

ON THE CONVERGENCE OF ZETA FUNCTIONS OF PREHOMOGENEOUS VECTOR SPACES

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Abstract. We prove a general convergence result for zeta functions of prehomogeneous vector spaces extending results of H. Saito, F. Sato and Yukie. Our analysis points to certain subspaces which yield boundary terms. We study it further in the setup arising from nilpotent orbits. In certain cases we determine the residue at the rightmost pole of the zeta function.

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§1. Introduction

The theory of prehomogeneous vector spaces (PVSs) and their zeta functions was initiated by M. Sato and Shintani [41], [47]. (See [23] for history and references.) One of the early motivations was to study asymptotics of arithmetic invariants [43], an endeavor followed up by a number of authors (e.g., [9], [24], [51], [53]). For these applications, the alternative approach of Bhargava using innovative geometry of numbers techniques was very successful (see [1] and the references therein). Nonetheless, zeta functions of PVSs are still useful for these applications [5]. Our motivation for the study of the zeta functions comes from their appearance in Arthur’s trace formula [7], [10], [16], [17], [19], although this will not play a direct role in the current paper.

The purpose of this paper is to provide a general basic convergence result for zeta functions of regular PVSs over numbers fields. It generalizes earlier results in this direction

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by F. Satō, H. Saito, Yukie, and others. More importantly, we hope that the method will enhance our understanding of the singularities of the zeta function and ultimately will be incorporated to Arthur's trace formula. As a first result in this direction, we determine the structure of the first pole (at the edge of the domain of absolute convergence) for a restricted, but natural class of PVSs.

In order to state our main result, let V be a reductive PVS over a number field F , i.e., an F -rational representation ρ of a reductive group G over F on V with an open orbit X . Let $H = G_v$ be the generic stabilizer and let $\chi_1, \dots, \chi_r \in X^*(G)$ be the fundamental F -rational characters of G . Let \mathbb{A} be the ring of adèles of F . For any rapidly decreasing continuous function ϕ on $V(\mathbb{A})$ let θ_ϕ be the automorphic function

$$\theta_\phi(g) = \sum_{v \in X(F)} \phi(\rho(g)^{-1}v), \quad g \in G(F) \backslash G(\mathbb{A}).$$

Roughly speaking, the zeta function is the Mellin transform of θ_ϕ . In the simplest case, our result is the following.

THEOREM 1.1. *Suppose that the connected component H° of H is semisimple. Then, the zeta function*

$$Z(\phi, (s_1, \dots, s_r)) = \int_{G(F) \backslash G(\mathbb{A})} \theta_\phi(g) |\det \rho(g)|^{-1} \prod_{i=1}^r |\chi_i(g)|^{-s_i} dg,$$

converges absolutely for $\Re s_i > 0$, $i = 1, \dots, r$.

Theorem 1.1 cannot hold as stated for a general PVS. The first issue is that assuming convergence, as a function of $x \in G(\mathbb{A})$ the integral

$$\int_{G(F) \backslash G(\mathbb{A})^1} \theta_\phi(gx) dg, \tag{1.1}$$

is invariant not just under $G(\mathbb{A})^1$ but also under the adelic points of the group $G^* = HG^{\text{der}}$. Therefore, in the case where $X^*(H)$ is infinite (or, equivalently $r < \dim X^*(G)$), we need to modify the definition of the zeta function as follows:

$$Z(\phi, (s_1, \dots, s_r)) = \int_{G(\mathbb{A})^1 G^*(\mathbb{A}) \backslash G(\mathbb{A})} \int_{G(F) \backslash G(\mathbb{A})^1} \theta_\phi(ga) dg |\det \rho(a)|^{-1} \prod_{i=1}^r |\chi_i(a)|^{-s_i} da. \tag{1.2}$$

We will show that this integral converges absolutely for $\Re s_i > 0$, $i = 1, \dots, r$ provided that H is connected reductive.

In the non-connected case, there is yet a more fundamental difficulty, namely that already the integral (1.1) may diverge. This happens if $\text{vol}(G_v(F) \backslash (G_v(\mathbb{A}) \cap G(\mathbb{A})^1)) = \infty$, or equivalently, if the group of connected components G_v/G_v° acts nontrivially on $X^*(G_v^\circ)$ for some $v \in X(F)$. We call such elements v isotropic. For instance, this is the case for the PVS of binary quadratic forms (with respect to the action of GL_2) in which the centralizer (an orthogonal group in dimension 2) may split over F .

To remedy the situation, we modify θ_ϕ , and consequently the zeta function, by simply removing the sum over isotropic $v \in X(F)$. Our main result is that the modified zeta function converges for $\Re s_i > 0$, $i = 1, \dots, r$ provided that H° is reductive (see Theorem 4.3 below).

In fact, using different methods, the convergence of this modified zeta function in *some* orthant $\Re s_i \gg 0$ was proved by H. Saito under the condition that $X^*(H)$ is finite [38].

We also remark that meromorphic continuation to \mathbb{C}^r and functional equations for the zeta function (1.2) were established by F. Satō [39] under the assumptions that H° is reductive, $X^*(H)$ is finite and that there are no isotropic elements in $X(F)$, extending the classical work by Sato and Shintani [48]. For the PVS of binary quadratic forms, meromorphic continuation of (1.2) to \mathbb{C} and the functional equation of a modified zeta function, which includes a contribution from the isotropic elements, were obtained by Yukie [52] (following the work of Shintani [45]). We will not deal with these questions here.

The contents of the paper are as follows. We start with recalling general facts, mostly well known, about PVSs (§2). We review examples of PVSs, especially those arising from nilpotent orbits, in §3. The main result is stated in §4. In §5, we give a general bound for the function θ_ϕ (Proposition 5.1). The analysis is based on the examination of the support sets of elements in $X(F)$ with respect to the weights of a maximal F -split torus of G , an idea which goes back to Yukie [53], [54]. The linchpin of the argument is a certain nonnegativity statement pertaining to these support sets (Corollary 5.7). Further analysis is carried out in §6 where the key notion of *special* subspaces of a PVS is introduced. In §7, we prove the main convergence result. An important aspect in the proof is that the existence of isotropic elements is closely related to the existence of special subspaces with additional properties, which we call exceptional (Proposition 7.4). The nonexceptional subspaces exhibit a crucial positivity property (Proposition 7.9), which strengthens Corollary 5.7 *supra* and guarantees convergence. In §8, we improve the convergence result for PVSs without nontrivial special subspaces (under an additional technical condition which is satisfied in the F -irreducible case) and establish meromorphic continuation to the left of the polar hyperplanes $s_i = 0$, $i = 1, \dots, r$ together with an explicit description of the residues. Finally, in §9, we give a general construction of special subspaces for regular PVSs coming from nilpotent orbits.

§2. Prehomogeneous vector spaces

We start with basic facts about PVSs. (See [23, Chapter 2] and [39, §1] for more details.)

2.1. General notation.

Throughout the paper F is a field of characteristic 0 with algebraic closure \bar{F} . (From §4 onward, F will be a number field.)

We denote by \mathbf{G}_m the multiplicative group.

Whenever a group G acts on a set A , we denote by A^G the set of fixed points of G in A .

We denote by $X_*(T)$ the lattice of F -rational cocharacters of an F -torus T .

For a linear algebraic group G over F , we use the following notation.

- G° —the identity component of G .
- $Z(G)$ —the center of G .
- G^{der} —the derived group of G .
- $G^{\text{ab}} = G/G^{\text{der}}$ —the abelianization of G .
- N_G —the unipotent radical of G .
- $G^{\text{lev}} = G/N_G$ —the Levi part of G , so that $G^{\text{red}} := (G^{\text{lev}})^\circ = (G^\circ)^{\text{lev}}$ is reductive.
- $G^{\text{ss}} = (G^{\text{red}})^{\text{der}}$ —the semisimple part of G° .

- $G^{\text{tor}} = (G^{\text{red}})^{\text{ab}}$ —the largest toric quotient of G° .
- $G^{\text{ssu}} = \text{Ker}(G^\circ \rightarrow G^{\text{tor}})$ —an extension of G^{ss} by N_G .
- $\mathfrak{g} = \text{Lie } G$ —the Lie algebra of G (and similarly for other groups).
- $\text{Ad} = \text{Ad}_G$ —the adjoint representation of G on \mathfrak{g} .
- $X^*(G) = X^*(G^{\text{ab}}) = X^*(G^{\text{lev}})$ —the finitely generated abelian group of F -rational characters of G .
- $\Delta_G = \det \text{Ad} \in X^*(G)$ —the modular character of G .
- T_G —the maximal F -split torus of $Z(G^{\text{red}})$.
- $\mathfrak{a}_G = X_*(T_G) \otimes_{\mathbb{Z}} \mathbb{R}$ —the \mathbb{R} -vector space generated by $X_*(T_G)$.

We often view G^{lev} as an F -subgroup (the Levi subgroup) of G , uniquely defined up to conjugation by $N_G(F)$ ([27] or [15, Chapter VIII, Theorem 4.3]).

If G is connected, then $X^*(G) = X^*(G^{\text{tor}})$ is torsion-free, i.e., it is a lattice.

If G is reductive, then $G^{\text{ssu}} = G^{\text{ss}} = G^{\text{der}}$.

If T is an F -split torus, then $X^*(T)$ can be identified with the dual lattice of $X_*(T)$. In general, if T' is the maximal F -split subtorus of T , then we can identify $X_*(T)$ with $X_*(T')$, while the restriction map $X^*(T) \rightarrow X^*(T')$ is injective with finite cokernel. Thus, we have isomorphisms

$$X^*(T) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X^*(T') \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathfrak{a}_{T'}^* = \mathfrak{a}_T^*.$$

The F -split rank of T , i.e. the dimension of T' , is the rank of $X^*(T)$ (and $X_*(T)$).

More generally, if G is connected, then the restriction map $X^*(G) = X^*(G^{\text{red}}) \rightarrow X^*(T_G)$ is injective with finite cokernel and hence yields an isomorphism

$$X^*(G) \otimes_{\mathbb{Z}} \mathbb{R} \simeq X^*(T_G) \otimes \mathbb{R} = \mathfrak{a}_G^*.$$

For any finite subset A of a real vector space we denote by $\mathbb{R}_{\geq 0}A$ the polyhedral cone generated by A , i.e., the set of linear combinations $\sum_{a \in A} x_a a$, where $x_a \geq 0$ for all $a \in A$. Similarly, we use the notation $\mathbb{R}_{> 0}A$, $\mathbb{Z}_{\geq 0}A$, $\mathbb{Z}_{> 0}A$ where we require the x_a to be strictly positive (resp., nonnegative integers, strictly positive integers) for all a .

Let G be an algebraic F -group and let H be an F -subgroup of G . We record a few basic results.

LEMMA 2.1. *Suppose that G is connected. Then, the following two conditions are equivalent.*

1. *The restriction map $X^*(G) \rightarrow X^*(H^\circ)$ has finite cokernel.*
2. *The torus $(Z(H^{\text{red}}) \cap G^{\text{ssu}})^\circ$ is F -anisotropic.*

Proof. We may assume that H is connected. Fix a Levi subgroup $L \simeq H^{\text{red}}$ of H . Then, the restriction map $X^*(H) \simeq X^*(L) \rightarrow X^*(T_L)$ is injective with finite cokernel. Therefore, we may rephrase condition 1 by saying that the restriction map $X^*(G) = X^*(G^{\text{tor}}) \rightarrow X^*(T_L)$ has finite cokernel, or equivalently, that $X_*(T_L) \rightarrow X_*(G/G^{\text{ssu}})$ is injective. In turn, this is equivalent to the triviality of $(T_L \cap G^{\text{ssu}})^\circ$, which amounts to condition 2. \square

LEMMA 2.2. *Suppose that $H = G^\circ$ and let $\Gamma = G/H$. Then, the kernel of the restriction map $X^*(G) \xrightarrow{r} X^*(H)$ is finite and its image is a finite index sublattice of $X^*(H)^G = X^*(H)^\Gamma$. Thus, the cokernel of r is finite if and only if G acts trivially on $X^*(H)$.*

Proof. The first part follows from the fact that the homomorphism $H/[G, H] \rightarrow G^{\text{ab}}$ has finite kernel and cokernel. In fact we have (a $\text{Gal}(\bar{F}/F)$ -equivariant) exact sequence

$$H_2(\Gamma, \mathbb{Z}) \rightarrow H/[G, H] \rightarrow G^{\text{ab}} \rightarrow \Gamma^{\text{ab}} \rightarrow 1,$$

[18, Corollary VI.8.2]. The second part follows from the first part and the fact that if a finite group Γ acts nontrivially on a lattice L , then L^Γ is of infinite index in L . \square

We denote the Picard group of a variety X by $\text{Pic } X = H^1(X, \mathcal{O}_X^*)$. If X is smooth, then we can identify $\text{Pic } X$ with the (Weil) divisor class group $\text{Cl } X$ of X . (See [14, §II.6] for basic facts about divisors and the Picard group.)

