*Ergod. Th. & Dynam. Sys.*, (2022), **42**, 491–499 © The Author(s), 2021. Published by Cambridge University Press.
doi:10.1017/etds.2021.117 

# Non-differentiable irrational curves for $C^1$ twist map

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(*Received* 25 *September* 2019 *and accepted in revised form* 26 *July* 2020)

*Abstract.* We construct a $C^1$ symplectic twist map $g$ of the annulus that has an essential invariant curve $\Gamma$ such that $\Gamma$ is not differentiable and $g$ restricted to $\Gamma$ is minimal.

Key words: twist maps, invariant curves, Birkhoff theory
2020 Mathematics Subject Classification: 37E40, 37E10 (Primary)

We denote $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ the circle and $\mathbb{A} = \mathbb{T} \times \mathbb{R}$ the annulus. An important class of area-preserving maps of the annulus are the so-called twist maps or maps that deviate the vertical, because this class of maps describes the behavior of area preserving surface diffeomorphisms in the neighborhood of a generic elliptic periodic point (see [**C85**]).

More precisely, a $C^1$ diffeomorphism $g$ of the annulus that is isotopic to identity is a positive twist map (respectively negative twist map) if for any given lift $\tilde{g} : \mathbb{R}^2 \to \mathbb{R}^2$ of $g$ and for every $\theta \in \mathbb{R}$, the maps $r \mapsto \tilde{\pi} \circ \tilde{g}(\theta, r)$ are increasing (respectively decreasing) diffeomorphisms. A twist map may be positive or negative. Here $\tilde{\pi}$ denotes the lift of the first projection map.

An *essential invariant curve* by a diffeomorphism $g$ of the annulus is a homotopically non-trivial simple loop that is invariant by $g$.

When $f$ is a $C^1$ twist map, it is known from Birkhoff theory that any invariant essential curve by $g$ is the graph of a Lipschitz map over $\mathbb{T}$. Furthermore, it was proven by Arnaud in [**A09**] that the latter map must be $C^1$ on a $G_\delta$ set of $\mathbb{T}$ of full measure.

Numerical experiments show that the invariant curves of a smooth twist map are actually more regular than just what the Birkhoff theory predicts. Moreover, in the perturbative setting of quasi-integrable twist maps, the Kolmogorov–Arnold–Moser (KAM) theory provides a large measure set of smooth invariant curves. The question of the regularity of the invariant curves of twist maps is thus a natural question that is also related to the study of how the KAM curves disappear as the perturbation of integrable curves becomes large.

Another natural and related question is that of the regularity of the boundaries of Birkhoff instability zones. A Birkhoff instability zone of a twist map $g$ is an open set of the annulus that is homeomorphic to the annulus, that does not contain any invariant essential curve, and that is maximal for these properties. By Birkhoff's theory, the boundary of an instability zone is an invariant curve that is a Lipschitz graph. We refer to the nice introduction of [**A11**], where many features and questions are discussed about the boundaries of Birkhoff instability zones.

In [**EKMY98**], the following question was asked.

*Question 1.* (J. Mather [**EKMY98**, Problem 3.1.1]) Does there exist an example of a symplectic $C^r$ twist map with an essential invariant curve that is not $C^1$ and that contains no periodic point?

In [**H83**, §III], Herman gave an example of a $C^2$ twist map of the annulus that has a $C^1$ invariant curve on which the dynamic is conjugated to that of a Denjoy counterexample. By Denjoy's theorem on topological conjugacy of $C^2$ circle diffeomorphisms with irrational rotation number, such a curve cannot be $C^2$.

In [**A13**], Arnaud gave an example of a $C^2$ twist map $g$ of the annulus that has an invariant curve $\Gamma$ that is non-differentiable on which the dynamic is conjugated to that of a Denjoy counterexample. In addition, she showed that in any $C^1$ neighborhood of $g$, there exist twist maps with Birkhoff instability zones having $\Gamma$ for a boundary and having the same dynamics as $g$ on $\Gamma$.

In [**A11**], the following natural question was raised.

*Question 2.* (M.-C. Arnaud) Does there exist a regular symplectic twist map of the annulus that has an essential invariant curve that is non-differentiable on which the restricted dynamics is minimal?

In this note, we give a positive answer to this question in low regularity.

