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Application of Measure of Noncompactness to Infinite Systems of Differential Equations

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Abstract. In this paper we determine the Hausdorff measure of noncompactness on the sequence space $n(\phi)$ of W. L. C. Sargent. Further we apply the technique of measures of noncompactness to the theory of infinite systems of differential equations in the Banach sequence spaces $n(\phi)$ and $m(\phi)$. Our aim is to present some existence results for infinite systems of differential equations formulated with the help of measures of noncompactness.

1 Introduction and Preliminaries

We shall write w for the set of all complex sequences $x = (x_k)_{k=0}^{\infty}$. Let φ, l_∞, c and c_0 denote the sets of all finite, bounded, convergent and null sequences respectively; and cs be the set of all convergent series. We write $l_p := \{x \in w : \sum_{k=0}^{\infty} |x_k|^p < \infty\}$ for $1 \le p < \infty$. By e and $e^{(n)}$ $(n \in \mathbb{N})$, we denote the sequences such that $e_k = 1$ for $k = 0, 1, \ldots$, and $e_n^{(n)} = 1$ and $e_k^{(n)} = 0$ $(k \neq n)$. For any sequence $x = (x_k)_{k=0}^{\infty}$, let $x^{[n]} = \sum_{k=0}^{n} x_k e^{(k)}$ be its n-section.

A sequence $(b^{(n)})_{n=0}^{\infty}$ in a linear metric space X is called a *Schauder basis* if for every $x \in X$, there is a unique sequence $(\lambda_n)_{n=0}^{\infty}$ of scalars such that $x = \sum_{n=0}^{\infty} \lambda_n b^{(n)}$. A sequence space X with a linear topology is called a *K-space* if each of the maps $p_i \colon X \to \mathbb{C}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$. A K-space is called an *FK-space* if X is complete linear metric space; a *BK-space* is a normed FK-space. An FK-space $X \supset \varphi$ is said to have AK if every sequence $x = (x_k)_{k=0}^{\infty} \in X$ has a unique representation $x = \sum_{k=0}^{\infty} x_k e^{(k)}$, that is, $x = \lim_{n\to\infty} x^{[n]}$.

The β-*duals* of a subset X of w is defined by

$$X^\beta = \Big\{ a = (a_k) \in w : \sum_{k=0}^{\infty} a_k x_k \text{ converges for all } x = (x_k) \in X \Big\}.$$

Let $(X, \|\cdot\|)$ be a normed space. Then the unit sphere and closed unit ball in X are denoted by $S_X := \{x \in X : \|x\| = 1\}$ and $\bar{B}_X := \{x \in X : \|x\| \le 1\}$. If X and Y are Banach spaces, then $\mathcal{B}(X, Y)$ is the set of all bounded linear operators $L \colon X \to Y$; $\mathcal{B}(X, Y)$ is a Banach space with the operator norm given by $\|L\| = \sup_{x\in S_X} \|L(x)\|$. In

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particular, if $Y = \mathbb{C}$ then we write X^* for the set of all continuous linear functionals on X with the norm $\|f\| = \sup_{x \in S_X} |f(x)|$.

Let $\mathcal{C}$ denote the space whose elements are finite sets of distinct positive integers. Given any element σ of $\mathcal{C}$, we denote by $c(\sigma)$ the sequence $\{c_n(\sigma)\}$ such that $c_n(\sigma) = 1$ for $n \in \sigma$; and $c_n(\sigma) = 0$ otherwise. Further

$$\mathcal{C}_s = \Big\{ \sigma \in \mathcal{C} : \sum_{n=1}^{\infty} c_n(\sigma) \leq s \Big\},$$

that is, $\mathcal{C}_s$ is the set of those σ whose support has cardinality at most s, and define

$$\Phi = \{\phi = (\phi_k) \in w : 0 < \phi_1 \leq \phi_n \leq \phi_{n+1} \text{ and } (n+1)\phi_n \geq n\phi_{n+1}\}.$$

For $\phi \in \Phi$, the following sequence spaces were introduced by Sargent [14] and further studied in [8].

$$m(\phi) = \Big\{ x = (x_k) \in w : \|x\|_{m(\phi)} = \sup_{s \geq 1} \sup_{\sigma \in \mathcal{C}_s} \Big(\frac{1}{\phi_s} \sum_{k \in \sigma} |x_k| \Big) < \infty \Big\},$$

$$n(\phi) = \Big\{ x = (x_k) \in w : \|x\|_{n(\phi)} = \sup_{u \in S(x)} \Big(\sum_{k=1}^{\infty} |u_k| \Delta\phi_k \Big) < \infty \Big\},$$

where $S(x)$ denotes the set of all sequences that are rearrangements of x.