LEMMA 2.3 ([33], Proposition VII.1.5; cf. also the proof of [31], Theorem 4). *Suppose that G is connected. Then, there is an exact sequence*

$$X^*(G) \rightarrow X^*(H) \rightarrow \text{Pic}(G/H).$$

2.2

Recall that a *prehomogeneous vector space* (PVS) over F is a finite-dimensional F -representation

$$\rho: G \rightarrow \text{GL}(V),$$

of a connected linear algebraic F -group G on a finite-dimensional F -vector space V , for which there exists a (necessarily unique) open orbit X . Note that X is defined over F and $X(F)$ is $G(F)$ -invariant. If F is algebraically closed, then $X(F)$ is a $G(F)$ -orbit and the stabilizers G_v , $v \in X(F)$ are conjugate by $G(F)$. In general, given $u \in X(F)$, the $G(F)$ -orbits in $X(F)$ are in bijection with the elements of the kernel of $H^1(F, G_u) \rightarrow H^1(F, G)$ (a map of pointed sets) [42, I §5.4, Corollary 1 of Proposition 36]. Correspondingly, the F -groups G_v , $v \in X(F)$ are pure inner forms of G_u .

Clearly, a quotient of a PVS by an invariant subspace is also a PVS.

We will write $H = G_v$ (the “generic stabilizer”) when $v \in X(F)$ is immaterial, e.g., when we refer to the group $X^*(H)$ or to properties pertaining to $H(\bar{F})$. Thus, we can identify X with G/H . The F -subgroup $G^* = G^{\text{der}} H$ of G is well defined. Note that in general, H may be disconnected and the groups G_v° , $v \in X(F)$ are *not* necessarily in the same inner class. In fact, the lattice $X^*(G_v^\circ)$ may depend on the choice of $v \in X(F)$. For this reason, we will refrain from using the notation $X^*(H^\circ)$.

For $v \in X(F)$, the linear map

$$D_v: \mathfrak{g} \rightarrow V, \quad x \mapsto d\rho(x)v, \tag{2.1}$$

is surjective, since it is the differential of the dominant map $G \rightarrow V$, $g \mapsto \rho(g)v$. The kernel \mathfrak{g}_v of D_v is the Lie algebra of G_v [2, Lemma II.7.4]. We thus get a G_v -equivariant isomorphism $\mathfrak{g}/\mathfrak{g}_v \rightarrow V$. In particular,

$$\Delta_G \det \rho^{-1}|_H = \Delta_H. \tag{2.2}$$

A *relative invariant* with respect to a character $\chi \in X^*(G)$ is an F -rational function $f \neq 0$ on V such that $f \circ \rho(g) = \chi(g)f$ for all $g \in G$. We will write $f = \text{RI}_\chi$.

The following basic properties hold.

1. RI_χ , if it exists, is unique up to a nonzero scalar and it is regular and nonvanishing on X .
2. The sublattice $X^*(G/G^*)$, which is the kernel of the restriction map $X^*(G) \rightarrow X^*(H)$, consists of the characters with respect to which there exists a relative invariant.
3. Let χ_1, \dots, χ_r be a basis of $X^*(G/G^*)$. Then $\text{RI}_{\chi_1}, \dots, \text{RI}_{\chi_r}$ are algebraically independent.
4. The F -irreducible polynomials that are nowhere vanishing on X (if any) are relative invariants. They are called the *fundamental relative invariants*. The corresponding characters are called *fundamental characters*. They form a basis for $X^*(G/G^*)$. Denote the set of all fundamental characters by \mathfrak{X} .

If G is reductive, then the following conditions are equivalent.

1. H° is reductive.
2. X is the complement of a hypersurface.
3. X is the complement of the hypersurface defined by $\text{RI}_{\chi_1} \cdots \text{RI}_{\chi_r}$ where $\mathfrak{X} = \{\chi_1, \dots, \chi_r\}$.
4. X is an affine variety.
5. V is regular (see [23, Definition 2.14]).

In this case, $(\det \rho)^2 \in X^*(G/G^*)$. (As far as we know, it is unknown whether $\text{RI}_{\det \rho^2}$ is a polynomial in general.)

We give some more basic properties of PVSs.

LEMMA 2.4. *There is a short exact sequence*

$$0 \rightarrow X^*(G/G^*) \rightarrow X^*(G) \rightarrow X^*(H) \rightarrow 0.$$

Thus, $\text{rk} X^*(G) - r = \text{rk} X^*(H)$.

Proof. Clearly, $X^*(G/G^*)$ is the kernel of the restriction map $X^*(G) \rightarrow X^*(H)$. The surjectivity of this map follows from Lemma 2.3 and the vanishing of $\text{Pic} X$. More generally, this is true for any open subvariety X of $Y = \text{Spec} A$ where A is a Noetherian UFD since $\text{Cl} Y = 1$ and $\text{Cl} Y \rightarrow \text{Cl} X$ is onto. \square

Let $\bar{\mathfrak{a}}_G^*$ be the subspace $X^*(G/G^*) \otimes \mathbb{R}$ of \mathfrak{a}_G^* . Thus, \mathfrak{X} forms a basis for $\bar{\mathfrak{a}}_G^*$.

For completeness, we also note the following fact, which can also be proved by a topological argument for $F = \mathbb{C}$.

LEMMA 2.5. *Suppose that V has no relative invariants. Then H is connected.*

Proof. We may assume that F is algebraically closed. The condition on V means that $V \setminus X$ has codimension > 1 in V . It follows from purity of the branch locus that X is simply connected [28, Exp. X, Corollaire 3.3], i.e., there is no nontrivial étale covering of X . On the other hand,

$$G/H^\circ \rightarrow X,$$

is an étale covering. Thus, H is connected. \square

LEMMA 2.6. *The following conditions on $v \in X(F)$ are equivalent.*

1. *The cokernel of the restriction map $X^*(G) \rightarrow X^*(G_v^\circ)$ is infinite.*

2. The cokernel of the restriction map $X^*(G_v) \rightarrow X^*(G_v^\circ)$ is infinite.
3. There is a non-trivial F -split torus in $Z(G_v^{\text{red}}) \cap G_v^{\text{ssu}}$.
4. The action of G_v (via the component group G_v/G_v°) on $X^*(G_v^\circ)$ is nontrivial.

Proof. 1 \iff 2: Lemma 2.4.

1 \iff 3: Lemma 2.1.

2 \iff 4: Lemma 2.2. □

DEFINITION 2.7. An element $v \in X(F)$ is *isotropic* if it satisfies the conditions of Lemma 2.6. Otherwise, v is *nonisotropic*.

We denote the subset of nonisotropic elements in $X(F)$ by $X(F)^{\text{noniso}}$. It is invariant under $G(F)$.

A PVS V is isotropic if it contains isotropic elements. Otherwise, V is nonisotropic.

By Lemma 2.6, if H is connected, then V is nonisotropic.

The basic example of an isotropic PVS is the space V of binary quadratic forms with respect to the action of GL_2 . Here, H is an orthogonal group in dimension 2; $v \in X(F)$ is isotropic if and only if it is isotropic in the usual sense of quadratic forms. We caution however, that the PVS of quadratic forms in $n > 2$ variables (with respect to the GL_n -action) is nonisotropic.

In general, it is possible that all elements of $X(F)$ are isotropic (for instance, if V is isotropic and $F = \bar{F}$), although we expect that this is never the case if F is a number field.

Assume now that G is reductive, so that every representation of G is completely reducible. We single out two important classes of PVSs in the opposite extremes.

DEFINITION 2.8. Let V be a reductive PVS. We say that

1. V is *basic* if for every $v \in X(F)$, G_v° is not contained in any proper parabolic F -subgroup of G .¹
2. V is *distinguished* if $H/\text{Ker } \rho$ is finite.²

Note that every basic PVS is regular, since any connected nonreductive F -subgroup of G is contained in a proper parabolic F -subgroup of G [4]. Moreover, a basic PVS is necessarily nonisotropic. It is also clear that every subrepresentation of a basic PVS is also basic.

A distinguished PVS is regular and nonisotropic.

§3. PVSs of DK-type, examples

A main source of PVSs are gradations on reductive Lie algebras and nilpotent orbits.

3.1

Let G' be a reductive group over F and let $\lambda : \mathbf{G}_m \rightarrow G'$ be a one-parameter subgroup. Let $h = (d\lambda)(1) \in \mathfrak{g}'$. Decompose

$$\mathfrak{g}' = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}'_i, \quad \text{where } \mathfrak{g}'_i = \{x \in \mathfrak{g}' \mid \text{Ad}(\lambda(t))x = t^i x \ \forall t \in \mathbf{G}_m\} = \{x \in \mathfrak{g}' \mid [h, x] = ix\}.$$

This is a \mathbb{Z} -gradation of \mathfrak{g}' . (If G' is semisimple and adjoint, then every gradation of \mathfrak{g}' is of this form.) Let $\mathcal{F}_i = \bigoplus_{j \geq i} \mathfrak{g}'_j$, $i \in \mathbb{Z}$ be the corresponding (decreasing) filtration of \mathfrak{g}' . The

¹ This condition depends on v . A strictly stronger condition would be that H° is not contained in any proper parabolic subgroup of G (not necessarily defined over F).

² The terminology comes from nilpotent orbits—see §3 below.

stabilizer P of \mathcal{F} in G under Ad is the parabolic F -subgroup of G whose Lie algebra is \mathcal{F}_0 . The Lie algebra of the unipotent radical of P is \mathcal{F}_1 . The centralizer G of λ is a Levi subgroup of P whose Lie algebra is \mathfrak{g}'_0 . For every $i \neq 0$, the vector space \mathfrak{g}'_i , which consists of nilpotent elements, comprises finitely many (geometric) orbits with respect to the adjoint action of G [49]. In particular, it is a PVS. In the terminology of [13], these are PVSs of V-type (for Vinberg). They are not necessarily regular. We also say that \mathcal{F} is a filtration of \mathfrak{g}' of V-type. (Note that we may restrict here to the case $i = 1$, since the general case follows from it by considering the graded Lie subalgebra $\oplus_j \mathfrak{g}'_{ji}$, which is reductive.)

As a special case, let P be a parabolic subgroup of G' with Levi factor G and $\mathfrak{n} = \text{Lie } N_P$. Then, $V = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is a PVS with respect to the adjoint action of G . In fact, \mathfrak{n} is a PVS with respect to the adjoint action of P by a well-known result of Richardson. We say that V is a PVS of R-type. We denote the nilpotent orbit containing the open orbit of P on \mathfrak{n} (the Richardson orbit with respect to P) by $\text{Rich}(P)$. The number of irreducible components of V is the F -corank of P in G .