THEOREM 1. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists a symplectic $C^1$ twist map of the annulus that has a non differentiable essential invariant curve $\Gamma$ such that the restriction of $g$ to $\Gamma$ is $C^0$ conjugated to the circle rotation of angle $\alpha$, and hence minimal.*

Owing to [**A11**, Theorem 2], we can deduce the following.

COROLLARY 2. *There exists a symplectic $C^1$ twist map $g$ of the annulus that has a non-differentiable essential invariant curve $\Gamma$ such that the restriction of $g$ to $\Gamma$ is minimal and such that $\Gamma$ is at the boundary of an instability zone of $g$.*

The derivation of Corollary 2 from Theorem 1 follows from the general result of [**A11**, Theorem 2], which asserts that any essential invariant curve of a $C^1$ twist map $g$ that has an irrational rotation number can be viewed as a boundary dynamics of Birkhoff instability zone of an arbitrarily nearby $C^1$ twist map. This result relies on a perturbation argument involving the Hayashi $C^1$-closing lemma.

Owing to a construction by Herman in [H83, §II.2], Theorem 1 can be derived from the following result on circle homeomorphisms.

THEOREM 3. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exists a non-differentiable orientation preserving (minimal) homeomorphism of the circle f, topologically conjugate to the circle rotation of angle $\alpha$, such that $f + f^{-1}$ is of class $C^1$.*

*Proof of Theorem 1.* To prove that Theorem 3 implies Theorem 1, we use Herman's beautiful trick that associates to some circle homeomorphisms $f$ a twist map of the annulus which preserves a circle that is determined by $f$ and on which the restricted dynamics is given by $f$. For completeness, we state and prove Herman's observation.

First of all, note that if $\tilde{f}$ is a lift of a circle homeomorphism $f$ and if there exists a $C^1$ function $\phi : \mathbb{T} \to \mathbb{R}$, which we also view as a function from $\mathbb{R}$ to $\mathbb{R}$ of period 1, satisfying

$$\mathrm{Id}_{\mathbb{R}} + \frac{\phi(\cdot)}{2} = \frac{1}{2}(\tilde{f}(\cdot) + \tilde{f}^{-1}(\cdot)), \tag{1}$$

then the map

$$g : \mathbb{A} \to \mathbb{A} : (\theta, r) \mapsto (\theta + r, r + \phi(\theta + r)) \tag{2}$$

is a $C^1$ Hamiltonian twist map of the annulus (this follows from the fact that $\int_{\mathbb{T}} \phi(\theta) d\theta = 0$, as proved in [H83, §II.2.3]). Moreover, with $\psi := \tilde{f} - id_{\mathbb{R}}$, which we view as a function from $\mathbb{T}$ to $\mathbb{R}$, or from $\mathbb{R}$ to $\mathbb{R}$ of period 1, the following holds.

CLAIM. [H83, §II.2] *The graph $\Gamma$ of $\theta \mapsto \psi(\theta)$ is invariant by g and the dynamics of g restricted to $\Gamma$ is conjugated (via the first projection) to f.*

*Proof.* From equation (1), we have that

$$2\tilde{f} + \phi \circ \tilde{f} = \tilde{f} \circ \tilde{f} + \mathrm{Id}_{\mathbb{R}} = \tilde{f} + \psi \circ \tilde{f} + \mathrm{Id}_{\mathbb{R}},$$

so

$$\psi \circ f - \psi = \phi \circ f,$$

and finally equation (2) implies

$$\begin{aligned} g(\theta, \psi(\theta)) &= (f(\theta), \psi(\theta) + \phi(f(\theta))) \\ &= (f(\theta), \psi(f(\theta))). \end{aligned}$$ □

The proof that Theorem 3 implies Theorem 1 is now straightforward from the claim. □

To prove Theorem 3, we will work with a special class of circle diffeomorphisms that we call $C$-great.

*Definition 1.* Let $f : \mathbb{T} \to \mathbb{T}$ be a $C^1$ minimal diffeomorphism. For $C > 1$, we say that $x$ is $C$-good if $\sum_{n \geq 0} |Df^n(x)|^{-2} < C$, while $\sup_{n<0} |Df^n(x)| = \infty$ and $\sum_{n<0} |Df^n(x)|^{-2} = \infty$.

We say that $f$ is $C$-great if the set of $C$-good points is uncountable and $\sup_{x \in \mathbb{T}} |Df(x) + Df^{-1}(x)| < C$.