Remark 1.1

(i) The spaces $m(\phi)$ and $n(\phi)$ are BK spaces with their respective norms.

(ii) If $\phi_n = 1$ for all $n \in \mathbb{N}$ then $m(\phi) = l_1$, $n(\phi) = l_\infty$; and if $\phi_n = n$ for all $n \in \mathbb{N}$ then $m(\phi) = l_\infty$, $n(\phi) = l_1$.

(iii) $l_1 \subseteq m(\phi) \subseteq l_\infty$ $[l_\infty \supseteq n(\phi) \supseteq l_1]$ for all ϕ of Φ.

(iv) $\big(m(\phi)\big)^\beta = n(\phi)$ and $\big(n(\phi)\big)^\beta = m(\phi)$.

Infinite systems of ordinary differential equations describe numerous real world problems that can be encountered in the theory of branching processes, the theory of neural nets, the theory of dissociation of polymers and so on (*cf.* [3], [4], [6], [18]). Let us also mention that several problems investigated in mechanics lead to infinite systems of differential equations [12], [13], [19]. Moreover, infinite systems of differential equations can also be used in solving some problems for parabolic differential equations investigated via semidiscretization [15], [16], [17].

Recently the theory of measures of noncompactness has been used in determining the compact operators of matrices on various BK spaces, *e.g.*, [7], [10], [11].

In this paper we apply the technique of measures of noncompactness to the theory of infinite systems of differential equations in some Banach sequence spaces. Our aim is to present some existence results for infinite systems of differential equations formulated with the help of measures of noncompactness. The results of this paper extend those of obtained by Banaś and Lecko [2] and we determine the sufficient conditions for the solvability of infinite systems of differential equations in BK space $n(\phi)$ analogous to those of Banaś and Lecko who considered the classical Banach sequence spaces c_0, c and l_1.

2 Measures of Noncompactness

We recall a few definitions and notations (*cf.* [1], [5]).

Let (X,d) be a metric space. We denote by $\mathcal{M}_X$ the class of all bounded subsets of X. If M and S are subsets of a metric space (X,d) and $\epsilon > 0$, then the set S is called an *ϵ-net* of M if for any $x \in M$ there exists an $s \in S$ such that $d(x,s) < \epsilon$; an ϵ-net S is said to be finite if S is a finite set. The set M is said to be totally bounded if it has a finite ϵ-net. It is well known that a subset M of a metric space is compact if every sequence (x_n) in M has a convergent subsequence, and in this case the limit of this subsequence is in M. A linear operator L from X into Y is said to be *compact* or *completely continuous* if its domain is all of X and the sequence $\big(L(x_n)\big)$ of images of any bounded sequence (x_n) in X has a convergent subsequence. If X and Y are Banach spaces, we denote by $\mathcal{C}(X,Y)$ the subset of all compact operators in $\mathcal{B}(X,Y)$.

Now we give the definition of the Hausdorff measure of noncompactness.

Definition 2.1 Let (X,d) be a metric space and

$$B(x,r) := \{y \in X : d(x,y) < r\}$$

denote the open ball of radius $r > 0$ with centre x. The *Hausdorff measure of noncompactness* is the function $\chi\colon \mathcal{M}_X \to \mathbb{R}$ and $\chi(Q)$ is the *Hausdorff measure of noncompactness of the set* $Q \in \mathcal{M}_X$, where

$$\chi(Q) := \inf\Big\{\epsilon > 0 : Q \subset \bigcup_{i=1}^{n} B(x_i, r_i), x_i \in X, r_i < \epsilon\ (i = 1, 2, \dots), n \in \mathbb{N}_0\Big\}.$$

Definition 2.2 Let X and Y be Banach spaces and χ_1 and χ_2 be the Hausdorff measures of noncompactness on X and Y, respectively. An operator $L\colon X \to Y$ is said to be *(χ_1,χ_2)-bounded* if $L(Q) \in \mathcal{M}_Y$ for all $Q \in \mathcal{M}_X$ and there exist a constant $C \geq 0$ such that $\chi_2\big(L(Q)\big) \leq C\chi_1(Q)$ for all $Q \in \mathcal{M}_X$. If an operator L is (χ_1,χ_2)-bounded then the number $\|L\|_{(\chi_1,\chi_2)} := \inf\big\{C \geq 0 : \chi_2\big(L(Q)\big) \leq C\chi_1(Q)$ for all $Q \in \mathcal{M}_X\big\}$ is called the *(χ_1,χ_2)-measure of noncompactness of L*. If $\chi_1 = \chi_2 = \chi$, then we write $\|L\|_{(\chi_1,\chi_2)} = \|L\|_\chi$.