Other important special cases, this time of regular PVSs, arise from the Dynkin–Kostant classification of nilpotent orbits. In more detail, let (e, h, f) be an \mathfrak{sl}_2 -triple in $[\mathfrak{g}', \mathfrak{g}']$. (By the Jacobson–Morozov theorem, every nilpotent $e \in \mathfrak{g}'$ is a part of an \mathfrak{sl}_2 -triple.) It corresponds to an F -homomorphism $\text{SL}_2 \rightarrow G'^{\text{der}}$. Let $\mathfrak{g}' = \oplus_i \mathfrak{g}'_i$ be the grading induced by h . The corresponding (decreasing) filtration $\mathcal{F}_i = \oplus_{j \geq i} \mathfrak{g}'_j$, $i \in \mathbb{Z}$ of \mathfrak{g}' depends only on e and in fact only on the P -orbit of e , where $P = P(e)$ is the stabilizer of $\mathcal{F} = (\mathcal{F}_i)_{i \in \mathbb{Z}}$ in G' under the adjoint action. Recall that P is the parabolic subgroup of G' with Lie algebra \mathcal{F}_0 . The Lie algebra of the unipotent radical of P is \mathcal{F}_1 . (That \mathcal{F} depends only on e follows from the fact that two \mathfrak{sl}_2 -triples (e, h, f) and (e, h', f') with the same first element e are conjugate by an element of $N_P(F) \cap G'_e(F)$ [3, Lemma VIII.11.1.4].)

Let G be the Levi quotient of P , so that $\mathfrak{g} = \mathcal{F}_0/\mathcal{F}_1 \simeq \mathfrak{g}'_0$. Let $V = \mathcal{F}_2/\mathcal{F}_3 \simeq \mathfrak{g}'_2$ with the G -action ρ induced by the adjoint representation. Let \mathfrak{o} be the geometric G' -orbit of e and let $\mathfrak{o}_{\mathcal{F}} = \mathfrak{o} \cap \mathcal{F}_2$.

THEOREM 3.1. *With the notation above:*

1. V is a regular PVS with respect to ρ .
2. $\mathfrak{o}_{\mathcal{F}} = X + \mathcal{F}_3$.
3. $\mathfrak{o}_{\mathcal{F}}$ is a P -orbit.
4. For every $g \in G'$, $\text{Ad}(g)\mathfrak{o}_{\mathcal{F}} \cap \mathfrak{o}_{\mathcal{F}} \neq \emptyset$ if and only if $g \in P$. In particular, $G'_v \subset P$ for any $v \in \mathfrak{o}_{\mathcal{F}}$.

These are the PVSs of *DK-type* (Dynkin–Kostant) in the language of [13]. (We quickly sketch how to derive these assertions from the literature. Note first that $\mathfrak{o} \cap \mathfrak{g}'_2$ is the open G -orbit X of V by [3, Proposition VIII.11.3.6]. V is regular by [22], see below. In addition, the N_P -orbit of e is $e + \mathcal{F}_3$ [34, Lemma 1]. This shows the second and third assertions. The fourth assertion easily reduces to the special case $G'_e \subset P$. Let $g \in G'_e$. Then $(e, \text{Ad}(g)h, \text{Ad}(g)f)$ is an \mathfrak{sl}_2 -triple with the same first element as (e, h, f) , and therefore conjugate to (e, h, f) by an element $n \in N_P \cap G'_e$. Since ng centralizes h , we have $ng \in G$, and therefore $g \in P$.)

If we fix in addition to e the \mathfrak{sl}_2 -triple (e, h, f) , then we can identify G with the centralizer of h in G' and V with the subspace \mathfrak{g}'_2 of \mathfrak{g}' . We have then $X = V \cap \mathfrak{o}$. We may identify \mathfrak{g}'_{-2} with the dual of $V = \mathfrak{g}'_2$ via the Killing form. Fixing a vector space isomorphism between \mathfrak{g}'_{-2} and \mathfrak{g}'_2 , $\text{RI}_{(\det \rho)^2}$ is given by the determinant of the linear map $\text{ad}(e) \circ \text{ad}(e) : \mathfrak{g}'_{-2} \rightarrow \mathfrak{g}'_2$, a polynomial in $e \in V$ whose nonzero locus is X [22, Proposition 1.1, 1.2, 2.1].

Fix a minimal parabolic F -subgroup P'_0 of G' and a maximal split F -torus T'_0 contained in P'_0 . Then, we may choose (e, h, f) in its $G'(F)$ -orbit so that $h \in \mathfrak{t}'_0$ is dominant and $P = P(e)$ is standard. This is the canonical parabolic subgroup pertaining to \mathfrak{o} . G is then identified with the standard Levi subgroup of P .

An orbit \mathfrak{o} is called *even* if $\mathfrak{g}'_i = 0$ for all odd i (or equivalently, if $\mathfrak{g}'_1 = 0$). For even orbits clearly $\mathfrak{o} = \text{Rich}(P)$. A particular case is given by *distinguished* orbits, where the connected stabilizer $(G'_e)^\circ$, $e \in \mathfrak{o}$ is modulo the center of G' a unipotent group. In this case, the PVS V is distinguished (and the converse holds for even orbits \mathfrak{o}). In general, not every standard parabolic subgroup is the canonical parabolic subgroup of some nilpotent orbit, let alone an even one. (By [22, Proposition 2.1] and [13, Theorem 2.4], the Richardson orbit of a standard parabolic subgroup P has P as its canonical parabolic subgroup if and only if the associated PVS $V = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$ is regular and $\delta_{\mathfrak{n}^{(i)}/\mathfrak{n}^{(i+1)}} \in \bar{\mathfrak{a}}^*_G$ for all $i \geq 1$, where $\mathfrak{n} = \mathfrak{n}^{(0)} \supset \mathfrak{n}^{(1)} \supset \dots$ denotes the descending central series of \mathfrak{n} .)

Any PVS of DK-type arises from an even nilpotent orbit for the reductive Lie algebra $\oplus_i \mathfrak{g}'_{2i}$. Hence, any PVS of DK-type is also of R-type, although not necessarily with the same G' .

3.2

Let us give some concrete examples of DK-type arising from even nilpotent orbits. For each case we provide the following data:

- The numbering in the Sato–Kimura classification (appendix of [23]) in the irreducible case,
- the group G' ,
- the (even) nilpotent orbit \mathfrak{o} in \mathfrak{g}' (either the corresponding partition for classical groups or the notation in the Bala–Carter classification),
- the canonical Levi G ,
- the regular PVS $V = \mathfrak{n}/[\mathfrak{n}, \mathfrak{n}]$,
- the representation ρ of G on V ,
- the fundamental relative invariant polynomials (FRIPs),
- the generic stabilizer H .
- If V is basic, distinguished, or isotropic, we indicate that.

For simplicity we only consider split G' .

In the following m, n, k are positive integers (possibly with some restrictions). We denote by $\text{Mat}_{m,n}$ the space of $m \times n$ matrices and by Mat_n (resp., Sym_n , Skew_n) the spaces of square (resp., symmetric, skew-symmetric) $n \times n$ -matrices. We denote by Pf the Pfaffian of a skew-symmetric matrix.

EXAMPLE 3.2 (#1 for $k = 2$).

- $G' = \text{GL}_{n_1 + \dots + n_k}$, $k > 1$, $n_i = n_{k+1-i}$ for all i , $n_i \leq n_{i+1}$ for $i \leq l := \lfloor \frac{k-1}{2} \rfloor$.
- \mathfrak{o} corresponds to the partition

$$\left(\overbrace{k, \dots, k}^{n_1}, \overbrace{k-2, \dots, k-2}^{n_2-n_1}, \dots, \overbrace{k-2l}^{n_{l+1}-n_l} \right).$$

- $G = \text{GL}_{n_1} \times \dots \times \text{GL}_{n_k}$.
- $V = \text{Mat}_{n_1, n_2} \oplus \dots \oplus \text{Mat}_{n_{k-1}, n_k}$.

- $\rho(g_1, \dots, g_k)(x_1, \dots, x_{k-1}) = (g_1 x_1 g_2^{-1}, \dots, g_{k-1} x_{k-1} g_k^{-1})$.
- FRIPs: $\begin{cases} \det x_i & \text{if } n_i = n_{i+1} \\ \det(x_i \dots x_{k-i}) & \text{if } n_i < n_{i+1} \end{cases}$
- $H = \text{GL}_{n_1} \times \text{GL}_{n_2-n_1} \times \dots \times \text{GL}_{n_{l+1}-n_l}$ embedded as $(g_1, \dots, g_{l+1}) \mapsto (g_1, \text{diag}(g_1, g_2), \dots, \text{diag}(g_1, \dots, g_{l+1}), \dots, \text{diag}(g_1, g_2), g_1)$.
- Identifying $X^*(G)$ with \mathbb{Z}^k (and consequently \mathfrak{a}_G^* with \mathbb{R}^k) by $(\lambda_1, \dots, \lambda_k) \mapsto \prod_{i=1}^k \det g_i^{\lambda_i}$, the subspace $\bar{\mathfrak{a}}_G^*$ is given by the equations $\lambda_{i+1} + \dots + \lambda_{k-i} = 0$ whenever $n_i < n_{i+1}$ or $i = 0$.
- V is basic if and only if $n_1 = \dots = n_k$. V is distinguished if and only if $n_1 = \dots = n_k = 1$.

EXAMPLE 3.3 (#13 for $k = 1$, up to castling).

- $G' = \text{Sp}_{2(n_1+\dots+n_k)+n_{k+1}}$, $n_1 \leq \dots \leq n_{k+1}$ even. Set $m_i = n_i - n_{i-1}$ (with $n_0 = 0$).
- \mathfrak{o} corresponds to the partition

$$\left(\overbrace{2k+1, \dots, 2k+1}^{m_1}, \overbrace{2k-1, \dots, 2k-1}^{m_2}, \dots, \overbrace{1, \dots, 1}^{m_{k+1}} \right).$$

- $G = \text{GL}_{n_1} \times \dots \times \text{GL}_{n_k} \times \text{Sp}_{n_{k+1}}$.
- $V = \text{Mat}_{n_1, n_2} \oplus \dots \oplus \text{Mat}_{n_k, n_{k+1}}$.
- $\rho(g_1, \dots, g_{k+1})(x_1, \dots, x_k) = (g_1 x_1 g_2^{-1}, \dots, g_k x_k g_{k+1}^{-1})$.
- FRIPs: $\begin{cases} \det x_i & \text{if } n_i = n_{i+1}, \quad i = 1, \dots, k+1 \text{ where } J_{n_{k+1}} \in \\ \text{Pf}(x_i \dots x_{k+1} J_{n_{k+1}} x_{k+1}^t \dots x_i^t) & \text{otherwise,} \end{cases}$
- $\text{Skew}_{n_{k+1}}$ defines $\text{Sp}_{n_{k+1}}$.
- $H = \text{Sp}_{m_1} \times \dots \times \text{Sp}_{m_{k+1}}$ embedded as

$$(h_1, \dots, h_{k+1}) \mapsto (h_1, \iota_2(h_1, h_2), \dots, \iota_{k+1}(h_1, \dots, h_{k+1}))$$

where ι_j is an embedding $\text{Sp}_{m_1} \times \dots \times \text{Sp}_{m_j} \hookrightarrow \text{Sp}_{n_j}$.

- V is basic if and only if $n_1 = \dots = n_k$.

EXAMPLE 3.4 (#2 for $k = 1$).

- $G' = \text{Sp}_{2(n_1+\dots+n_k)}$, $n_1 \leq \dots \leq n_k$. Set $m_i = n_i - n_{i-1}$ (with $n_0 = 0$).
- \mathfrak{o} corresponds to the partition

$$\left(\overbrace{2k, \dots, 2k}^{m_1}, \overbrace{2k-2, \dots, 2k-2}^{m_2}, \dots, \overbrace{2, \dots, 2}^{m_k} \right).$$

- $G = \text{GL}_{n_1} \times \dots \times \text{GL}_{n_k}$.
- $V = \text{Mat}_{n_1, n_2} \oplus \dots \oplus \text{Mat}_{n_{k-1}, n_k} \oplus \text{Sym}_{n_k}$.
- $\rho(g_1, \dots, g_{k+1})(x_1, \dots, x_k) = (g_1 x_1 g_2^{-1}, \dots, g_{k-1} x_{k-1} g_k^{-1}, g_k x_k g_k^t)$.
- FRIPs: $\begin{cases} \det x_i & \text{if } n_i = n_{i+1} \text{ or } i = k, \\ \det x_i \dots x_k x_{k-1}^t \dots x_i^t & \text{otherwise,} \end{cases} \quad i = 1, \dots, k.$
- $H = \text{O}_{m_1} \times \dots \times \text{O}_{m_k}$ embedded as

$$(h_1, \dots, h_k) \mapsto (h_1, \iota_2(h_1, h_2), \dots, \iota_k(h_1, \dots, h_k)),$$

where ι_j is an embedding $\text{O}_{m_1} \times \dots \times \text{O}_{m_j} \hookrightarrow \text{O}_{n_j}$.