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Moreover, we say that $x$ is $C$-great if it is accumulated by an uncountable set of $C$-good points. We then say that the pair $(f, x)$ is $C$-great.

It is not hard to see that if $f$ is $C$-great, then it has a $C$-great point $x$. Moreover, a $C$-great point is actually accumulated by an uncountable set of $C$-great points.

Let $0 < u < v$. For an injective map $f$ defined from $[x - v, x + v]$ to $\mathbb{R}$, we define

$$\Delta(f, x, u, v) = \frac{v}{u} \frac{f(x + u) - f(x)}{f(x + v) - f(x)}.$$

The following is a straightforward criterion of non-differentiability of $f$.

LEMMA 4. *If $f$ is a differentiable circle diffeomorphism, then it must satisfy for every $x \in \mathbb{T}$, the limit, as $u, v$ go to 0, of $\Delta(f, x, u, v)$ exists and equals* 1.

The proof of Theorem 3 is based on the following two lemmas.

LEMMA 5. *For every $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, there exist $C > 1$ and a circle diffeomorphism $f$ topologically conjugate to the circle rotation of angle $\alpha$ that is $C$-great.*

We will prove Lemma 5 at the end of the note.

We first show how, starting from a $C$-great diffeomorphism, we can prove Theorem 3. The proof is based on an inductive application of the following lemma.

LEMMA 6. *For every $C > 1$, there exists $\epsilon_0 > 0$ with the following property. Let $(f, x)$ be $C$-great. Then for every $\epsilon > 0$, there exists a $C^1$-diffeomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ such that $g = h \circ f \circ h^{-1}$ satisfies:*

- *$h$ is $\epsilon$-close to $\mathrm{Id}_{\mathbb{T}}$ in the $C^0$ topology;*
- *$g + g^{-1}$ is $\epsilon$-close to $f + f^{-1}$ in the $C^1$ topology;*
- *there exists $y \in \mathbb{T}$ such that $|y - x| < \epsilon$ and $(g, y)$ is $C$-great, and there exists $u, v \in (0, \epsilon)$ such that $\Delta(g, y, u, v) > 1 + \epsilon_0$.*

*Proof that Lemmas 5 and 6 imply Theorem 3.* Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. We want to construct a sequence of circle diffeomorphisms $(f_n)$ that converge to a circle homeomorphism $f_\infty$ that is topologically conjugated to $R_\alpha$ and such that $f_\infty$ is not a diffeomorphism while $f_\infty + f_\infty^{-1}$ is $C^1$. We will use the criterion of Lemma 4 to guarantee the non-differentiability of $f_\infty$.

In light of Lemma 5, we can start with a $C^1$ circle diffeomorphism $f_0$ and $x_0$ such that $(f_0, x_0)$ is $C$-great and $f_0$ is topologically conjugate to the circle rotation of angle $\alpha$. Using Lemma 6, we construct inductively, for $n \geq 1$, sequences $(h_n)$ of circle diffeomorphisms, $(x_n)$ of points in $\mathbb{T}$, and $(u_n)$ and $(v_n)$ of positive numbers, such that for

$$f_n := h_n \circ f_{n-1} \circ h_n^{-1}, \quad F_n := f_n + f_n^{-1}, \quad H_n := h_n \circ \cdots \circ h_1,$$

the following hold:

- $(f_n, x_n)$ is $C$-great;
- $u_n + v_n \leq 2^{-n}$;
- $\|x_{n+1} - x_n\| \leq 2^{-n}$;
- $\Delta(f_n, x_n, u_n, v_n) > 1 + \epsilon_0$;

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- $\|H_n - H_{n-1}\|_{C^0} \leq 2^{-n}$;
- $\|F_{n+1} - F_n\|_{C^1} \leq 2^{-n}$.

We also ask that the convergence of $(H_n)$ and $(x_n)$ be sufficiently fast so that $\Delta(f_m, x_m, u_n, v_n) > 1 + \epsilon_0$ for all $m \geq n$.

From the convergence of $H_n$, we get that $f_n$ converges to a circle homeomorphism $f_\infty$ that is topologically conjugated to $R_\alpha$. In addition, the $C^1$ limit of $F_n$ must be $f_\infty + f_\infty^{-1}$. Because $\Delta(f_m, x_m, u_n, v_n) > 1 + \epsilon_0$ for all $m \geq n$, we guarantee that the limit point $x_\infty$ of $(x_n)$ satisfies $\Delta(f_\infty, x_\infty, u_n, v_n) > 1 + \epsilon_0$ for every $n$. Hence, $f_\infty$ is not differentiable by Lemma 4.

