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REDUCTION TECHNIQUES FOR PROVING DECIDABILITY IN LOGICS AND THEIR MEET–COMBINATION

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Abstract. Satisfaction systems and reductions between them are presented as an appropriate context for analyzing the satisfiability and the validity problems. The notion of reduction is generalized in order to cope with the meet-combination of logics. Reductions between satisfaction systems induce reductions between the respective satisfiability problems and (under mild conditions) also between their validity problems. Sufficient conditions are provided for relating satisfiability problems to validity problems. Reflection results for decidability in the presence of reductions are established. The validity problem in the meet-combination is proved to be decidable whenever the validity problem for the components are decidable. Several examples are discussed, namely, involving modal and intuitionistic logics, as well as the meet-combination of K modal logic and intuitionistic logic.

§1. Introduction. A logical decision problem is a pair (L,Γ) where L is the set of formulas of a certain logic and Γ is a subset of L. When looking at a logic from a semantic perspective we can consider two main decision problems. The satisfiability problem is the pair

$$(L, \{\gamma \in L : \gamma \text{ is a satisfiable formula}\})$$

and the validity problem is the pair

$$(L, \{\gamma \in L : \gamma \text{ is a valid formula}\}).$$

Logic decision problems were firstly stated by David Hilbert (Entscheidungsproblem; see [18, 19]) for first-order logic in the following way. The Satisfiability Problem: Given a first-order formula φ, is φ satisfiable? The Validity Problem: Given a first-order formula φ, is φ valid? and the Provability Problem: Given a first-order formula φ, is φ provable (in a given proof system)?

The main interest in decision problems is related to their decidability. Intuitively speaking, a decision problem (L,Γ) is decidable if there is an algorithm (a recipe or a finite set of rules) that when applied to an argument $\varphi \in L$ returns either 1 if $\varphi \in \Gamma$ or 0 otherwise. When dealing with a decision problem in logic, we have two different options. Either we work with formulas all over or we convert our logical decision problem to the universe

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of the natural numbers (see, for instance, [17, 26, 33]). The latter option was followed by Kurt Gödel while proving the Incompleteness Theorems, by giving what we now call Gödelization maps (see [11, 13, 28]). The essential requirement for a Gödelization map is to guarantee that the logical decision problem is decidable if and only if the corresponding problem in the natural numbers' setting is decidable. When working with natural numbers, Turing machines and recursive functions are often adopted as the computation model (see [20, 34]). If we decide to work with other universes it seems interesting to use an abstract high level language (see [31]).

Proving or disproving decidability is not always a simple task. However, there are some techniques that can help besides the direct way of producing a program that computes the characteristic map of the set under consideration. The finite model property technique has been used in the context of first-order logic (see [6, 10, 23]) and modal logic. In first-order logic the technique was used for showing that the satisfiability problems in $\mathrm{FO}(\exists^*\forall^*)$ (Bernays–Schönfinkel class), $\mathrm{FO}(\exists^*\forall\exists^*)$ (Ackermann class), and FO^2 are decidable. In normal modal logic, filtrations have been used for showing that several normal modal logics have the finite model property (see [5]).

Another very important technique is reduction. Intuitively, a problem is reducible to another problem whenever a method (an algorithm) for solving the latter provides a method (an algorithm) for solving the former. The concept was introduced by Alan Turing (see [35]) in terms of oracles. Later on Stephen Kleene gave a notion of reduction using recursive functions (see [21]) on the natural numbers.

Nowadays there is an abstract notion of many-to-one reducibility over the set of natural numbers (see [26, 34]). This notion can be adapted to logical decision problems, by saying that a reduction from (L,Γ) to (L',Γ') is a computable map $\tau : L \to L'$ such that, for each $\varphi \in L$,

$$\varphi \in \Gamma \quad \text{if and only if} \quad \tau(\varphi) \in \Gamma'.$$

However, we need a generalized version of this concept appropriate for meet-combination of logics (that provides an axiomatization of the product of matrix logics; see [25, 30]). In the meet-combination of two logics a connective is a pair composed by a connective of each logic inheriting only the common logical properties of the component connectives. So, to transfer decidability results from the component logics to the meet-combination we need to extend the notion of reduction from one logical decision problem to a non-empty finite collection of logical decision problems.

The reductions described in the literature are for specific logics and no general mechanism is presented for reduction between logical decision problems. Moreover, in most of the cases the reduction is between the same kind of decision problems, that is, either between two satisfiability problems, or between two validity problems. One of the exceptions is related to the equivalence between the satisfiability problem and the validity problem for first-order logic, that is, the satisfiability problem for first-order logic is decidable if and only if the validity problem for first-order logic is decidable (see [18, 19]).

The paper is organized as follows. In Section 2, we adopt the notion of satisfaction system (introduced in [2]) as the appropriate framework for setting-up logical decision problems based on semantics. We define decision problem over a satisfaction system and provide examples of the satisfiability (respectively, the dual of the satisfiability) and of the validity (respectively, the dual of the validity) problems, proving that decidability is reflected by reduction. We end the section by establishing a sufficient condition for having a reduction from the validity problem in a satisfaction system to the satisfiability in another satisfaction system.

Section 3 concentrates on the important concept of reduction between a satisfaction system and a finite collection of satisfaction systems. Two main results are established: decidability of the satisfiability problem for the source satisfaction system whenever the satisfiability problem for each target satisfaction system is decidable and, under some mild conditions, decidability of the validity problem for the source satisfaction system whenever the validity problem for each target satisfaction system is decidable. Both results capitalize on the fact that a satisfaction system reduction induces a reduction between the corresponding satisfiability problems (similarly for validity problems). Three examples are provided. (1) Decidability of the satisfiability problem in K modal logic endowed with local Kripke semantics from the decidability of the satisfiability problem in FO^2 first-order logic endowed with contextual satisfaction. (2) Decidability of the validity problem in K modal logic endowed with algebraic semantics from the decidability of the validity problem in K modal logic endowed with global Kripke semantics using Stone Representation Theorem and Jonssón–Tarski Theorem (see [4, 9]). (3) Finally, decidability of the validity problem in intuitionistic logic endowed with algebraic semantics from the decidability of the validity problem in intuitionistic logic endowed with global Kripke semantics using Stone Representation Theorem (see [9, 27]).

Finally, in Section 4 we start by introducing the meet-combination of two (matrix) satisfaction systems. We show that there is a reduction from the satisfaction system for meet-combination to the collection composed by the component satisfaction systems and prove that the validity problem in meet-combination is decidable provided that the validity problem in each component logic is decidable as well. We end up the section by discussing the meet-combination of K modal logic and intuitionist logic both endowed with algebraic semantics.

§2. Satisfaction system decision problems. The objective of this section is to introduce some important decision problems on logics presented by a satisfaction system.

Definition 2.1. A *satisfaction system* is a triple

$$S = (L, \mathcal{M}, \Vdash),$$

where L is a non-empty *set of formulas*, $\mathcal{M}$ is a class of *semantic structures*, and $\Vdash \subseteq \mathcal{M} \times L$ is a binary relation called the *satisfaction relation*.

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EXAMPLE 2.2. The local Kripke satisfaction system

$$S_\Pi^{\mathrm{lk,K}} = (L_\Pi^{\mathrm{ML}}, \mathcal{M}_\Pi^{\mathrm{lk,K}}, \Vdash_\Pi^{\mathrm{lk,K}})$$

for K modal logic over a set Π of propositional symbols is such that

- L_Π^{ML} is the set of modal formulas inductively defined as follows:
  - $\Pi \subseteq L_\Pi^{\mathrm{ML}}$;
  - $\neg\varphi, \Diamond\varphi \in L_\Pi^{\mathrm{ML}}$ provided that $\varphi \in L_\Pi^{\mathrm{ML}}$;
  - $\varphi_1 \supset \varphi_2 \in L_\Pi^{\mathrm{ML}}$ provided that $\varphi_1, \varphi_2 \in L_\Pi^{\mathrm{ML}}$;
- $\mathcal{M}_\Pi^{\mathrm{lk,K}}$ is the class of all pointed Kripke structures, that is, pairs $((W,R,V),w)$ where (W,R,V) is a Kripke structure, i.e.,
  - W is a non-empty set whose elements are called worlds;
  - $R \subseteq W^2$ is a relation called the accessibility relation;
  - $V : \Pi \to \wp W$ is a map called valuation;

  and $w \in W$;
- $\Vdash_\Pi^{\mathrm{lk,K}} \subseteq \mathcal{M}_\Pi^{\mathrm{lk,K}} \times L_\Pi^{\mathrm{lk,K}}$ is the local satisfaction relation of a formula φ by $((W,R,V),w)$, written

  $$(W,R,V),w \Vdash_\Pi^{\mathrm{lk,K}} \varphi$$

  inductively defined as follows:
  - $(W,R,V),w \Vdash_\Pi^{\mathrm{lk,K}} p$ whenever $w \in V(p)$ for $p \in \Pi$;
  - $(W,R,V),w \Vdash_\Pi^{\mathrm{lk,K}} \neg\varphi$ whenever $(W,R,V),w \not\Vdash_\Pi^{\mathrm{lk,K}} \varphi$;
  - $(W,R,V),w \Vdash_\Pi^{\mathrm{lk,K}} \varphi_1 \supset \varphi_2$ whenever either $(W,R,V),w \not\Vdash_\Pi^{\mathrm{lk,K}} \varphi_1$ or $(W,R,V),w \Vdash_\Pi \varphi_2$;
  - $(W,R,V),w \Vdash_\Pi^{\mathrm{lk,K}} \Diamond\varphi$ whenever $(W,R,V),w' \Vdash_\Pi^{\mathrm{lk,K}} \varphi$ for some $w' \in W$ such that $w\,R\,w'$.

We now present another modal satisfaction system using the previous example.

EXAMPLE 2.3. The global satisfaction system

$$S_\Pi^{\mathrm{gk,K}} = (L_\Pi^{\mathrm{ML}}, \mathcal{M}_\Pi^{\mathrm{gk,K}}, \Vdash_\Pi^{\mathrm{gk,K}})$$

for K modal logic is such that

- $\mathcal{M}_\Pi^{\mathrm{gk,K}}$ is the class of all Kripke structures;
- $\Vdash_\Pi^{\mathrm{gk,K}} \subseteq \mathcal{M}_\Pi^{\mathrm{gk,K}} \times L_\Pi^{\mathrm{ML}}$ is the global satisfaction relation of a formula φ by (W,R,V), written

  $$(W,R,V) \Vdash_\Pi^{\mathrm{gk,K}} \varphi$$

  that holds when $(W,R,V),w \Vdash_\Pi^{\mathrm{lk,K}} \varphi$ for every $w \in W$.

DEFINITION 2.4. A *decision problem on a satisfaction system*

$$S = (L, \mathcal{M}, \Vdash)$$

is a pair (L, Γ) where $\Gamma \subseteq L$.

Informally, such a decision problem can be stated as

$$\text{given } \varphi \in L, \quad \text{is } \varphi \in \Gamma?$$

In order to discuss decision problems induced by the satisfaction system S we need some notions. A formula $\varphi \in L$ is *satisfiable* if there is $M \in \mathcal{M}$ such that $M \Vdash \varphi$. Moreover, φ is *valid* whenever $M \Vdash \varphi$ for every $M \in \mathcal{M}$.

DEFINITION 2.5. The *satisfiability problem* is the pair

$$\mathrm{Sat}_S = (L, \{\varphi \in L : \varphi \text{ is satisfiable}\}).$$

The *co-satisfiability problem* is the pair

$$\text{co-Sat}_S = (L, \{\varphi \in L : \varphi \text{ is not satisfiable}\}).$$

The *validity problem* is the pair

$$\mathrm{Val}_S = (L, \{\varphi \in L : \varphi \text{ is valid}\}).$$

The *co-validity problem* is the pair

$$\text{co-Val}_S = (L, \{\varphi \in L : \varphi \text{ is not valid}\}).$$

A decision problem (L,Γ) is *decidable* whenever the characteristic map

$$\chi_{(L,\Gamma)} : L \to \{0,1\}$$

defined as follows:

$$\chi_{(L,\Gamma)}(\varphi) = \begin{cases} 1 & \text{whenever } \varphi \in \Gamma, \\ 0 & \text{otherwise} \end{cases}$$

is computable (see [31]).