- V is basic if and only if $n_1 = \dots = n_k \neq 2$. V is isotropic if and only if $m_i = 2$ for some i . V is distinguished if and only if $m_i = 0$ or 1 for all i .

EXAMPLE 3.5 (#15 for $k = 1$, up to castling).

- $G' = \text{SO}_{2(n_1+\dots+n_k)+n_{k+1}}$, $n_1 \leq \dots \leq n_{k+1}$. Set $m_i = n_i - n_{i-1}$ (with $n_0 = 0$).
- \mathfrak{o} corresponds to the partition

$$\left(\overbrace{2k+1, \dots, 2k+1}^{m_1}, \overbrace{2k-1, \dots, 2k-1}^{m_2}, \dots, \overbrace{1, \dots, 1}^{m_{k+1}} \right).$$

- $G = \text{GL}_{n_1} \times \dots \times \text{GL}_{n_k} \times \text{SO}_{n_{k+1}}$.
- $V = \text{Mat}_{n_1, n_2} \oplus \dots \oplus \text{Mat}_{n_k, n_{k+1}}$.
- $\rho(g_1, \dots, g_{k+1})(x_1, \dots, x_k) = (g_1 x_1 g_2^{-1}, \dots, g_k x_k g_{k+1}^{-1})$.
- FRIPs: $\begin{cases} \det x_i & \text{if } n_i = n_{i+1}, \\ \det(x_i \dots x_{k+1} J_{n_{k+1}} x_{k+1}^t \dots x_i^t) & \text{otherwise,} \end{cases}$, $i = 1, \dots, k+1$ where $J_{n_{k+1}} \in \text{Sym}_{n_{k+1}}$ defines $\text{SO}_{n_{k+1}}$.
- $H = S(\text{O}_{m_1} \times \dots \times \text{O}_{m_{k+1}})$ embedded as

$$(h_1, \dots, h_{k+1}) \mapsto (h_1, \iota_2(h_1, h_2), \dots, \iota_{k+1}(h_1, \dots, h_{k+1})),$$

where ι_j is an embedding $\text{O}_{m_1} \times \dots \times \text{O}_{m_j} \hookrightarrow \text{O}_{n_j}$.

- V is basic if and only if $n_1 = \dots = n_k \neq 2$ and $m_{k+1} \neq 2$. V is isotropic if and only if $m_i = 2$ for some i . V is distinguished if and only if $m_i = 0$ or 1 for all i .

EXAMPLE 3.6 (#3 for $k = 1$).

- $G' = \text{SO}_{2(n_1+\dots+n_k)}$, $n_1 \leq \dots \leq n_k$ even. Set $m_i = n_i - n_{i-1}$ (with $n_0 = 0$).
- \mathfrak{o} corresponds to the partition

$$\left(\overbrace{2k, \dots, 2k}^{m_1}, \overbrace{2k-2, \dots, 2k-2}^{m_2}, \dots, \overbrace{2, \dots, 2}^{m_k} \right).$$

- $G = \text{GL}_{n_1} \times \dots \times \text{GL}_{n_k}$.
- $V = \text{Mat}_{n_1, n_2} \oplus \dots \oplus \text{Mat}_{n_{k-1}, n_k} \oplus \text{Skew}_{n_k}$.
- $\rho(g_1, \dots, g_{k+1})(x_1, \dots, x_k) = (g_1 x_1 g_2^{-1}, \dots, g_{k-1} x_{k-1} g_k^{-1}, g_k x_k g_k^t)$.
- FRIPs: $\begin{cases} \det x_i & \text{if } n_i = n_{i+1}, \\ \text{Pf}(x_i \dots x_k x_{k-1}^t \dots x_i^t) & \text{otherwise,} \end{cases}$, $i = 1, \dots, k$.
- $H = \text{Sp}_{m_1} \times \dots \times \text{Sp}_{m_k}$ embedded as

$$(h_1, \dots, h_k) \mapsto (h_1, \iota_2(h_1, h_2), \dots, \iota_k(h_1, \dots, h_k)),$$

where ι_j is an embedding $\text{Sp}_{m_1} \times \dots \times \text{Sp}_{m_j} \hookrightarrow \text{Sp}_{n_j}$.

- V is basic if and only if $n_1 = \dots = n_k$.

EXAMPLE 3.7 (#4).

- $G' = G_2$.
- \mathfrak{o} is the subregular (10-dimensional) orbit.
- $G = \text{GL}_2$, with Δ_0^G consisting of the short simple root.
- V is the space of binary cubic forms
- ρ is the symmetric cube representation twisted by \det^{-1} .
- FRIP: the discriminant.

Table 1. Irreducible basic PVSs of DK-type for exceptional groups

| # | G' | G | ρ | H° | $H^\circ \hookrightarrow G$ |
|----|-------|---|---------------------------------|----------------------------|---|
| 16 | F_4 | GSpin_7 | $\mathrm{spin}(8)$ | G_2 | ϕ |
| 6 | E_7 | GL_7 | \wedge^3 | G_2 | π |
| 10 | E_7 | $\mathrm{SL}_5 \times \mathrm{GL}_3$ | $\wedge^2 \otimes \mathrm{Std}$ | SL_2 | $\mathrm{Sym}^4 \times \mathrm{Sym}^2$ |
| 27 | E_7 | GE_6 | 27-dim. | F_4 | σ |
| 20 | E_7 | $\mathrm{Spin}_{10} \times \mathrm{GL}_2$ | half-spin(16) \otimes Std | $G_2 \times \mathrm{SL}_2$ | ι_1 |
| | E_7 | $\mathrm{GL}_2 \times \mathrm{SL}_3 \times \mathrm{SL}_4$ | Std \otimes Std \otimes Std | SL_2 | $\mathrm{id} \times \mathrm{Sym}^2 \times \mathrm{Sym}^3$ |
| 7 | E_8 | GL_8 | \wedge^3 | PGL_3 | adjoint rep. |
| 21 | E_8 | $\mathrm{Spin}_{10} \times \mathrm{GL}_3$ | half-spin(16) \otimes Std | SO_4 | ι_2 |
| 24 | E_8 | GSpin_{14} | half-spin(64) | $G_2 \times G_2$ | ι_3 |

- H is (an F -form of) the symmetric group S_3 (with its faithful two-dimensional representation).
- V is distinguished.

EXAMPLE 3.8 (#8).

- $G' = F_4$, $\Delta_0 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, α_1, α_2 long, α_3, α_4 short.
- $\mathfrak{o} = F_4(a_3)$ (40-dimensional)
- $G = (\mathrm{GL}_2 \times \mathrm{GL}_3) / \{(\lambda^2 I_2, \lambda^{-1} I_3)\}$, $\Delta_0^G = \{\alpha_1, \alpha_3, \alpha_4\}$.
- $V = \mathrm{Hom}_F(F^2, \mathrm{Sym}_3)$.
- $\rho(g_1, g_2)(A)(\xi) = g_2 A(\xi g_1) g_2^t$
- FRIP: the discriminant of the binary cubic form $\xi \mapsto \det A(\xi)$.
- H is (an F -form of) of the symmetric group S_4 , embedded in G by its faithful three-dimensional representation on the GL_3 -factor and by its irreducible two-dimensional representation (which factors through the surjection $S_4 \rightarrow S_3$) on the GL_2 -factor.
- V is distinguished.

EXAMPLE 3.9 (#12).

- $G' = E_6$.
- $\mathfrak{o} = D_4(a_1)$ (58-dimensional)
- $G = \mathrm{SL}_3 \times \mathrm{SL}_3 \times \mathrm{GL}_2$.
- $V = \mathrm{Hom}_F(F^2, \mathrm{Mat}_3)$.
- $\rho(g_1, g_2, g_3)(A)(\xi) = g_1 A(\xi g_3) g_2^{-1}$, $\xi \in F^2$.
- FRIP: the discriminant of the binary cubic form $\xi \mapsto \det A(\xi)$.
- H is the normalizer of a maximal torus in SL_3 , embedded diagonally into the SL_3 -factors and mapped to the GL_2 -factor via the faithful two-dimensional representation of its component group S_3 .
- V is isotropic.

Other examples of basic PVSs of DK-type are summarized in Table 1 which refers to the numbering in the list in the appendix of [23] whenever ρ is reduced. (See also [21].)

Here, π is the embedding $G_2 \hookrightarrow \mathrm{SO}_7 \hookrightarrow \mathrm{GL}_7$ and ϕ is its lifting to an embedding $G_2 \hookrightarrow \mathrm{Spin}_7$. Also, σ is the embedding of F_4 as the fixed point subgroup of the outer involution of E_6 . Finally,

Table 2. PVSSs of DK-type pertaining to the Freudenthal–Tits magic square

| # | G' | G | L | K | ρ | ρ' |
|----|-------|-----------------------|---|------------------|---------------|-------------------------------------|
| 14 | F_4 | GSp_6 | GL_3 | PGL_2 | 14-dim. | Sym^2 |
| 5 | E_6 | GL_6 | $(\mathrm{GL}_3 \times \mathrm{GL}_3) \cap \mathrm{SL}_6$ | SL_3 | \wedge^3 | $\mathrm{Std} \otimes \mathrm{Std}$ |
| 23 | E_7 | GSpin_{12} | GL_6 | Sp_6 | half-spin(32) | \wedge^2 |
| 29 | E_8 | GE_7 | GE_6 | F_4 | 56-dim. | 27-dim. |

$$\begin{aligned} \iota_1(x, y) &= (\psi_{7,3}(\phi(x), y), y), \\ \iota_2(x, y) &= (\psi_{6,4}(\mathrm{Sym}^3(y), (x, y)), \mathrm{Sym}^2(y)), \\ \iota_3(x, y) &= \psi_{7,7}(\phi(x), \phi(y)), \end{aligned}$$

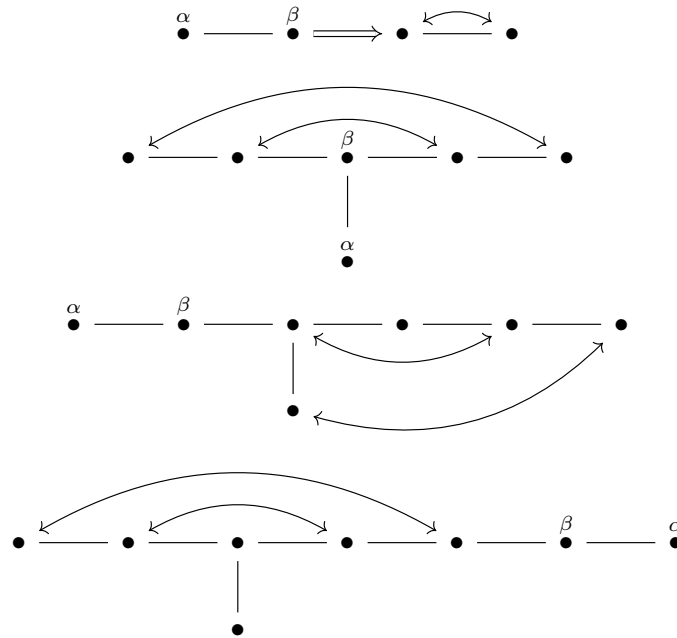
where $\psi_{m,n}$ is the homomorphism $\mathrm{Spin}_m \times \mathrm{Spin}_n \rightarrow \mathrm{Spin}_{m+n}$ and we identify $\mathrm{Spin}_3 \simeq \mathrm{SL}_2$, $\mathrm{Spin}_6 \simeq \mathrm{SL}_4$, $\mathrm{Spin}_4 \simeq \mathrm{SL}_2 \times \mathrm{SL}_2$, $\mathrm{SO}_4 \simeq (\mathrm{SL}_2 \times \mathrm{SL}_2) / \{\pm 1\}$.