In conclusion, $f_\infty$ satisfies all the properties of Theorem 3. □

*Proof of Lemma 6.* To motivate what follows, note that if $g = h \circ f \circ h^{-1}$ with $Dh(x) = 1 + e(x)$, then, to have $g + g^{-1} = f + f^{-1}$, one must have

$$h \circ f + h \circ f^{-1} = f \circ h + f^{-1} \circ h.$$

The latter implies that

$$e(f(x))Df(x) + \frac{e(f^{-1}(x))}{Df(f^{-1}(x))} = e(x)\bigg(Df(h(x)) + \frac{1}{Df(f^{-1}(h(x))}\bigg) + \Delta,$$

with

$$\Delta = Df \circ h - Df + Df^{-1} \circ h - Df^{-1}.$$

Assuming $h$ is $C^0$ close to id, we see that for $g + g^{-1}$ to be $C^1$-close to $f + f^{-1}$, we should choose $e$ such that $(e(f(x)) - e(x))Df(x)Df(f^{-1}(x))$ is close to $e(x) - e(f^{-1}(x))$.

To construct the conjugacy $h$ of Lemma 6, we start by defining a sequence $(e_j)_{j\in\mathbb{Z}}$ that will serve to define $h$ along the orbit of a point that is $C$-good for $f$.

Let $x$ be a $C$-good point for $f$. Let $b_j = Df(f^j(x))$, $j \in \mathbb{Z}$. Define $c_j$, $j \in \mathbb{Z}$ by $c_0 = 1$ and $c_j b_j b_{j-1} = c_{j-1}$.

SUBLEMMA 7. *For any $\nu > 0$, we can define a sequence $(e_j)_{j\in\mathbb{Z}}$ such that $e_j = 0$ for $j \notin [-N', N]$ for some $N, N' \in \mathbb{N}^*$ as well as the following:*

- $e_j \geq 0$, *for all $j \in \mathbb{Z}$;*
- $e_1 - e_0 = -1$*;*
- $e_0 \leq C^3$*;*
- $\eta_j := |(e_{j+1} - e_j)b_j b_{j-1} + (e_{j-1} - e_j)| \leq \nu$, *for all $j \in \mathbb{Z}$.*

*Proof.* We will need the following properties on the orbit of $x$ that will be a consequence from the fact that $x$ is $C$-good for $f$.

CLAIM. *We have that $\sum_{j\geq 0} c_j \leq C^3$ and $\sum_{j\leq 0} c_j = \infty$,* $\liminf_{j\leq 0} c_j = 0$.

*Proof.* We have that for $j \geq 0$, $c_j = (Df(x)Df(f^j(x)))/(Df^{j+1}(x))^2$. Hence, it follows from Definition 1 that $\sum_{j\geq 0} c_j \leq C^3$.

For $j \leq -1$, we have that $c_j = (Df(x)/Df(f^j(x)))1/(Df^j(x))^2$. Hence, it follows from Definition 1 that $\sum_{j<0} c_j = \infty$ as well as $\liminf_{j\leq 0} c_j = 0$. □

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Let us now define $e_j$, $j \in \mathbb{Z}$ as follows. Fix $N$ large such that $c_{-N}$ is small. Note that by taking $N$ sufficiently large, we will have that $c_N$ is also small. We let $e_j = 0$ for $j > N$. For $|j| \leq N$, we define $e_j$ so that $e_{j+1} - e_j = -c_j$. We now let $N'$ be much larger than $N$ such that $c_{-N'+1}$ is small and such that $\alpha = \sum_{-N' \leq j < -N} c_j / e_{-N}$ is large. We set then $e_{j+1} - e_j = c_j/\alpha$ for $-N' \leq j \leq -N-1$. We now define $e_j = 0$ for $j < -N'$. By construction, $e_1 - e_0 = -c_0 = -1$.

Note the following.

- For $j \in [-N, N]$, we have that $e_j = \sum_{k=j}^{N} c_k$.
- For $j \in [-N', -N)$, we have that $e_j = e_{-N} - (1/\alpha) \sum_{k=j}^{-N-1} c_k$.