If X is a normed linear space and $a = (a_k) \in w$, we write

$$\|a\|^* = \|a\|^*_X = \sup_{x \in S_X}\Big|\sum_{k=1}^{\infty} a_k x_k\Big|.$$

We have the following result.

Lemma 2.3 *Let Q be a bounded subset of the normed space X, where X is l_p ($1 \leq p < \infty$) or c_0. If $P_n\colon X \to X$ is the operator defined by $P_n(x) = (x_0, x_1, \dots, x_n, 0, 0, \dots)$, then*

$$\chi(Q) = \lim_{n\to\infty}(\sup_{x\in Q}\|(I - P_n)x\|).$$

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In this section, we determine the Hausdorff measure of noncompactness on $n(\phi)$, while for $m(\phi)$, it was given in [9].

Theorem 2.4 *Let Q be a bounded subset of $n(\phi)$. Then*

$$\chi(Q) = \lim_{k\to\infty} \sup_{x\in Q}\Big(\sup_{u\in S(x)} \Big(\sum_{n=k}^{\infty} |u_n|\Delta\phi_n \Big) \Big) \tag{2.1}$$

Proof Define the operator $P_k \colon n(\phi) \to n(\phi)$ by $P_k(x) = (x_1, x_2, \dots, x_k, 0, 0, \dots)$ for $x = (x_1, x_2, \dots) \in n(\phi)$. Then clearly

$$Q \subset P_kQ + (I - P_k)Q. \tag{2.2}$$

It follows from (2.2) and the properties of χ that

$$\begin{aligned} \chi(Q) &\le \chi(P_kQ) + \chi\big((I - P_k)Q\big) = \chi\big((I - P_k)Q\big) \\ &\le \operatorname{diam}\big((I - P_k)Q\big) = \sup_{x\in Q} \|(I - P_k)x\|, \end{aligned} \tag{2.3}$$

where for a subset A of a metric space (X, d), $\operatorname{diam}(A) = \sup\{d(x, y) : x, y \in A\}$, and

$$\|(I - P_k)x\| = \sup_{u\in S(x)} \Big(\sum_{n=k}^{\infty} \|u_n\|\Delta\phi_n \Big).$$

So we have

$$\chi(Q) \le \lim_{k\to\infty} \sup_{x\in Q} \|(I - P_k)x\|. \tag{2.4}$$

Conversely, let $\varepsilon > 0$ and $\{z_1, z_2, \dots, z_j\}$ be a $[\chi(Q) + \varepsilon]$-net of Q. Then

$$Q \subset \{z_1, z_2, \dots, z_j\} + [\chi(Q) + \varepsilon]B_{n(\phi)}(0, 1). \tag{2.5}$$

Hence

$$\sup_{x\in Q} \|(I - P_k)x\| \le \sup_{1\le i\le j} \|(I - P_k)z_i\| + [\chi(Q) + \varepsilon],$$

which implies that

$$\lim_{k\to\infty} \sup_{x\in Q} \|(I - P_k)x\| \le \chi(Q) + \varepsilon. \tag{2.6}$$

Since ε was arbitrary, (2.4) and (2.6) together imply (2.1). ∎

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3 Infinite Systems of Differential Equations

Consider the ordinary differential equation

$$x' = f(t,x) \tag{3.1}$$

with the initial condition

$$x(0) = x_0 \tag{3.2}$$

Then the following result for the existence of the Cauchy problem (3.1)–(3.2) was given in [2] and is a slight modification of the result proved in [1].

Assume that E is a real Banach space with the norm $\|\cdot\|$. Denote by $B(x_0, r)$ the closed ball in E centered at x_0 and with radius r.

Theorem A ([2]) *Assume that $f(t,x)$ is a function defined on $I \times E$ with values in E such that*

$$\|f(t,x)\| \leq P + Q\|x\|,$$

for any $x \in E$, where P and Q are nonnegative constants. Further, let f be uniformly continuous on the set $I_1 \times B(x_0, r)$, where $I_1 = [0, T_1] \subset I$, $QT_1 < 1$ and $r = (PT_1 + QT_1\|x_0\|)/(1 - QT_1)$. Moreover, assume that for any nonempty set $X \subset B(x_0, r)$ and for almost all $t \in I$, the following inequality holds:

$$\mu\big(f(t,X)\big) \leq p(t)\mu(X), \tag{3.3}$$

with a sublinear measure of noncompactness μ such that $\{x_0\} \in \ker \mu$. Then problem (3.1)–(3.2) has a solution x such that $\{x(t)\} \in \ker \mu$ for $t \in I_1$; where $p(t)$ is an integrable function on I.