3.3

Another interesting set of examples of irreducible PVSSs of DK-type arises from the Freudenthal–Tits magic square (see Table 2). In these cases, G is a maximal Levi subgroup of G' corresponding to a simple root α which is a leaf in the Dynkin diagram (it is the unique simple root which is not orthogonal to the highest root $\tilde{\alpha}$ of G'). The highest weight μ of ρ is $\tilde{\alpha} - \alpha$ and its restriction to G^{der} is the fundamental weight corresponding to the simple root β adjacent to α (and of the same length) in the Dynkin diagram. Let $Q = LN'$ be the maximal parabolic subgroup of G^{der} corresponding to β . Then, there exists a nontrivial element w_0 in the normalizer of L in G^{der} . We have $\alpha = w_0\mu$, which is the lowest weight of V . Clearly, L stabilizes the root spaces V_α and V_μ . Remarkably, any $v \in V_\alpha + V_\mu$ outside $V_\alpha \cup V_\mu$ is regular. We have $H^\circ = L^{\mathrm{der}}$ and $[H : H^\circ] = 2$ with H containing a representative of w_0 . In particular, V is not basic. Since $V \simeq \mathfrak{g}/\mathfrak{h}$ as a representation of H , we see that the restriction of ρ to H is $\mathrm{Ind}_{H^\circ}^H(\rho' \oplus 1)$ where ρ' is the representation of L on \mathfrak{n}' . It turns out that N' is abelian. Therefore, $(L, \mathfrak{n}', \rho')$ is a PVS and in fact it is basic: the generic stabilizer (whose identity component is denoted by K in the table) is the centralizer of (a suitable choice of) w_0 in L , i.e., the stabilizer of an outer involution of L corresponding to w_0 . See drawings below.

The fundamental relative invariant polynomials in these cases (which are of degree 4) are called Freudenthal quartics (see [12], [20], [37]). They arise by considering $\mathrm{ad}(x)^4$, $x \in V \subset \mathfrak{g}'$ as a map between the one-dimensional spaces $\mathfrak{g}'_{-\tilde{\alpha}}$ and $\mathfrak{g}'_{\tilde{\alpha}}$ (cf. [13, Table II]). The orbit structure of V is described in [32].

What is more, the Richardson orbit with respect to the maximal parabolic subgroup corresponding to β is also even. These orbits give rise to the PVSSs #8 (Example 3.8), #12 (Example 3.9), #9 and #28, respectively, in the list of the appendix of [23]. They all have a fundamental relative invariant polynomial of degree 12, obtained as the determinant of $\mathrm{ad}(x)^6 : \mathfrak{g}'_{-6} \rightarrow \mathfrak{g}'_6$, $x \in V$, where $\mathfrak{g}'_6 = \mathfrak{g}'_{\tilde{\alpha}-\alpha} \oplus \mathfrak{g}'_{\tilde{\alpha}}$ is two-dimensional. Alternatively, the PVS $(L, \mathfrak{n}', \rho')$ mentioned above has a cubic fundamental relative invariant polynomial, and the PVS V associated to β is the direct sum of \mathfrak{n}' and of its isomorphic image under the simple reflection w_α , which enables one to construct the fundamental relative invariant of V as the discriminant of the cubic form associated to pairs of elements of \mathfrak{n}' .



Sato and Kimura classified the irreducible PVSs over \bar{F} up to a suitable notion of equivalence weaker than isomorphism [46].

There are 29 essential cases of regular PVSs, listed in the appendix of [23].³ All these examples are either of DK-type or restrictions thereof to suitable subgroups.⁴ Most of these examples are non-isotropic. The exceptions are

- The symmetric square representation of GL_2 (Example 3.4 for $k = 1, n_1 = 2$).
- Example 3.9.
- The representation $GL_2 \times SO_{k+2}$ on $Mat_{2,k+2}$ (Example 3.5 for $k = 1, n_1 = 2$).
- The restriction of the latter to $GL_2 \times Spin_7$ (resp., $GL_2 \times G_2$) for $k = 6$ (resp., $k = 5$) via the Spin representation $Spin_7 \hookrightarrow SO_8$ (resp., the 7-dimensional representation $G_2 \hookrightarrow SO_7$).

The non-regular irreducible PVSs over \bar{F} are (up to castling equivalence) mostly of R-type or restrictions thereof. The notable exception is the representation ρ of $SL_2 \times SL_n$ on $Hom_F(F^2, Skew_n)$ given by $\rho(g_1, g_2)A(\xi) = g_2A(\xi g_1)g_2^t$, which is a PVS for odd n but not of R-type for $n > 7$. Another curious example is the restriction of the representation of $GL_3 \times Sp_{2n}$ (of R-type) on $Mat_{3,2n}, n > 1$ given by $\rho(g_1, g_2)A = g_1A g_2^{-1}$ to the image of the symmetric square representation of GL_2 times Sp_{2n} . This irreducible, non-regular PVS has a relative invariant.

§4. Statement of main result

From now on assume that F is a number field. Let $\mathbb{A} = F_\infty \times \mathbb{A}_{\text{fin}}$ be the ring of adeles of F . For any connected F -group G we write

$$G(\mathbb{A})^1 = \cap_{\chi \in X^*(G)} \text{Ker} |\chi|,$$

³ The general case is obtained from these by castling transforms, see [23, §7.1]. These preserve irreducibility and the generic stabilizer but may yield PVSs which are far removed from nilpotent orbits.

⁴ The examples not of DK-type are # 17, # 18, # 19, # 22, # 25 and # 26. Example # 1 can be of DK-type or not.

where $|\chi| : G(\mathbb{A}) \rightarrow \mathbb{R}_{>0}$ is the composition $G(\mathbb{A}) \xrightarrow{\chi} \mathbb{A}^* \xrightarrow{|\cdot|} \mathbb{R}_{>0}$. The automorphic space $G(F)\backslash G(\mathbb{A})^1$ is of finite volume.

Let

$$\mathfrak{a}_{G,\mathbb{C}}^* = \mathfrak{a}_G^* \otimes_{\mathbb{R}} \mathbb{C} = X^*(G) \otimes_{\mathbb{Z}} \mathbb{C}.$$

Every $\lambda \in \mathfrak{a}_{G,\mathbb{C}}^*$ defines a (topological) quasi-character $g \mapsto g^\lambda$ of $G(\mathbb{A})$ by

$$g^{\sum c_i \chi_i} = \prod_i |\chi_i(g)|^{c_i}, \quad \chi_i \in X^*(G), \quad c_i \in \mathbb{C}.$$

All quasi-characters of $G(\mathbb{A})/G(\mathbb{A})^1$ are obtained this way. In particular, we get an isomorphism of topological groups

$$\mathfrak{ia}_G^* \xrightarrow{\text{Exp}_G} \text{PD}(G(\mathbb{A})/G(\mathbb{A})^1), \quad \lambda \mapsto (g \mapsto g^\lambda),$$

where PD denotes the Pontryagin dual.

The following is a supplement to Lemma 2.1.

LEMMA 4.1. *Suppose that G is connected and H is an F -subgroup of G . Write $H(\mathbb{A})^{(1)} = H(\mathbb{A}) \cap G(\mathbb{A})^1$. Then, the following two conditions are equivalent.⁵*

1. *The restriction map $X^*(G) \xrightarrow{r} X^*(H^\circ)$ has finite cokernel.*
2. *The volume of the quotient $H(F)\backslash H(\mathbb{A})^{(1)}$ is finite.*

Proof. We may assume without loss of generality that H is connected, since $[H(F) : H^\circ(F)]$ is finite and $H(\mathbb{A})/H^\circ(\mathbb{A})$ is compact. Now, $\text{vol}(H(F)\backslash H(\mathbb{A})^1) < \infty$. Therefore, $\text{vol}(H(F)\backslash H(\mathbb{A})^{(1)}) < \infty$ if and only if $Y = H(\mathbb{A})^{(1)}/H(\mathbb{A})^1$ is compact. Note that Y is the kernel of the homomorphism

$$H(\mathbb{A})/H(\mathbb{A})^1 \rightarrow G(\mathbb{A})/G(\mathbb{A})^1.$$

Passing to the Pontryagin dual, the lemma follows from the commutative diagram

$$\begin{array}{ccc} \mathfrak{ia}_G^* = \mathfrak{i}(X^*(G) \otimes \mathbb{R}) & \xrightarrow{\text{Exp}_G} & \text{PD}(G(\mathbb{A})/G(\mathbb{A})^1) \\ \downarrow r \otimes \text{id} & & \downarrow \\ \mathfrak{ia}_H^* = \mathfrak{i}(X^*(H) \otimes \mathbb{R}) & \xrightarrow{\text{Exp}_H} & \text{PD}(H(\mathbb{A})/H(\mathbb{A})^1). \end{array}$$

In fact, Y is compact if and only if $Y = 1$ if and only if the restriction map $\mathfrak{a}_G^* \rightarrow \mathfrak{a}_H^*$ is surjective. □

Henceforth, (G, V, ρ) is a reductive PVS (i.e., G is reductive). We apply the criterion of Lemma 4.1 to the stabilizer G_v of a vector $v \in X(F)$. Recall Definition 2.7.

LEMMA 4.2. *Let $v \in X(F)$. Then,*

1. *The group $G_v(\mathbb{A})^{(1)} = G_v(\mathbb{A}) \cap G(\mathbb{A})^1$ is unimodular.*
2. *The volume $\kappa_v = \text{vol}(G_v(F)\backslash G_v(\mathbb{A})^{(1)})$ is finite if and only if v is non-isotropic.*

Proof. The first assertion follows from the fact that modular character of G_v is the restriction of a character of G by (2.2). The second assertion follows from Lemma 4.1 and Lemma 2.6. □

⁵ Note that both conditions imply that $H(\mathbb{A}) \cap G(\mathbb{A})^1$ is unimodular.

We say that a continuous function ϕ on $V(\mathbb{A})$ is rapidly decreasing, if it is supported in $V(F_\infty)K$ for some compact subgroup K of $V(\mathbb{A}_{\text{fin}})$ and $\sup_{v \in V(\mathbb{A})} |f(v)|(1 + \|v_\infty\|)^N < \infty$ for every nonnegative integer N , where $\|\cdot\|$ denotes a norm on the real vector space $V(F_\infty)$. Let $C_{\text{rd}}(V(\mathbb{A}))$ be the space of continuous rapidly decreasing functions on $V(\mathbb{A})$. It is a (strict) LF-space, i.e., the strict (locally convex) inductive limit of Fréchet spaces.

For any $\phi \in C_{\text{rd}}(V(\mathbb{A}))$ we form the theta function

$$\theta_\phi(g) = \theta_\phi^V(g) = \sum_{v \in X(F)} \phi(\rho(g)^{-1}v), \quad g \in G(\mathbb{A}).$$

The sum is locally uniformly absolutely convergent and defines a continuous function on $G(F) \backslash G(\mathbb{A})$. We would like to study this function and the corresponding zeta function which is roughly speaking the Mellin transform of θ_ϕ .

