next sections, we analyze the cost components one at a time. To express these costs, let $\mathbf{\Lambda_o} = \{\Lambda_o(t) : t \geq 0\}$ be the counting process of order arrivals, $\mathbf{\Lambda_u} = \{\Lambda_u(t) : t \geq 0\}$ be the counting process of the unsatisfied demand, $\mathbf{\Lambda_d} = \{\Lambda_d(t) : t \geq 0\}$ be the counting process of the effective perishability events. Let $F(\cdot)$ denote the steady-state probability distribution of the inventory level (this exists as the system is regenerative).

We use three different methodologies to express the different costs. First, we express the LST of T, U, and Z by considering the different stopping times related to $\mathbf{W}$. We derive these LSTs using the laws of the stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$, and $\tilde{\tau}_S$ for $\bar{\mathbf{W}}$. A critical building block in this derivation is the memoryless property of $\mathbf{W}$; a property that follows because the arrival process is a compound Poisson and both the lead time and time to perishability are exponentially distributed. Second, to express the holding cost, we derive $F(\cdot)$, which allows us to calculate $E(W)$ using level crossing theory. This derivation requires solving the functional equation for $F(\cdot)$. For the Poisson demand case, we also derive the holding costs rate from the Markov chain representation of the inventory level process.

Our derivation, in the next section, holds for any general demand size distribution Y_i. As mentioned above, we focus on the two easiest discrete and continuous demand cases: Poisson demand where $Y_i = 1$ at each arrival and the compound Poisson case with $Y_i \sim \exp(\mu)$. For these cases, we provide the exact expressions for the different LSTs and the solution for $F(\cdot)$. We note that while in both cases the solutions results in closed-form expressions for the cost function in Eq. (5), these expressions are cumbersome and, therefore, some of them are not explicitly given below. Still, the availability of closed-form expressions allows us to easily optimize the inventory control parameters, the base stock level, S, and the reorder point, s.

3. GENERAL DERIVATION OF THE COST

Here, we derive the cost functionals for any compound Poisson demand case in terms of the LST of the different stopping times defined for the well studied $\bar{\mathbf{W}}$ process: $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$,

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and $\tilde{\tau}_S$. In the next sections, we express the LST for the cases of Poisson and a compound Poisson (with exponential demand sizes) processes to derive the corresponding costs. We remind that the expected value of an RV X, denoted by $E(X)$ can be obtained from the LST of X, $E\left(e^{-\beta X}\right)$ as $E(X) = -\lim_{\beta\to 0} d\left(E\left(e^{-\beta X}\right)\right)/d\beta$.

3.1. Cost of Ordering

We consider the time between two order arrivals, T, as a generic cycle. We have:

PROPOSITION 3.1: *The LST of the length of cycles is*

$$E\left(e^{-\beta T}\right) = \frac{\xi}{\xi+\beta}\left(\frac{\eta}{\beta+\eta} + \frac{\beta E(e^{-(\beta+\eta)\tilde{\tau}_{S-s}})}{\beta+\eta}\right).$$

As expected, the length of the cycle is a convolution of the length of the time to order with the exponentially distributed lead time (with rate ξ). The time until an order is placed is either the time to perishability or the time until the inventory level downcrosses s for the process without perishability, $\tilde{\tau}_{S-s}$.

3.2. Cost of Unsatisfied Demand

It follows by the strong Markov property that the counting process of the unsatisfied demands is a renewal process. And let U be generic inter-arrival time between unsatisfied demands. To express $E\left(e^{-\beta U}\right)$ define the random variable X as the time it takes from the instant of order arrival until the next unsatisfied demand. We then have:

CRITERION 3.2:

$$X = \begin{cases} D_\eta + \hat{U}, & D_\eta \le \min\{T, \tau_S\}, \\ \tau_S + \hat{U}\,\mathbf{1}_{\{W(\tilde{\tau}_S)=0\}}, & \tau_S < \min\{D_\eta, T\}, \\ T + \hat{X}, & T \le \tau_S, \end{cases} \quad \textbf{(6)}$$

where $\hat{U}$ is stochastically equal to U, $\hat{X}$ is stochastically equal to X and all the random variables on the right-hand side of Eq. (6) are independent.

To understand the idea behind Eq. (6), we consider the three stopping times D_η, τ_S, and T. Suppose first that the event $\{D_\eta \le \min\{T, \tau_S\}\}$ occurred; this event means that the system becomes empty due to perishability and not due to an unsatisfied demand, so that the system is empty before the cycle ends. From that instant the time to the next unsatisfied demand is a probabilistic replication of the time between two unsatisfied demands. Thus, when $\{D_\eta \le \min\{T, \tau_S\}\}$, X is stochastically equal to $D_\eta + U$. Next, suppose that $\{\tau_S < \min\{D_\eta, T\}\}$; this event means that level 0 is downcrossed before perishability and also before the cycle ends, so that X equals to τ_S. Note that the indicator function is required for the cases where level 0 is not strictly downcrossed. In these cases, the time to the next unsatisfied demand is a probabilistic replication of the time between two unsatisfied demands. (Such cases can only occur if demand size is discrete.) Finally, suppose that $\{T \le \tau_S\}$; this event means that the cycle ends without an unsatisfied demand, the inventory level is filled up to level S and the time to the next unsatisfied demand is stochastically equal to X.

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Proposition 3.3: *The LST's of X and U satisfy the two equations:*

$$E\left(e^{-\beta U}\right)=\frac{\lambda}{\lambda+\beta+\xi}+\frac{\xi}{\lambda+\beta+\xi}E\left(e^{-\beta X}\right). \tag{7}$$

$$E\left(e^{-\beta X}\right)=\frac{\left(E\left(e^{-\beta U}\right)\eta/\eta+\beta\right)\left(1-E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\left(\xi+(\beta+\eta)E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)/\beta+\eta+\xi\right)\right)\right)}{\left(1-E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})(\xi\left(1-E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\right)/\xi+\eta+\beta)\right)}$$
$$+\frac{E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\left(P\left(W\left(\tilde{\tau}_S\right)<0\right)+E\left(e^{-\beta U}\right)P\left(W\left(\tilde{\tau}_S\right)=0\right)\right)}{\left(1-E(e^{-(\eta+\beta)\tilde{\tau}_{S-s}})\left(\xi\left(1-E(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}})\right)/\xi+\eta+\beta\right)\right)}. \tag{8}$$

The intuition behind Eq. (7) is that after an unsatisfied demand there are two competing independent exponential RVs to the next possible event: the next demand arrival, with rate λ, and the lead time, with rate ξ. If (i) the next demand arrives first, with probability $\lambda/\left(\lambda+\xi\right)$, then the next unsatisfied demand occurs after an exponential time with rate $(\lambda+\xi)$. Then, the partial LST of U (together with the event that it comes before the lead time) is $\lambda/\left(\lambda+\xi\right)*\left(\lambda+\xi\right)/\left(\lambda+\xi+\beta\right))$. If (ii) the next event is the lead time, with probability $\xi/\left(\lambda+\xi\right)$, then the time until the next un-met demand is a convolution of the time for the lead time (with LST $\xi/\left(\lambda+\xi\right)*\left(\lambda+\xi\right)/\left(\lambda+\xi+\beta\right))$ and the RV X as defined above.