From the definition of $\alpha$, we deduce that $e_{-N'} = 0$ and that $e_j \geq 0$ for all $j \in \mathbb{Z}$. Also, $-e_0 = e_{N+1} - e_0 = -\sum_{0 \leq j \leq N} c_j$, so $e_0 = \sum_{0 \leq j \leq N} c_j \leq C^3$.

Note that by construction, because $c_j b_j b_{j-1} = c_{j-1}$, we have that

$$\eta_j = (e_{j+1} - e_j) b_j b_{j-1} + (e_{j-1} - e_j) = 0,$$

for all $j \in \mathbb{Z}$, except for $j = -N'$, $j = -N$, and $j = N+1$. We also have, because $e_{-N'-1} = e_{-N'} = 0$, that $\eta_{-N'} = c_{-N'}/\alpha$ is small as $c_{-N'}$ is small and $\alpha$ is large. Next, we compute $\eta_{-N} = -c_{-N-1} - (c_{-N-1})/\alpha$, which is also small because $c_N$ is small and $\alpha$ is large. Finally, $\eta_{N+1} = c_N$ is also small by our choice of $N$. □

We want now to modify $f$ by conjugation along a neighborhood of the orbit of $x$ between the times $-N'$ and $N$ to obtain the diffeomorphism $g$ with the required properties of Lemma 6. We will look for the conjugacy under the form $Dh(x) = 1 + e(x)$. The choice of the sequence $(e_j)$ in Sublemma 7, would essentially allow to get a good control on $g$ and $g + g^{-1}$ along the orbit of the point $x$ if the function $e$ takes the values $e_j$ along the orbit of $x$. We need however to define $h$ in the neighborhood of the orbit of $x$ without losing the required properties on $g$ and $g + g^{-1}$ and this will will require some additional technicalities that we now address.

To guarantee a good control of $g + g^{-1}$ everywhere, we start by slightly modifying $f$ to make it affine along the $-N'$ to $N$ orbit of a small interval around $x$. For this, choose a small interval $I_0$ centered around $x$, and let $I_j = f^j(I_0)$, $j \in \mathbb{Z}$. We may assume that $I_j \cap I_{j'} = \emptyset$ if $0 < |j - j'| \leq 2N'$. Then for every $\delta > 0$, we can define a diffeomorphism $h' : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, which is the identity outside $\cup_{-N'+1 \leq j \leq N+1} I_j$ such that $\sup_y |Dh'(y) - 1| < \delta$, and such that letting $f' = h' \circ f \circ h'^{-1}$, we have $f'(y) = f^{j+1}(x) + b_j(y - f^j(x))$ whenever $y$ is near $f^j(x)$ and $-N' \leq j \leq N$.

Let us now select an interval $I'_0 \subset I_0$ centered around $x$ such that letting $I'_j = f'^{\,j}(I'_0)$, $j \in \mathbb{Z}$, we have that $f'|I'_j$ is affine for $-N' \leq j \leq N$.

Note that by choosing $\delta > 0$ small, we get that $F' = f' + f'^{-1}$ is $C^1$ close to $F = f + f^{-1}$.

We now define another diffeomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ which is the identity outside $\cup_{-N'+1 \leq j \leq N} I'_j$ as follows. To specify $h$, it is enough to define $Dh$ on $I'_j$ for $-N' + 1 \leq j \leq N$. We let $\phi : \mathbb{R} \to [-\delta, 1]$ be a smooth function supported on $(-1/2, 1/2)$, symmetric around 0, and such that $\phi(0) = 1$ and $\int \phi(x)dx = 0$. We then let $Dh =$

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$1 + e_j\phi \circ A_j$, where $A_j : I'_j \to [-1/2, 1/2]$ is an affine homeomorphism. If $\delta > 0$ is chosen sufficiently small, $h$ is indeed a diffeomorphism because all the $e_j$ are positive.