In this section we study of the solvability of the infinite systems of differential equations in the Banach sequence space $n(\phi)$. We will be interested in the existence of solutions $x(t) = \big(x_i(t)\big)$ of the infinite systems of differential equations

$$x_i' = f_i(t, x_1, x_2, \dots), \tag{3.4}$$

with the initial condition

$$x_i(0) = x_0, \quad (i = 1, 2, \dots), \tag{3.5}$$

which are defined on the interval $I = [0, T]$ and such that $x(t) \in n(\phi)$ for each $t \in I$.

An existence theorem for problem (3.4)–(3.5) in the space $n(\phi)$ can be formulated by making the following assumptions:

(1) $x_0 = (x_i^0) \in n(\phi)$;

(2) $f_i \colon I \times \mathbb{R}^\infty \to \mathbb{R}$ $(i = 1, 2, \dots)$ maps continuously the set $I \times n(\phi)$ into $n(\phi)$;

(3) there exist nonnegative functions $p_i(t)$ and $q_i(t)$ defined on I such that

$$\|f_i(t,u)\| = \|f_i(t, u_1, u_2, \dots)\| \leq p_i(t) + q_i(t)|u_i|$$

for $t \in I$; $x = (x_i) \in n(\phi)$ and $i = 1, 2, \dots$; where $u = (u_i)$ is a sequence of rearrangement of $x = (x_i)$;

(4) the functions $p_i(t)$ are continuous on I and the function series $\sum_{i=1}^{\infty} p_i(t)\Delta\phi_i$ converges uniformly on I;

(5) the sequence $\big(q_i(t)\big)$ is equibounded on the interval I and the function $q(t) = \limsup_{i\to\infty} q_i(t)$ is integrable on I.

Now, we prove the following result.

Theorem 3.1 *Under the assumptions* (1)–(5)*, problem* (3.4)–(3.5) *has a solution* $x(t) = \big(x_i(t)\big)$ *defined on the interval* $I = [0, T]$ *whenever* $QT < 1$*, where* Q *is defined as the number*

$$Q = \sup\{q_i(t) : t \in I, i = 1, 2, \dots\}.$$

Moreover, $x(t) \in n(\phi)$ for any $t \in I$.

Proof For any $x(t) \in n(\phi)$ and $t \in I$, under the above assumptions, we have

$$\begin{aligned}
\|f(t,x)\|_{n(\phi)} &= \sup_{u\in S(x)} \sum_{i=1}^{\infty} \|f_i(t,u)\|\Delta\phi_i \\
&\le \sup_{u\in S(x)} \sum_{i=1}^{\infty} [p_i(t) + q_i(t)|u_i|]\Delta\phi_i \\
&\le \sum_{i=1}^{\infty} p_i(t)\Delta\phi_i + \big(\sup_i q_i(t)\big)\Big(\sup_{u\in S(x)} \sum_{i=1}^{\infty} |u_i|\Delta\phi_i\Big) \\
&\le P + Q\|x\|_{n(\phi)},
\end{aligned}$$

where $P = \sup_{t\in I} \sum_{i=1}^{\infty} p_i(t)\Delta\phi_i$.

Now choose the number r defined according to Theorem A, *i.e.*,

$$r = \frac{PT + QT\|x_0\|_{n(\phi)}}{1 - QT}.$$

Consider the operator $f = (f_i)$ on the set $I \times B(x_0; r)$. Let us take a set $X \in \mathcal{M}_{n(\phi)}$. Then by using (2.1), we get

$$\begin{aligned}
\chi\big(f(t,X)\big) &= \lim_{k\to\infty} \sup_{x\in X}\Big(\sup_{u\in S(x)}\Big(\sum_{n=k}^{\infty} |f_n(t, u_1, u_2, \dots)|\Delta\phi_n\Big)\Big) \\
&\le \lim_{k\to\infty}\Big(\sum_{n=k}^{\infty} p_n(t)\Delta\phi_n + \big(\sup_{n\ge k} q_n(t)\big)\Big(\sup_{u\in S(x)} \sum_{n=k}^{\infty} |u_n|\Delta\phi_n\Big)\Big).
\end{aligned}$$

Hence by assumptions (4)–(5), we get

$$\chi\big(f(t,X)\big) \le q(t)\chi(X),$$

i.e., the operator f satisfies condition (3.3) of Theorem A. Hence the problem (3.4)–(3.5) has a solution $x(t) = \big(x_i(t)\big)$. ■

Remark 3.2 Accordingly, we can reformulate assumptions (1)–(5) to prove the result for the space $m(\phi)$ analogous to Theorem 3.1.

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