J. Fluid Mech. (2015), *vol.* 777, *pp.* 591–603.
doi:10.1017/jfm.2015.384

Scaling bounds on dissipation in turbulent flows

Christian Seis†

Institut für Angewandte Mathematik, Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

(Received 5 January 2015; revised 3 June 2015; accepted 30 June 2015; first published online 22 July 2015)

We propose a new rigorous method for estimating statistical quantities in fluid dynamics such as the (average) energy dissipation rate directly from the equations of motion. The method is tested on shear flow, channel flow, Rayleigh–Bénard convection and porous medium convection.

Key words: turbulent convection, turbulent flows

1. Introduction

One of the most fascinating features in turbulent flows is the emergence of complicated chaotic structures involving a wide range of length scales behind which 'typical' flow patterns are still recognizable. The state of motion is too complex to allow a detailed description of the fluid velocity and experimental or numerical measurements of certain system quantities appear disorganized and unpredictable. Yet, some statistical properties are reproducible (Frisch 1995). One of the challenges in theoretical fluid dynamics is thus the derivation of quantitative statements on turbulent flows. Many of the approaches consist of various approximation procedures, the imposition of physically motivated but *ad hoc* assumptions (e.g. 'closure') or the introduction of scaling hypotheses. Rigorous results beginning with the equations of motion directly are therefore indispensable for checking the validity of the imposed simplifications and for justifying secondary models and theories.

The idea of extracting information about driven turbulent flows via bounds on physical quantities through mathematically justifiable operations and without imposing *ad hoc* assumptions goes back to the pioneering works of Malkus (1954), Howard (1972) and Busse (1970, 1979). These authors applied variational approaches for the derivation of bounds on the energy dissipation rate in models for shear flow and heat convection. In the 1990s, Constantin and Doering introduced a practical framework for estimating physical quantities rigorously and directly from the equations of motion, which they called the 'background flow method' (Doering & Constantin 1994, 1996; Constantin & Doering 1995).

The background flow method is an extremely robust method for constructing bounds in fluid dynamics and builds on techniques developed by Hopf (1941) to generalize Leray solutions of the Navier–Stokes equations to finite geometries with physical boundary conditions. In this method one manipulates the equations of motion relative to a steady trial background state, which satisfies the forcing conditions. On decomposing the quantity of interest, e.g. the energy dissipation, into a background

† Email address for correspondence: seis@iam.uni-bonn.de

and a fluctuating component, the background part yields an upper bound if the fluctuation term satisfies a certain non-negativity condition, which is often referred to as the spectral constraint. In a certain sense, finding the least upper bound using Constantin and Doering's method resembles a variational saddle point problem. The equations imposed on the fluctuations necessarily include the equations of motion. To simplify the derivation of the Euler–Lagrange equations corresponding to the variational problem for the fluctuation term, however, fluctuations are often chosen in a much larger class of functions. In other words, the spectral constraint in the background method is required to hold for an infinite-dimensional set of vector fields, that strictly contains the solutions of the equations of motion. In that case, enforcing the spectral constraint may yield an overestimation of the quantity of primal interest.

After its introduction, the background flow method was the upper bound method with applications ranging from various problems in turbulent heat convection and boundary-force-driven and body-force-driven turbulence to idealized models in magnetohydrodynamics. The method was soon improved by Nicodemus, Grossmann & Holthaus (1997) who introduced an additional balance parameter, and Kerswell (1998) showed that this improved method is actually 'equivalent' to the approaches of Busse and Howard.

Apart from its practical performance, for many years the background flow method was considered as a rigorous manifestation of Malkus's marginally stable boundary layer theory (Malkus 1954). The latter is based on the assumption that turbulent boundary-driven flows organize themselves into marginally stable configurations. If the well-mixed core is bounded by thin laminar boundary layers, the thickness of these layers is determined by the condition of marginal stability. The association of the background flow method with Malkus's theory relies on the surprising observation that the spectral condition in the background flow method resembles a nonlinear stability condition on the background flow. A recent work of C. Nobili & F. Otto (Personal observations), however, proves the failure of this association – at least in the context of infinite Prandtl number Rayleigh–Bénard convection: the authors compute the least upper bound on the Nusselt number (the quantity of interest in Rayleigh–Bénard convection) within the framework of the background flow method. This bound, however, exceeds the bound derived by Otto & Seis (2011) using completely different methods. In the context of Rayleigh–Bénard convection, it thus seems that this physical interpretation of the background flow method is misleading. Whether the background flow method indeed gives physically relevant information (apart from scaling bounds) in different problems of fluid dynamics can only be speculated.

In this paper, we focus on the energy dissipation rate as an example of one specific physical quantity and present a new method for its rigorous estimation directly from the equations of motion. (In fact, the method has already been introduced in Otto & Seis (2011), but its universality was not seen at that time.) To allow a straight comparison with the background flow method, the method is tested on the problems considered by Constantin and Doering in Doering & Constantin (1994), Constantin & Doering (1995), Doering & Constantin (1996, 1998). More precisely, we study the following classical fluid dynamics problems: shear flow, channel flow, Rayleigh–Bénard convection and porous medium convection. All of these problems can be considered as model problems for boundary-force-driven or body-force-driven flows or for thermal convection. In fact, we will recover the same results as Constantin and Doering in the above mentioned papers.