For example, for a continuous demand case, when $P\left(W\left(\tau_S=0\right)\right)=0$, the solution of Proposition 3.3 gives

$$E\left(e^{-\beta U}\right)=\frac{\lambda}{\lambda+\beta+\xi}+\frac{\frac{\xi}{\lambda+\beta+\xi}\frac{\lambda}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1-\frac{E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\xi+(\beta+\eta)E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)\right)}{\beta+\eta+\xi}\right)}{1-\frac{E\left(e^{-(\eta+\beta)\tilde{\tau}_{S-s}}\right)\left(\xi-\xi E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right)\right)}{\xi+\eta+\beta}-\frac{\xi}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1-\frac{E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\xi+(\beta+\eta)E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)\right)}{\beta+\eta+\xi}\right)}$$
$$+\frac{\frac{\xi}{\lambda+\beta+\xi}E\left(e^{-(\eta+\beta)\tilde{\tau}_{S-s}}\right)E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right)}{1-\frac{E\left(e^{-(\eta+\beta)\tilde{\tau}_{S-s}}\right)\left(\xi-\xi E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right)\right)}{\xi+\eta+\beta}-\frac{\xi}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1-\frac{E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\xi+(\beta+\eta)E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)\right)}{\beta+\eta+\xi}\right)}. \tag{9}$$

3.3. Cost of Perishability

Potential perishability instances arrive according to a Poisson process with rate η, but effective perishability events occur only when the system is not empty. It is clear that the effective time to perishability process is a renewal process. Recalling that Z denotes the time between effective perishability events:

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PROPOSITION 3.4: *The LST of Z is given by*

$$E\left(e^{-\beta Z}\right)=\frac{\xi}{\xi+\beta}\frac{\eta}{\eta+\beta}\frac{\left(1-E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\frac{\xi+(\beta+\eta)E\left(e^{-(\beta+\xi+\eta)\tilde{\tau}_{s-Y}}\right)}{\beta+\xi+\eta}\right)}{1-\xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\frac{\eta E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)}{(\xi+\beta)(\xi+\beta+\eta)}+\frac{1}{\xi+\beta+\eta}\right)}. \tag{10}$$

It is seen from Eq. (10) that the time between effective perishability events is the convolution of (i) the lead time, exponential with rate ξ, in which there are no perishability (because after each perishability instance the lead time starts afresh and there are no items on the shelf), with (ii) the time to perishability, exponential with rate η, and with (iii) an RV that corrects for the probability that the next time to perishability occurs when the inventory level is 0 again. In this third case, the perishability is not effective and the time to the next effective perishability event has the same distribution as before. (It can be verified that the third fraction represents the LST of an RV by taking its limit as $\beta \to 0$, which is indeed 1.)

3.4. Holding Cost

By the dominated convergence theorem $\lim_{t\to\infty} E\left(W(t)\right) = E\left(W\right)$, since the inventory level is bounded in $[0, S]$. To calculate $E\left(W\right)$ we first evaluate the steady-state distribution $F(\cdot)$ or in terms of the steady-state density $dF(\cdot)$ and then take expectation. It should be noted that $F(\cdot)$ if the distribution of the demand size, $F_Y\left(\cdot\right)$, is an absolutely continuous distribution, so is $F\left(\cdot\right)$ for all $0 < x < S$ but, for any demand size distribution, $F\left(\cdot\right)$ has an atom π_S at S and an atom π_0 at 0, so that $dF\left(S\right) = \pi_S$ and $dF\left(0\right) = \pi_0$. In the next proposition, we introduce the balance equation generated by level crossing theory that can be used to express $dF\left(\cdot\right)$. We first remind that $F_Y\left(\cdot\right)$ denotes the distribution of the demand size Y.

PROPOSITION 3.5: *We have*

$$\eta\int_x^S dF\left(w\right)+\lambda\int_x^S\left(1-F_Y\left(w-x\right)\right)dF\left(w\right)=\begin{cases}\xi\int_0^x dF\left(w\right), & x\le s,\\ \xi\int_0^s dF\left(w\right), & s<x\le S,\end{cases} \tag{11}$$

alternatively, letting $f\left(\cdot\right)$ be the density of $F\left(\cdot\right)$ in $(0, S)$, we have

$$\eta\int_x^S f(w)dw+\eta\pi_S+\lambda\int_x^S\left(1-F_Y\left(w-x\right)\right)f(w)dw+\lambda\pi_S e^{-\mu(S-x)}$$
$$=\begin{cases}\xi\pi_0+\xi\int_0^x f(w)dw, & x\le s,\\ \xi\pi_0+\xi\int_0^s f(w)dw, & s<x\le S.\end{cases}$$

The boundary conditions can be found from the normalizing condition, and by taking $x = S$, $x = s$, and $x = 0$, in Eq. (11):

$$\int_0^S f\left(w\right)dw = 1-\pi_S-\pi_0, \tag{12}$$

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$$(\lambda+\eta)\pi_S = \xi\pi_0 + \xi\int_0^s f(w)\,dw, \tag{13}$$

$$\eta\int_s^S f(w)\,dw + \eta\pi_S + \lambda\int_s^S (1-F_Y(w-s))\,f(w)\,dw + \lambda\pi_S F_Y(S-s) = \xi\pi_0 + \xi\int_0^s f(w)\,dw, \tag{14}$$

$$\eta(1-\pi_0) + \lambda\int_0^S (1-F_Y(w))\,f(w)\,dw + \lambda\pi_S(1-F_Y(S)) = \xi\pi_0. \tag{15}$$

Now, $E(W) = \int_0^S wf(w)\,dw + S\pi_S$ can be calculated in closed form.

4. EXAMPLES OF EXACT COST ANALYSIS

Here we consider two examples for the demand size at each arrival: a Poisson demand, with unit demand sizes, and a compound Poisson demand, with exponential demand sizes. The case of Poisson demand can be directly solved from an analysis of a Markov chain. However, it is brought here to demonstrate the generality of our approach.

4.1. Poisson Demand Case

We assume that the demand follows a Poisson process, i.e., that the size of each demand is one unit. The stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_s$, (here $s-Y=s$), and $\tilde{\tau}_S$ that are required for substituting in the results of Section 3 are all $Erlang(n,\lambda)$ where n is the number of demands realized

$$E\left(e^{-\beta\tilde{\tau}_n}\right) = \left(\frac{\lambda}{\lambda+\beta}\right)^n,$$

where $n=S-s$, $n=s$, and $n=S$ for the LST of $\tilde{\tau}_{S-s}$, $\tilde{\tau}_s$, and $\tilde{\tau}_S$, respectively. The results below follow by substitution of the relevant LST in the results derived in Section 3, noting that in this case $P(W(\tau_S)=0)=1$, and some (tedious) algebra, no further proof is provided. Because expressing the steady-state distribution for the inventory level (and the implied holding cost) for this case can be also done directly from the Markov chain representation of the inventory level process, we provide this direct derivation in the next subsection.

Proposition 4.1: *The LST of the cycles is*

$$E\left(e^{-\beta T}\right) = \frac{\xi}{\xi+\beta}\,\frac{(\lambda/\lambda+\beta+\eta)^{S-s}\beta+\eta}{\beta+\eta}, \tag{16}$$

so that

$$E(T) = \frac{\eta+\xi-(\lambda/\lambda+\eta)^{S-s}\xi}{\xi\eta} = \frac{1}{\eta}+\frac{1}{\xi}-\frac{(\lambda/\lambda+\eta)^{S-s}}{\eta}. \tag{17}$$

The LST for the time between unsatisfied demand is

$$E(e^{-\beta U}) = \frac{\lambda(\eta+\beta)\left(\eta+\beta+\xi+\xi\left(\frac{\lambda}{\lambda+\beta+\eta}\right)^{S-s}\left(\left(\frac{\lambda}{\lambda+\beta+\eta+\xi}\right)^{s}-1\right)\right)}{\left(\beta+\eta+\xi-\xi\left(\frac{\lambda}{\lambda+\beta+\eta}\right)^{S-s}\right)(\beta(\beta+\xi+\eta)+(\eta+\beta)\lambda) + \xi(\eta+\beta)\lambda(\frac{\lambda}{\lambda+\beta+\eta})^{S-s}(\frac{\lambda}{\lambda+\beta+\eta+\xi})^{s}}, \tag{18}$$

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so that

$$E(U) = \left(\lambda\left(\frac{\eta\xi}{((\eta+\xi)-\xi(\lambda/\lambda+\eta)^{S-s})(\lambda+\eta)}\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\right.\right.$$
$$\left.\left.\times\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s}\frac{\lambda}{\xi+\eta}+\frac{\eta}{\xi+\eta}\right)\right)^{-1}. \quad (19)$$