PROPOSITION 2.6. *Let (L,Γ) be a decision problem. Then*

$$(L,\Gamma) \textit{ is decidable} \quad \textit{if and only if} \quad (L, L \setminus \Gamma) \textit{ is decidable}.$$

PROOF. Observe that

$$\chi_{(L,L\setminus\Gamma)}(\varphi) = 1 - \chi_{(L,\Gamma)}(\varphi) \quad \text{and} \quad \chi_{(L,\Gamma)}(\varphi) = 1 - \chi_{(L,L\setminus\Gamma)}(\varphi).$$

So $\chi_{(L,\Gamma)}$ is computable if and only if $\chi_{(L,L\setminus\Gamma)}$ is computable. ⊣

As a consequence,

$$\mathrm{Sat}_S \text{ is decidable} \quad \text{if and only if} \quad \text{co-Sat}_S \text{ is decidable},$$

and

$$\mathrm{Val}_S \text{ is decidable} \quad \text{if and only if} \quad \text{co-Val}_S \text{ is decidable}.$$

DEFINITION 2.7. Let $k \in \mathbb{N}^+$, $D_S = (L,\Gamma)$ be a decision problem on $S = (L, \mathcal{M}, \Vdash)$, and $D_{S^i} = (L^i, \Gamma^i)$ a decision problem on $S^i = (L^i, \mathcal{M}^i, \Vdash^i)$ for each $i = 1,\dots,k$. A collection of computable maps $\tau^i : L \to L^i$ for each $i = 1,\dots,k$ is a *reduction* from D_S to $D_{S^1},\dots,D_{S^k}$, denoted by

$$(\tau^1,\dots,\tau^k) : D_S \to D_{S^1} \times \cdots \times D_{S^k},$$

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whenever

$$\varphi \in \Gamma \quad \text{if and only if} \quad \tau^i(\varphi) \in \Gamma^i \quad \text{for each } i = 1, \dots, k.$$

PROPOSITION 2.8. *Let* $(\tau^1, \dots, \tau^k) : D_S \to D_{S^1} \times \cdots \times D_{S^k}$ *be a reduction. Then,* D_S *is decidable whenever* D_{S^i} *is decidable for each* $i = 1, \dots, k$.

PROOF. Assume that D_{S^i} is decidable for each $i = 1, \dots, k$. Then, $\chi_{D_{S^i}}$ is a computable map for each $i = 1, \dots, k$. Observe that

$$\chi_{D_S}(\varphi) = \prod_{i=1}^{k} \chi_{D_{S^i}} \circ \tau_i(\varphi),$$

since $(\tau^1, \dots, \tau^k)$ is a reduction. Hence, χ_{D_S} is a computable map. Therefore, D_S is decidable. ⊣

It is worthwhile to discuss the relationship between the satisfiability and the validity problems. For that we need to introduce the following notion.

DEFINITION 2.9. A satisfaction system $S' = (L', \mathcal{M}', \Vdash')$ has a (*standard*) *negation* $\neg$ if L' is closed for $\neg$, that is, if $\varphi \in L'$ then $\neg\varphi \in L'$, and

$$M' \Vdash \varphi \quad \text{if and only if} \quad M' \nVdash \neg\varphi.$$

Observe that not all negations have the property above (see [7]).

EXAMPLE 2.10. The local Kripke satisfaction system

$$S_{\Pi}^{\text{lk,K}}$$

defined in Example 2.2 has the negation $\neg$.

PROPOSITION 2.11. *Let* $S = (L, \mathcal{M}, \Vdash)$ *and* $S' = (L, \mathcal{M}', \Vdash')$ *be satisfaction systems such that* S' *has a negation* $\neg$. *Assume that there is a map* $f : \mathcal{M} \to \wp\mathcal{M}'$ *such that* $f(\mathcal{M}) = \mathcal{M}'$ *and*

$$M \Vdash \varphi \quad \textit{if and only if} \quad M' \Vdash' \varphi \textit{ for each } M' \in f(M).$$

Then, there is a reduction from Val_S *to co-*$\mathit{Sat}_{S'}$. *Moreover,*

$$\mathit{Sat}_{S'} \textit{ is decidable} \quad \textit{implies} \quad \mathit{Val}_S \textit{ is decidable}.$$

PROOF. We start by showing that the map $\tau : L \to L$ such that $\psi \mapsto \neg\psi$ is a reduction from Val_S to co-$\text{Sat}_{S'}$.

(1) It is immediate to see that τ is a computable map.
(2) We must show that

$$\varphi \text{ is valid in } S \quad \text{if and only if} \quad \neg\varphi \text{ is not satisfiable in } S'.$$

($\to$) Assume that φ is valid in S. Thus, $M \Vdash \varphi$ for every $M \in \mathcal{M}$. Therefore, by hypothesis, $M' \Vdash' \varphi$ for every $M' \in f(M)$ and $M \in \mathcal{M}$. Hence, $M' \nVdash' \neg\varphi$ for every $M' \in f(M)$ and $M \in \mathcal{M}$. Since $f(\mathcal{M}) = \mathcal{M}'$ then there is no $M' \in \mathcal{M}'$ such that $M' \Vdash' \neg\varphi$. Thus $\neg\varphi$ is not satisfiable in S'.

($\leftarrow$) Assume that $\neg\varphi$ is not satisfiable in S'. Then $M' \not\Vdash' \neg\varphi$ for every $M' \in \mathcal{M}'$. Hence, $M' \Vdash' \varphi$ for each $M' \in \mathcal{M}'$. Let $M \in \mathcal{M}$. Then $M' \Vdash' \varphi$ for every $M' \in f(M)$ and so $M \Vdash \varphi$.

Assume that $\text{Sat}_{S'}$ is decidable. Then, co-$\text{Sat}_{S'}$ is decidable (see Proposition 2.6). Therefore, Val_S is decidable by Proposition 2.8. ⊣

Proposition 2.12. *The problem* $Val_{S_\Pi^{gk,K}}$ *is decidable for every set* Π *of propositional symbols.*

Proof. We start by observing that $S_\Pi^{\text{gk},\text{K}}$ and $S_\Pi^{\text{lk},\text{K}}$ satisfy the requirements of Proposition 2.11 by taking $f(W,R,V) = \{((W,R,V),w) : w \in W\}$. Again, by Proposition 2.11,

$$\text{Sat}_{S_\Pi^{\text{lk},\text{K}}} \text{ is decidable} \quad \text{implies} \quad \text{Val}_{S_\Pi^{\text{gk},\text{K}}} \text{ is decidable.}$$

The result follows since $\text{Sat}_{S_\Pi^{\text{lk},\text{K}}}$ is decidable as we will show in Proposition 3.11. ⊣

§3. Satisfaction system reductions. We now introduce the concept of reduction from one satisfaction system to a non-empty finite collection of satisfaction systems and discuss its impact on reductions between decision problems.

Definition 3.1. A *reduction* from satisfaction system $(L, \mathcal{M}, \Vdash)$ to satisfaction systems $(L^1, \mathcal{M}^1, \Vdash^1), \dots, (L^k, \mathcal{M}^k, \Vdash^k)$ where $k \in \mathbb{N}^+$, is a tuple

$$(\tau^1, \dots, \tau^k, g^1, \dots, g^k, h),$$

where, for each $i = 1, \dots, k$,

- $\tau^i : L \to L^i$ is a computable map;
- $g^i : \mathcal{M} \to \mathcal{M}^i$ is a map such that

$$\text{if } M \Vdash \varphi \text{ then } g^i(M) \Vdash^i \tau^i(\varphi)$$

  for every $M \in \mathcal{M}$;

and $h : \mathcal{M}^1 \times \cdots \times \mathcal{M}^k \to \mathcal{M}$ is a map such that

$$\text{if} \quad M_1 \Vdash^1 \tau^1(\varphi) \ \dots \ M_k \Vdash^k \tau^k(\varphi) \quad \text{then} \quad h(M_1, \dots, M_k) \Vdash \varphi$$

for every $M_i \in \mathcal{M}^i$ with $i = 1, \dots, k$.

In the sequel we denote such a reduction by

$$(\tau^1, \dots, \tau^k, g^1, \dots, g^k, h) : (L, \mathcal{M}, \Vdash) \to (L^1, \mathcal{M}^1, \Vdash^1) \times \cdots \times (L^k, \mathcal{M}^k, \Vdash^k).$$

We compare the notion of reduction for $k = 1$ with the usual definition of satisfaction system morphism (see [2, 29]).

Definition 3.2. A *satisfaction system morphism* from $S = (L, \mathcal{M}, \Vdash)$ to $S' = (L', \mathcal{M}', \Vdash')$ is a pair $(\overline{h}, \underline{h})$ where $\overline{h} : L \to L'$ and $\underline{h} : \mathcal{M}' \to \mathcal{M}$ are maps such that

$$M' \Vdash' \overline{h}(\varphi) \quad \text{if and only if} \quad \underline{h}(M') \Vdash \varphi.$$

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Hence, without further assumptions, these two notions do not coincide. We now show that a satisfaction system reduction induces a reduction over the respective satisfiability problems.

PROPOSITION 3.3. *Let* $(\tau^1, \dots, \tau^k, g^1, \dots, g^k, h) : S \to S^1 \times \cdots \times S^k$ *be a satisfaction system reduction. Then,*

$$(\tau^1, \dots, \tau^k)$$

is a reduction from Sat_S *to* $Sat_{S_1}, \dots, Sat_{S_k}$.

PROOF. We must show that

$$\varphi \text{ is satisfiable in } S \quad \text{iff} \quad \tau^i(\varphi) \text{ is satisfiable in } S^i \text{ for each } i = 1, \dots, k.$$

$(\to)$ Assume that $\tau^i(\varphi)$ is not satisfiable in S^i for some $i = 1, \dots, k$. Then, there is no $M_i \in \mathcal{M}^i$ such that $M_i \Vdash^i \tau^i(\varphi)$. Hence, there is no $M \in \mathcal{M}$ such that $g^i(M) \Vdash^i \tau^i(\varphi)$. Therefore, there is no $M \in \mathcal{M}$ such that $M \Vdash \varphi$. Thus, φ is not satisfiable in S.
$(\leftarrow)$ Assume that $\tau^i(\varphi)$ is satisfiable in S^i for each $i = 1, \dots, k$. Then, there are $M_1 \in \mathcal{M}^1, \dots, M_k \in \mathcal{M}^k$ such that $M_i \Vdash^i \tau^i(\varphi)$ for each $i = 1, \dots, k$. Hence, $h(M_1, \dots, M_k) \Vdash \varphi$. Therefore, φ is satisfiable in S. ⊣

PROPOSITION 3.4. *Let* $(\tau^1, \dots, \tau^k, g^1, \dots, g^k, h) : S \to S^1 \times \cdots \times S^k$ *be a satisfaction system reduction. Then,* Sat_S *is decidable provided that* $Sat_{S^1}, \dots, Sat_{S^k}$ *are decidable.*

PROOF. Assume $\mathrm{Sat}_{S^1}, \dots, \mathrm{Sat}_{S^k}$ are decidable. By Proposition 3.3,

$$(\tau^1, \dots, \tau^k) : \mathrm{Sat}_S \to \mathrm{Sat}_{S_1} \times \cdots \times \mathrm{Sat}_{S_k}$$

is a reduction. Hence, by Proposition 2.8, Sat_S is decidable. ⊣

Similarly, a satisfaction system reduction induces a reduction over the respective validity problems whenever the semantic translation maps are surjective up to satisfaction, as we define now.

DEFINITION 3.5. Let $(\tau^1, \dots, \tau^k, g^1, \dots, g^k, h) : S \to S^1 \times \cdots \times S^k$ be a satisfaction system reduction. The maps $g^1, \dots, g^k, h$ are *surjective up to satisfaction* whenever

- for every $M \in \mathcal{M}$,
$$h(g_1(M), \dots, g_k(M)) \Vdash \varphi \quad \text{if and only if} \quad M \Vdash \varphi;$$
- for every $M_1 \in \mathcal{M}^1, \dots, M_k \in \mathcal{M}^k$,
$$g_i(h(M_1, \dots, M_k)) \Vdash^i \tau^i(\varphi) \quad \text{if and only if} \quad M_i \Vdash^i \tau^i(\varphi)$$
for each $i = 1, \dots, k$.

PROPOSITION 3.6. *Let* $(\tau^1, \dots, \tau^k, g^1, \dots, g^k, h) : S \to S^1 \times \cdots \times S^k$ *be a satisfaction system reduction such that* $g^1, \dots, g^k, h$ *are surjective up to satisfaction. Then,*

$$(\tau^1, \dots, \tau^k)$$

is a reduction from Val_S *to* $Val_{S_1}, \dots, Val_{S_k}$.