Our new approach presented in this paper is entirely different from the background flow method in that it is based on (local) conservation laws. More precisely, at the heart of our method conservation laws for certain flux components are established. The conservation laws are local in the direction of the symmetry axis and the conserved quantity can be explicitly expressed in terms of the energy dissipation rate. Averaging over small boundary layers and applying elementary estimates yields a bound on the energy dissipation rate. How the new method applies to fluids with different boundary conditions, e.g. no-stress, remains to be seen.

The author believes that the strength of the new method relies on the fact that it uses the equations of motion and some secondary derived physical laws directly instead of working with a rigid 'upper bound construction' which in some cases is too restrictive to yield the optimal result, cf. Otto & Seis (2011). The mathematical operations involved are elementary.

In order to advertise the new method as an alternative to the most widely used background flow method, we choose quick applications at the expense of not optimizing the numerical constants in our bounds. To give a flavour of how competitive results can be obtained by the method, we compute the numerical prefactor only in the first example (shear flow). We moreover caution the reader that our results are only 'formally' true in the sense that many of the mathematical operations performed apply only to sufficiently smooth solutions of the Navier–Stokes equations. Of course, the results can be made rigorous if the analysis is performed on suitable weak solutions, e.g. Leray solutions, in the appropriate framework.

The article is organized as follows: after fixing the notation in § 2, we will derive upper bounds on the energy dissipation for shear flows (§ 3), channel flows (§ 4), Rayleigh–Bénard convection (§ 5) and porous medium convection (§ 6).

2. Notation

In what follows, we will try to be as consistent as possible with regard to the notation even though different physical problems will be considered.

Our models are non-dimensionalized.

Our system is a layer of fluid in the box $[0, L)^{d-1} \times [0, 1]$ where L is an arbitrary positive number that will not enter our analysis. Throughout the article, we will refer to the first $d-1$ coordinates as the horizontal ones and the last coordinate as the vertical one. We write $x = (y, z) \in \boldsymbol{R}^{d-1} \times \boldsymbol{R}$ accordingly, and denote by $\{e_1, \ldots, e_{d-1}, e_d\}$ the canonical basis of $\boldsymbol{R}^d = \boldsymbol{R}^{d-1} \times \boldsymbol{R}$.

We assume periodic boundary conditions in the horizontal directions for all variables involved. The horizontal boundaries are rigid and the imposed conditions will depend on the particular physical problem under consideration.

The fluid velocity is denoted by u, and we write $u = (v, w) \in \boldsymbol{R}^{d-1} \times \boldsymbol{R}$ to distinguish the horizontal velocity vector from the vertical component. The hydrodynamic pressure is p and T is the temperature field.

We consider the rate of energy dissipation

$$\varepsilon = \int_0^1 \langle |\nabla u|^2 \rangle \, \mathrm{d}z, \tag{2.1}$$

where $\langle \cdot \rangle$ denotes the horizontal and time average, that is,

$$\langle f \rangle = \lim_{\tau \uparrow \infty} \frac{1}{\tau} \int_0^\tau \frac{1}{L^{d-1}} \int_{[0,L)^{d-1}} f(t, y) \, \mathrm{d}y \, \mathrm{d}t. \tag{2.2}$$

In general, the long-time average need not exist, even if finite-time averages are bounded, and we could be more careful at this point by choosing the lim sup instead of the lim. However, for the sake of a clearer statement and to simplify the subsequent analysis, we will be quite formal in most of our computations.

3. Shear flow

As the simplest example of a boundary-driven flow, we consider a fluid which is confined between two parallel plates that are moving at a constant speed relative to each other. The equations of motion are the Navier–Stokes equations in a box $[0, L)^{d-1} \times [0, 1]$,

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = 0, \tag{3.1}$$

$$\nabla \cdot u = 0, \tag{3.2}$$

and, as stated in the previous section, the velocity u and the pressure p are both periodic in the horizontal variables. At the horizontal plates, we assume no-slip boundary conditions for the velocity field. If the upper boundary plate is moving with constant speed Re in direction e_1 while the lower plate is at rest, we must have

$$u = \begin{cases} 0 & \text{for } z = 0 \\ Re\, e_1 & \text{for } z = 1. \end{cases} \tag{3.3}$$

Notice that all quantities are non-dimensionalized and Re is the Reynolds number.

The scaling of the energy dissipation rate ε as a function of the Reynolds number is of fundamental importance in many engineering applications since, for steady states, the energy dissipation rate measures the rate at which work must be done by an agent to keep the upper plate moving.