Let $\bar{\mathfrak{a}}_{G,\mathbb{C}}^* = \bar{\mathfrak{a}}_G^* \otimes \mathbb{C} = X^*(G/G^*) \otimes \mathbb{C}$. Ideally, we would like to study the double integral

$$\int_{G(\mathbb{A})^1 G^*(\mathbb{A}) \backslash G(\mathbb{A})} \int_{G(F) \backslash G(\mathbb{A})^1} \theta_\phi(ga) \, dg \, a^{-\lambda} |\det \rho(a)|^{-1} \, da, \tag{4.1}$$

for λ in a suitable region of convergence in $\bar{\mathfrak{a}}_{G,\mathbb{C}}^*$. Note that the outer integral formally makes sense, since the inner integral can be written as the sum over the $G(F)$ -orbits in $X(F)$, and the contribution of an orbit $\rho(G(F))v$ is

$$\int_{G_v(F) \backslash G(\mathbb{A})^1} \phi(\rho(ga)^{-1}v) \, dg = \kappa_v \int_{(G_v(\mathbb{A}) \cap G(\mathbb{A})^1) \backslash G(\mathbb{A})^1} \phi(\rho(ga)^{-1}v) \, dg,$$

where $\kappa_v = \text{vol}(G_v(F) \backslash (G_v(\mathbb{A}) \cap G(\mathbb{A})^1))$ as above. Note that the images of $G_v(\mathbb{A})$ and $G^*(\mathbb{A})$ in $G(\mathbb{A})^1 \backslash G(\mathbb{A}) = \mathfrak{a}_G = X_*(G^{\text{ab}}) \otimes \mathbb{R}$ coincide—they are both equal to the image of $X_*(G^*/G^{\text{der}}) \otimes \mathbb{R}$ in \mathfrak{a}_G . Therefore, we can write the above as

$$\kappa_v \int_{G_v(\mathbb{A}) \backslash G(\mathbb{A})^1 G^*(\mathbb{A})} \phi(\rho(ga)^{-1}v) |\det \rho(g)|^{-1} \, dg.$$

This also explains why the more straightforward integral $\int_{G(F) \backslash G(\mathbb{A})} \theta_\phi(g) g^{-\lambda} \, dg$ would not converge (for any λ) if $\bar{\mathfrak{a}}_G^* \neq \mathfrak{a}_G^*$ (i.e., if $\text{rk } X^*(H) > 0$ by Lemma 2.4).

A necessary condition for the convergence of this orbital integral for all ϕ is that κ_v is finite, which by Lemma 4.2 is equivalent to v being non-isotropic. Define

$$\theta_\phi^{\text{niso}}(g) = \sum_{v \in X(F)^{\text{niso}}} \phi(\rho(g)^{-1}v), \quad g \in G(\mathbb{A}).$$

Thus, $\theta_\phi^{\text{niso}}$ omits the $G(F)$ -orbits in $X(F)$ whose contributions to (4.1) clearly diverge.

Our main result, which will be ultimately proved in §7, is the following.

THEOREM 4.3. *Assume that (G, V, ρ) is a regular PVS. Then, the double integral*

$$Z(\phi, \lambda) = \int_{G(\mathbb{A})^1 G^*(\mathbb{A}) \backslash G(\mathbb{A})} \int_{G(F) \backslash G(\mathbb{A})^1} \theta_\phi^{\text{niso}}(ga) \, dg \, a^{-\lambda} |\det \rho(a)|^{-1} \, da,$$

is absolutely convergent and holomorphic for any $\lambda \in \bar{\mathfrak{a}}_{G,\mathbb{C}}^$ such that $\Re \lambda \in \mathbb{R}_{>0} \mathfrak{X}$. Moreover, the integral is continuous in $\phi \in C_{\text{rd}}(V(\mathbb{A}))$.*

Roughly speaking, Theorem 4.3 means that the only problematic terms in (4.1) are those arising from the orbits $\rho(G(F))v$ where $\kappa_v = \infty$.

The proof of Theorem 4.3 is based on a direct estimation of the theta function $\theta_\phi^{\text{niso}}$ using standard lattice-point counting methods. In fact, this idea has already been used previously by Yukie [53], [54] to obtain partial results in the direction of Theorem 4.3, mostly via a case-by-case analysis, for absolutely irreducible representations ρ .⁶ Our approach is completely general and avoids any classification result. Also, the possibility to define a zeta function in the case where $\bar{\mathfrak{a}}_G^* \neq \mathfrak{a}_G^*$ does not seem to have been noticed in the literature.

An alternative classification-free method was developed by Sato [40] and H. Saito [38], among others, using the decomposition of $X(F)$ into $G(F)$ -orbits (Galois cohomology) and an analysis of the resulting Euler products. Saito proves Theorem 4.3 under the additional assumption that H° is semisimple and the component group H/H° is abelian. Under the weaker assumption $\bar{\mathfrak{a}}_G^* = \mathfrak{a}_G^*$, i.e., $X^*(H)$ finite, he proves convergence for all $\Re\lambda$ in some cone of the form $\lambda_0 + \mathbb{R}_{>0}\mathfrak{X}$.⁷

We expect Theorem 4.3 to hold without the regularity assumption on V . In the case where there are no relative invariants (in which V is called *special*), the set $G(\mathbb{A})X(F)$ is open in $V(\mathbb{A})$, and the integral $\int_{G(F)\backslash G(\mathbb{A})^1} \theta_\phi(g) dg$ converges to a constant multiple of $\int_{G(\mathbb{A})X(F)} \phi(x) dx$. (See [29], [30] and the discussion in [17, §2.4]. The set $G(\mathbb{A})X(F)$ is described in [36, Proposition 1.7].)

§5. Bounding the theta function

We start with a general bound on $\theta_\phi(g)$ (Proposition 5.1 below) which is valid without the regularity assumption, but is by itself insufficient to prove convergence of the zeta integral of $\theta_\phi^{\text{niso}}(g)$. To state it, we need to set some more notation which will be used throughout the paper.

5.1

Recall that G is a reductive group over a number field F . Fix a maximal F -split torus T_0 of G and a minimal parabolic F -subgroup P_0 of G containing T_0 . We have a Levi decomposition $P_0 = M_0 \ltimes N_0$ where M_0 is the centralizer of T_0 in G and $N_0 = N_{P_0}$, which is a maximal unipotent F -subgroup of G .

If P is a standard parabolic subgroup of G , then we denote by M_P its standard Levi subgroup.

For simplicity we write $\mathfrak{a}_0 = \mathfrak{a}_{P_0} = \mathfrak{a}_{M_0} \simeq \mathfrak{a}_{T_0}$. Let

$$H_0 : M_0(\mathbb{A}) \rightarrow \mathfrak{a}_0,$$

be the group homomorphism defined by

$$e^{\langle \chi, H_0(m) \rangle} = |\chi(m)|, \quad m \in M_0(\mathbb{A}), \quad \chi \in X^*(M_0).$$

Let $\Delta_0 \subset X^*(T_0)$ be the set of simple roots of T_0 on $\text{Lie } N_0$.

Fix a maximal compact subgroup K of $G(\mathbb{A})$ which is in good position with respect to P_0 .

⁶ Yukie proved a general qualitative convergence result for distinguished, absolutely irreducible PVSSs.

⁷ Strictly speaking, Saito assumes that there exists $v \in X(F)$ such that $X^*(G_v^\circ) = 1$. By Lemma 2.2 and Lemma 2.6 this implies that $v \in X(F)^{\text{niso}}$ and $X^*(H)$ is finite. Conversely, if $X^*(H)$ is finite, then $X(F)^{\text{niso}} = \{v \in X(F) \mid X^*(G_v^\circ) = 1\}$. Also, he considers a more general situation where G is a connected linear algebraic group and X the complement of a hypersurface.

For any $g \in G(\mathbb{A})$, we write $m_0(g) \in M_0(\mathbb{A})/(M_0(\mathbb{A}) \cap K)$ for the $M_0(\mathbb{A})$ -part in the Iwasawa decomposition $G(\mathbb{A}) = N_0(\mathbb{A})M_0(\mathbb{A})K$ of g .

We extend H_0 to a left- $N_0(\mathbb{A})$ and right- K -invariant function on $G(\mathbb{A})$. Thus,

$$H_0(g) = H_0(m_0(g)).$$

Fix a Siegel set \mathfrak{S} in $G(\mathbb{A})$ of the form

$$\mathfrak{S} = \{g \in G(\mathbb{A}) \mid \langle \alpha, H_0(g) \rangle > c_0 \ \forall \alpha \in \Delta_0\},$$

for some fixed constant c_0 . Thus,

1. \mathfrak{S} is left $P_0(F)N_0(\mathbb{A})$ -invariant.
2. For any $g \in G(\mathbb{A})$, the set $\{\gamma \in P_0(F) \backslash G(F) \mid \gamma g \in \mathfrak{S}\}$ is finite.
3. There exists a compact subset $\Omega \subset G(\mathbb{A})$ such that

$$\mathfrak{S} \subset P_0(F)(T_0(\mathbb{A}) \cap \mathfrak{S})\Omega. \tag{5.1}$$

4. $G(F)\mathfrak{S} = G(\mathbb{A})$ provided that c_0 is sufficiently negative.

Let δ_{P_0} be the modulus function of $P_0(\mathbb{A})$.

5.2

Recall that V is a PVS. (For now, we do not assume that V is regular.) The first order of business is the following result, which will be proved in the rest of this section.

PROPOSITION 5.1. *There exists a continuous seminorm ν on $C_{\text{rd}}(V(\mathbb{A}))$ such that*

$$|\theta_\phi(g)| \leq \delta_{P_0}(m_0(g)) |\det \rho(g)| \nu(\phi),$$

for all $\phi \in C_{\text{rd}}(V(\mathbb{A}))$ and $g \in \mathfrak{S}$.

Note that $|\det \rho|$ is the modulus function for the action of $G(\mathbb{A})$ on $V(\mathbb{A})$, that is,

$$\int_{V(\mathbb{A})} f(\rho(g)^{-1}v) \, dv = |\det \rho(g)| \int_{V(\mathbb{A})} f(v) \, dv,$$

for any $f \in L^1(V(\mathbb{A}))$ and $g \in G(\mathbb{A})$.

5.3

Let Φ_G be the set of roots of T_0 on G and let Ψ_V be the set of weights of T_0 on V . The set Φ_G is a (not necessarily reduced) root system and the co-roots are in $X_*(T_0)$. We decompose \mathfrak{g} and V according to the roots and weights:

$$\mathfrak{g} = \bigoplus_{\alpha \in \Phi_G \cup \{0\}} \mathfrak{g}_\alpha, \tag{5.2a}$$

$$V = \bigoplus_{\beta \in \Psi_V} V_\beta. \tag{5.2b}$$

The vector spaces \mathfrak{g}_α , $\alpha \in \Phi_G$, are not necessarily one-dimensional unless G is split. Moreover, \mathfrak{g}_α is not necessarily a commutative Lie subalgebra (namely, if $2\alpha \in \Phi_G$). The spaces \mathfrak{g}_α and V_β are invariant under M_0 . We write

$$n_\beta = \dim V_\beta, \quad \beta \in \Psi_V,$$

and

$$\delta_V = \det \rho|_{T_0} = \sum_{\beta \in \Psi_V} n_\beta \beta \in X^*(T_0).$$

By convention, $V_\gamma = 0$ if $\gamma \notin \Psi_V$. Note that

$$d\rho(\mathfrak{g}_\alpha)(V_\beta) \subset V_{\alpha+\beta}, \quad \alpha \in \Phi_G, \beta \in \Psi_V, \tag{5.3}$$

where $d\rho : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ is the Lie algebra representation obtained by differentiating ρ .

Let

$$\delta_0 = \sum_{\alpha \in \Phi_G^+} (\dim \mathfrak{g}_\alpha) \alpha \in X^*(T_0),$$

where $\Phi_G^+ \subset \Phi_G$ is the subset of positive roots with respect to P_0 , so that

$$\delta_{P_0}(m) = e^{\langle \delta_0, H_0(m) \rangle}, \quad m \in M_0(\mathbb{A}).$$

We enumerate the roots $\Phi_G \cup \{0\}$ (including multiplicities) as $\alpha_i, i \in \mathcal{I}_G$ for some index set \mathcal{I}_G , and fix a basis $\mathcal{B}_G = (x_i)_{i \in \mathcal{I}_G}$ for \mathfrak{g} such that $x_i \in \mathfrak{g}_{\alpha_i}$ for all i . Similarly, we enumerate the weights Ψ_V including multiplicities as $\beta_j, j \in \mathcal{J}_V$, and fix a basis $\mathcal{B}_V = (y_j)_{j \in \mathcal{J}_V}$ for V such that $y_j \in V_{\beta_j}$ for all j .