The LST of the time between effective perishability events is

$$E\left(e^{-\beta Z}\right) = \frac{\xi}{\xi+\beta}\frac{\eta}{\eta+\beta}\frac{\left(1-\left(\frac{\lambda}{\lambda+\beta+\eta}\right)^{S-s}\frac{\xi+(\beta+\eta)\left(\frac{\lambda}{\lambda+\beta+\xi+\eta}\right)^{s}}{\beta+\xi+\eta}\right)}{1-\xi\left(\frac{\lambda}{\lambda+\beta+\eta}\right)^{S-s}\left(\frac{\eta\left(\frac{\lambda}{\lambda+\beta+\xi+\eta}\right)^{s}}{(\xi+\beta)(\xi+\beta+\eta)}+\frac{1}{\xi+\beta+\eta}\right)}, \quad (20)$$

so that

$$E(Z) = \frac{\xi+\eta}{\xi\eta}\frac{\xi+\eta-\xi(\lambda/\lambda+\eta)^{S-s}}{\xi+\eta-\xi(\lambda/\lambda+\eta)^{S-s}-\eta(\lambda/\lambda+\eta)^{S-s}(\lambda/\lambda+\xi+\eta)^{s}}. \quad (21)$$

Recalling our discussion after Proposition 3.1, the cycle, T, is a convolution of the length of time to order with the lead time. For this demand case, the LST of the time to order for a model with no lead time was derived in Eq. (19) of Baron et al. [3]. (It agrees with Eq. (16), after substituting $\alpha \to 0$, $\xi \to \eta$, and $S \to S-s$ in their result.)

4.1.1. Holding cost. To calculate $E(W)$ we first evaluate the steady-state distribution $F(i) = \sum_{j=0}^{i} P_j$, where P_j is the steady-state probability that the inventory level is j, and then take expectation. To this end, we consider the Markov chain representation of the inventory level to get

$$\begin{aligned}(\lambda+\eta)P_S &= \xi F(s) \\ (\lambda+\eta)P_i &= \lambda P_{i+1} \quad i = s+1,\ldots,S-1 \\ (\lambda+\eta+\xi)P_i &= \lambda P_{i+1} \quad i = 1,\ldots,s \\ (\xi+\eta)P_0 &= \eta+\lambda P_1.\end{aligned} \quad (22)$$

Solving (22) we get

$$P_i = \begin{cases} P_S\left(\dfrac{\lambda}{\lambda+\eta}\right)^{S-i} & i = s+1,\ldots,S \\ P_S\left(\dfrac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\dfrac{\lambda}{\lambda+\eta+\xi}\right)^{s+1-i} & i = 1,\ldots,s \\ P_S\left(\dfrac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\dfrac{\lambda}{\lambda+\eta+\xi}\right)^{s}\dfrac{\lambda}{\xi+\eta}+\frac{\eta}{\xi+\eta} & i = 0, \end{cases}$$

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where P_S is given from the solution of (after some tedious algebra)

$$\frac{1-(\eta/\xi+\eta)}{P_S} = \sum_{i=s+1}^{S}\left(\frac{\lambda}{\lambda+\eta}\right)^{S-i} + \sum_{i=1}^{s}\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s+1-i}$$
$$+\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s}\frac{\lambda}{\xi+\eta},$$
$$P_S = \frac{\eta\xi}{\left((\eta+\xi)-(\lambda/\lambda+\eta)^{S-s}\xi\right)(\lambda+\eta)}.$$

It can be verified that $(\lambda+\eta)P_S = \xi F(s)$.

Now

$$E(W) = P_S\left(\sum_{i=s+1}^{S} i\left(\frac{\lambda}{\lambda+\eta}\right)^{S-i} + \sum_{i=1}^{s} i\left(\frac{\lambda}{\lambda+\eta}\right)^{S-s-1}\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s+1-i}\right) \tag{23}$$
$$= P_S\left(\frac{\eta S-\lambda}{\eta^2} + \left(\frac{\lambda}{\lambda+\eta}\right)^{S-s}\left(\frac{s(\xi+\eta)+\lambda\left(\left(\frac{\lambda}{\lambda+\eta+\xi}\right)^{s}-1\right)}{(\xi+\eta)^2} - \frac{(\eta s-\lambda)}{\eta^2}\right)\right).$$

By Poisson arrivals see time averages and since all unsatisfied demand arrives while there is no inventory, i.e., during P_0 of the time (recall with Poisson demand there is no unsatisfied demand if inventory is positive), the rate of unsatisfied demand $1/E(U)$ equals the rate of arrivals that see an empty system, i.e.,

$$\lambda P_0 = \frac{1}{E(U)},$$

which, of course, agrees with Eq. (19).

4.2. Compound Poisson with Exponential Demand

Here we assume that the demand follows a compound Poisson process where the size of each demand is exponentially(μ) distributed. The stopping times $\tilde{\tau}_{S-s}$, $\tilde{\tau}_{s-Y}$, and $\tilde{\tau}_S$ that are required for substituting in the results of Section 3 are known (see, e.g., Baron et al. [3]. Specifically, setting $\alpha = 0$ and substituting $S-s$ rather than S in Eq. (7) of Baron et al. [3] we get

$$E\left(e^{-\beta\tilde{\tau}_S}\right) = \frac{\lambda}{\lambda+\beta}e^{-(\mu\beta S/\lambda+\beta)}, \quad \text{and} \tag{24}$$
$$E\left(e^{-\beta\tilde{\tau}_{S-s}}\right) = \frac{\lambda}{\lambda+\beta}e^{-(\mu\beta(S-s)/\lambda+\beta)}.$$

Using that $\bar{W}(\tilde{\tau}_{S-s}) = s - Y$ and Eq. (24) we get

$$E\left(e^{-\beta\tilde{\tau}_{s-Y}}\right) = \int_0^s \frac{\lambda}{\lambda+\beta}e^{-(\mu\beta(s-y)/\lambda+\beta)}\mu e^{-\mu y}dy + \int_s^\infty \mu e^{-\mu y}dy$$
$$= e^{-\mu\frac{\beta}{\lambda+\beta}s}. \tag{25}$$

Note that the independence of $\tilde{\tau}_S$ and $\tilde{\tau}_{S-s}$ implies that $E(e^{-\beta\tilde{\tau}_{s-Y}}) = E\left(e^{-\beta\tilde{\tau}_S}\right)/E(e^{-\beta\tilde{\tau}_{S-s}})$, which leads to Eq. (25) as well. Substituting Eqs. (24) and (25) into the derivation in Section 3 gives (no further proof is provided).

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PROPOSITION 4.2: *The LST of the cycles is*

$$E\left(e^{-\beta T}\right) = \frac{\xi}{\xi+\beta}\frac{\eta+\beta(\lambda/\lambda+\beta+\eta)e^{-(\mu(\beta+\eta)/\lambda+\beta+\eta)(S-s)}}{\beta+\eta}, \tag{26}$$

so that

$$E\left(T\right) = \frac{1}{\eta}+\frac{1}{\xi}-\frac{\lambda e^{(-(\eta\mu(S-s)/\eta+\lambda)}}{\eta\left(\lambda+\eta\right)}. \tag{27}$$

The LST for the time between unsatisfied demand is

$$E\left(e^{-\beta U}\right) = \frac{\lambda}{\lambda+\beta+\xi}+\frac{\xi}{\lambda+\beta+\xi}$$
$$\times\frac{\frac{\lambda}{\lambda+\beta+\xi}\frac{\eta}{\eta+\beta}\left(1-\frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\frac{\xi+(\beta+\eta)e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{\beta+\eta+\xi}\right)+\frac{\lambda e^{-\frac{\mu(\eta+\beta)(S-s)}{\lambda+\eta+\beta}}}{\lambda+\eta+\beta}e^{-\frac{\mu(\xi+\beta+\eta)s}{\lambda+\xi+\beta+\eta}}}{\left(1-\frac{\lambda e^{-\left(\frac{\mu(\eta+\beta)(S-s)}{\lambda+\eta+\beta}\right)}}{\lambda+\eta+\beta}\frac{\xi\left(1-e^{-\frac{\mu(\xi+\beta+\eta)s}{\lambda+\xi+\beta+\eta}}\right)}{\xi+\eta+\beta}-\frac{\eta}{\eta+\beta}\frac{\xi}{\lambda+\beta+\xi}\left(1-\frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\frac{\xi+(\beta+\eta)e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{\beta+\eta+\xi}\right)\right)}. \tag{28}$$