The interest in understanding the energy dissipation rate in boundary-driven fluid flows as a function of the Reynolds number dates back to Stokes. It is well-known (Serrin 1959; Keller, Rubenfeld & Molyneux 1967) that solutions to the Stokes equation minimize the dissipation rate among all divergence-free vector fields with a fixed velocity at the boundary. In many situations of physical interest, this solution is laminar and thus continues to exist as a solution of the Navier–Stokes equation. In our case, the laminar solution is the so-called Couette flow $u_C(z) = Reze_1$ which dissipates energy $\varepsilon_C = Re^2$. For small Reynolds numbers, u_C is a stable solution of the Navier–Stokes equation in the sense that any sufficiently small perturbation will decrease in time. For large Reynolds numbers, however, perturbations of the laminar steady state are unstable and the flows can become chaotic or turbulent. The energy dissipation rate is a monotone function of the Reynolds number and acts as a measure of how turbulent the flow is. With regard to the presence of steady regular solutions in the high Reynolds number regime also, absolute lower bounds on the energy dissipation will be dictated by non-turbulent flows: $\varepsilon \geqslant \varepsilon_C = Re^2$. A rigorous scaling theory for the dissipation rate can hence only be an upper bound theory. An upper bound, however, sets limits on the possible turbulent structures of the flow and is thus an indispensable piece of information in the study of turbulence.

Developing a conventional statistical turbulence theory for high Reynolds numbers, Doering & Constantin (1994, appendix A) predict the 'logarithmic friction law'

$$\varepsilon \sim \frac{Re^3}{(\log Re)^2} \quad \text{for } Re \gg 1, \tag{3.4}$$

which is in accordance with the experimental data derived e.g. by Lathrop, Fineberg & Swinney (1992). In the same paper, the authors derive a first upper bound which proves the conjectured rate up to the logarithmic factor, that is, they prove that $\varepsilon \lesssim Re^3$ using the background flow method. In the following, we reproduce Constantin and

Doering's bound using our new method. We will first derive a quick scaling bound and then compute numerical constants in a second step.

As a starting point of our analysis, we recall that the energy dissipation can equally be expressed, for instance, as the trace of the vertical derivative of $v_1 = \boldsymbol{u}\cdot\boldsymbol{e}_1$ at the top plate, that is,

$$\varepsilon = Re\langle \partial_z|_{z=1} v_1\rangle. \tag{3.5}$$

This is the averaged energy balance. The identity follows from testing (3.1) with $\boldsymbol{u}$, integrating by parts and using (3.2) and (3.3). Moreover, multiplying (3.1) by $\boldsymbol{e}_1$, taking the horizontal and time average, and using (3.2) and the periodicity, we see that

$$\langle wv_1 - \partial_z v_1\rangle = \text{const.} \quad \text{for all } z \in [0, 1], \tag{3.6}$$

and thus, in view of (3.3) and (3.5), we obtain the formula

$$\varepsilon = Re\langle \partial_z v_1 - wv_1\rangle \quad \text{for all } z \in [0, 1]. \tag{3.7}$$

In particular, averaging this identity over some boundary layer $[0, \ell]$, where $0 \leqslant \ell \leqslant 1$ has to be determined later, we have

$$\varepsilon = \frac{Re}{\ell}\int_0^\ell \langle \partial_z v_1\rangle\,\mathrm{d}z - \frac{Re}{\ell}\int_0^\ell \langle wv_1\rangle\,\mathrm{d}z. \tag{3.8}$$

On the one hand, an application of Jensen's inequality yields

$$\frac{1}{\ell}\int_0^\ell \langle \partial_z v_1\rangle\,\mathrm{d}z \leqslant \left(\frac{1}{\ell}\int_0^\ell \langle (\partial_z v_1)^2\rangle\,\mathrm{d}z\right)^{1/2}. \tag{3.9}$$

On the other hand, with the help of the Cauchy–Schwarz and Poincaré inequalities, we estimate

$$\begin{aligned}\frac{1}{\ell}\int_0^\ell \langle wv_1\rangle\,\mathrm{d}z &\leqslant \frac{1}{\ell}\left(\int_0^\ell \langle w^2\rangle\,\mathrm{d}z\int_0^\ell \langle v_1^2\rangle\,\mathrm{d}z\right)^{1/2}\\ &\leqslant \frac{4\ell}{\pi^2}\left(\int_0^\ell \langle (\partial_z w)^2\rangle\,\mathrm{d}z\int_0^\ell \langle (\partial_z v_1)^2\rangle\,\mathrm{d}z\right)^{1/2}.\end{aligned} \tag{3.10}$$

The optimal Poincaré constant $4\ell^2/\pi^2$ is computed in the appendix A. Notice, moreover, that the divergence-free condition (3.2) reveals that $\langle(\partial_z w)^2\rangle \leqslant \langle|\nabla_y v|^2\rangle$. The above estimates can thus be combined into

$$\frac{\varepsilon}{Re} \leqslant \left(\frac{1}{\ell^{1/2}} + \frac{4\ell}{\sqrt{2}\pi^2}\left(\int_0^{1/2}\langle(\partial_z w)^2 + |\nabla_y v|^2\rangle\,\mathrm{d}z\right)^{1/2}\right)\left(\int_0^{1/2}\langle(\partial_z v_1)^2\rangle\,\mathrm{d}z\right)^{1/2}. \tag{3.11}$$

To obtain a quick bound, we estimate the integral expressions by ε and obtain

$$\varepsilon \lesssim \frac{Re}{\ell^{1/2}}\varepsilon^{1/2} + Re\ell\varepsilon. \tag{3.12}$$

Optimizing the last expression in ℓ yields $\ell \sim \varepsilon^{-1/3} \ll 1$, and thus $\varepsilon \lesssim Re\varepsilon^{2/3}$, which entails

$$\varepsilon \lesssim Re^3. \tag{3.13}$$

This estimate agrees with the bound derived by Doering & Constantin (1994).