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PROOF. We must show that

$$\varphi \text{ is valid in } S \quad \text{if and only if} \quad \tau^i(\varphi) \text{ is valid in } S^i \text{ for each } i = 1,\dots,k.$$

($\rightarrow$) Let $M_i \in \mathcal{M}^i$ for $i = 1,\dots,k$. Then $h(M_1,\dots,M_k) \in \mathcal{M}$ and so

$$h(M_1,\dots,M_k) \Vdash \varphi,$$

because φ is valid in S. Since $(\tau^1,\dots,\tau^k,g^1,\dots,g^k,h)$ is a reduction,

$$g^i(h(M_1,\dots,M_k)) \Vdash^i \tau^i(\varphi)$$

for every $i = 1,\dots,k$. Therefore, by surjectivity up to satisfaction,

$$M_i \Vdash^i \tau^i(\varphi)$$

for every $i = 1,\dots,k$.

($\leftarrow$) Let $M \in \mathcal{M}$. Then $g^i(M) \in \mathcal{M}^i$ for each $i = 1,\dots,k$. Hence

$$g^i(M) \Vdash^i \tau^i(\varphi),$$

because $\tau^i(\varphi)$ is valid for each $i = 1,\dots,k$. Since $(\tau^1,\dots,\tau^k,g^1,\dots,g^k,h)$ is a reduction,

$$h(g^1(M),\dots,g^k(M)) \Vdash \varphi.$$

Moreover, by surjectivity up to satisfaction, $M \Vdash \varphi$. ⊣

PROPOSITION 3.7. *Let $(\tau^1,\dots,\tau^k,g^1,\dots,g^k,h) : S \to S^1 \times \cdots \times S^k$ be a satisfaction system reduction such that $g^1,\dots,g^k,h$ are surjective up to satisfaction. Then, Val_S is decidable provided that $Val_{S^1},\dots,Val_{S^k}$ are decidable.*

PROOF. By Proposition 3.6,

$$(\tau^1,\dots,\tau^k) : \mathrm{Val}_S \to \mathrm{Val}_{S_1} \times \cdots \times \mathrm{Val}_{S_k}$$

is a reduction. So, by Proposition 2.8, Val_S is decidable if $\mathrm{Val}_{S^1},\dots,\mathrm{Val}_{S^k}$ are decidable. ⊣

3.1. ***Satisfiability of $S^{\mathrm{lk,K}}_{\Pi}$ from satisfiability of $S^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}}$.*** We want to show, using the results in the previous section and [4], that the satisfiability problem for K modal logic with local Kripke semantics is decidable by using the fact that the satisfiability problem for a fragment FO^2 of FOL consisting of formulas using only a pair of variables is decidable (see [16]).

EXAMPLE 3.8 (FO^2-two variable first-order logic.). Let Σ be a first-order logic signature with no function symbols and with a set P_n of predicate symbols of arity n for each $n \in \mathbb{N}^+$ and $X = \{x,y\}$. A satisfaction system for FO^2

$$S^{\mathrm{FO}^2}_{\Sigma} = (L^{\mathrm{FO}^2}_{\Sigma}, \mathcal{M}^{\mathrm{FO}^2}_{\Sigma}, \Vdash^{\mathrm{FO}^2}_{\Sigma})$$

over signature Σ and X is such that

- $L^{\mathrm{FO}^2}_{\Sigma}$ is the set of formulas inductively defined as follows:
  - $p(z_1,\dots,z_n) \in L^{\mathrm{FO}^2}_{\Sigma}$ for every $p \in P_n$ and $z_1,\dots,z_n \in X$;

  - $\neg\varphi \in L^{\mathrm{FO}^2}_{\Sigma}$ whenever $\varphi \in L^{\mathrm{FO}^2}_{\Sigma}$;
  - $\varphi_1 \supset \varphi_2 \in L^{\mathrm{FO}^2}_{\Sigma}$ whenever $\varphi_1, \varphi_2 \in L^{\mathrm{FO}^2}_{\Sigma}$;
  - $\exists z\, \varphi \in L^{\mathrm{FO}^2}_{\Sigma}$ whenever $z \in X$ and $\varphi \in L^{\mathrm{FO}^2}_{\Sigma}$;
- $\mathcal{M}^{\mathrm{FO}^2}_{\Sigma}$ is the class of all pairs

$$(I, \rho),$$

where I is a tuple

$$(D, \{\{p^I\}_{p \in P_n}\}_{n \in \mathbb{N}^+}),$$

called an interpretation structure, such that
  - D is a non-empty set;
  - $p^I : D^n \to \{0,1\}$ is a map for each $p \in P_n$ and $n \in \mathbb{N}^+$;

and $\rho : X \to D$ is a map called an assignment;
- $\Vdash^{\mathrm{FO}^2}_{\Sigma} \subseteq \mathcal{M}^{\mathrm{FO}^2}_{\Sigma} \times L^{\mathrm{FO}^2}_{\Sigma}$ is the satisfaction relation inductively defined as follows:
  - $I, \rho \Vdash^{\mathrm{FO}^2}_{\Sigma} p(z_1, \dots, z_n)$ whenever $p^I(\rho(z_1), \dots, \rho(z_n)) = 1$;
  - $I, \rho \Vdash^{\mathrm{FO}^2}_{\Sigma} \neg\varphi$ whenever $I, \rho \not\Vdash^{\mathrm{FO}^2}_{\Sigma} \varphi$;
  - $I, \rho \Vdash^{\mathrm{FO}^2}_{\Sigma} \varphi_1 \supset \varphi_2$ whenever either $I, \rho \not\Vdash^{\mathrm{FO}^2}_{\Sigma} \varphi_1$ or $I, \rho \Vdash^{\mathrm{FO}^2}_{\Sigma} \varphi_2$;
  $I, \rho \Vdash^{\mathrm{FO}^2}_{\Sigma} \exists z\, \varphi$ whenever $I, \rho' \Vdash^{\mathrm{FO}^2}_{\Sigma} \varphi$ for some $\rho' \equiv_z \rho$, i.e., assignment ρ' such that $\rho'(z') = \rho(z')$ for every $z' \in X \setminus \{z\}$.

Let $\Sigma_{\mathrm{ML},\Pi}$ be the FO^2 signature induced by the K modal logic over Π with no function symbols, set of predicate symbols $\{\bar{p} : p \in \Pi\}$ of arity 1, and predicate symbol $\bar{R}$ of arity two. In order to discuss the reduction from K modal logic to FO^2 logic, we start by introducing an auxiliary map for each $z \in X$. Let

$$\tau_z : L^{\mathrm{ML}}_{\Pi} \to L^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}}$$

be inductively defined as follows:

- $\tau_z(p) = \bar{p}(z)$;
- $\tau_z(\neg\varphi) = \neg\tau_z(\varphi)$;
- $\tau_z(\varphi_1 \supset \varphi_2) = \tau_z(\varphi_1) \supset \tau_z(\varphi_2)$;
- $\tau_z(\Diamond\varphi) = \exists z'(\bar{R}(z, z') \wedge \tau_{z'}(\varphi))$, where $z' \in X \setminus \{z\}$.

Furthermore, for each $z \in X$, consider the map

$$g_z : \mathcal{M}^{\mathrm{lk,K}}_{\Pi} \to \mathcal{M}^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}}$$

defined as follows:

$$g_z((W, R, V), w) = (I, \rho_z),$$

where $I = (W, \{\bar{p}^I\}_{p \in \Pi}, \bar{R}^I)$ such that, for every $w_1, w_2 \in W$,

- $\bar{p}^I(w_1) = 1$ if and only if $w_1 \in V(p)$;
- $\bar{R}^I(w_1, w_2) = 1$ if and only if $w_1 \, R \, w_2$;

and ρ_z is an assignment such that $\rho_z(z) = w$.

We now show that local satisfaction carries over from K modal logic to FO^2 logic.

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PROPOSITION 3.9. *Let* $((W,R,V),w) \in \mathcal{M}^{lk,K}_{\Pi}$ *and* $\varphi \in L^{ML}_{\Pi}$. *Then, for each* $z \in X$,

$$(W,R,V),w \Vdash^{lk,K}_{\Pi} \varphi \quad \textit{if and only if} \quad I,\rho_z \Vdash^{FO^2}_{\Sigma_{ML,\Pi}} \tau_z(\varphi),$$

where $g_z((W,R,V),w) = (I,\rho_z)$.

PROOF. We prove the result by induction on φ.

(Base) Let φ be $p \in \Pi$. Hence, $(W,R,V),w \Vdash^{\mathrm{lk,K}}_{\Pi} p$ if and only if $w \in V(p)$ if and only if $\bar{p}^I(w) = 1$ if and only if $I,\rho_z \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \bar{p}(z)$ if and only if $I,\rho_z \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \tau_z(p)$.

(Step) We only consider the case that φ is $\Diamond\psi$. We start by showing that

$$(W,R,V),w \Vdash^{\mathrm{lk,K}}_{\Pi} \Diamond\psi \quad \text{implies} \quad I,\rho_z \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \tau_z(\Diamond\psi).$$

Assume that $(W,R,V),w \Vdash^{\mathrm{lk,K}}_{\Pi} \Diamond\psi$. Then, there is $w' \in W$ with $w\,R\,w'$ and $(W,R,V),w' \Vdash^{\mathrm{lk,K}}_{\Pi} \psi$. We must show that

$$I,\rho_z \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \exists z'(\bar{R}(z,z') \wedge \tau_{z'}(\psi)),$$

where $z' \in X \setminus \{z\}$. Let $\rho_{z'} \equiv_{z'} \rho_z$ be such that $\rho_{z'}(z') = w'$. Then,

$$I,\rho_{z'} \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \bar{R}(z,z'),$$

since $\bar{R}^I(w,w') = 1$. Moreover,

$$I,\rho_{z'} \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \tau_{z'}(\psi)$$

by the induction hypothesis, since $(W,R,V),w' \Vdash^{\mathrm{lk,K}}_{\Pi} \psi$. We now prove that

$$I,\rho_z \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \tau_z(\Diamond\psi) \quad \text{implies} \quad (W,R,V),w \Vdash^{\mathrm{lk,K}}_{\Pi} \Diamond\psi.$$

Assume that $I,\rho_z \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \tau_z(\Diamond\psi)$. Hence

$$I,\rho_z \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \exists z'(\bar{R}(z,z') \wedge \tau_{z'}(\psi)).$$

Thus, there is $\rho_{z'} \equiv_{z'} \rho_z$ such that

$$(\dagger) \quad I,\rho_{z'} \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \bar{R}(z,z')$$

and

$$(\ddagger) \quad I,\rho_{z'} \Vdash^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \tau_{z'}(\psi).$$

From $(\dagger)$, we conclude that $\rho_{z'}(z)\,R\,\rho_{z'}(z')$ holds in (W,R,V). So $w\,R\,\rho_{z'}(z')$ because $\rho_{z'}(z) = \rho_z(z) = w$. On the other hand, from $(\ddagger)$ we can conclude, by the induction hypothesis, that $(W,R,V),\rho_{z'}(z') \Vdash^{\mathrm{lk,K}}_{\Pi} \psi$. Therefore, $(W,R,V),w \Vdash^{\mathrm{lk,K}}_{\Pi} \Diamond\psi$. ⊣

Finally, for each $z \in X$, consider the map

$$h_z : \mathcal{M}^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}} \to \mathcal{M}^{\mathrm{lk,K}}_{\Pi}$$

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defined as follows:

$$h_z(I,\rho) = ((W,R,V),\rho(z)),$$

where

- W is the domain of I;
- $R = \{(w_1,w_2) \in W^2 : \bar{R}^I(w_1,w_2) = 1\}$;
- $V(p) = \{w \in W : \bar{p}^I(w) = 1\}$.

The following result shows that h_z preserves and reflects local satisfaction. We omit its proof since it follows the same steps as the proof of Proposition 3.9.

PROPOSITION 3.10. *Let $(I,\rho) \in \mathcal{M}^{FO^2}_{\Sigma_{ML,\Pi}}$ and $\varphi \in L^{ML}_{\Pi}$. Then, for each $z \in X$,*

$$(W,R,V),\rho(z) \Vdash^{lk,K}_{\Pi} \varphi \quad \textit{if and only if} \quad I,\rho \Vdash^{FO^2}_{\Sigma_{ML,\Pi}} \tau_z(\varphi),$$

where $h_z(I,\rho) = ((W,R,V),\rho(z))$.