The LST of the time between effective perishability events is

$$E\left(e^{-\beta Z}\right) = \frac{\xi}{\xi+\beta}\frac{\eta}{\eta+\beta}\frac{\left(1-\frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\frac{\xi+(\beta+\eta)e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{\beta+\eta+\xi}\right)}{1-\xi\frac{\lambda e^{-\left(\frac{\mu(\beta+\eta)(S-s)}{\lambda+\beta+\eta}\right)}}{\lambda+\beta+\eta}\left(\frac{\eta e^{-\mu\frac{\beta+\eta+\xi}{\lambda+\beta+\eta+\xi}s}}{(\xi+\beta)(\xi+\beta+\eta)}+\frac{1}{\xi+\beta+\eta}\right)}. \tag{29}$$

Recall that the cycle, T, is a convolution of the time to order with the lead time. In a model with perishability but no lead time Eq. (16) of Baron et al. [3] gives the LST of the time to order for this demand case. Indeed, their Eq. (16) (with the required corrections as in the Poisson case) agrees with Eq. (26).

4.2.1. Holding cost. Following our discussion in Section 3.4 we calculate $E\left(W\right)$. The next proposition that follows from Proposition 3.5 (no detailed proof is provided) and substitution of the demand distribution, we introduce the balance equation generated by level crossing theory that can be used to express $dF\left(\cdot\right)$ in closed form as explained below.

PROPOSITION 4.3: *We have*

$$\eta\int_x^S dF\left(w\right)+\lambda\int_x^S e^{-\mu(w-x)}dF\left(w\right) = \begin{cases}\xi\int_0^x dF\left(w\right), & x\leq s,\\ \xi\int_0^s dF\left(w\right), & s<x\leq S,\end{cases} \tag{30}$$

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or

$$\eta\int_{x}^{S^-} f(w)dw+\eta\pi_S+\lambda\int_{x}^{S^-} e^{-\mu(w-x)}f(w)dw+\lambda\pi_S e^{-\mu(S-x)}$$
$$=\begin{cases}\xi\pi_0+\xi\int_{0+}^{x} f(w)dw, & x\le s,\\ \xi\pi_0+\xi\int_{0+}^{s} f(w)dw, & s<x\le S,\end{cases}$$

where $S^- = \sup\{x|x<S\}$ *and* $0^+ = \inf\{x|x>0\}$.

Solving for $dF(\cdot)$ in Eq. (30) we get

$$dF(x)=\begin{cases}k_0e^{ax} & 0<x\le s,\\ k_1e^{bx} & s<x<S,\end{cases} \tag{31}$$

where $a=(\mu(\xi+\eta)/\lambda+\xi+\eta)$ and $b=(\mu\eta/\lambda+\eta)$. We now have four unknowns: π_0, π_S, k_0, and k_1. To find these unknowns we use perishability events

$$\int_{0+}^{s} k_0e^{aw}dw+\int_{s}^{S^-} k_1e^{bw}dw=1-\pi_S-\pi_0,$$
$$(\lambda+\eta)\pi_S=\xi\pi_0+\xi\int_{0+}^{s}k_0e^{aw}dw,$$
$$\eta\int_{s}^{S^-} k_1e^{bw}dw+\eta\pi_S+\lambda\int_{s}^{S^-} e^{-\mu(w-s)}k_1e^{bw}dw+\lambda\pi_Se^{-\mu(S-s)}=\xi\pi_0+\xi\int_{0+}^{s}k_0e^{aw}dw,$$
$$\eta\,(1-\pi_0)+\lambda k_0\int_{0+}^{s} e^{-(\mu-a)w}dw+\lambda k_1\int_{s}^{S^-} e^{-(\mu-b)w}dw+\lambda\pi_Se^{-\mu S}=\xi\pi_0.$$

These equations are equivalent to

$$k_0\frac{e^{as}-1}{a}+k_1\frac{e^{bS}-e^{bs}}{b}=1-\pi_S-\pi_0,$$
$$(\lambda+\eta)\pi_S=\xi\pi_0+\xi k_0\frac{e^{as}-1}{a},$$
$$\eta k_1\frac{e^{bS}-e^{bs}}{b}+\eta\pi_S+\lambda e^{\mu s}k_1\frac{e^{(b-\mu)S}-e^{(b-\mu)s}}{b-\mu}+\lambda\pi_Se^{-\mu(S-s)}=\xi\pi_0+\xi k_0\frac{e^{as}-1}{a},$$
$$\eta\,(1-\pi_0)+\lambda k_0\frac{e^{(-\mu+a)s}-1}{-\mu+a}+\lambda k_1\frac{e^{(-\mu+b)S}-e^{(-\mu+b)s}}{-\mu+b}+\lambda\pi_Se^{-\mu S}=\xi\pi_0.$$

These four linear equations in four unknowns can be solved in closed form (which is too cumbersome to include here). We then have

$$E\,(W)=k_0\int_{0}^{s} xe^{ax}dx+k_1\int_{s}^{S} xe^{bx}dx+S\pi_S \tag{32}$$
$$=k_0\frac{e^{as}\,(as-1)+1}{a^2}+k_1\frac{e^{bS}\,(bS-1)-e^{bs}\,(bs-1)}{b^2}+S\pi_S.$$

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5. OPTIMIZATION

In this section, we discuss the optimization of the order up-to level, S, and the re-order level, s, for the Poisson and compound Poisson demand distributions. For both cases, we consider the problem (see Eq. (5))

$$\min_{S,s} R = \frac{K_0}{E(T)} + c\frac{\lambda}{\mu} + c\eta E(W) + hE(W) + \frac{k_u - c}{\mu E(U)} + \frac{K_d}{E(Z)},$$

where for the Poisson demand cases $E(T)$ and $E(W)$ are given, respectively, in Eqs. (17), (23) and $E(U)$ and $E(Z)$ are derived from Eqs. (18) and (21) and for the compound Poisson demand cases $E(T)$ and $E(W)$ are given, respectively, in Eqs. (27) and (32) and $E(U)$ and $E(Z)$ are derived from Eqs. (28) and (29). We provide numerical results on the sensitivity of parameters for the optimal s^* and S^*.

Two important asymptotics, leading to relevant special cases, are the one when there is no lead time, i.e., $\xi \to \infty$, analyzed by Baron et al. in [3], and the one when there is no perishability, i.e., $\eta \to 0$. To capture the value of considering perishability, we compared the value of the exact analysis of the system with perishability to the one that ignores perishability. We refer to the system without perishability as the heuristic. We note that the alternative heuristic of ignoring the lead time and use the results of Baron et al. [3], is not sensible because having no safety stock, i.e., choosing $s = 0$, is clearly not optimal in the presence of lead times.

We note that the expressions for the objective function allows no theoretical derivation of structural properties for it. However, our exact, closed-form analysis allowed us to use an off the shelf non-linear optimization (the solver in Maple) to numerically find the optimal policy (controls and cost). For our numerical results, we considered a base case with $K_0 = 50$, $h = 1$, $K_u = 10$, $K_d = 50$, $c = 5$, $\xi = 0.2$, $\eta = 0.05$, and $\lambda = 50$ (and $\mu = 1$ in the compound Poisson case) and derived the optimal policy. We then varied each of the parameters ξ, η, and λ one at a time under the optimal policy and compared the values of the exact analysis with perishability to the system with no perishability which we refer to as the heuristic. We observed that it is beneficial to provide the solver with a starting point (values for both S and s). We found that providing the starting point as the solution of the previous optimization run allowed the solver to find the optimal solution in “no time” (few seconds for ten runs, for both demand scenarios). To make sure that the solution provided by Maple is (close to) optimal, we used standard methodology such as plotting the objective function for different choices of (s, S) and starting the optimization from different starting points. We used similar optimization approach for the heuristic. As is common, we let S^*, s^*, and R^*, denote the optimal order up-to level, reorder point, and cost, respectively. We let R^H denote the cost of the heuristic and report the $Loss\ \% = \left(R^H - R^*\right)/R^*$.