To compute a numerical constant, we optimize (3.11) in ℓ and obtain

$$\frac{\varepsilon}{Re} \leqslant \frac{3}{2^{1/6}\pi^{2/3}} \left(\int_0^{1/2} \langle (\partial_z w)^2 + |\nabla_y v|^2 \rangle \, \mathrm{d}z \right)^{1/6} \left(\int_0^{1/2} \langle (\partial_z v_1)^2 \rangle \, \mathrm{d}z \right)^{1/2}. \tag{3.14}$$

With the help of Young's inequality $ab \leqslant (1/p)a^p + (1/q)b^q$ for any $p, q \in (1, \infty)$ with $1/p + 1/q = 1$, we further have

$$\begin{aligned} \frac{\varepsilon}{Re} &\leqslant \frac{9\delta}{2^{4/3}\pi^{4/3}} \left(\int_0^{1/2} \langle (\partial_z w)^2 + |\nabla_y v|^2 \rangle \, \mathrm{d}z \right)^{1/3} + \frac{1}{2\delta} \int_0^{1/2} \langle (\partial_z v_1)^2 \rangle \, \mathrm{d}z \\ &\leqslant \sqrt{\frac{27}{2} \frac{\delta^2}{\pi^2}} + \frac{1}{2\delta} \int_0^{1/2} \langle (\partial_z w)^2 + |\nabla_y v|^2 + (\partial_z v_1)^2 \rangle \, \mathrm{d}z \\ &\leqslant \sqrt{\frac{27}{2} \frac{\delta^2}{\pi^2}} + \frac{1}{2\delta} \int_0^{1/2} \langle |\nabla u|^2 \rangle \, \mathrm{d}z, \end{aligned} \tag{3.15}$$

where $\delta > 0$ is an arbitrary constant. For symmetry reasons, an analogous estimate can be derived in the upper half of the box $[0, L)^{d-1} \times [0, 1]$. Altogether we have

$$\frac{\varepsilon}{Re} \leqslant \sqrt{\frac{27}{2} \frac{1}{\pi^2} \delta^2} + \frac{1}{4\delta} \varepsilon. \tag{3.16}$$

Optimizing in δ finally yields

$$\frac{\varepsilon}{Re^3} \leqslant \frac{81\sqrt{6}}{128\pi^2} \approx 0.157. \tag{3.17}$$

This bound is a factor of two larger than the bound derived by Doering & Constantin (1994). To keep the present paper concise, we do not make any attempt at improving the constant.

4. Channel flow

As the simplest example of a body-force-driven flow, we consider a fluid in a rectangular domain, which is driven by a pressure gradient in the direction of one of the horizontal boundary plates. The problem is modelled by the forced Navier–Stokes equation

$$\partial_t u + u \cdot \nabla u - \Delta u + \nabla p = Gre_1, \tag{4.1}$$

$$\nabla \cdot u = 0, \tag{4.2}$$

in $[0, L)^{d-1} \times [0, 1]$, supplemented with no-slip boundary conditions

$$u = 0 \quad \text{for } z \in \{0, 1\}. \tag{4.3}$$

Here, Gr denotes the non-dimensional Grashof number. We remark that when the force is specified *a priori* as in the present situation, the Reynolds number is an emergent quantity.

The channel flow problem has the laminar solution $u_P(z) = (Gr(z^2 - z)e_1)/2$, the so-called Poiseuille flow, for which the energy dissipation rate is $\varepsilon_P = Gr^2/12$.

The Poiseuille flow plays a similar role in the channel flow problem as the Couette flow in the shear flow problem: u_P solves the corresponding Stokes equation and is an unstable solution of the Navier–Stokes equation in the high Grashof number regime. When bounds are expressed in terms of the Grashof number instead of the Reynolds number, the Stokes limit represents an upper bound on the energy dissipation rate: $\varepsilon \leqslant \varepsilon_P = Gr^2/12$.

The scaling of the energy dissipation rate in the high Grashof number regime is expected to obey the modified logarithmic friction law

$$\varepsilon \sim \frac{Gr^{3/2}}{(\log Gr)^2}, \tag{4.4}$$

similar to the shear flow. The best we can hope for is a lower bound on the energy dissipation rate since the Poiseuille flow always provides an upper bound. A first rigorous lower bound on the energy dissipation rate was established by Constantin & Doering (1995), $\varepsilon \gtrsim Gr^{3/2}$, which is optimal up to the logarithm. For further improvements we refer to Petrov, Lu & Doering (2005) and references therein. In the following, we will recover this scaling bound by applying our new method.