Let $\Gamma(V)$ be the (finite) set of subspaces of V that are sums of weight subspaces. Thus, every $U \in \Gamma(V)$ is of the form

$$U = \bigoplus_{\beta \in \Psi_U} V_\beta,$$

where Ψ_U is a subset of Ψ_V . We will also write

$$\mathcal{J}_{V/U} = \{j \in \mathcal{J}_V \mid \beta_j \notin \Psi_U\}. \tag{5.4}$$

LEMMA 5.2. *Let $\psi \in \mathbb{Z}_{\geq 0}\mathfrak{X}$, so that $f = \text{RI}_\psi$ is a polynomial. Let $U \in \Gamma(V)$ be such that $f|_U \neq 0$. Then,*

1. $\psi \in \mathbb{Z}_{\geq 0}\Psi_U$.
2. *There exists $U' \in \Gamma(V), U' \subset U$ such that $f|_{U'} \neq 0$ and $\psi \in \mathbb{R}_{> 0}\Psi_{U'}$.*

Proof. Let $\mathcal{B}_{V^*} = (y_j^*)_{j \in \mathcal{J}_V}$ be the basis of V^* dual to \mathcal{B}_V and let $\mathcal{B}_{F[V]}$ be the basis of $F[V]$ consisting of monomials in \mathcal{B}_{V^*} . By definition, the weight of a monomial $y_{j_1}^* \cdots y_{j_k}^*$ in $\mathcal{B}_{F[V]}$ is the sum $\beta_{j_1} + \cdots + \beta_{j_k}$. We view $F[U]$ as a subalgebra of $F[V]$ using the unique projection $V \rightarrow U$ with kernel in $\Gamma(V)$. The basis $\mathcal{B}_{F[V]}$ contains $\mathcal{B}_{F[U]}$ (defined in the obvious way).

Let $h = f|_U$. Expand h in the basis $\mathcal{B}_{F[U]}$. Since $f = \text{RI}_\psi$, the weight of any basis element with nonzero coefficient is ψ . The first part follows.

Let U' be the smallest subspace in $\Gamma(V)$ such that $h \in F[U']$ (viewed as a subspace of $F[U]$ via the projection $U \rightarrow U'$ with kernel in $\Gamma(V)$). Then, $\Psi_{U'}$ is the union of $\{\beta_{j_1}, \dots, \beta_{j_k}\}$ as we range over all the monomials $y_{j_1}^* \cdots y_{j_k}^*$ in $\mathcal{B}_{F(U)}$ with nonzero coefficients in the expansion of h . Once again, since $f = \text{RI}_\psi$, for any basis element with nonzero coefficient as above we have $\beta_{j_1} + \cdots + \beta_{j_k} = \psi$. Averaging over all these expressions of ψ , we deduce that $\psi \in \mathbb{R}_{> 0}\Psi_{U'}$. □

Consider the set $\Gamma_{\text{reg}}(V)$ of subspaces in $\Gamma(V)$ that contain a regular element:

$$\Gamma_{\text{reg}}(V) = \{U \in \Gamma(V) \mid U \cap X \neq \emptyset\}.$$

COROLLARY 5.3. *Let $U \in \Gamma_{\text{reg}}(V)$. Then,*

1. $\mathfrak{X} \subset \mathbb{Z}_{\geq 0}\Psi_U$.
2. *Assume that V is regular. Let $\psi \in \mathbb{Z}_{> 0}\mathfrak{X}$. Then there exists $U' \in \Gamma_{\text{reg}}(V)$, $U' \subset U$ such that $\psi \in \mathbb{R}_{> 0}\Psi_{U'}$.*

Proof. Let $\psi \in \mathbb{Z}_{\geq 0}\mathfrak{X}$ and let $f = \text{RI}_\psi$. Since $U \in \Gamma_{\text{reg}}(V)$, the restriction of f to U is not identically zero. By Lemma 5.2, $\psi \in \mathbb{Z}_{\geq 0}\Psi_U$, which implies the first part, and there exists $U' \subset U$, $U' \in \Gamma(V)$ such that $f|_{U'} \neq 0$ and $\psi \in \mathbb{R}_{> 0}\Psi_{U'}$.

If V is regular and $\psi \in \mathbb{Z}_{> 0}\mathfrak{X}$, then $X = \{v \in V \mid f(v) \neq 0\}$. Hence, $U' \cap X \neq \emptyset$. The second part follows. □

A connected algebraic F -subgroup R of G is called T_0 -saturated if its Lie algebra is of the form $\bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$ for some subset S of $\Phi_G \cup \{0\}$. In this case, we write $\Phi_R = S \setminus \{0\}$. Examples of T_0 -saturated subgroups are parabolic F -subgroups containing T_0 and their unipotent radicals, and semistandard Levi subgroups. If R is T_0 -saturated, then we write

$$\mathcal{I}_R = \{i \in \mathcal{I}_G \mid \mathfrak{g}_{\alpha_i} \subset \text{Lie } R\}. \tag{5.5}$$

For any subspace $U \subset V$ denote by $\text{Stab}(U) = \{g \in G \mid \rho(g)U = U\}$ the stabilizer of U . The Lie algebra of $\text{Stab}(U)$ is $\{x \in \mathfrak{g} \mid d\rho(x)U \subset U\}$ [2, Lemma II.7.4].

Let $U \in \Gamma(V)$ and let R be a T_0 -saturated subgroup of G . We denote by $R \star U$ the smallest R -stable space in $\Gamma(V)$ containing U .

The following is an immediate consequence of (5.3).

LEMMA 5.4. *Suppose that S is a subset of Ψ_V containing Ψ_U such that for all $\beta \in S$ and $\alpha \in \Phi_R$ with $\beta + \alpha \in \Psi_V$, we have $\beta + \alpha \in S$. Then, $\Psi_{R \star U} \subset S$. In particular, for every $\beta \in \Psi_{R \star U}$ there exist $\gamma \in \Psi_U$ and $\alpha_1, \dots, \alpha_k \in \Phi_R$ such that $\beta = \gamma + \alpha_1 + \dots + \alpha_k$.*

REMARK 5.5. It will follow from Lemma 6.9 below that $\Psi_{R \star U}$ is the smallest subset S of Ψ_V containing Ψ_U with the property that for every $\beta \in S$ and $\alpha \in \Phi_R$ such that $\beta + \alpha \in \Psi_V$, we have $\beta + \alpha \in S$.

5.4

For any $U \in \Gamma(V)$ define the gauge

$$\lambda(U) = \delta_0 + \sum_{\beta \in \Psi_V \setminus \Psi_U} n_\beta \beta = \delta_0 + \delta_V - \sum_{\beta \in \Psi_U} n_\beta \beta \in X^*(T_0). \tag{5.6}$$

It will turn out to be a key parameter governing the growth of the subsum of θ_ϕ pertaining to U .

The following combinatorial-geometric lemma is crucial.

LEMMA 5.6. *Let $U \in \Gamma_{\text{reg}}(V)$. Then, there exists a one-to-one function*

$$\iota : \mathcal{J}_V \rightarrow \mathcal{I}_G,$$

such that

$$\beta_j + \alpha_{\iota(j)} \in \Psi_U,$$

for all $j \in \mathcal{J}_V$.

Proof. Let $v_0 \in U \cap X$. Recall the surjection D_{v_0} (2.1). Since $v_0 \in U$, $D_{v_0}\mathfrak{g}_\alpha \subset V_{\alpha+\Psi_U}$ for all $\alpha \in \Phi_G \cup \{0\}$ by (5.3). Writing the matrix of D_{v_0} with respect to the bases \mathcal{B}_G and \mathcal{B}_V , a nonzero entry at a position (j, i) , $j \in \mathcal{J}_V$, $i \in \mathcal{I}_G$, can only occur if $\beta_j - \alpha_i \in \Psi_U$. A non-zero minor of order $\dim V$ therefore gives rise to a function ι as required. \square

Note that we used that G is reductive in the proof, namely that $-\Phi_G = \Phi_G$.

COROLLARY 5.7. *Let $U \in \Gamma_{\text{reg}}(V)$. Then, for any $J \subset \mathcal{J}_V$ we have*

$$\delta_0 + \sum_{j \in J} \beta_j \in \mathbb{R}_{\geq 0}(\Psi_U \cup \Delta_0).$$

In particular, we have

$$\lambda(U) \in \mathbb{R}_{\geq 0}(\Psi_U \cup \Delta_0). \tag{5.7}$$

Proof. We write

$$\delta_0 + \sum_{j \in J} \beta_j = \delta_0 - \sum_{j \in J} \alpha_{\iota(j)} + \sum_{j \in J} (\beta_j + \alpha_{\iota(j)}),$$

and observe that $\delta_0 - \sum_{j \in J} \alpha_{\iota(j)} \in \mathbb{R}_{\geq 0}\Delta_0$ since ι is injective.

Taking $J = \mathcal{J}_{V/U}$ (cf. (5.4) and (5.6)) we obtain (5.7). \square

5.5

The proof of Proposition 5.1 is based on a standard bound for lattice sums.

For any $\xi \in V$ denote by $\xi_\beta \in V_\beta$, $\beta \in \Psi_V$ the components of ξ with respect to the decomposition (5.2b).

We write $x_\pm = \max(0, \pm x)$ for the positive and negative parts of a real number x , so that $x = x_+ - x_-$.

For simplicity, we write $A(\cdot, \phi) \ll_{N, \phi}^{\text{rd}} B_N(\cdot)$ if for every $N \geq 0$ there exists a continuous seminorm ν_N on $C_{\text{rd}}(V(\mathbb{A}))$ such that $|A(\cdot, \phi)| \leq B_N(\cdot)\nu_N(\phi)$ for every $\phi \in C_{\text{rd}}(V(\mathbb{A}))$.

LEMMA 5.8. *Let Ω be a compact subset of $G(\mathbb{A})$. Let $U \in \Gamma(V)$. Then,*

$$\sum_{\xi \in U: \xi_\beta \neq 0 \forall \beta \in \Psi_U} |\phi(\rho(g)^{-1}\xi)| \ll_{N, \phi}^{\text{rd}} e^{\sum_{\beta \in \Psi_U} (n_\beta \langle \beta, H_0(g) \rangle_+ - N \langle \beta, H_0(g) \rangle_-)}, \quad g \in T_0(\mathbb{A})\Omega.$$

Equivalently,

$$e^{\langle \lambda(U), H_0(g) \rangle} |\det \rho(g)|^{-1} \sum_{\xi \in U: \xi_\beta \neq 0 \forall \beta \in \Psi_U} |\phi(\rho(g)^{-1}\xi)| \ll_{N, \phi}^{\text{rd}} \delta_{P_0}(m_0(g)) e^{-N \sum_{\beta \in \Psi_U} \langle \beta, H_0(g) \rangle_-}.$$

In particular, for $U = V$

$$|\det \rho(g)|^{-1} \sum_{\xi \in V: \xi_\beta \neq 0 \forall \beta \in \Psi_V} |\phi(\rho(g)^{-1}\xi)| \ll_{N, \phi}^{\text{rd}} \min(1, |\det \rho(g)|)^N.$$

Proof. The first two statements are clearly equivalent by the definition (5.6) of $\lambda(U)$. Since ρ acts continuously on $C_{\text{rd}}(V(\mathbb{A}))$, we may assume without loss of generality that $g \in T_0(\mathbb{A})$. (In fact, since both sides are left $T_0(F)$ -invariant, we may assume that $g \in T_0(\mathbb{R})$ where \mathbb{R} is viewed as a subring of $\mathbb{A} = \mathbb{A}_\mathbb{Q} \otimes F$ via $x \mapsto x \otimes 1$.)