5.1. Poisson Demand Case

We note that when demand is Poisson, there is no value in ordering a non-integer number of units, so after finding the optimal continuous controls, if needed, we run a quick search comparing the costs for any round up and round down combination of S and s. The same procedure was used with the heuristic (where, of course, we used the heuristic cost function in this comparison). We note that the optimal S^* and s^* were the round up results of the continuous solution in all our numerical experiments.

From Table 1 we observe that when ξ increases, as expected, R^*, S^*, s^*, $E(T)$, and $E(Z)$ decrease while $E(W)$ and $E(U)$ increase.

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Table 1. Results when varying ξ-Poisson demand.

ξ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
S^*	169	159	151	145	139	134	130	126	122	119
s^*	106	96	88	81	76	71	67	63	59	56
R^*	483	469	458	449	440	434	427	422	417	413
ET	21.22	11.22	7.89	6.24	5.22	4.55	4.08	3.72	3.44	3.22
EZ	140.87	80.58	60.41	50.19	44.17	40.15	37.21	35.09	33.52	32.16
EW	12.63	21.04	26.92	31.35	34.39	36.65	38.53	39.73	40.34	41.09
EU	0.02	0.03	0.03	0.03	0.04	0.04	0.04	0.05	0.05	0.05
R^H	485	472	461	452	444	437	430	425	420	416
Loss %	0.40	0.58	0.66	0.71	0.69	0.72	0.70	0.67	0.65	0.66

Table 2. Results when varying η-Poisson demand.

η	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
S^*	145	120	102	88	78	70	63	57	52	48
s^*	81	62	49	39	32	26	22	18	15	12
R^*	449	449	467	473	477	481	484	487	489	491
ET	6.24	6.09	5.98	5.89	5.82	5.77	5.71	5.67	5.63	5.6
EZ	50.19	30.26	23.66	20.48	18.47	17.15	16.27	15.68	15.22	14.83
EW	31.35	21.57	15.67	11.73	9.25	7.46	6.08	5	4.19	3.58
EU	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.02	0.02	0.02
R^H	452	469	485	500	512	524	535	545	554	562
Loss %	0.71	2.14	3.83	5.57	7.2	8.9	10.42	11.85	13.17	14.39

From Table 2 we observe that when varying η, as expected, when η increases, because the time to perishability decreases, S^* and s^*, decrease, R^* increases, and $E(T)$, $E(Z)$, $E(W)$, and $E(U)$ decrease.

From Table 3 we observe that when varying λ, as expected, when λ increases, S^*, s^*, R^*, and $E(W)$ increase, and $E(T)$, and $E(U)$ decrease. The value of $E(Z)$ does not change much and it may increase or decrease with λ (see the case where $\lambda = 60$). The gap between the optimal and heuristic costs is less than 1%; again this is despite that the heuristic may be considerably off in calculating its controls.

As expected the no-perishability heuristic is only close to optimal for small η. Consider the second column in Table 2 where $\eta = 0.1$ and recall that here $\xi = 0.2$. That is in this

Table 3. Results when varying λ-Poisson demand.

λ	10	20	30	40	50	60	70	80	90	100
S^*	32	61	89	117	145	172	200	228	256	284
s^*	4	21	40	60	81	103	126	148	171	194
R^*	96.45	184.77	272.80	360.71	448.57	536.40	624.21	712.00	799.79	887.56
ET	7.61	6.90	6.57	6.37	6.24	6.12	6.03	5.98	5.92	5.88
EZ	51.85	50.52	50.35	50.26	50.19	50.29	50.20	50.16	50.12	50.08
EW	6.56	12.90	19.02	25.16	31.35	37.23	43.50	49.70	55.94	62.18
EU	0.16	0.08	0.06	0.04	0.03	0.03	0.02	0.02	0.02	0.02
R^H	97.36	186.26	275.14	363.30	451.75	541.29	629.64	717.83	806.17	893.45
Loss %	0.94	0.81	0.86	0.72	0.71	0.91	0.87	0.82	0.80	0.66

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column perishability occurs on average every two lead times. The results in columns 3–10 are for shorter time to perishability. Therefore, ignoring perishability is costly in these settings; nevertheless, we believe these settings are not common in practice.

5.2. Compound Poisson with Exponential Demand

Here, we report the results for the compound Poisson case where as mentioned above we choose $\mu = 1$ in the base case.

The results when varying ξ, η, and λ are depicted in Tables 4–6, respectively. The direction of changes of s^*, S^*, R^*, $E(T)$, $E(W)$, and $E(U)$ in these tables is similar to the direction of changes in Tables 1–3. Similarly, $E(Z)$ in Tables 4 and 5 decreases, as

TABLE 4. Results when varying ξ-compound Poisson arrival with $\exp(\mu)$ demand.

ξ	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
S^*	196.5	142.4	113.5	95.7	83.6	75.1	68.7	63.9	59.2	54.9
s^*	134	79.8	50.9	33.0	21.0	12.5	6.1	1.3	0	0
R^*	459	445	438	435	433	432	431	431	431	432
ET	21.23	11.23	7.90	6.23	5.23	4.56	4.09	3.73	3.39	3.09
EZ	123.73	87.76	75.49	69.22	65.39	62.79	60.91	59.46	58.07	56.80
EW	16.87	17.17	15.81	14.43	13.27	12.34	11.61	11.03	10.47	9.93
EU	0.02	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03	0.03
R^H	491.7	514.8	538.6	558.9	575.6	589.3	600.8	610.3	618.4	625.4
Loss %	7.07	15.70	22.92	28.59	33.03	36.53	39.31	41.55	43.37	44.84

TABLE 5. Results when varying η-compound Poisson arrival with $\exp(\mu)$ demand.

η	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
S^*	95.65	52.59	34.79	25.86	20.52	16.98	14.46	12.58	11.12	9.95
s^*	33.04	0	0	0	0	0	0	0	0	0
R^*	435	467	481	488	492	495	497	499	500	501
ET	6.23	6.01	5.68	5.51	5.41	5.34	5.29	5.26	5.23	5.21
EZ	69.22	59.28	55.91	54.27	53.25	52.51	51.93	51.44	51.02	50.64
EW	14.43	4.60	2.17	1.26	0.82	0.58	0.43	0.33	0.27	0.22
EU	0.03	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02	0.02
R^H	559	742	864	945	997	1029	1049	1060	1064	1065
Loss %	28.59	58.84	79.85	93.69	102.53	107.91	110.95	112.39	112.76	112.42

TABLE 6. Results when varying λ-compound Poisson arrival with $\exp(\mu)$ demand.

λ	10	20	30	40	50	60	70	80	90	100
S^*	22.07	41.54	59.67	77.69	95.65	113.59	131.50	149.39	167.28	185.15
s^*	0	2.14	11.29	21.74	33.04	44.93	57.30	70.03	83.06	96.35
R^*	93.37	178.89	264.22	349.43	434.59	519.71	604.82	689.90	774.98	860.05
ET	7.17	6.92	6.58	6.37	6.23	6.13	6.05	5.98	5.92	5.88
EZ	66.10	68.79	69.03	69.15	69.22	69.27	69.31	69.33	69.35	69.37
EW	3.55	6.30	8.99	11.71	14.43	17.16	19.89	22.63	25.37	28.11
EU	0.14	0.07	0.05	0.04	0.03	0.02	0.02	0.02	0.02	0.01
R^H	118	229	339	449	559	669	779	889	999	1109
Loss %	26.96	27.76	28.2	28.44	28.59	28.7	28.78	28.83	28.88	28.92

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in Tables 1 and 2. In Table 6, $E(Z)$ increases with λ, whereas in Table 3, $E(Z)$ is not monotone; but this increase is quite mild. We further observe that the heuristic performance is deteriorating when the lead time and the demand rate increase and when the time to perishability decreases.