Finally, let

$$g := h \circ f' \circ h^{-1} = h \circ h' \circ f \circ h'^{-1} \circ h^{-1}.$$

CLAIM. *The diffeomorphism g satisfies the requirements of Lemma 6.*

*Proof.* Let us show that $G = g + g^{-1}$ is $C^1$ close to $F' = f' + f'^{-1}$. Note that $G = F'$ in the complement of $\bigcup_{-N' \leq j \leq N} I'_j$, so it is enough to show that $DG - DF'$ is small in each $I'_j$, $-N' \leq j \leq N$. Indeed for $y \in I'_j$, letting $\kappa = \phi \circ A_j(y)$, we have

$$DG(h(y)) - DF'(h(y)) = \frac{\kappa}{(1 + e_j\kappa)b_{j-1}}((e_{j+1} - e_j)b_j b_{j-1} + e_{j-1} - e_j),$$

which is small because the term $\kappa/(1 + e_j\kappa)b_{j-1}$ is bounded. Indeed, $-\delta \leq \kappa \leq 1$, and $1 + e_j\kappa \geq 1 - e_j\delta \geq 1/2$ provided $\delta$ is chosen sufficiently small, and finally $b_{j-1} > C^{-1}$ because $\sup_{x\in\mathbb{T}} |Df(x) + Df^{-1}(x)| < C$.

Moreover, we have $Dg(x) - Df(x) = ((1 + e_1)b_0)/(1 + e_0) - b_0 = -b_0/(1 + e_0) < -(1/C(1 + C^3))$. Because $((g(x + u) - g(x))/u) \sim Dg(x)$ for $u \ll |I_0|$, while $((g(x + v) - g(x))/v) \sim Df(x)$ for $1 \gg u \gg |I_0|$, this allows to exhibit $u$ and $v$ such that $\Delta(g, x, u, v) \geq 1 + \epsilon_0$ with $\epsilon_0 = 1/(2C^2(1 + C^3))$.

To conclude, we must show that there exists arbitrary close to $x$ a point $y$ such that $(g, y)$ is $C$-great. It suffices for this to show that in any interval $J$ around $x$, there is an uncountable set of $C$-good points for $g$. By definition of $x$, we know that the latter is true for $f$. Note that if $y$ is $C$-good for $f$, then $y' = h(h'(y))$ is $C$-good for $g$ if $\sum_{n\geq 0} |Dg^n(y')|^{-2} < C$.

Fix some $\Lambda > 0$ much larger than $\sup e_j$. Notice that for small $\lambda > 0$, we can choose $m > 0$ such that there is an uncountable set $K' \subset J$ of $y$ such that

$$\sum_{n\geq 0} |Df^n(y)|^{-2} < C - \lambda, \quad \text{and} \quad \sum_{n\geq m} |Df^n(y)|^{-2} < \lambda/\Lambda.$$

Now, notice that $h \circ h' = \text{id}$ except in $\cup_{-N'+1\leq j\leq N+1} I_j$, and the derivative of $h \circ h'$ is bounded by $(1 + \delta)(1 + \sup e_j)$. In particular, if $y \in K'$, then $h(h'(y))$ will be $C$-good for $g$ provided $g^n(y) \notin \cup_{-N'+1\leq j\leq N+1} I_j$ for $0 \leq n \leq m$. If we choose the size of $I_0$ sufficiently small, we will have the latter property for uncountably many $y \in K' \subset J$. □

The proof of Lemma 6 is thus accomplished. □

To finish we still need to show the existence of $C$-great diffeomorphisms.

*Proof of Lemma 5.* Given an interval $I = [a, b]$, let $l(I) = a + \frac{3}{8}(b - a)$ and $r(I) = a + \frac{5}{8}(b - a)$.

Our construction will depend on a sequence of integers $m_n \in \mathbb{N}$, $n \geq 0$, such that $m_0$ is large and $m_{n+1}$ is much larger than $m_n$.

Let $\Omega$ be the set of all finite sequences $\omega = (\omega_1, \ldots, \omega_n)$ of $l$ and $r$ and length $|\omega| \geq 0$. For $\omega \in \Omega$, we define intervals $I_\omega$ inductively as follows. Let $n = |\omega|$. If $n = 0$, we let $I_\omega$ be the interval of length $2^{-m_0}$ centered on 0. If $n \geq 1$, let $\omega'$ consist of $\omega$ stripped of its

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last digit, and let $I_\omega$ be the interval of length $2^{-m_n}$ centered on $t(I_{\omega'})$, where $t \in \{l, r\}$ is the last digit of $\omega$.

Let $\phi : \mathbb{R} \to [0, 1]$ be a smooth function symmetric around 0, supported on $[-1/2, 1/2]$ and such that $\phi|[-1/4, 1/4] = 1$.