We see from testing (4.1) with u, integrating by parts and using (4.2) and (4.3) that the energy dissipation rate is proportional to the average flow velocity

$$\varepsilon = Gr \int_0^1 \langle v_1 \rangle \, \mathrm{d}z. \tag{4.5}$$

Moreover, multiplying (4.1) by e_1, taking the horizontal and time average, and using (4.2) and periodicity, we obtain

$$\partial_z \langle w v_1 - \partial_z v_1 \rangle = Gr \quad \text{for all } z \in [0, 1]. \tag{4.6}$$

From the no-slip boundary conditions (4.3) we thus infer that

$$\langle w v_1 - \partial_z v_1 \rangle = Gr z - \langle \partial_z|_{z=0} v_1 \rangle \quad \text{for all } z \in [0, 1]. \tag{4.7}$$

On the one hand, averaging in z over the strip $[1-\ell, 1]$ of width $\ell \ll 1$ yields

$$Gr \sim \frac{1}{\ell} \int_{1-\ell}^1 \langle v_1 w - \partial_z v_1 \rangle \, \mathrm{d}z + \langle \partial_z|_{z=0} v_1 \rangle. \tag{4.8}$$

On the other hand, averaging (4.7) over $[0, \ell]$ yields

$$\langle \partial_z|_{z=0} v_1 \rangle \sim \frac{1}{\ell} \int_0^\ell \langle \partial_z v_1 - v_1 w \rangle \, \mathrm{d}z + \ell Gr. \tag{4.9}$$

Since $\ell \ll 1$, a combination of the previous two estimates gives

$$Gr \lesssim \frac{1}{\ell} \int_0^\ell \langle \partial_z v_1 - v_1 w \rangle \, \mathrm{d}z + \frac{1}{\ell} \int_{1-\ell}^1 \langle v_1 w - \partial_z v_1 \rangle \, \mathrm{d}z. \tag{4.10}$$

We can apply the same arguments as in the shear flow case considered in the previous section to deduce

$$Gr \lesssim \frac{\varepsilon^{1/2}}{\ell^{1/2}} + \ell \varepsilon. \tag{4.11}$$

The optimal ℓ is of size $\ell \sim \varepsilon^{-1/3}$, which implies that

$$Gr^{3/2} \lesssim \varepsilon. \tag{4.12}$$

In view of (4.5), this bound is equivalent to

$$Gr^{1/2} \lesssim \int_0^1 \langle v_1 \rangle \, \mathrm{d}z, \tag{4.13}$$

which agrees with the bound derived by Constantin & Doering (1995) and appears to be sharp to within logarithms.

5. Rayleigh–Bénard convection

Rayleigh–Bénard convection is the transport of heat by thermal convection in a fluid layer that is heated from below and cooled from above. The problem is modelled by the equations of the Boussinesq approximation

$$\partial_t T + u \cdot \nabla T - \Delta T = 0, \tag{5.1}$$

$$\nabla \cdot u = 0, \tag{5.2}$$

$$\frac{1}{Pr}(\partial_t u + u \cdot \nabla u) - \Delta u + \nabla p = Ra\, T e_d. \tag{5.3}$$

The system is non-dimensionalized and admits two controlling parameters, the Rayleigh number Ra and the Prandtl number Pr. The equations are complemented by the boundary conditions

$$T = 1 \quad \text{on } z = 0, \quad \text{and} \quad T = 0 \quad \text{on } z = 1, \tag{5.4a,b}$$

representing heating at the bottom and cooling at the top, and no-slip boundary conditions for the velocity field,

$$u = 0 \quad \text{on } z \in \{0, 1\}. \tag{5.5}$$

The quantity of interest in this model is the so-called Nusselt number, a measure for the average upward heat flux. It is defined by

$$Nu = \int_0^1 \langle (uT - \nabla T) \cdot e_d \rangle \, \mathrm{d}z = \int_0^1 \langle wT - \partial_z T \rangle \, \mathrm{d}z. \tag{5.6}$$

The scaling of the Nusselt number in terms of Ra and Pr has been the subject of enormous experimental, numerical and theoretical research for over fifty years. For a recent review, we refer to Ahlers, Grossmann & Lohse (2009) and references therein. For all values of Ra and Pr, a laminar solution is given by $u_c = 0$ and $T_c = 1 - z$, which corresponds to pure conduction. The corresponding Nusselt number is $Nu_c = 1$, and the laminar solution is unstable for large Rayleigh numbers.

In the high Rayleigh number regime, $Ra \gg 1$, the scaling of the Nusselt number is proportional to the scaling of the energy dissipation rate. Indeed, testing (5.3) with u, using the incompressibility assumption (5.2) and invoking the boundary conditions (5.4) and (5.5) for T and u yields

$$\varepsilon = Ra(Nu - 1) \sim Ra\, Nu. \tag{5.7}$$

The bound on the Nusselt number in the ultimate turbulent regime is expected to be

$$Nu \sim Ra^{1/2}, \tag{5.8}$$

if $Ra \gg 1$, uniformly in Pr. In the following, we derive an upper bound on this scaling with the help of different representations of the Nusselt number, similar to the approach in the previous two sections. Averaging the heat equation (5.1) and using periodicity and (5.2), it follows that the heat flux is constant on every horizontal slice, that is

$$Nu = \langle Tw - \partial_z T \rangle \quad \text{for all } z \in [0, 1]. \tag{5.9}$$