PROPOSITION 3.11. *The satisfiability problem $Sat_{S^{lk,K}_{\Pi}}$ for K modal logic is decidable for every set Π of propositional symbols.*

PROOF. We start by observing that (τ_x, g_x, h_x) is a reduction from $S^{\mathrm{lk,K}}_{\Pi}$ to $S^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}}$ by Propositions 3.9 and 3.10 and since τ_x is computable. The satisfiability problem $\mathrm{Sat}_{S^{\mathrm{lk,K}}_{\Pi}}$ is decidable by Proposition 3.4 since the satisfiability problem $\mathrm{Sat}_{S^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}}}$ is decidable because $\mathrm{Sat}_{S^{\mathrm{FO}^2}_{\Sigma}}$ is decidable for every signature Σ of FO^2 (see [16, 23, 24]). ⊣

3.2. *Validity of $S^{\mathrm{ga,K}}_{\Pi}$ from the validity of $S^{\mathrm{gk,K}}_{\Pi}$*. Herein, we prove that the validity problem of K modal logic endowed with an (global) algebraic semantics (see [14, 27]) is decidable taking into account the decidability of the validity problem of K modal logic endowed with a global Kripke semantics (see Proposition 2.12).

EXAMPLE 3.12. The (global) algebraic satisfaction system

$$S^{\mathrm{ga,K}}_{\Pi} = (L^{\mathrm{ML}}_{\Pi}, \mathcal{M}^{\mathrm{ga,K}}_{\Pi}, \Vdash^{\mathrm{ga,K}}_{\Pi})$$

for K modal logic over a set Π of propositional symbols is such that

- L^{ML}_{Π} is as defined in Example 2.2;
- $\mathcal{M}^{\mathrm{ga,K}}_{\Pi}$ is the class of all modal algebras with distinguished value, that is, pairs $(\mathfrak{A}, D)$ such that
  - $\mathfrak{A} = (A, \sqcap, \sqcup, \sqsupset, -, \top, \underline{\Box}, V)$ where $(A, \sqcap, \sqcup, \sqsupset, -, \top)$ is a Boolean algebra, $\underline{\Box} : A \to A$ satisfies the following identities:

$$\underline{\Box}(a_1 \sqcap a_2) = (\underline{\Box} a_1 \sqcap \underline{\Box} a_2) \quad \text{and} \quad \underline{\Box}\top = \top,$$

    and $V : \Pi \to A$ is a map;
  - $D = \{\top\}$;

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- $\Vdash_{\Pi}^{\text{ga,K}} \subseteq \mathcal{M}_{\Pi}^{\text{ga,K}} \times L_{\Pi}^{\text{ga,K}}$ is the satisfaction relation such that

$$(\mathfrak{A}, D) \Vdash_{\Pi}^{\text{ga,K}} \varphi$$

whenever $[\![\varphi]\!]^{\mathfrak{A}} = \top$ where

$$[\![\varphi]\!]^{\mathfrak{A}} \in A$$

is inductively defined as follows:
  - $[\![p]\!]^{\mathfrak{A}} = V(p)$ for $p \in \Pi$;
  - $[\![\neg \psi]\!]^{\mathfrak{A}} = -[\![\psi]\!]^{\mathfrak{A}}$;
  - $[\![\psi_1 \supset \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \sqsupset [\![\psi_2]\!]^{\mathfrak{A}}$;
  - $[\![\Box \psi]\!]^{\mathfrak{A}} = \underline{\Box}[\![\psi]\!]^{\mathfrak{A}}$.

We need to consider a restricted class of Kripke structures for K modal logic in order to provide a reduction from the algebraic semantics to the Kripke semantics.

Example 3.13. The descriptive global Kripke satisfaction system for K modal logic is the tuple

$$S_{\Pi}^{\text{dgk,K}} = (L_{\Pi}^{\text{ML}}, \mathcal{M}_{\Pi}^{\text{dgk,K}}, \Vdash_{\Pi}^{\text{dgk,K}})$$

over a set Π of propositional symbols obtained from $S_{\Pi}^{\text{gk,K}}$ (see Example 2.3) by taking $\mathcal{M}_{\Pi}^{\text{dgk,K}}$ as the subclass of $\mathcal{M}_{\Pi}^{\text{gk,K}}$ composed by Kripke structures that are descriptive (that is, differentiated, tight, and compact; (see [9, 14])) and $\Vdash_{\Pi}^{\text{dgk,K}}$ as the restriction of $\Vdash_{\Pi}^{\text{gk,K}}$ to $\mathcal{M}_{\Pi}^{\text{dgk,K}}$. Similarly for $S_{\Pi}^{\text{dlk,K}}$.

We now show that there is a map from modal algebras to descriptive Kripke structures that preserves and reflects satisfaction. Before we recall some notions.

Definition 3.14. A *filter in a modal algebra* $\mathfrak{A}$ is a set $F \subseteq A$ such that:

- $\top \in F$;
- if $a, b \in F$ then $a \sqcap b \in F$;
- if $a \in F$ and $a \leq b$ then $b \in F$ where $a \leq b$ whenever $a \sqcap b = a$.

A filter F is a *ultrafilter* whenever:

- $\bot \notin F$;
- for every $a \in A$ either $a \in F$ or $-a \in F$.

Proposition 3.15. *Let* $g : \mathcal{M}_{\Pi}^{\text{ga,K}} \to \mathcal{M}_{\Pi}^{\text{dgk,K}}$ *be such that*

$$g(\mathfrak{A}, \{\top\}) = (W, R, \underline{V}),$$

where

- W *is* $\{U \subseteq A : U$ *is an ultrafilter of* $\mathfrak{A}\}$;
- $U\,R\,U'$ *whenever for every* $a \in A$ *if* $\underline{\Box}a \in U$ *then* $a \in U'$;
- $\underline{V}(p) = \{U \in W : V(p) \in U\}$.

Then, for every $U \in W$,

$$[\![\varphi]\!]^{\mathfrak{A}} \in U \quad \textit{if and only if} \quad (W,R,\underline{V}),U \Vdash^{\mathrm{dlk,K}}_{\Pi} \varphi.$$

Furthermore,

$$(\mathfrak{A},\{\top\}) \Vdash^{\mathrm{ga,K}}_{\Pi} \varphi \quad \textit{if and only if} \quad (W,R,\underline{V}) \Vdash^{\mathrm{dgk,K}}_{\Pi} \varphi.$$

Proof. We start by observing that $(W,R,\underline{V})$ is a descriptive global Kripke structure (see [14, Theorem 1.10.5]). It is immediate to see that the second assertion about satisfaction of formulas follows from the first (taking into account Theorem 5.38 in [5]). The proof of the first statement follows by induction on φ and we only consider the step when φ is $\Box\psi$.

Before, we show that:

(†) $\underline{\Box}a \in U$ provided that $a \in U'$ for every $U' \in W$ such that $U\,R\,U'$

for every $U \in W$ and $a \in A$. Assume, by contradiction, that $a \in U'$ for every $U' \in W$ such that $U\,R\,U'$ and $\underline{\Box}a \notin U$. Consider

$$F = \{b \in A : \underline{\Box}b \in U\}.$$

Observe that $a \notin F$. Moreover, F is a filter. Indeed,

(1) $\top \in F$. Since U is an ultrafilter $\top \in U$. Since $\underline{\Box}\top = \top$ then $\underline{\Box}\top \in U$ and so $\top \in F$.
(2) Assume that $b,b' \in F$. Hence $\underline{\Box}b, \underline{\Box}b' \in U$. Therefore, $\underline{\Box}b \sqcap \underline{\Box}b' \in U$ and so $\underline{\Box}(b \sqcap b') \in U$ because $\underline{\Box}(b \sqcap b') = (\underline{\Box}b) \sqcap (\underline{\Box}b')$. Thus, $b \sqcap b' \in F$.
(3) Suppose that $b \in F$ and $b \leq b'$. Thus, $b \sqcap b' = b$ and so $\underline{\Box}(b \sqcap b') = \underline{\Box}b$. Then, $(\underline{\Box}b) \sqcap (\underline{\Box}b') = \underline{\Box}b$. Hence, $(\underline{\Box}b) \leq (\underline{\Box}b')$. Therefore, $\underline{\Box}b' \in U$ since $\underline{\Box}b \in U$ and U is a filter. Hence, $b' \in F$.

Then, (see Proposition 5.38 of [5]) there is $U'' \in W$ extending F such that $a \notin U''$. Moreover, $U\,R\,U''$ by definition of R. The existence of such U'' contradicts the initial assumption.

We are ready to prove the step when φ is $\Box\psi$.
$(\rightarrow)$ Assume that $[\![\Box\psi]\!]^{\mathfrak{A}} \in U$. Then, $\underline{\Box}[\![\psi]\!]^{\mathfrak{A}} \in U$. Let $U' \in W$ be such that $U\,R\,U'$. Thus, by definition of R, $[\![\psi]\!]^{\mathfrak{A}} \in U'$. Hence, by the induction hypothesis $(W,R,\underline{V}),U' \Vdash^{\mathrm{dlk,K}}_{\Pi} \psi$. Therefore, $(W,R,\underline{V}),U \Vdash^{\mathrm{dlk,K}}_{\Pi} \varphi$.
$(\leftarrow)$ Assume that $(W,R,\underline{V}),U \Vdash^{\mathrm{dlk,K}}_{\Pi} \Box\psi$. Then, $(W,R,\underline{V}),U' \Vdash^{\mathrm{dlk,K}}_{\Pi} \psi$ for every $U' \in W$ such that $U\,R\,U'$. Thus, by the induction hypothesis, $[\![\psi]\!]^{\mathfrak{A}} \in U'$ for every $U' \in W$ such that $U\,R\,U'$. Therefore, by (†), we can conclude that $\underline{\Box}[\![\psi]\!]^{\mathfrak{A}} \in U$. ⊣

We now define a map from descriptive Kripke structures to modal algebras that preserves and reflects satisfaction.

Proposition 3.16. *Let* $h : \mathcal{M}^{\mathrm{dgk,K}}_{\Pi} \to \mathcal{M}^{\mathrm{ga,K}}_{\Pi}$ *be such that*

$$h(W,R,V) = ((\wp W, \cap, \cup, \sqsupset, -, W, \underline{\Box}, V), \{W\}),$$

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where

- $-Z = W \setminus Z$;
- $Z_1 \sqsupset Z_2 = -Z_1 \cup Z_2$;
- $\underline{\Box} Z = \{w \in W : w' \in Z \textit{ whenever } w\, R\, w'\}$;

for $Z, Z_1, Z_2 \subseteq W$. *Then, for every* $w \in W$,

$$(W, R, V), w \Vdash^{\mathrm{dlk,K}}_{\Pi} \varphi \quad \textit{if and only if} \quad w \in [\![\varphi]\!]^{\mathfrak{A}}.$$

Furthermore,

$$(W, R, V) \Vdash^{\mathrm{dgk,K}}_{\Pi} \varphi \quad \textit{iff} \quad ((\wp W, \cap, \cup, \sqsupset, -, W, \underline{\Box}, V), \{W\}) \Vdash^{\mathrm{ga,K}}_{\Pi} \varphi.$$

Proof. It is immediate to see that the second assertion follows from the first, so we only prove the first assertion by induction on φ. Consider the case of $\Box$.
(Step) φ is $\Box\psi$.
$(\rightarrow)$ Assume that $(W, R, V), w \Vdash^{\mathrm{dlk,K}}_{\Pi} \Box\psi$. Then, for every $w' \in W$ such that $w\, R\, w'$,

$$(W, R, V), w' \Vdash^{\mathrm{dlk,K}}_{\Pi} \psi.$$

So by the induction hypothesis,

$$w' \in [\![\psi]\!]^{\mathfrak{A}}$$

for every $w' \in W$ such that $w\, R\, w'$. Therefore, $w \in \underline{\Box}[\![\psi]\!]^{\mathfrak{A}} = [\![\varphi]\!]^{\mathfrak{A}}$.

$(\leftarrow)$ Assume that $w \in [\![\Box\psi]\!]^{\mathfrak{A}}$. Then $w' \in [\![\psi]\!]^{\mathfrak{A}}$ for every $w' \in W$ such that $w\, R\, w'$. Hence, by the induction hypothesis, $(W, R, V), w' \Vdash^{\mathrm{dlk,K}}_{\Pi} \psi$ for every $w' \in W$ such that $w\, R\, w'$. Thus, $(W, R, V), w \Vdash^{\mathrm{dlk,K}}_{\Pi} \varphi$. ⊣

In summary, we have the following reductions between decision problems:

$$\mathrm{Val}_{S^{\mathrm{gk,K}}_{\Pi}} \to_{\text{Prop 2.12}} \mathrm{Sat}_{S^{\mathrm{lk,K}}_{\Pi}} \qquad \mathrm{Sat}_{S^{\mathrm{lk,K}}_{\Pi}} \to_{\text{Prop 3.11}} \mathrm{Sat}_{S^{\mathrm{FO}^2}_{\Sigma_{\mathrm{ML},\Pi}}}$$

and

$$\mathrm{Val}_{S^{\mathrm{ga,K}}_{\Pi}} \to_{\text{Prop 3.17}} \mathrm{Val}_{S^{\mathrm{dgk,K}}_{\Pi}} \qquad \mathrm{Val}_{S^{\mathrm{dgk,K}}_{\Pi}} =_{\text{Cor 1.10.6 of [14]}} \mathrm{Val}_{S^{\mathrm{gk,K}}_{\Pi}}.$$

We are ready to prove that the validity problem of K modal logic with an algebraic semantics is decidable.