Let $A^\omega : I_\omega \to [-1/2, 1/2]$ be an affine homeomorphism. Let us now define functions smooth functions $\phi^\omega : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ supported inside the intervals $I_\omega + k\alpha$, $1 \le k \le 2n - 1$, such that

$$\phi^\omega(x + k\alpha) = (n - |n - k|)\phi(A^\omega(x)).$$

Note that by selecting $m_n$ sufficiently large, we may assume the following property. *Low recurrence.* For any $n \in \mathbb{N}$, for all $|\omega| = |\omega'| = n$ for any $0 \le |k| \le 2^{2^n}$, $I_\omega + k\alpha$ does not intersect $I_{\omega'}$. In particular, the supports of $\phi^\omega$ and $\phi^{\omega'}$ do not intersect.

We let $\phi_n = \sum_{|\omega|=n} \phi_\omega$. Note that low recurrence implies that, for any $x \in \mathbb{R}/\mathbb{Z}$, the following hold:

(L1) $|\phi_n(x + \alpha) - \phi_n(x)| \le 1$;

(L2) $\phi_n(x - m\alpha) = 0$ if $m \ge 0$ and $2n - 1 + m \le 2^{2^n}$;

(L3) $\phi_n(x) \le n$.

Finally, define a non-decreasing sequence of smooth function $\Phi_N : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$ by

$$\Phi_N = \sum_{0 \le n \le N} \frac{1}{(n+1)^{4/3}} \phi_n.$$

Define the uncountable set $K := \bigcap_n \bigcup_{|\omega|=n} I_\omega$. Note that for $x \in K$, we have the following for $N \ge 1$:

(K1) $\Phi_N(x) = 0$;

(K2) $\Phi_N(x + N\alpha) \ge (1/10)N^{2/3}$;

(K3) for $m \in [0, 2^{2^n} - 2n + 1]$, $\Phi_N(x - m\alpha) \le n(n+1)/2$.

To see the last item, just observe that if $\Phi_N(x - m\alpha) \ge n(n+1)/2$, then $\phi_{n'}(x - m\alpha) > 0$ for some $n \le n' \le N$ [because $\phi_n \le n$ by item (L3)], and by item (L2), we must have $m > 2^{2^n} - 2n + 1$.

Next, observe that item (L1) implies that $\Phi_N(\cdot + \alpha) - \Phi_N(\cdot)$ converges in the $C^0$ topology to some continuous function $\Theta : \mathbb{R}/\mathbb{Z} \to \mathbb{R}$.

Introduce $\xi_N := \int e^{\Phi_N(x)} dx$ and $\Psi_N := e^{\Phi_N}/\xi_N$. Again, by letting the integers $m_n$ grow fast, we obtain that $\xi_N$ is close to 1 for all $N \ge 1$, and that if we consider the circle homeomorphism $h_n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ given by $Dh_n = \Psi_n$ and $h_n(0) = 0$, we get that $h_n$ converges in the $C^0$ topology to some homeomorphism $h : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$.

Because $\Phi_N(x + \alpha) - \Phi_N(x)$ converges in $C^0$ to a continuous function $\Theta$, we get that $f_n(x) = h_n(h_n^{-1}(x) + \alpha)$ is converging in the $C^1$ topology to some $f$ satisfying $f(x) = h(h^{-1}(x) + \alpha)$ and $\ln Df = \Theta \circ h^{-1}$. In particular, $f$ is minimal.

Note that all $x \in h(K)$ are $C$-good for some absolute $C$, because $Df^n(x) \ge e^{n^{2/3}/10}$ for $x \in h(K)$ by item (K2).

However, item (K2) also implies that for each $x$, from time to time, $h^{-1}(x) - n\alpha$, for $n \ge 0$, will visit regions where $\sup_N \Phi_N$ is large, so for $x \in h(K)$, $Df^{-n}(x)$ will be large,

https://doi.org/10.1017/etds.2021.117 Published online by Cambridge University Press

because $\Phi_N(h^{-1}(x)) = 0$ by item (K1). Moreover, if $x \in h(K)$, item (K3) implies that $\Phi_N(h^{-1}(x) - n\alpha)$ is at most of order $(\ln \ln n)^2$, so $\sum_{n\geq 1} |Df^{-n}(x)|^{-2} = \infty$. □

*Acknowledgement.* We are grateful to the referee for many useful comments on the first versions of this paper.

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