Now, averaging this identity over a boundary layer of thickness $\ell \in [0, 1]$, and using the maximum principle on the temperature, $\max |T| \leqslant 1$, which is enforced by the boundary conditions (if not initially, then exponentially fast in time), we obtain that

$$Nu \leqslant \frac{1}{\ell} \int_0^\ell \langle wT \rangle \, \mathrm{d}z + \frac{1}{\ell} \leqslant \frac{1}{\ell} \int_0^\ell \langle |w| \rangle \, \mathrm{d}z + \frac{1}{\ell}. \tag{5.10}$$

We use Poincaré's and Hölder's inequalities and (5.5) to bound the integral over the vertical velocity component by the energy dissipation rate ε, that is

$$\int_0^\ell \langle |w| \rangle \, \mathrm{d}z \leqslant \ell \int_0^\ell \langle |\partial_z w| \rangle \, \mathrm{d}z \leqslant \ell^{3/2} \left(\int_0^\ell \langle (\partial_z w)^2 \rangle \, \mathrm{d}z \right)^{1/2} \leqslant \ell^{3/2} (Ra\, Nu)^{1/2} \tag{5.11}$$

by (5.7), so that

$$Nu \leqslant \ell^{1/2} (Ra\, Nu)^{1/2} + \frac{1}{\ell}. \tag{5.12}$$

Optimizing in ℓ yields that $\ell \sim (Ra\, Nu)^{-1/3}$, which entails that

$$Nu \lesssim Ra^{1/2}. \tag{5.13}$$

This is precisely the same scaling law as derived by Doering & Constantin (1996).

Notice that Whitehead & Doering (2011) prove the bound $Nu \lesssim Ra^{5/12}$ for two-dimensional Rayleigh–Bénard convection with free-stress boundary conditions. The proof relies heavily on the two-dimensional structure (no vortex stretching) and the free-slip boundary conditions (homogeneous vorticity boundary conditions). Whether such a bound can be extended to our problem at hand is not clear to the author. Recent numerical simulations at least indicate that the 5/12 scaling should be expected for any solution of (5.1)–(5.3) with finite energy dissipation rate (Hassanzadeh, Chini & Doering 2014).

Applying the same method combined with sophisticated maximal regularity arguments, Chuffrut, Nobili & Otto (2014) recently obtained new bounds on Nu which improve this bound in certain Ra–Pr regimes. The results in particular apply to the large Prandtl number regime. The Nusselt number bound can be interpreted as a bound on the average temperature gradient, cf. (6.9) below. Developing techniques similar to those presented in this paper, the author has derived bounds on higher-order derivatives of the temperature field in infinite Prandtl number convection and estimated deviations of the average vertical temperature profile from linearity (Seis 2013).

6. Porous medium convection

We finally consider thermal convection in a porous medium. In this case, Darcy's law approximates the Navier–Stokes equations, and the Rayleigh–Bénard system (5.1)–(5.3) reduces to

$$\partial_t T + u \cdot \nabla T - \Delta T = 0, \tag{6.1}$$

$$\nabla \cdot u = 0, \tag{6.2}$$

$$u + \nabla p = (Ra) T e_d. \tag{6.3}$$

As before, the non-dimensional number Ra is the Rayleigh number. The boundary condition satisfied by the fluid velocity is

$$w = 0 \quad \text{on } z \in \{0, 1\}, \tag{6.4}$$

and we suppose that the container is cooled from above and heated from below, modelled by

$$T = \begin{cases} 1 & \text{on } z = 0, \\ 0 & \text{on } z = 1. \end{cases} \tag{6.5}$$

Again, we assume periodic boundary conditions in all horizontal directions for all quantities involved.

As in the case of classical Rayleigh–Bénard convection, the quantity of interest here is the Nusselt number

$$Nu = \int_0^1 \langle wT - \partial_z T \rangle \, \mathrm{d}z. \tag{6.6}$$

Before estimating the energy dissipation rate in this example, we start with a study of the Nusselt number. Experiments and numerics suggest that

$$Nu \sim Ra \quad \text{for } Ra \gg 1, \tag{6.7}$$

cf. Hewitt, Neufeld & Lister (2012). Because of the existence of laminar solutions, we can only expect to prove the upper bound $Nu \lesssim Ra$, previously established by Doering & Constantin (1998). We first need to establish some alternative identities for Nu. We first recall that on averaging (6.1), we obtain $\langle wT - \partial_z T \rangle = \text{const.}$, and thus

$$Nu = \langle wT - \partial_z T \rangle \quad \text{for all } z \in [0, 1]. \tag{6.8}$$

In particular, $Nu = -\langle \partial_z |_{z=0} T \rangle$ thanks to (6.4). Now testing (6.1) with T, integrating by parts and using (6.2), (6.4) and (6.5), we see that

$$Nu = \int_0^1 \langle |\nabla T|^2 \rangle \, \mathrm{d}z. \tag{6.9}$$

On the other hand, by the definition of Nu and (6.5), testing (6.3) with u and using (6.2) and (6.4), we obtain

$$Nu = \frac{1}{Ra} \int_0^1 \langle |u|^2 \rangle \, \mathrm{d}z + 1. \tag{6.10}$$