6. CONCLUSIONS: INSIGHTS ON MANAGEMENT OF PERISHABLE INVENTORY

In this paper, we presented a continuous review (s, S) model with lost sales and perishability with stochasticity due to four factors: time to perishability, lead time, demand arrival time, and demand size (at each arrival). We provided exact, closed-form expressions for the cost of the system under the average cost criterion. Using these expressions, we minimize the total cost and obtained optimal re-order and order up-to levels. We compared the cost resulting from the exact model to the one of a heuristic that ignores perishability.

There are several insights on the management of perishable items from our analysis and the numerical results. These insights are focused on the changes in s, S, and the efficiency of the heuristic along the four uncertainties considered. We remind that we slightly abuse the definitions of cycle and safety stock: the cycle stock is controlled by both S and s, thus we call $S-s$ the "cycle stock;" similarly, the safety stock is dictated by s, thus we call s the "safety stock" (note that in most of our examples, the safety stock, $s-\lambda/(\mu\xi)$, is negative). From Tables 1–6 we observe:

The effect of time to perishability: As the time to perishability increases both s and S increase at a faster than linear rate (s grows a bit faster), the total cost decreases at a rate slower linear, and the heuristic that ignores perishability gets closer to optimal. These results are intuitive because perishability increases the cost in the case of having high inventory and therefore decreasing the total level of inventory in the system (including the safety stock) makes sense. That the ratio s/S increases implies that reducing the safety stock is more effective in combating perishability than reducing the cycle stock. Without perishability safety stock is carried most of the time, whereas only half of the cycle stock is carried on average, thus in events of perishability the relative cost savings from decreasing the safety stock are higher than the savings when decreasing the cycle stock. As expected, considering perishability is important when perishability events are quite frequent.

The effect of lead time: As the lead time increases both s and S increase in the same amount and the cost of the heuristic that ignores perishability gets closer to optimal. Without perishability longer lead time increases the safety stock and do not effect the cycle stock. Apparently, the effect of lead time in the presence of perishability is identical. Shorter lead time helps to mitigate the effects of perishability explaining the changes in the cost. The improved effectiveness of the heuristic follows because longer lead times increase the length of cycles between inventory orders and increase the total inventory such that the relative error of the heuristic decreases.

The effect of demand arrival rate: As the arrival rate increases S increases linearly and s grows faster than linear, but with an absolute growth that is lower than that of S, the total cost increases (linearly), and the efficiency of the heuristic that ignores perishability remains fixed (i.e., its cost also linearly increases). While in the non perishable case a linear growth of the cost with the demand rate is expected, this is a somewhat surprising result when there is perishability. In the presence of perishability, when more inventory is carried (due to higher demand), we expect that any perishability instance will affect a higher inventory and the overall cost to increase faster than in the non perishable case. As it turns out, the optimal (s, S) are such that the rate of event instances slowly decreases

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with the demand rate such that the total effect on the cost of the increase demand rate remains linear.

The effect of demand size variability: As the demand size variability increases (from a unit to an exponential demand size) s and S and the total cost may either increase or decrease, and the efficiency of the heuristic that ignores perishability significantly decreases. A reduction of the inventory level for a riskier business is expected; the fact that this reduction may offset the costs such that the cost in the more variable demand case ends up lower than the lower demand variability case is surprising.

Efficiency of the heuristic: For the Poisson demand case, the heuristic is quite effective, with a typical relative (compared with the optimal cost) error of less than 1%, if perishability is not too frequent. This is true despite that its controls may be quite far from optimal (e.g., for the base case, $S^* = 145$, $s^* = 81$, whereas the heuristic chooses $S = 183$ and $s = 108$), implying that the cost function is quite flat. In contrast, in the compound Poisson demand case, the heuristic performs quite poorly. Its best performance are in the case where the lead time is quite long, $\xi = 0.05$. In all other cases the heuristic is much worse with a maximum error of about 45, 113, and 29% in Tables 1–3, respectively. Therefore, we conclude that considering perishability in inventory control is important as the variability of the demand size increases.

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APPENDIX – PROOFS OF PROPOSITIONS

Appendix A: Proof of Proposition 3.1

To calculate the functional $E\left(e^{-\beta T}\right)$, we notice that L_ξ and τ_{S-s} in Eq. (3) are independent. Thus,

$$E\left(e^{-\beta T}\right) = \frac{\xi}{\xi+\beta} E\left(e^{-\beta\tau_{S-s}}\right),$$

where $\xi/\xi+\beta$ is the LST of L_ξ. So we only need to calculate the LST of τ_{S-s}, $E\left(e^{-\beta\tau_{S-s}}\right) = E\left(e^{-\beta\min\{\tilde{\tau}_{S-s},D_\eta\}}\right)$. By definition

$$E\left(e^{-\beta\tau_{S-s}}\right) = \beta\int_0^\infty e^{-\beta t} d\Pr(\tau_{S-s}\le t).$$

Thus

$$\begin{aligned} E\left(e^{-\beta\tau_{S-s}}\right) &= E\left(e^{-\beta\min\{\tilde{\tau}_{S-s},D_\eta\}}\right) \qquad \textbf{(A.1)} \\ &= \beta\int_0^\infty e^{-\beta t}\left(1-\Pr(\min\{\tilde{\tau}_{S-s},D_\eta\}>t)\right)dt \\ &= \beta\int_0^\infty e^{-\beta t}\left(1-e^{-\eta t}\Pr(\tilde{\tau}_{S-s}>t)\right)dt, \end{aligned}$$

so that

$$\begin{aligned} E\left(e^{-\beta\tau_{S-s}}\right) &= 1-\frac{\beta}{\beta+\eta}\int_0^\infty (\beta+\eta)\,e^{-(\eta+\beta)t}\left(1-\Pr(\tilde{\tau}_{S-s}\le t)\right)dt \\ &= 1-\frac{\beta}{\beta+\eta}+\frac{\beta E(e^{-(\beta+\eta)\tilde{\tau}_{S-s}})}{\beta+\eta} = \frac{\eta+\beta E(e^{-(\beta+\eta)\tilde{\tau}_{S-s}})}{\beta+\eta}. \end{aligned}$$

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Appendix B: Proof of Proposition 3.3

Let V denote the time between two consecutive demands. Using the memoryless of the lead time we have

$$E\left(e^{-\beta U}\right) = E\left(e^{-\beta V}\mathbf{1}_{\{V<L_\xi\}}\right) + E\left(e^{-\beta L_\xi}\mathbf{1}_{\{L_\xi\le V\}}\right)E\left(e^{-\beta X}\right),$$

where $\mathbf{1}_{\{V<L_\xi\}}$ is a competition between two exponentials, so on this event V the time between consecutive demands is $\exp(\lambda+\xi)$ and

$$E\left(e^{-\beta V}\mathbf{1}_{\{V<L_\xi\}}\right) = \frac{\lambda}{\lambda+\beta+\xi}.$$

Similarly, we have

$$E\left(e^{-\beta L_\xi}\mathbf{1}_{\{L_\xi\le V\}}\right) = \frac{\xi}{\lambda+\beta+\xi}.$$

Establishing Eq. (7).