Proposition 3.17. *The validity problem* $\mathrm{Val}_{S^{\mathrm{ga,K}}_{\Pi}}$ *is decidable for every set* Π *of propositional symbols.*

Proof. Observe that

$$(\mathsf{id}_{L_{\mathrm{ML}}}, g, h) : S^{\mathrm{ga,K}}_{\Pi} \to S^{\mathrm{dgk,K}}_{\Pi},$$

where g and h are defined in Propositions 3.15 and 3.16, respectively, is a satisfaction system reduction since $\mathsf{id}_{L_{\mathrm{ML}}}$ is computable. Observe that g and h are surjective up to satisfaction because for every modal algebra $\mathfrak{A}$,

$$(\mathfrak{A}, \{\top\}) \Vdash^{\mathrm{ga,K}}_{\Pi} \varphi \quad \text{if and only if} \quad h(g(\mathfrak{A}, \{\top\})) \Vdash^{\mathrm{dgk,K}}_{\Pi} \varphi$$

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(see [14, p. 17]) and, for every descriptive Kripke structure (W,R,V),

$$(W,R,V) \Vdash_{\Pi}^{\mathrm{dgk,K}} \varphi \quad \text{if and only if} \quad g(h(W,R,V) \Vdash_{\Pi}^{\mathrm{ga,K}} \varphi$$

(see [14, Theorem 1.10.7]). Hence, by Proposition 3.7 $\mathrm{Val}_{S_{\Pi}^{\mathrm{ga,K}}}$ is decidable if $\mathrm{Val}_{S_{\Pi}^{\mathrm{dgk,K}}}$ is decidable. Note also that

$$\varphi \text{ is valid in } S_{\Pi}^{\mathrm{dgk,K}} \quad \text{if and only if} \quad \varphi \text{ is valid in } S_{\Pi}^{\mathrm{gk,K}}$$

(see [14, Corollary 1.10.6]). Thus, $\mathrm{Val}_{S_{\Pi}^{\mathrm{ga,K}}}$ is decidable if $\mathrm{Val}_{S_{\Pi}^{\mathrm{gk,K}}}$ is decidable. The thesis follows since $\mathrm{Val}_{S_{\Pi}^{\mathrm{gk,K}}}$ is decidable, by Proposition 2.12. ⊣

3.3. *Validity in $S_{\Pi}^{\mathrm{ga,Int}}$ from Hintikka systems.* The objective of this subsection is to show that the validity problem for intuitionistic logic with an algebraic semantics is decidable. We start by showing that the dual of the validity problem for intuitionistic logic with a global Kripke semantics is decidable using Hintikka systems. After that, we show that there is a reduction from the satisfaction system for intuitionistic logic with a Heyting finite algebra semantics to the satisfaction system with finite Kripke structures.

Example 3.18 (Intuitionistic logic). Let

$$S_{\Pi}^{\mathrm{gk,Int}} = (L_{\Pi}^{\mathrm{Int}}, \mathcal{M}_{\Pi}^{\mathrm{gk,Int}}, \Vdash_{\Pi}^{\mathrm{gk,Int}})$$

be a satisfaction system for intuitionistic logic where:

- the set of formulas L_{Π}^{Int} is inductively defined as follows:
  - $\Pi \cup \{\mathsf{ff}\} \subseteq L_{\Pi}^{\mathrm{Int}}$;
  - $\varphi_1 \Rightarrow \varphi_2, \varphi_1 \wedge \varphi_2, \varphi_1 \vee \varphi_2 \in L_{\Pi}^{\mathrm{Int}}$ provided that $\varphi_1, \varphi_2 \in L_{\Pi}^{\mathrm{Int}}$.

  Moreover, $\sim \varphi \in L_{\Pi}^{\mathrm{Int}}$ is defined as an abbreviation of $\varphi \Rightarrow \mathsf{ff}$ provided that $\varphi \in L_{\Pi}^{\mathrm{Int}}$;
- $\mathcal{M}_{\Pi}^{\mathrm{gk,Int}}$ is the class of all Kripke structures (W,R,V) such that
  - R is reflexive, transitive, and anti-symmetric;
  - if $w \in V(p)$ and $w\,R\,w'$ then $w' \in V(p)$;
- $\Vdash_{\Pi}^{\mathrm{gk,Int}}$ is such that

  $$(W,R,V) \Vdash_{\Pi}^{\mathrm{gk,Int}} \varphi$$

  whenever

  $$(W,R,V), w \Vdash_{\Pi}^{\mathrm{lk,Int}} \varphi \quad \text{for each } w \in W,$$

  where $\Vdash_{\Pi}^{\mathrm{lk,Int}}$ is inductively defined as follows:
  - $(W,R,V), w \Vdash_{\Pi}^{\mathrm{lk,Int}} p$ whenever $w \in V(p)$;
  - $(W,R,V), w \not\Vdash_{\Pi}^{\mathrm{lk,Int}} \mathsf{ff}$;
  - $(W,R,V), w \Vdash_{\Pi}^{\mathrm{lk,Int}} \varphi_1 \Rightarrow \varphi_2$ whenever if $(W,R,V), w' \Vdash_{\Pi}^{\mathrm{lk,Int}} \varphi_1$ then $(W,R,V), w' \Vdash_{\Pi}^{\mathrm{lk,Int}} \varphi_2$ for every w' such that $w\,R\,w'$;

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– $(W,R,V),w \Vdash^{\mathrm{lk,Int}}_{\Pi} \varphi_1 \wedge \varphi_2$ whenever $(W,R,V),w \Vdash^{\mathrm{lk,Int}}_{\Pi} \varphi_i$ for $i = 1,2$;
– $(W,R,V),w \Vdash^{\mathrm{lk,Int}}_{\Pi} \varphi_1 \vee \varphi_2$ whenever either $(W,R,V),w \Vdash^{\mathrm{lk,Int}}_{\Pi} \varphi_1$ or $(W,R,V),w \Vdash^{\mathrm{lk,Int}}_{\Pi} \varphi_2$.

We want to investigate the validity problem in $S^{\mathrm{gk,Int}}_{\Pi}$. For that we need to introduce the concepts of tableau and Hintikka system (see [9]).

DEFINITION 3.19. A *tableau* is a pair (Γ,Δ) where $\Gamma,\Delta \subseteq L^{\mathrm{Int}}_{\Pi}$. A tableau (Γ,Δ) is *saturated* if it fulfils the following closure conditions:

- if $\gamma_1 \wedge \gamma_2 \in \Gamma$ then $\gamma_1 \in \Gamma$ and $\gamma_2 \in \Gamma$;
- if $\delta_1 \wedge \delta_2 \in \Delta$ then either $\delta_1 \in \Delta$ or $\delta_2 \in \Delta$;
- if $\gamma_1 \vee \gamma_2 \in \Gamma$ then either $\gamma_1 \in \Gamma$ or $\gamma_2 \in \Gamma$;
- if $\delta_1 \vee \delta_2 \in \Delta$ then $\delta_1 \in \Delta$ and $\delta_2 \in \Delta$;
- if $\delta \Rightarrow \gamma \in \Gamma$ then either $\delta \in \Delta$ or $\gamma \in \Gamma$.

A saturated tableau (Γ,Δ) is *disjoint* if $\Gamma \cap \Delta = \emptyset$ and $\mathsf{ff} \notin \Gamma$. A *Hintikka system* is a pair $(\mathcal{T}, \preceq)$ where $\mathcal{T}$ is a non-empty set of disjoint saturated tableaux and $\preceq$ is a partial order on $\mathcal{T}$ such that:

- if $(\Gamma,\Delta),(\Gamma',\Delta') \in \mathcal{T}$ and $(\Gamma,\Delta) \preceq (\Gamma',\Delta')$ then $\Gamma \subseteq \Gamma'$ (*hereditarity*);
- if $(\Gamma,\Delta) \in \mathcal{T}$ and $\gamma \Rightarrow \delta \in \Delta$ then there is $(\Gamma',\Delta') \in \mathcal{T}$ such that $(\Gamma,\Delta) \preceq (\Gamma',\Delta')$, $\gamma \in \Gamma'$, and $\delta \in \Delta'$.

A pair $(\mathcal{T}, \preceq)$ is a *Hintikka system for* (Γ,Δ) if there is $(\Gamma',\Delta') \in \mathcal{T}$ such that $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$. Moreover, $(\mathcal{T}, \preceq)$ is a Hintikka system for φ whenever $(\mathcal{T}, \preceq)$ is a Hintikka system for $(\emptyset,\{\varphi\})$.

For instance, $(\emptyset,\{\varphi\})$ is a tableau provided that $\varphi \in L^{\mathrm{Int}}_{\Pi}$.

In the sequel, we denote by $\mathrm{sub}(\varphi)$ the set of subformulas of φ. For the next result we need to consider the following decision problem:

$$\mathrm{HS}_{S^{\mathrm{gk,Int}}_{\Pi}} = (L^{\mathrm{Int}}_{\Pi}, \{\varphi : \text{exists a Hintikka system } \mathcal{T} \text{ for } \varphi, |\mathcal{T}| \leq 2^{|\mathrm{sub}(\varphi)|}\}).$$

PROPOSITION 3.20. *The decision problem* co–$\mathrm{Val}_{S^{\mathrm{gk,Int}}_{\Pi}}$ *is decidable, for every set* Π *of propositional symbols.*

PROOF. We show that $\mathsf{id}_{L^{\mathrm{Int}}_{\Pi}}$ is a reduction from co–$\mathrm{Val}_{S^{\mathrm{gk,Int}}_{\Pi}}$ to $\mathrm{HS}_{S^{\mathrm{gk,Int}}_{\Pi}}$. It is immediate that $\mathsf{id}_{L^{\mathrm{Int}}_{\Pi}}$ is computable. It remains to show that

$$\varphi \text{ is not valid in } S^{\mathrm{gk,Int}}_{\Pi} \quad \text{if and only if} \quad \begin{array}{l}\text{there is a Hintikka system } (\mathcal{T}, \preceq)\\ \text{for } \varphi \text{ and } |\mathcal{T}| \leq 2^{|\mathrm{sub}(\varphi)|}.\end{array}$$

$(\rightarrow)$ Let $(W,R,V) \in \mathcal{M}^{\mathrm{gk,Int}}_{\Sigma}$ and $w \in W$ be such that $(W,R,V),w \nVdash^{\mathrm{lk,Int}}_{\Pi} \varphi$. Consider the family

$$\mathcal{T} = \{(\Gamma_u,\Delta_u)\}_{u \in W}$$

of tableaux such that

$$\begin{cases}\Gamma_u = \{\psi \in \mathrm{sub}(\varphi) : (W,R,V),u \Vdash^{\mathrm{lk,Int}}_{\Pi} \psi\},\\ \Delta_u = \{\eta \in \mathrm{sub}(\varphi) : (W,R,V),u \nVdash^{\mathrm{lk,Int}}_{\Pi} \eta\}.\end{cases}$$

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Define the relation $\preceq$ over $\mathcal{T}$ such that $(\Gamma_u,\Delta_u) \preceq (\Gamma_{u'},\Delta_{u'})$ whenever $\Gamma_u \subseteq \Gamma_{u'}$. We now prove that $(\mathcal{T}, \preceq)$ is a Hintikka system for φ with $|\mathcal{T}| \leq 2^{|\mathrm{sub}(\varphi)|}$.