We are now able to estimate the Nusselt number. Letting $\ell \in (0, 1)$ be an arbitrary number, we estimate

$$Nu \leqslant \frac{1}{\ell} \int_0^\ell \langle wT \rangle \, dz + \frac{1}{\ell} \tag{6.11}$$

as in the previous section. Because $|T| \leqslant 1$ by the maximum principle for the temperature, we obtain via Jensen's inequality

$$Nu \leqslant \left(\frac{1}{\ell} \int_0^\ell \langle w^2 \rangle \, dz \right)^{1/2} + \frac{1}{\ell}. \tag{6.12}$$

By the Nusselt number representation (6.10), this yields

$$Nu \lesssim \frac{(Ra\, Nu)^{1/2}}{\ell^{1/2}} + \frac{1}{\ell}. \tag{6.13}$$

This bound is optimized by $\ell \sim 1$, so that

$$Nu \lesssim Ra \tag{6.14}$$

because $Ra \gg 1$.

In a final step, we would like to relate the current bound on the Nusselt number to a bound on the viscous dissipation rate. In fact, we will show that $\varepsilon \lesssim (Ra)^2 Nu$, so that the above statement turns into

$$\varepsilon \lesssim (Ra)^3. \tag{6.15}$$

We start noting that Darcy's law (6.3) together with the boundary conditions (6.4) and (6.5) provides us with Neumann boundary conditions for the pressure: $\partial_z|_{z=0} p = Ra$ and $\partial_z|_{z=1} p = 0$. In particular, differentiating the horizontal velocity components with respect to z, $\partial_z v = -\nabla_y \partial_z p$, multiplying by v, averaging and integrating by parts yields

$$\langle v \cdot \partial_z v \rangle = -\langle v \cdot \nabla_y \partial_z p \rangle = \langle (\nabla_y \cdot v) \partial_z p \rangle. \tag{6.16}$$

In particular, the above values for $\partial_z p$ and the horizontal periodicity imply that

$$\langle v \cdot \partial_z v \rangle|_{z=0,1} = 0. \tag{6.17}$$

It thus follows via integration by parts that

$$\int_0^1 \langle |\nabla v|^2 \rangle \, dz = \langle v \cdot \partial_z v \rangle|_{z=0}^{z=1} - \int_0^1 \langle v \cdot \Delta v \rangle \, dz \overset{(6.17)}{=} - \int_0^1 \langle v \cdot \Delta v \rangle \, dz. \tag{6.18}$$

Now notice that $-\Delta v = (Ra) \nabla_y \partial_z T$. Indeed, taking the divergence of (6.3) yields $\Delta p = (Ra) \partial_z T$, and thus $-\Delta v = \nabla_y \Delta p = (Ra) \nabla_y \partial_z T$. Therefore, (6.18) becomes

$$\int_0^1 \langle |\nabla v|^2 \rangle \, dz = Ra \int_0^1 \langle v \cdot \nabla_y \partial_z T \rangle \, dz = -Ra \int_0^1 \langle (\nabla_y \cdot v) \partial_z T \rangle \, dz. \tag{6.19}$$

Since $|\nabla_y \cdot v| \lesssim |\nabla_y v| \leqslant |\nabla v|$, we can use the Cauchy–Schwarz inequality to obtain

$$\int_0^1 \langle |\nabla v|^2 \rangle \, dz \lesssim Ra \left(\int_0^1 \langle |\nabla v|^2 \rangle \, dz \int_0^1 \langle |\nabla T|^2 \rangle \, dz \right)^{1/2}, \tag{6.20}$$

and thus, via (6.9),

$$\int_0^1 \langle |\nabla v|^2 \rangle \, \mathrm{d}z \lesssim (Ra)^2 Nu. \tag{6.21}$$

The estimate of the vertical velocity component is easier because of (6.4): testing $\Delta w = (Ra)\Delta T - \partial_z \Delta p = (Ra)\Delta_y T$ with w, integrating by parts and using the Cauchy–Schwarz inequality yields

$$\int_0^1 \langle |\nabla w|^2 \rangle \, \mathrm{d}z \lesssim (Ra)^2 Nu. \tag{6.22}$$

Combining the last two estimates finally yields $\varepsilon \lesssim (Ra)^2 Nu$ as desired.

Acknowledgements

Part of this work was performed when the author was visiting the MPI MIS in Leipzig. The author is grateful for the hospitality. He further acknowledges stimulating discussions with C. Nobili, C. Doering, J. Whitehead and R. Wittenberg.

Appendix A

In this appendix, we derive the Poincaré inequality

$$\int_0^\ell f^2 \, \mathrm{d}z \leqslant \frac{4\ell^2}{\pi^2} \int_0^\ell (f')^2 \, \mathrm{d}z, \tag{A 1}$$

for all functions f such that $f(0) = 0$. This estimate is sharp as can be seen by choosing $f(z) = \sin(\pi/2\ell)z$. Notice that (A 1) is equivalent to the Poincaré inequality

$$\int_{-\ell}^\ell g^2 \, \mathrm{d}z \leqslant \frac{4\ell^2}{\pi^2} \int_{-\ell}^\ell (g')^2 \, \mathrm{d}z, \tag{A 2}$$

for all functions g satisfying $\int_{-\ell}^\ell g \, \mathrm{d}z = 0$ in the sense that the Poincaré constants agree. Indeed, if f satisfies (A 1) and g is obtained from f by odd reflection at $z = 0$, then g has mean zero on $(-\ell, \ell)$ and satisfies (A 2). On the other hand, if g is an odd function for which (A 2) holds and f is the restriction of g to $(0, \ell)$ then f satisfies (A 1).