By Eq. (6)

$$\begin{aligned} E\left(e^{-\beta X}\right) &= E\left(e^{-\beta D_\eta}e^{-\beta U}\mathbf{1}_{\{D_\eta\le\min\{T,\tau_S\}\}}\right) \\ &\quad + E\left(\left(e^{-\beta\tau_S}\left(\mathbf{1}_{\{W(\tilde\tau_S)<0\}} + e^{-\beta U}\mathbf{1}_{\{W(\tilde\tau_S)=0\}}\right)\right)\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right) \\ &\quad + E\left(e^{-\beta(T+X)}\mathbf{1}_{\{T\le\tau_S\}}\right). \end{aligned} \tag{B.1}$$

The functional $E\left(e^{-\beta D_\eta}e^{-\beta U}\mathbf{1}_{\{D_\eta\le\min\{T,\tau_S\}\}}\right)$. We first note that due to the definitions in Eqs. (2)–(4) the following events are equivalent:

$$\begin{aligned} \{D_\eta\le\min\{T,\tau_S\}\} &= \left\{D_\eta\le\min\left\{\tau_{S-s}+L_\xi,\tilde\tau_S\right\}\right\} = \left\{D_\eta\le\min\left\{\tilde\tau_{S-s}+L_\xi,\tilde\tau_{S-s}+\tilde\tau_{s-Y}\right\}\right\} \\ &= \left\{D_\eta\le\tilde\tau_{S-s}+\min\left\{L_\xi,\tilde\tau_{s-Y}\right\}\right\}, \end{aligned}$$

so that

$$\begin{aligned} E\left(e^{-\beta D_\eta}e^{-\beta U}\mathbf{1}_{\{D_\eta\le\min\{T,\tau_S\}\}}\right) &= E\left(e^{-\beta U}\right)E\left(\int_0^{\tilde\tau_{S-s}+\min\{L_\xi,\tilde\tau_{s-Y}\}} e^{-\beta z}\eta e^{-\eta z}dz\right) \\ &= \frac{E\left(e^{-\beta U}\right)\eta}{\eta+\beta}\left(1-E\left(e^{-(\eta+\beta)(\tilde\tau_{S-s}+\min\{L_\xi,\tilde\tau_{s-Y}\})}\right)\right). \end{aligned}$$

We remind that $\tilde\tau_{S-s}$, L_ξ, and $\tilde\tau_{s-Y}$ are all independent, so that $E\left(e^{-(\eta+\beta)(\tilde\tau_{S-s}+\min\{L_\xi,\tilde\tau_{s-Y}\})}\right) = E\left(e^{-(\eta+\beta)\tilde\tau_{S-s}}\right)E\left(e^{-(\eta+\beta)\min\{L_\xi,\tilde\tau_{s-Y}\}}\right)$. To calculate $E\left(e^{-\beta\min\{L_\xi,\tilde\tau_{s-Y}\}}\right)$, we use a similar argument to the one in Eq. (A.1), replacing $\tilde\tau_{S-s}$ with $\tilde\tau_{s-Y}$ and η with ξ, to get

$$E\left(e^{-\beta\min\{L_\xi,\tilde\tau_{s-Y}\}}\right) = \frac{\xi+\beta E\left(e^{-(\beta+\xi)\tilde\tau_{s-Y}}\right)}{\beta+\xi}. \tag{B.2}$$

The functional $E\left(\left(e^{-\beta\tau_S}\left(\mathbf{1}_{\{W(\tilde\tau_S)<0\}} + e^{-\beta U}\mathbf{1}_{\{W(\tilde\tau_S)=0\}}\right)\right)\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)$.

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We notice that the following events are equivalent:

$$\{\tau_S < \min\{D_\eta, T\}\} = \{\{\tilde{\tau}_{s-Y} \le L_\xi\} \cup \{\tilde{\tau}_S \le D_\eta\}\} = \{\{\tilde{\tau}_{s-Y} \le L_\xi\} \cup \{\tilde{\tau}_{s-Y} + \tilde{\tau}_{S-s} \le D_\eta\}$$

and that on $\{\tau_S < \min\{D_\eta, T\}\}$ we have

$$\tau_S = \tilde{\tau}_{s-Y} + \tilde{\tau}_{S-s}.$$

So that by the independence of U, $\tilde{\tau}_{s-Y}$ and $\tilde{\tau}_{S-s}$ we have

$$\begin{aligned}
&E\left(e^{-\beta\tau_S}\left(\mathbf{1}_{\{W(\tilde{\tau}_S)<0\}} + e^{-\beta U}\mathbf{1}_{\{W(\tilde{\tau}_S)=0\}}\right)\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)\\
&\quad = E\left(e^{-\beta\tau_S}\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)E\left(\mathbf{1}_{\{W(\tilde{\tau}_S)<0\}} + e^{-\beta U}\mathbf{1}_{\{W(\tilde{\tau}_S)=0\}}\right)\\
&\quad = E\left(e^{-\beta\tau_S}\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right)\left(P\left(W\left(\tilde{\tau}_S\right)<0\right) + E\left(e^{-\beta U}\right)P\left(W\left(\tilde{\tau}_S\right)=0\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
E\left(e^{-\beta\tau_S}\mathbf{1}_{\{\tau_S<\min\{D_\eta,T\}\}}\right) &= E\left(e^{-\beta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}\mathbf{1}_{\{\tilde{\tau}_{s-Y}\le L_\xi\}}\mathbf{1}_{\{\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s}\le D_\eta\}}\right)\\
&= E\left(e^{-\beta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}\int_{\tilde{\tau}_{s-Y}}^{\infty}\xi e^{-\xi y}dy\int_{\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s}}^{\infty}\eta e^{-\eta x}dx\right)\\
&= E\left(e^{-\beta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}e^{-\xi\tilde{\tau}_{s-Y}}e^{-\eta(\tilde{\tau}_{s-Y}+\tilde{\tau}_{S-s})}\right)\\
&= E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right).
\end{aligned}$$

The functional $E\left(e^{-\beta(T+X)}\mathbf{1}_{\{T\le\tau_S\}}\right)$.

We notice that the following events are equivalent:

$$\begin{aligned}
\{T \le \tau_S\} &= \left\{\tau_{S-s} + L_\xi \le \min\{\tilde{\tau}_S, D_\eta\}\right\} = \left\{\min\{\tilde{\tau}_{S-s}, D_\eta\} + L_\xi \le \min\{\tilde{\tau}_S, D_\eta\}\right\}\\
&= \left\{\tilde{\tau}_{S-s} + L_\xi \le \min\{\tilde{\tau}_S, D_\eta\}\right\} = \left\{\tilde{\tau}_{S-s} + L_\xi \le D_\eta\right\} \cup \left\{L_\xi \le \tilde{\tau}_{s-Y}\right\},
\end{aligned}$$

so that

$$E\left(e^{-\beta(T+X)}\mathbf{1}_{\{T\le\tau_S\}}\right) = E\left(e^{-\beta X}\right)E\left(e^{-\beta T}\mathbf{1}_{\{\tilde{\tau}_{S-s}+L_\xi\le D_\eta\}}\mathbf{1}_{\{L_\xi\le\tilde{\tau}_{s-Y}\}}\right).$$

Next, since on $\{T \le \tau_S\}$ we have $T = \tau_{S-s} + L_\xi = \tilde{\tau}_{S-s} + L_\xi$ and due to the independence of D_η, $\tilde{\tau}_{S-s}$, and L_ξ we have

$$\begin{aligned}
&E\left(e^{-\beta T}\mathbf{1}_{\{\tilde{\tau}_{S-s}+L_\xi\le D_\eta\}}\mathbf{1}_{\{L_\xi\le\tilde{\tau}_{s-Y}\}}\right)\\
&\quad = E\left(e^{-\beta\tilde{\tau}_{S-s}}\int_0^{\tilde{\tau}_{s-Y}}e^{-\beta L_\xi}\left(\int_{\tilde{\tau}_{S-s}+L_\xi}^{\infty}\eta e^{-\eta x}dx\right)\xi e^{-\xi L_\xi}dL_\xi\right)\\
&\quad = E\left(e^{-\beta\tilde{\tau}_{S-s}}\int_0^{\tilde{\tau}_{s-Y}}e^{-\beta L_\xi}e^{-\eta(\tilde{\tau}_{S-s}+L_\xi)}\xi e^{-\xi L_\xi}dL_\xi\right)\\
&\quad = \xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)E\left(\int_0^{\tilde{\tau}_{s-Y}}e^{-(\xi+\beta+\eta)L_\xi}dL_\xi\right)\\
&\quad = \frac{\xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(1 - E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right)\right)}{\xi+\beta+\eta}.
\end{aligned} \qquad \textbf{(B.3)}$$

Substituting these functionals into Eq. (B.1), establishes Eq. (8).