(1) Assume that $(\Gamma_u,\Delta_u),(\Gamma_{u'},\Delta_{u'}) \in \mathcal{T}$ and $(\Gamma_u,\Delta_u) \preceq (\Gamma_{u'},\Delta_{u'})$. Then, by definition of $\preceq$, $\Gamma_u \subseteq \Gamma_{u'}$.
(2) Suppose that $(\Gamma_u,\Delta_u) \in \mathcal{T}$ and $\gamma \Rightarrow \delta \in \Delta_u$. So, $(W,R,V),u \not\Vdash^{\mathrm{lk,Int}}_{\Pi} \gamma \Rightarrow \delta$. Therefore, there is $u' \in W$ such that $u\,R\,u'$, $\gamma \in \Gamma_{u'}$ and $\delta \in \Delta_{u'}$. Moreover, by hereditarity (see [27]), $\Gamma_u \subseteq \Gamma_{u'}$ and so $(\Gamma_u,\Delta_u) \preceq (\Gamma_{u'},\Delta_{u'})$.
(3) $(\mathcal{T}, \preceq)$ is a Hintikka system for φ since $\emptyset \subseteq \Gamma_w$ and $\varphi \in \Delta_w$.
(4) By construction, each Γ_u and Δ_u are subsets of $\mathrm{sub}(\varphi)$. Moreover, $\Delta_u = \mathrm{sub}(\varphi) \setminus \Gamma_u$. Therefore, $|\mathcal{T}| \leq 2^{|\mathrm{sub}(\varphi)|}$.

$(\leftarrow)$ Let $(\mathcal{T}, \preceq)$ be a Hintikka system for $(\emptyset,\{\varphi\})$ with $|\mathcal{T}| \leq 2^{|\mathrm{sub}(\varphi)|}$. Consider the Kripke structure $(\mathcal{T},\preceq,V)$ where V is such that

$$V(p) = \{(\Gamma,\Delta) : (\Gamma,\Delta) \in \mathcal{T} \text{ and } p \in \Gamma\}.$$

Observe that if $(\Gamma,\Delta) \in V(p)$ and $(\Gamma,\Delta) \preceq (\Gamma',\Delta')$ then, by the first property of Hintikka system, $(\Gamma',\Delta') \in V(p)$. We now show, for every $(\Gamma,\Delta) \in \mathcal{T}$, that

$$\begin{cases} \text{if } \eta \in \Gamma \text{ then } (\mathcal{T},\preceq,V),(\Gamma,\Delta) \Vdash^{\mathrm{lk,Int}}_{\Pi} \eta, \\ \text{if } \eta \in \Delta \text{ then } (\mathcal{T},\preceq,V),(\Gamma,\Delta) \not\Vdash^{\mathrm{lk,Int}}_{\Pi} \eta, \end{cases}$$

by induction on η. The base and the cases where η is either a conjunction or a disjunction are immediate. We concentrate on η being $\eta_1 \Rightarrow \eta_2$.

(1) Assume, by contradiction, $\eta_1 \Rightarrow \eta_2 \in \Gamma$ and $(\mathcal{T},\preceq,V),(\Gamma,\Delta) \not\Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_1 \Rightarrow \eta_2$. Therefore, there exists $(\Gamma',\Delta') \in \mathcal{T}$ such that $(\Gamma,\Delta) \preceq (\Gamma',\Delta')$, $(\mathcal{T},\preceq,V),(\Gamma',\Delta') \Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_1$, and $(\mathcal{T},\preceq,V),(\Gamma',\Delta') \not\Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_2$. Thus, by the first property of the Hintikka system, $\eta_1 \Rightarrow \eta_2 \in \Gamma'$. Then, either $\eta_1 \in \Delta'$ or $\eta_2 \in \Gamma'$ because (Γ',Δ') is saturated. Hence, by the induction hypothesis, either $(\mathcal{T},\preceq,V),(\Gamma',\Delta') \Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_2$ or $(\mathcal{T},\preceq,V),(\Gamma',\Delta') \not\Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_1$ which is a contradiction.
(2) Assume that $\eta_1 \Rightarrow \eta_2 \in \Delta$. Then, by the second property of the Hintikka system, there is $(\Gamma',\Delta') \in \mathcal{T}$ such that $(\Gamma,\Delta) \preceq (\Gamma',\Delta')$, $\eta_1 \in \Gamma'$, and $\eta_2 \in \Delta'$. Therefore, by the induction hypothesis, $(\mathcal{T},\preceq,V),(\Gamma',\Delta') \Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_1$ and $(\mathcal{T},\preceq,V),(\Gamma',\Delta') \not\Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_2$. So $(\mathcal{T},\preceq,V),(\Gamma,\Delta) \not\Vdash^{\mathrm{lk,Int}}_{\Pi} \eta_1 \Rightarrow \eta_2$.

Hence, we can conclude that co–$\mathrm{Val}^{\mathrm{Int}}_{\Pi}$ is decidable, since $\mathrm{HS}^{\mathrm{Int}}_{\Pi}$ is decidable for every set Π (see [9, p. 39]) and there is a reduction from the former to the latter (see Proposition 2.8). ⊣

The following result is a direct consequence of Propositions 2.6 and 3.20.

Proposition 3.21. *The decision problem* $\mathrm{Val}_{S^{\mathrm{gk,Int}}_{\Pi}}$ *is decidable, for every set* Π *of propositional symbols.*

We now describe an algebraic satisfaction system for intuitionistic logic.

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Example 3.22. The (global) algebraic satisfaction system

$$S_\Pi^{\text{ga, Int}} = (L_\Pi^{\text{Int}}, \mathcal{M}_\Pi^{\text{ga, Int}}, \Vdash_\Pi^{\text{ga, Int}})$$

for intuitionistic logic over a set Π of propositional symbols is such that

- L_Π^{Int} is as defined in Example 3.18;
- $\mathcal{M}_\Pi^{\text{ga, Int}}$ is the class of all Heyting or pseudo-Boolean algebras with distinguished value, that is, pairs $(\mathfrak{A}, D)$ such that
  - $\mathfrak{A} = (A, \underline{\wedge}, \underline{\vee}, \underline{\Rightarrow}, \bot, V)$ where $(A, \underline{\wedge}, \underline{\vee}, \underline{\Rightarrow}, \bot)$ satisfies the following identities, where $a_1 \leq a_2$ whenever $a_1 \underline{\wedge} a_2 = a_1$:
    * $a_1 \underline{\wedge} a_2 = a_2 \underline{\wedge} a_1$ and $a_1 \underline{\vee} a_2 = a_2 \underline{\vee} a_1$;
    * $(a_1 \underline{\wedge} a_2) \wedge a_3 = a_1 \underline{\wedge} (a_2 \wedge a_3)$ and $(a_1 \underline{\vee} a_2) \underline{\vee} a_3 = a_1 \underline{\vee} (a_2 \underline{\vee} a_3)$;
    * $(a_1 \underline{\wedge} a_2) \underline{\vee} a_2 = a_2$ and $a_1 \underline{\wedge} (a_1 \underline{\vee} a_2) = a_1$;
    * $a_1 \underline{\wedge} a_2 \leq a_3$ if and only if $a_1 \leq a_2 \underline{\Rightarrow} a_3$;
    * $\bot \leq a$;

    and $V : \Pi \to A$ is a map;
  - $D = \{\top\}$ where $\top$ is $\bot \underline{\Rightarrow} \bot$;
- $\Vdash_\Pi^{\text{ga, Int}} \subseteq \mathcal{M}_\Pi^{\text{ga, Int}} \times L_\Pi^{\text{Int}}$ is the satisfaction relation such that

$$(\mathfrak{A}, \{\top\}) \Vdash_\Pi^{\text{ga, Int}} \varphi$$

  whenever $[\![\varphi]\!]^{\mathfrak{A}} = \top$ and

$$[\![\varphi]\!]^{\mathfrak{A}} \in A$$

  is inductively defined as follows:
  - $[\![p]\!]^{\mathfrak{A}} = V(p)$ for $p \in \Pi$;
  - $[\![\mathsf{ff}]\!]^{\mathfrak{A}} = \bot$;
  - $[\![\psi_1 \wedge \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \underline{\wedge} [\![\psi_2]\!]^{\mathfrak{A}}$;
  - $[\![\psi_1 \vee \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \underline{\vee} [\![\psi_2]\!]^{\mathfrak{A}}$;
  - $[\![\psi_1 \Rightarrow \psi_2]\!]^{\mathfrak{A}} = [\![\psi_1]\!]^{\mathfrak{A}} \underline{\Rightarrow} [\![\psi_2]\!]^{\mathfrak{A}}$.

We denote by

$$S_\Pi^{\text{fga, Int}}$$

the satisfaction system obtained from $S_\Pi^{\text{ga, Int}}$ by taking the elements of $\mathcal{M}_\Pi^{\text{ga, Int}}$ where the set A is finite. Similarly, we denote by

$$S_\Pi^{\text{fgk, Int}}$$

the satisfaction system obtained from $S_\Pi^{\text{gk, Int}}$ (see Example 3.18) by taking the elements of $\mathcal{M}_\Pi^{\text{gk, Int}}$ where the set W is finite.

Observe that from the point of view of validity it is equivalent to work with Heyting algebras or finite Heyting algebras (see [9, Theorem 7.21]). We start by introducing two relevant concepts.

Definition 3.23. A *filter U in a Heyting algebra* $\mathfrak{A}$ is a subset of A such that:

- $\top \in U$;
- if $a, a \underline{\Rightarrow} b \in U$ then $b \in U$.

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A filter U is *prime* whenever $U \neq A$ and if $a \underline{\vee} b \in U$ then either $a \in U$ or $b \in U$.

As a consequence we have that if $a \in U$ and $a \leq b$ then $b \in U$.

PROPOSITION 3.24. *Let* $g : \mathcal{M}_{\Pi}^{\text{fga,Int}} \to \mathcal{M}_{\Pi}^{\text{fgk,Int}}$ *be such that*

$$g(\mathfrak{A}, \{\top\}) = (W, R, \underline{V}),$$

where

- *W is the set of all prime filters in $\mathfrak{A}$;*
- *R is such that $U\, R\, U'$ whenever $U \subseteq U'$;*
- *$\underline{V}(p)$ is $\{U \in W : V(p) \in U\}$.*

Then, for every $U \in W$,

$$[\![\varphi]\!]^{\mathfrak{A}} \in U \quad \textit{if and only if} \quad (W, R, \underline{V}), U \Vdash_{\Pi}^{\text{fgk,Int}} \varphi.$$

Furthermore,

$$(\mathfrak{A}, \{\top\}) \Vdash_{\Pi}^{\text{fga,Int}} \varphi \quad \textit{if and only if} \quad (W, R, \underline{V}) \Vdash_{\Pi}^{\text{fgk,Int}} \varphi.$$

PROOF. It is immediate to see that the second assertion follows from the first (taking into account Theorem 7.41 of [9]). The proof of the first statement follows by induction on φ. We only prove the step where φ is $\varphi_1 \Rightarrow \varphi_2$.
($\rightarrow$) Assume that $[\![\varphi_1 \Rightarrow \varphi_2]\!]^{\mathfrak{A}} \in U$, $U\, R\, U'$ and $(W, R, \underline{V}), U' \Vdash_{\Pi}^{\text{fgk,Int}} \varphi_1$. Hence, by the induction hypothesis, $[\![\varphi_1]\!]^{\mathfrak{A}} \in U'$. Thus, $[\![\varphi_2]\!]^{\mathfrak{A}} \in U'$ because U' is a filter and $[\![\varphi_1 \Rightarrow \varphi_2]\!]^{\mathfrak{A}} = [\![\varphi_1]\!]^{\mathfrak{A}} \underline{\Rightarrow} [\![\varphi_2]\!]^{\mathfrak{A}} \in U'$. So, by the induction hypothesis, $(W, R, \underline{V}), U' \Vdash_{\Pi}^{\text{fgk,Int}} \varphi_2$. Hence, $(W, R, \underline{V}), U \Vdash_{\Pi}^{\text{fgk,Int}} \varphi_1 \Rightarrow \varphi_2$.
($\leftarrow$) Assume that $(W, R, \underline{V}), U \Vdash_{\Pi}^{\text{fgk,Int}} \varphi_1 \Rightarrow \varphi_2$. Then, for every $U' \in W$ such that $U\, R\, U'$ if $(W, R, \underline{V}), U' \Vdash_{\Pi}^{\text{fgk,Int}} \varphi_1$ then $(W, R, \underline{V}), U' \Vdash_{\Pi}^{\text{fgk,Int}} \varphi_2$. Consider two cases:

(1) Assume that $[\![\varphi_2]\!]^{\mathfrak{A}} \in U$. We show that $[\![\varphi_1 \Rightarrow \varphi_2]\!]^{\mathfrak{A}} \in U$. Observe that $[\![\varphi_2]\!]^{\mathfrak{A}} \leq [\![\varphi_1 \Rightarrow \varphi_2]\!]^{\mathfrak{A}}$ because $[\![\varphi_2]\!]^{\mathfrak{A}} \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq [\![\varphi_2]\!]^{\mathfrak{A}}$. So, the thesis follows.
(2) Assume that $[\![\varphi_2]\!]^{\mathfrak{A}} \notin U$. Let

$$F = \{a \in A : \exists b \in U\; b \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq a\}.$$

Observe that:

(a) $U \subseteq F$. It is enough to note that $u \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq u$ for every $u \in U$.
(b) $[\![\varphi_1]\!]^{\mathfrak{A}} \in F$ since $\top \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq [\![\varphi_1]\!]^{\mathfrak{A}}$ and $\top \in U$.
(c) F is a filter. Indeed, $\top \in F$ since $\top \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq \top$ and $\top \in U$. For the other condition, assume that $a_1, a_1 \underline{\Rightarrow} a_2 \in F$. Then there are $b_1, b \in U$ such that

$$b_1 \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq a_1 \quad \text{and} \quad b \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq a_1 \underline{\Rightarrow} a_2.$$

Hence,

$$b \underline{\wedge} a_1 \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq a_2.$$

Thus,

$$b \underline{\wedge} b_1 \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq a_2.$$

Since $b \underline{\wedge} b_1 \in U$ then $a_2 \in F$.