The statement in (A 2) is equivalent to the variational problem

$$\frac{\pi^2}{4\ell} = \min \left\{ \int_{-\ell}^\ell (g')^2 \, \mathrm{d}z : \int_{-\ell}^\ell g \, \mathrm{d}z = 0, \int_{-\ell}^\ell g^2 \, \mathrm{d}z = \ell \right\}, \tag{A 3}$$

and the expression on the right is minimized by $g(z) = \sin(\pi/2\ell)z$.

REFERENCES

AHLERS, G., GROSSMANN, S. & LOHSE, D. 2009 Heat transfer and large scale dynamics in turbulent Rayleigh–Bénard convection. *Rev. Mod. Phys.* **81**, 503–537.

BUSSE, F. 1979 The optimum theory of turbulence. *Adv. Appl. Mech.* **18**, 77–121.

BUSSE, F. H. 1970 Bounds for turbulent shear flows. *J. Fluid Mech.* **41** (1), 219–240.

CHUFFRUT, A., NOBILI, C. & OTTO, F. 2014 Bounds on Nusselt number at finite Prandtl number; Preprint arXiv:1412.4812.

CONSTANTIN, P. & DOERING, C. R. 1995 Variational bounds on energy dissipation in incompressible flows. II. Channel flow. *Phys. Rev.* E **51**, 3192–3198.

DOERING, C. R. & CONSTANTIN, P. 1994 Variational bounds on energy dissipation in incompressible flows: shear flow. *Phys. Rev.* E **49**, 4087–4099.

DOERING, C. R. & CONSTANTIN, P. 1996 Variational bounds on energy dissipation in incompressible flows. III. Convection. *Phys. Rev.* E **53**, 5957–5981.

DOERING, C. R. & CONSTANTIN, P. 1998 Bounds for heat transport in a porous layer. *J. Fluid Mech.* **376**, 263–296.

FRISCH, U. 1995 *Turbulence*. Cambridge University Press, (the legacy of A. N. Kolmogorov).

HASSANZADEH, P., CHINI, G. P. & DOERING, C. R. 2014 Wall to wall optimal transport. *J. Fluid Mech.* **751**, 627–662.

HEWITT, D. R., NEUFELD, J. A. & LISTER, J. R. 2012 Ultimate regime of high Rayleigh number convection in a porous medium. *Phys. Rev. Lett.* **108**, 224503.

HOPF, E. 1941 Ein allgemeiner Endlichkeitssatz der Hydrodynamik. *Math. Ann.* **117**, 764–775.

HOWARD, L. N. 1972 Bounds on flow quantities. *Annu. Rev. Fluid Mech.* **4** (1), 473–494.

KELLER, J. B., RUBENFELD, L. A. & MOLYNEUX, J. E. 1967 Extremum principles for slow viscous flows with applications to suspensions. *J. Fluid Mech.* **30** (1), 97–125.

KERSWELL, R. R. 1998 Unification of variational principles for turbulent shear flows: the background method of Doering–Constantin and the mean-fluctuation formulation of Howard–Busse. *Physica* D **121** (1–2), 175–192.

LATHROP, D. P., FINEBERG, J. & SWINNEY, H. L. 1992 Transition to shear-driven turbulence in Couette–Taylor flow. *Phys. Rev.* A **46**, 6390–6405.

MALKUS, W. V. R. 1954 The heat transport and spectrum of thermal turbulence. *Proc. R. Soc. Lond.* A **225** (1161), 196–212.

NICODEMUS, R., GROSSMANN, S. & HOLTHAUS, M. 1997 Improved variational principle for bounds on energy dissipation in turbulent shear flow. *Physica* D **101** (1–2), 178–190.

OTTO, F. & SEIS, C. 2011 Rayleigh–Bénard convection: improved bounds on the Nusselt number. *J. Math. Phys.* **52** (8), 083702,1–24.

PETROV, N. P., LU, L. & DOERING, C. R. 2005 Variational bounds on the energy dissipation rate in body-forced shear flow. *J. Turbul.* **6**, p. N17, http://dx.doi.org/10.1080/14685240500228262.

SEIS, C. 2013 Laminar boundary layers in convective heat transport. *Commun. Math. Phys.* **324** (3), 995–1031.

SERRIN, J. 1959 Mathematical principles of classical fluid mechanics. In *Fluid Dynamics I/Strömungsmechanik I* (ed. C. Truesdell), Encyclopedia of Physics/Handbuch der Physik, vol. 3/8/1, pp. 125–263. Springer.

WHITEHEAD, J. P. & DOERING, C. R. 2011 Ultimate state of two-dimensional Rayleigh–Bénard convection between free-slip fixed-temperature boundaries. *Phys. Rev. Lett.* **106**, 244501.