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Appendix C: Proof of Proposition 3.4

To compute the LST of Z we shift the origin to the time of an outdating and recall that

$$E\left(e^{-\beta Z}\right) = \frac{\xi}{\xi+\beta} E\left(e^{-\beta B}\right), \tag{C.1}$$

where the random variable B is the time it takes from the instant the inventory level is S until the next perishability instance. The explanation for Eq. (C.1) is simple. Since perishability does not occur when the system is empty, we should wait until the end of the lead time which is exponentially distributed(ξ). From that instant the time to the next outdating is B. We now have

$$\begin{aligned} E\left(e^{-\beta B}\right) &= E\left(e^{-\beta D_\eta} \mathbf{1}_{\{D_\eta < \tilde{\tau}_{S-s} + \min(\tilde{\tau}_{s-Y}, L_\xi)\}}\right) \\ &\quad + E\left(e^{-\beta \tilde{\tau}_S} \mathbf{1}_{\{\tilde{\tau}_S \le D_\eta\}} \mathbf{1}_{\{\tilde{\tau}_{s-Y} \le L_\xi)\}}\right) \frac{\xi}{\xi+\beta} E\left(e^{-\beta B}\right) \\ &\quad + E\left(e^{-\beta(\tilde{\tau}_{S-s}+L_\xi)} \mathbf{1}_{\{\tilde{\tau}_{S-s}+L_\xi \le D_\eta\}} \mathbf{1}_{\{L_\xi \le \tilde{\tau}_{S-Y})\}}\right) E\left(e^{-\beta B}\right). \end{aligned} \tag{C.2}$$

The first term on the right hand side of Eq. (C.2) indicates the partial LST of the time to the next perishability instance if it arrives before $\tilde{\tau}_{S-s} + \min(\tilde{\tau}_{S-Y}, L_\xi)$. The second term indicates that an unsatisfied demand occurs before the next perishability event and before the end of the lead time. Then, we have to wait until the end of the lead time and from that instant the time to the next outdating starts afresh. The third term on the right hand side of Eq. (C.2) is for the case when the time until the inventory level equals S is $\tilde{\tau}_{S-s} + L_\xi$, where $\tilde{\tau}_{S-s} + L_\xi$ occurs before perishability and also the lead time comes before $\tilde{\tau}_{S-Y}$. Again, since the inventory level is refilled up to level S, the time to the next outdating starts afresh.

For the first component of Eq. (C.2) we obtain (using the independence between $\tilde{\tau}_{S-s}$ and $\min(\tilde{\tau}_{s-Y}, L_\xi)$

$$\begin{aligned} &E\left(e^{-\beta D_\eta} \mathbf{1}_{\{D_\eta < \tilde{\tau}_{S-s} + \min(\tilde{\tau}_{s-Y}, L_\xi)\}}\right) \\ &= E\left(\int_0^{\tilde{\tau}_{S-s} + \min(\tilde{\tau}_{s-Y}, L_\xi)} e^{-\beta t} \eta e^{-\eta t} dt\right) \\ &= \frac{\eta}{\eta+\beta}\left(1 - E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right) E\left(e^{-(\beta+\eta)\min(\tilde{\tau}_{s-Y}, L_\xi)}\right)\right) \\ &= \frac{\eta}{\eta+\beta}\left(1 - E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right) \frac{\xi + (\beta+\eta) E\left(e^{-(\beta+\xi+\eta)\tilde{\tau}_{s-Y}}\right)}{\beta+\xi+\eta}\right), \end{aligned}$$

where $E\left(e^{-(\beta+\eta)\min(\tilde{\tau}_{s-Y}, L_\xi)}\right)$ is given in Eq. (B.2).

For the second component, using that $\tilde{\tau}_S = \tilde{\tau}_{S-s} + \tilde{\tau}_{s-Y}$ and that the last two RVs are independent we get

$$\begin{aligned} &E\left(e^{-\beta \tilde{\tau}_S} \mathbf{1}_{\{\tilde{\tau}_S \le D_\eta\}} \mathbf{1}_{\{\tilde{\tau}_{s-Y} \le L_\xi)\}}\right) \frac{\xi}{\xi+\beta} E\left(e^{-\beta B}\right) \\ &= \frac{\xi}{\xi+\beta} E\left(e^{-\beta B}\right) E\left(e^{-\beta \tilde{\tau}_S} \int_{\tilde{\tau}_S}^{\infty} \eta e^{-\eta x} dx \int_{\tilde{\tau}_{s-Y}}^{\infty} \xi e^{-\xi z} dz\right) \\ &= \frac{\xi E\left(e^{-\beta B}\right)}{\xi+\beta} E\left(e^{-\beta \tilde{\tau}_S} e^{-\eta \tilde{\tau}_S} e^{-\xi \tilde{\tau}_{s-Y}}\right) \\ &= \frac{\xi E\left(e^{-\beta B}\right)}{\xi+\beta} E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right) E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right). \end{aligned}$$

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For the third component, because on these events $T = (\tilde{\tau}_{S-s} + L_\xi)$ we can use Eq. (B.3)

$$E\left(e^{-\beta(\tilde{\tau}_{S-s}+L_\xi)}\mathbf{1}_{\{\tilde{\tau}_{S-s}+L_\xi \le D_\eta\}}\mathbf{1}_{\{L_\xi \le \tilde{\tau}_{s-Y})\}}\right) = \frac{\xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(1 - E\left(e^{-(\xi+\beta+\eta)\tilde{\tau}_{s-Y}}\right)\right)}{\xi+\beta+\eta}.$$

Solving for $E\left(e^{-\beta B}\right)$ we get

$$E\left(e^{-\beta B}\right) = \frac{\frac{\eta}{\eta+\beta}\left(1 - E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\frac{\xi+(\beta+\eta)E\left(e^{-(\beta+\xi+\eta)\tilde{\tau}_{s-Y}}\right)}{\beta+\xi+\eta}\right)}{1 - \xi E\left(e^{-(\beta+\eta)\tilde{\tau}_{S-s}}\right)\left(\frac{E\left(e^{-(\beta+\eta+\xi)\tilde{\tau}_{s-Y}}\right)\eta}{(\xi+\beta)(\xi+\beta+\eta)} + \frac{1}{\xi+\beta+\eta}\right)}.$$

Using Eq. (C.1), we obtain Eq. (10).

Appendix D: Proof of Proposition 3.5

The left hand side of Eq. (11) is the long run average number of downcrossings of level x. There are two types of downcrossings. The first occurs at times of perishability, so that when a perishability event arrives (at rate η) and the inventory level is above level x, level x is downcrossed. Therefore the long run average number of downcrossings that are generated by perishability is $\eta\int_x^S dF(w)$. The second type is due to the demands that arrive at rate λ. Every demand starting at inventory level w downcrosses x if the demand size is greater than $w - x$, i.e., with probability $1 - F_Y(w - x)$. Therefore, the second type of downcrossings of level x has a rate $\lambda\int_x^S (1 - F_Y(w - x))\, dF(w)$. The right hand side of Eq. (11) is the long run average upcrossing of level x. When $x \le s$, the upcrossings are generated by the replenishment (that arrive at rate ξ); every replenishment is an upcrossing if the inventory level is less than x. However, if $s < x \le S$ level x is upcrossed only by jumps that start below level s, since the lead time starts only after downcrossings of level $S - s$. Finally, the density $dF(\cdot)$ is the same in both sides of Eq. (11) because of poisson arrivals see time averages.

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