(d) We prove that $[\![\varphi_2]\!]^{\mathfrak{A}} \in F$. Assume, by contradiction, that $[\![\varphi_2]\!]^{\mathfrak{A}} \notin F$. Then (see [9, Theorem 7.41]) there is a prime filter U' such that $F \subseteq U'$ and $[\![\varphi_2]\!]^{\mathfrak{A}} \notin U'$. Thus, $U \subseteq U'$ by (a) and so $U\,R\,U'$. Therefore, by the induction hypothesis, $(W,R,\underline{V}), U' \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi_1$ by (b) but $(W,R,\underline{V}), U' \nVdash^{\mathrm{fgk,Int}}_{\Pi} \varphi_2$ which is a contradiction because $(W,R,\underline{V}), U \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi_1 \Rightarrow \varphi_2$. Thus, $[\![\varphi_2]\!]^{\mathfrak{A}} \in F$ and so there is $b \in U$ such that

$$b \underline{\wedge} [\![\varphi_1]\!]^{\mathfrak{A}} \leq [\![\varphi_2]\!]^{\mathfrak{A}}.$$

Hence, $b \leq [\![\varphi_1]\!]^{\mathfrak{A}} \underline{\Rightarrow} [\![\varphi_2]\!]^{\mathfrak{A}}$ and, consequently, $[\![\varphi_1]\!]^{\mathfrak{A}} \underline{\Rightarrow} [\![\varphi_2]\!]^{\mathfrak{A}} \in U$.⊣

We now define a map from finite Kripke structure for intuitionistic logics to finite pseudo-Boolean algebra that preserves and reflects satisfaction.

PROPOSITION 3.25. *Let* $h : \mathcal{M}^{\mathrm{fgk,Int}}_{\Pi} \to \mathcal{M}^{\mathrm{fga,Int}}_{\Pi}$ *be such that*

$$h(W,R,V) = ((\mathrm{Up}\,W, \cap, \cup, \underline{\Rightarrow}, \emptyset, \underline{V}), \{W\}),$$

where

- $\mathrm{Up}\,W$ *is the set of all subsets of* W *that are upwards closed with respect to* R;
- $X \underline{\Rightarrow} Y$ *is* $\{w \in W : \forall w' \in W$ *if* $w\,R\,w'$ *and* $w' \in X$ *then* $w' \in Y\}$;
- $\underline{V}(p) = \{w \in W : w \in V(p)\}$.

Then, for every $w \in W$,

$$(W,R,V), w \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi \quad \textit{if and only if} \quad w \in [\![\varphi]\!]^{\mathfrak{A}},$$

where $\mathfrak{A} = (\mathrm{Up}\,W, \cap, \cup, \underline{\Rightarrow}, \emptyset, \underline{V})$. *Furthermore,*

$$(W,R,V) \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi \quad \textit{if and only if} \quad ((\mathrm{Up}\,W, \cap, \cup, \underline{\Rightarrow}, \emptyset, \underline{V}), \{W\}) \Vdash^{\mathrm{fga,Int}}_{\Pi} \varphi.$$

PROOF. It is immediate to see that the second assertion follows from the first, so we only prove the first assertion by induction on φ. Consider the case of $\varphi_1 \Rightarrow \varphi_2$.

$(\rightarrow)$ Assume that $(W,R,V), w \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi_1 \Rightarrow \varphi_2$, $w' \in W$ such that $w\,R\,w'$ and $w' \in [\![\varphi_1]\!]^{\mathfrak{A}}$. Then, by the induction hypothesis, $(W,R,V), w' \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi_1$ and so $(W,R,V), w' \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi_2$. Thus, by the induction hypothesis, $w' \in [\![\varphi_2]\!]^{\mathfrak{A}}$. Hence, $w \in [\![\varphi_1 \Rightarrow \varphi_2]\!]^{\mathfrak{A}}$.

$(\leftarrow)$ Assume $w \in [\![\varphi_1 \Rightarrow \varphi_2]\!]^{\mathfrak{A}}$, $w' \in W$, $w\,R\,w'$, and $(W,R,V), w' \Vdash^{\mathrm{fgk,Int}}_{\Pi} \varphi_1$. Hence, by the induction hypothesis, $w' \in [\![\varphi_1]\!]^{\mathfrak{A}}$ and so $w' \in [\![\varphi_2]\!]^{\mathfrak{A}}$.

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Once again by the induction hypothesis, $(W,R,V),w' \Vdash_{\Pi}^{\mathrm{fgk,Int}} \varphi_2$. Therefore, $(W,R,V),w \Vdash_{\Pi}^{\mathrm{fgk,Int}} \varphi_1 \Rightarrow \varphi_2$. ⊣

In summary, we have the following reductions between decision problems:

$$\mathrm{Val}_{S_{\Pi}^{\mathrm{gk,Int}}} \to_{\text{Prop 3.21}} \text{co–}\mathrm{Val}_{S_{\Pi}^{\mathrm{gk,Int}}} \to_{\text{Prop 3.20}} \mathrm{HS}_{S_{\Pi}^{\mathrm{gk,Int}}}$$

and

$$\mathrm{Val}_{S_{\Pi}^{\mathrm{fga,Int}}} \to_{\text{Prop 3.26}} \mathrm{Val}_{S_{\Pi}^{\mathrm{fgk,Int}}}$$

and

$$\mathrm{Val}_{S_{\Pi}^{\mathrm{ga,Int}}} =_{\text{Thm 7.21 of [9]}} \mathrm{Val}_{S_{\Pi}^{\mathrm{fga,Int}}}, \quad \mathrm{Val}_{S_{\Pi}^{\mathrm{gk,Int}}} =_{\text{Thm 2.5.60 of [27]}} \mathrm{Val}_{S_{\Pi}^{\mathrm{fgk,Int}}}.$$

We are ready to prove that the validity problem for the intuitionistic logic endowed with algebraic semantics is decidable.

PROPOSITION 3.26. *The validity problem* $\mathrm{Val}_{S_{\Pi}^{\mathrm{ga,Int}}}$ *is decidable for every set* Π *of propositional symbols.*

PROOF. Observe that

$$(\mathsf{id}_{L_{\Pi}^{\mathrm{Int}}}, g, h),$$

where g and h are defined in Propositions 3.24 and 3.25, respectively, is a satisfaction system reduction. Moreover, g and h are surjective up to satisfaction because for every finite Heyting algebra $(\mathfrak{A}, D)$,

$$(\mathfrak{A}, D) \Vdash_{\Pi}^{\mathrm{fga,Int}} \varphi \quad \text{if and only if} \quad h(g(\mathfrak{A}, D)) \Vdash_{\Pi}^{\mathrm{fga,Int}} \varphi$$

(see [9, Theorem 8.18]) and, for every finite Kripke structure (W,R,V),

$$(W,R,V) \Vdash_{\Pi}^{\mathrm{fgk,Int}} \varphi \quad \text{if and only if} \quad g(h(W,R,V)) \Vdash_{\Pi}^{\mathrm{fgk,Int}} \varphi$$

(see [27, Theorem 2.5.60]). Therefore, by Proposition 3.7 $\mathrm{Val}_{S_{\Pi}^{\mathrm{fga,Int}}}$ is decidable if $\mathrm{Val}_{S_{\Pi}^{\mathrm{fgk,Int}}}$ is decidable. On the other hand, $\mathrm{Val}_{S_{\Pi}^{\mathrm{ga,Int}}}$ is decidable if and only if $\mathrm{Val}_{S_{\Pi}^{\mathrm{fga,Int}}}$ is decidable (see [9, Theorem 7.21]). Note also that

$$\varphi \text{ is valid in } S_{\Pi}^{\mathrm{fgk,Int}} \quad \text{if and only if} \quad \varphi \text{ is valid in } S_{\Pi}^{\mathrm{gk,Int}}$$

(see [3]). The thesis follows because $\mathrm{Val}_{S_{\Pi}^{\mathrm{gk,Int}}}$ is decidable, by Proposition 3.21. ⊣

§4. Reductions for meet-combination. Herein we discuss reductions between the satisfaction system resulting from meet-combination and their components. Meet-combination was introduced in [25, 30] and it provides an axiomatization for the product of two matrix logics.

Let Σ be a signature, that is, a family $\{\Sigma^{(n)}\}$ where $\Sigma^{(n)}$ is a set for every $n \in \mathbb{N}$ such that $\mathsf{tt}, \mathsf{ff} \in \Sigma^{(0)}$. Each element of $\Sigma^{(n)}$ is a *connective of arity n*.

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DEFINITION 4.1. A *matrix satisfaction system* over Σ is a triple

$$(L_\Sigma, \mathcal{A}_\Sigma, \Vdash_\Sigma),$$

where

- L_Σ is inductively defined as follows: $\Sigma^{(0)} \subseteq L_\Sigma$ and if $c \in \Sigma^{(n)}$ and $\varphi_1, \ldots, \varphi_n \in L_\Sigma$ then $c(\varphi_1, \ldots, \varphi_n) \in L_\Sigma$;
- $\mathcal{A}_\Sigma$ is a non-empty class of matrices over Σ, that is, pairs $(\mathfrak{A}, D)$ where
  - $\mathfrak{A}$ is an algebra over Σ, that is, a pair $(A, \{\{c^{\mathfrak{A}}\}_{c\in\Sigma^{(n)}}\}_{n\in\mathbb{N}})$ where A is a non-empty set and $c^{\mathfrak{A}} : A^n \to A$ is a map for each $c \in \Sigma^{(n)}$. Moreover, we denote by $[\![\varphi]\!]^{\mathfrak{A}}$ the denotation of φ in $\mathfrak{A}$;
  - D is a non-empty subset of A (the elements of D are called distinguished elements) such that $\mathrm{tt}^{\mathfrak{A}} \in D$ and $\mathrm{ff}^{\mathfrak{A}} \notin D$;
- $\Vdash_\Sigma \subseteq \mathcal{A}_\Sigma \times L_\Sigma$ is such that

$$(\mathfrak{A}, D) \Vdash_\Sigma c(\varphi_1, \ldots, \varphi_n)$$

  whenever $c^{\mathfrak{A}}([\![\varphi_1]\!]^{\mathfrak{A}}, \ldots, [\![\varphi_n]\!]^{\mathfrak{A}}) \in D$.

Observe that a matrix satisfaction system is also a satisfaction system where each semantic structure in $\mathcal{M}$ is a pair $(\mathfrak{A}, D)$.

Given signatures Σ_1 and Σ_2, let

$$\Sigma_{\lceil 12\rceil}$$

be the signature such that, for each $n \in \mathbb{N}$, $\Sigma^{(n)}_{\lceil 12\rceil}$ is

$$\{\lceil c_1c_2\rceil \mid c_1 \in \Sigma_1^{(n)}, c_2 \in \Sigma_2^{(n)}\} \cup \{\lceil c_1\mathrm{tt}_2\rceil \mid c_1 \in \Sigma_1^{(n)}\} \cup \{\lceil \mathrm{tt}_1c_2\rceil \mid c_2 \in \Sigma_2^{(n)}\},$$

where the constructor $\lceil c_1c_2\rceil$ is the *meet-combination* of c_1 and c_2. Observe that we look at signature $\Sigma_{\lceil 12\rceil}$ as an enrichment of Σ_1 via the embedding $\eta_1 : c_1 \mapsto \lceil c_1\mathrm{tt}_2\rceil$ for each $c_1 \in \Sigma_1^{(n)}$ and similarly for Σ_2. For the sake of lightness of notation, in the context of $\Sigma_{\lceil 12\rceil}$, from now on, we may write c_1 for $\lceil c_1\mathrm{tt}_2\rceil$ when $c_1 \in \Sigma_1^{(n)}$ and c_2 for $\lceil \mathrm{tt}_1c_2\rceil$ when $c_2 \in \Sigma_2^{(n)}$. In this vein, for $k = 1,2$, we look at L_{Σ_k} as a subset of $L_{\Sigma_{\lceil 12\rceil}}$. Given a formula φ over $\Sigma_{\lceil 12\rceil}$ and $k \in \{1,2\}$, we denote by $\varphi|_k$ the formula in L_{Σ_k} inductively defined as follows:

- $\varphi|_k$ is tt_k whenever φ is $\lceil c_1c_2\rceil(\varphi_1, \ldots, \varphi_n)$ and c_k is tt_k;
- $\varphi|_k$ is $c_k(\varphi_1|_k, \ldots, \varphi_n|_k)$ whenever φ is $\lceil c_1c_2\rceil(\varphi_1, \ldots, \varphi_n)$ and c_k is not tt_k.

DEFINITION 4.2. The *meet-combination* of matrix satisfaction systems $S_{\Sigma_1} = (L_{\Sigma_1}, \mathcal{A}_{\Sigma_1}, \Vdash_{\Sigma_1})$ and $S_{\Sigma_2} = (L_{\Sigma_2}, \mathcal{A}_{\Sigma_2}, \Vdash_{\Sigma_2})$ over signatures Σ_1 and Σ_2, respectively, denoted by

$$\lceil S_{\Sigma_1}\, S_{\Sigma_2}\rceil$$

is the matrix satisfaction system

$$(L_{\Sigma_{\lceil 12\rceil}}, \mathcal{A}_{\Sigma_{\lceil 12\rceil}}, \Vdash_{\Sigma_{\lceil 12\rceil}})$$

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over the signature $\Sigma_{\lceil 12 \rceil}$ such that $\mathcal{A}_{\Sigma_{\lceil 12 \rceil}}$ is the class of product matrices

$$\{(\mathfrak{A}_1, D_1) \times (\mathfrak{A}_2, D_2) : (\mathfrak{A}_1, D_1) \in \mathcal{A}_{\Sigma_1} \text{ and } (\mathfrak{A}_2, D_2) \in \mathcal{A}_{\Sigma_2}\}$$

over $\Sigma_{\lceil 12 \rceil}$ such that each $(\mathfrak{A}_1, D_1) \times (\mathfrak{A}_2, D_2) = (\mathfrak{A}_1 \times \mathfrak{A}_2, D_1 \times D_2)$ where

$$\mathfrak{A}_1 \times \mathfrak{A}_2 = (A_1 \times A_2, \{\{\lceil c_1 c_2 \rceil^{\mathfrak{A}_1 \times \mathfrak{A}_2}\}_{\lceil c_1 c_2 \rceil \in \Sigma^{(n)}_{\lceil 12 \rceil}}\}_{n \in \mathbb{N}})$$

with

$$\lceil c_1 c_2 \rceil^{\mathfrak{A}_1 \times \mathfrak{A}_2}((a_1, b_1), \dots, (a_n, b_n)) =$$

$$\begin{cases} (c_1{}^{\mathfrak{A}_1}(a_1, \dots, a_n), c_2{}^{\mathfrak{A}_2}(b_1, \dots, b_n)) & \text{if } c_1 \in \Sigma_1^{(n)} \text{ and } c_2 \in \Sigma_2^{(n)}; \\ (\mathrm{tt}_1{}^{\mathfrak{A}_1}, c_2{}^{\mathfrak{A}_2}(b_1, \dots, b_n)) & \text{if } c_1 \text{ is } \mathrm{tt}_1 \text{ and } c_2 \in \Sigma_2^{(n)}; \\ (c_1{}^{\mathfrak{A}_1}(a_1, \dots, a_n), \mathrm{tt}_2{}^{\mathfrak{A}_2}) & \text{if } c_2 \text{ is } \mathrm{tt}_2 \text{ and } c_1 \in \Sigma_1^{(n)}. \end{cases}$$

The following results were proved in [25, 30].

PROPOSITION 4.3. *Let* $\varphi \in L_{\Sigma_1} \cup L_{\Sigma_2}$, $(\mathfrak{A}_1, D_1) \in \mathcal{A}_{\Sigma_1}$, *and* $(\mathfrak{A}_2, D_2) \in \mathcal{A}_{\Sigma_2}$. *Then*

$$[\![\varphi]\!]^{\mathfrak{A}_1 \times \mathfrak{A}_2} = \begin{cases} ([\![\varphi]\!]^{\mathfrak{A}_1}, [\![\mathrm{tt}_2]\!]^{\mathfrak{A}_2}) & \text{if } \varphi \text{ is in } L_{\Sigma_1}, \\ ([\![\mathrm{tt}_1]\!]^{\mathfrak{A}_1}, [\![\varphi]\!]^{\mathfrak{A}_2}) & \text{if } \varphi \text{ is in } L_{\Sigma_2}. \end{cases}$$

PROPOSITION 4.4. *Let* $\varphi \in L_{\Sigma_{\lceil 12 \rceil}}$, $(\mathfrak{A}_1, D_1) \in \mathcal{A}_{\Sigma_1}$, *and* $(\mathfrak{A}_2, D_2) \in \mathcal{A}_{\Sigma_2}$. *Then*

$$[\![\varphi]\!]^{\mathfrak{A}_1 \times \mathfrak{A}_2} = \left(([\![\varphi|_1]\!]^{\mathfrak{A}_1 \times \mathfrak{A}_2})_1, ([\![\varphi|_2]\!]^{\mathfrak{A}_1 \times \mathfrak{A}_2})_2\right).$$

PROPOSITION 4.5. *Let* $\varphi \in L_{\Sigma_{\lceil 12 \rceil}}$, $(\mathfrak{A}_1, D_1) \in \mathcal{A}_{\Sigma_1}$, *and* $(\mathfrak{A}_2, D_2) \in \mathcal{A}_{\Sigma_2}$. *Then*

$$(\mathfrak{A}_1, D_1) \times (\mathfrak{A}_2, D_2) \Vdash_{\Sigma_{\lceil 12 \rceil}} \varphi \quad \textit{iff} \quad (\mathfrak{A}_1, D_1) \Vdash_{\Sigma_1} \varphi|_1 \textit{ and } (\mathfrak{A}_2, D_2) \Vdash_{\Sigma_2} \varphi|_2.$$

PROOF. Observe that $(\mathfrak{A}_1, D_1) \times (\mathfrak{A}_2, D_2) \Vdash_{\Sigma_{\lceil 12 \rceil}} \varphi$ iff $[\![\varphi]\!]^{\mathfrak{A}_1 \times \mathfrak{A}_2}$ is in $D_1 \times D_2$ if and only if $([\![\varphi|_1]\!]^{\mathfrak{A}_1 \times \mathfrak{A}_2})_1$ is in D_1, $([\![\varphi|_2]\!]^{\mathfrak{A}_1 \times \mathfrak{A}_2})_2$ is in D_2, by Proposition 4.4, iff $[\![\varphi|_1]\!]^{\mathfrak{A}_1}$ is in D_1, $[\![\varphi|_2]\!]^{\mathfrak{A}_2}$ is in D_2, by Proposition 4.3, if and only if $(\mathfrak{A}_1, D_1) \Vdash_{\Sigma_1} \varphi|_1$ and $(\mathfrak{A}_2, D_2) \Vdash_{\Sigma_2} \varphi|_2$. ⊣

We omit the proof of the following result since it is a straightforward consequence of Proposition 4.5.

PROPOSITION 4.6. *Let* $S_{\Sigma_1} = (L_{\Sigma_1}, \mathcal{A}_{\Sigma_1}, \Vdash_{\Sigma_1})$ *and* $S_{\Sigma_2} = (L_{\Sigma_2}, \mathcal{A}_{\Sigma_2}, \Vdash_{\Sigma_2})$ *be matrix satisfaction systems over signatures* Σ_1 *and* Σ_2, *respectively. Then*

$$(\tau^1, \tau^2, g^1, g^2, h),$$

where

- $\tau^k(\varphi) = \varphi|_k$ *for* $k = 1, 2$;
- $g^k((\mathfrak{A}_1, D_1) \times (\mathfrak{A}_2, D_2)) = (\mathfrak{A}_k, D_k)$ *for* $k = 1, 2$;
- $h((\mathfrak{A}_1, D_1), (\mathfrak{A}_2, D_2)) = (\mathfrak{A}_1, D_1) \times (\mathfrak{A}_2, D_2)$;

is a satisfaction system reduction from $\lceil S_{\Sigma_1} S_{\Sigma_2} \rceil$ *to* $S_{\Sigma_1} \times S_{\Sigma_2}$.

PROPOSITION 4.7. *Let* $S_{\Sigma_1} = (L_{\Sigma_1}, \mathcal{A}_{\Sigma_1}, \Vdash_{\Sigma_1})$ *and* $S_{\Sigma_2} = (L_{\Sigma_2}, \mathcal{A}_{\Sigma_2}, \Vdash_{\Sigma_2})$ *be matrix satisfaction systems over arbitrary signatures* Σ_1 *and* Σ_2, *respectively. Then* $\mathrm{Val}_{\lceil S_{\Sigma_1}\, S_{\Sigma_2} \rceil}$ *is decidable whenever* $\mathrm{Val}_{S_{\Sigma_1}}$ *and* $\mathrm{Val}_{S_{\Sigma_2}}$ *are decidable for every pair of signatures* Σ_1 *and* Σ_2.

PROOF. Assume that $\mathrm{Val}_{S_{\Sigma_1}}$ and $\mathrm{Val}_{S_{\Sigma_2}}$ are decidable. Since there is a satisfaction system reduction, by Proposition 4.6, from $\lceil S_{\Sigma_1}\, S_{\Sigma_2} \rceil$ to $S_{\Sigma_1} \times S_{\Sigma_2}$ where g^k for $k = 1,2$ and h are surjective up to satisfaction then $\mathrm{Val}_{\lceil S_{\Sigma_1}\, S_{\Sigma_2} \rceil}$ is decidable by Proposition 3.7. ⊣

We now consider the meet-combination

$$\lceil S_{\Pi}^{\mathrm{ga,K}}\, S_{\Pi}^{\mathrm{ga,Int}} \rceil$$

of the matrix satisfaction system for K modal logic and the matrix satisfaction system for intuitionistic logic both endowed with algebraic semantics.

PROPOSITION 4.8. *The validity problem in the meet-combination*

$$\lceil S_{\Pi}^{\mathrm{ga,K}}\, S_{\Pi}^{\mathrm{ga,Int}} \rceil$$

is decidable.

PROOF. We know from Propositions 3.17 and 3.26 that $\mathrm{Val}_{S_{\Pi}^{\mathrm{ga,K}}}$ and $\mathrm{Val}_{S_{\Pi}^{\mathrm{ga,Int}}}$ are decidable, respectively. Therefore, by Proposition 4.7, $\mathrm{Val}_{\lceil S_{\Pi}^{\mathrm{ga,K}}\, S_{\Pi}^{\mathrm{ga,Int}} \rceil}$ is decidable. ⊣

§5. Concluding remarks. We presented satisfaction systems as the right general abstraction for analyzing in a semantic way logical decision problems. The essential notion of a reduction from a satisfaction system to a finite collection of satisfaction systems was here introduced. We showed that reductions between satisfaction systems induce reductions between the satisfiability and the validity problems leading to general results on decidability. We also consider the meet-combination of logics and proved that the validity problem in the meet-combination is decidable provided that the validity problem in the components is also decidable. An illustration was provided for the meet-combination of K modal logic with algebraic semantics and intuitionistic logic with algebraic semantics.

We intend to extend the reduction technique proposed herein for obtaining results about entailment and deductive consequence problems (having hypotheses). The general setting should be defined over the notion of consequence system. In this case, preservation of semidecidability seems to play an important role. Moreover, it would also be interesting to use the fact that if some non-decidable problem can be reduced to another problem then the latter one is also not decidable.

We think that preservation of decidability for other forms of combination should also be considered. For instance, the preservation of decidability in fusion of modal logics (see [22]) was already addressed but not using

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reductions. The other case that comes to mind is to investigate reduction techniques in the fibring of logics (see [8, 12, 32, 36]).

Another challenging extension of this work is to adapt the reduction techniques to enumerable sets of logics, namely in the context of logics of formal inconsistency (see [7]) and many-valued logics (see [15]).

Moreover, we intend to obtain preservation results for complexity classes when in the presence of reductions. In particular we would like to relate the complexity class of logical decision problems for the meet-combination with the complexity classes of logical decision problems for the component logics.

Finally, following the recent work of [1], we intend to investigate logical decision problems and their reductions for logics presented by a polynomial ring calculus.

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