Upon replacing ϕ by $\prod_{\beta \in \Psi_V} (1 + \|v_{\beta, \infty}\|)^{-n} \mathbf{1}_{K_\beta}(v_{\beta, \text{fin}})$, $n \gg 0$ where K_β is a compact open subgroup of $V_\beta(\mathbb{A}_{\text{fin}})$ for every $\beta \in \Psi_V$, we may also assume that $\phi|_{U(\mathbb{A})}$ factors as a

product $\phi(v) = \prod_{\beta \in \Psi_U} \phi_\beta(v_\beta)$, $v \in U(\mathbb{A})$, where each factor $\phi_\beta \in C_{\text{rd}}(V_\beta(\mathbb{A}))$ is nonnegative real. Since T_0 acts by scalars on each space V_β , we are reduced to the assertion (for $t \in \mathbb{A}^\times$)

$$\sum_{\xi_\beta \neq 0} \phi_\beta(t^{-1}\xi_\beta) \ll_{N, \phi_\beta}^{\text{rd}} \begin{cases} |t|^{n_\beta} & |t| \geq 1, \\ |t|^N & |t| \leq 1. \end{cases}$$

In turn, this immediately reduces to the following elementary statement for $\varphi \in C_{\text{rd}}(\mathbb{A})$:

$$\sum_{\xi \in F^\times} \varphi(t^{-1}\xi) \ll_{N, \varphi}^{\text{rd}} \begin{cases} |t| & |t| \geq 1 \\ |t|^N & |t| \leq 1. \end{cases}$$

To prove this, note that given a compact subset K of \mathbb{A} , we have

$$\#(F^\times \cap tK) \ll |t|.$$

In particular, $F^\times \cap tK = \emptyset$ if $|t|$ is sufficiently small. Thus, given a compact subset K_{fin} of \mathbb{A}_{fin} ,

$$\#\{\xi \in F^\times \mid (t^{-1}\xi)_{\text{fin}} \in K_{\text{fin}} \text{ and } \|(t^{-1}\xi)_\infty\| \leq R\} \ll R^{[F:\mathbb{Q}]} |t|, \quad R \geq 0.$$

The statement above follows easily from this estimate and the fact that $\varphi \in C_{\text{rd}}(\mathbb{A})$. □

Proof of Proposition 5.1. Define the support of an element $\xi \in V$ to be

$$\text{supp } \xi = \{\beta \in \Psi_V \mid \xi_\beta \neq 0\},$$

so that for any $U \in \Gamma(V)$, $\xi \in U \iff \text{supp } \xi \subset \Psi_U$.

We decompose θ_ϕ according to the support of the summands, i.e., we write

$$\theta_\phi(g) = \sum_{U \in \Gamma(V)} \Theta_U(g) \text{ where } \Theta_U(g) = \sum_{\xi \in X(F): \text{supp } \xi = \Psi_U} \phi(\rho(g)^{-1}\xi). \tag{5.8}$$

Of course, Θ_U is no longer $G(F)$ -invariant or even $P_0(F)$ -invariant, but it is $T_0(F)$ -invariant.

At any rate, for the proof of Proposition 5.1 we may assume by (5.1) that $g \in (T_0(\mathbb{A}) \cap \mathfrak{S})\Omega$, which we will do throughout.

Fix $U \in \Gamma(V)$. By Lemma 5.8, we have

$$\Theta_U(g) \ll_{N, \phi}^{\text{rd}} \delta_{P_0}(m_0(g)) |\det \rho(g)| e^{-\langle \lambda(U), H_0(g) \rangle - N \sum_{\beta \in \Psi_U} \langle \beta, H_0(g) \rangle_-}. \tag{5.9}$$

Moreover, we may assume that $U \in \Gamma_{\text{reg}}(V)$, for otherwise $\Theta_U(g) = 0$. By (5.7), we have

$$\lambda(U) = \sum_{\beta \in \Psi_U} c_\beta \beta + \sum_{\alpha \in \Delta_0} d_\alpha \alpha,$$

with suitable coefficients $c_\beta, d_\alpha \geq 0$. Since $g \in (T_0(\mathbb{A}) \cap \mathfrak{S})\Omega$, we deduce that

$$\Theta_U(g) \ll_{N, \phi}^{\text{rd}} \delta_{P_0}(m_0(g)) |\det \rho(g)| e^{-\sum_{\beta \in \Psi_U} (c_\beta \langle \beta, H_0(g) \rangle_+ + N \langle \beta, H_0(g) \rangle_-)}.$$

In particular,

$$\Theta_U(g) \ll_{\phi}^{\text{rd}} \delta_{P_0}(m_0(g)) |\det \rho(g)|. \tag{5.10}$$

The proposition follows. □

§6. Special subspaces

In this section, we provide some geometric prerequisites to analyze further the behavior of θ_ϕ on the Siegel domain. A crucial role will be played by certain subspaces of V that are PVSs for parabolic subgroups of G (cf. Definition 6.5 below).

6.1

We start with the following elementary fact from convex geometry.

LEMMA 6.1. *Let W be a finite-dimensional real vector space and let $v_i, i \in I$ be finitely many vectors in W . Given $\lambda \in \mathbb{R}_{\geq 0}\{v_i, i \in I\}$ we write $\lambda = \sum_{i \in I} \lambda_i v_i$ with $\lambda_i \geq 0$ for all i and with maximal number of nonzero (i.e., positive) coefficients. Then,*

1. $I' := \{i \in I \mid \lambda_i > 0\}$ is uniquely determined by λ .
2. Suppose that $\lambda = \sum_{i \in I} \lambda'_i v_i$ with $\lambda'_i \in \mathbb{R}, i \in I$, such that $\lambda'_i \geq 0$ for all $i \notin I'$. Then, $\lambda'_i = 0$ for all $i \notin I'$.
3. In particular, for any $i \in I$, if v_i is in the linear span of $\{v_j\}_{j \in I'}$ then $i \in I'$.

In the situation of Lemma 6.1, we will say that $v_i, i \in I'$ is the *positive envelope* of λ with respect to $v_i, i \in I$.

Fix $U \in \Gamma_{\text{reg}}(V)$. Recall that $\lambda(U) \in X^*(T_0)$ was defined in (5.6) and that $\lambda(U) \in \mathbb{R}_{\geq 0}(\Delta_0 \cup \Psi_U)$ by (5.7). Let $\mathcal{C}_U \subset \mathfrak{a}_0$ be the cone dual to $\mathbb{R}_{\geq 0}(\Delta_0 \cup \Psi_U)$, that is,

$$\mathcal{C}_U = \{x \in \mathfrak{a}_{0,+} \mid \langle \beta, x \rangle \geq 0 \ \forall \beta \in \Psi_U\}. \tag{6.1}$$

Let \mathcal{F}_U be the face of \mathcal{C}_U defined by $\lambda(U)$, i.e.,

$$\mathcal{F}_U = \{x \in \mathcal{C}_U \mid \langle \lambda(U), x \rangle = 0\}.$$

Roughly speaking, \mathcal{F}_U describes the directions in $T_0(F) \setminus (T_0(\mathbb{A}) \cap \mathfrak{S})$ where the bounded function $\delta_{P_0}^{-1} |\det \rho|^{-1} \Theta_U$ (see (5.8)) does not decay. To analyze \mathcal{F}_U , let $A \subset \Delta_0 \cup \Psi_U$ be the positive envelope of $\lambda(U)$ with respect to $\Delta_0 \cup \Psi_U$. Let $I = A \cap \Delta_0$ and $J = A \cap \Psi_U$.

(Note that we do not assume at the outset that Δ_0 and Ψ_U are disjoint.)

DEFINITION 6.2. Let $\mathbf{E}(U)$ (the enveloping group of U) be the standard parabolic subgroup of G such that $I = \Delta_0^{\mathbf{E}(U)}$.

Then,

$$\mathcal{F}_U = \{x \in \mathcal{C}_U \mid \langle \alpha, x \rangle = \langle \beta, x \rangle = 0 \ \forall \alpha \in \Delta_0^{\mathbf{E}(U)}, \beta \in J\}.$$

REMARK 6.3. Let $U \in \Gamma_{\text{reg}}(V)$ and let ι be as in Lemma 5.6. Then, the image under ι of $\mathcal{J}_{V/U}$ contains $\mathcal{I}_{N_{\mathbf{E}(U)}}$ and is contained in $\mathcal{I}_{\mathbf{E}(U)}$.

Indeed, as in the proof of Corollary 5.7 we write

$$\lambda(U) = \sum_{j \in \mathcal{J}_{V/U}} (\beta_j + \alpha_{\iota(j)}) + \delta_0 - \sum_{j \in \mathcal{J}_{V/U}} \alpha_{\iota(j)}.$$

It follows from the definition of $\mathbf{E}(U)$ that $\delta_0 - \sum_{j \in \mathcal{J}_{V/U}} \alpha_{\iota(j)}$ can be written as a linear combination of $\Delta_0^{\mathbf{E}(U)}$. This precisely means that $\mathcal{I}_{N_{\mathbf{E}(U)}} \subset \iota(\mathcal{J}_{V/U}) \subset \mathcal{I}_{\mathbf{E}(U)}$.

6.2

To go further, we need a follow-up of Lemma 5.6. It will play a key role.

PROPOSITION 6.4. *Let $U \in \Gamma_{\text{reg}}(V)$. Assume that P_0 stabilizes U , so that the stabilizer $S = \text{Stab}(U)$ of U in G is a standard F -parabolic subgroup of G . Then,*

1. *There exists a one-to-one function*

$$\iota : \mathcal{J}_{V/U} \rightarrow \mathcal{I}_{N_S},$$

such that $\beta_j + \alpha_{\iota(j)} \in \Psi_U$ for all $j \in \mathcal{J}_{V/U}$. In particular, $\dim V/U \leq \dim G/S$.

2. $S \subset \mathbf{E}(U)$.
3. *The following conditions are equivalent.*
 - (a) $\dim V/U = \dim G/S$.
 - (b) ι is a bijection.
 - (c) $\lambda(U) = \sum_{j \in \mathcal{J}_{V/U}} (\beta_j + \alpha_{\iota(j)}) + \delta_0^S$.
 - (d) *For some (or every) $v \in U \cap X$, $\rho(S)v$ is dense in U and $G_v^\circ \subset S$.*
 - (e) *For some (or every) $v \in U \cap X$, the variety*

$$G_{v,U} = \{g \in G \mid \rho(g)^{-1}v \in U\},$$

is the union of finitely many left cosets of S in G .

4. *If the conditions in part 4 are satisfied, then U is a PVS with respect to S with open orbit $U^{\text{reg}} = U \cap X$. Moreover, $G_{v,U} = G_v S$ for any $v \in U^{\text{reg}}$.*

On the other hand, if the conditions in part 4 are not satisfied, then $\mathbf{E}(U) \neq S$.

Proof. Write $\mathfrak{s} = \text{Lie } S$ for the Lie algebra of S . Let $N = N_S$, and let \bar{N} be the unipotent radical of the parabolic subgroup of G opposite to S . Denote the corresponding Lie algebras by $\mathfrak{n} = \text{Lie } N$ and $\bar{\mathfrak{n}} = \text{Lie } \bar{N}$.

Let $v_0 \in U \cap X$. The composition of the surjection D_{v_0} (2.1) with the projection $V \rightarrow V/U$ factors as a surjective map

$$\tilde{D}_{v_0} : \bar{\mathfrak{n}} \simeq \mathfrak{g}/\mathfrak{s} \rightarrow V/U \simeq \bigoplus_{\beta \in \Psi_V \setminus \Psi_U} V_\beta.$$

We have $\tilde{D}_{v_0}(\mathfrak{g}_\alpha) \subset V_{\alpha + \Psi_U \setminus \Psi_U}$ for all $\alpha \in \Phi_{\bar{N}} = -\Phi_N$. The existence of ι follows as in the proof of Lemma 5.6. Moreover,

$$\lambda(U) = \sum_{\beta \in \Psi_V \setminus \Psi_U} n_\beta \beta + \delta_0 = \sum_{j \in \mathcal{J}_{V/U}} (\beta_j + \alpha_{\iota(j)}) + \delta_0 - \sum_{j \in \mathcal{J}_{V/U}} \alpha_{\iota(j)}.$$

Clearly,

$$\delta_0 - \sum_{j \in \mathcal{J}_{V/U}} \alpha_{\iota(j)} = \delta_0 - \sum_{\alpha \in \Phi_N} (\dim \mathfrak{g}_\alpha) \alpha + \mu = \delta_0^S + \mu,$$

where $\mu = \sum_{i \in \mathcal{I}_N \setminus \iota(\mathcal{J}_{V/U})} \alpha_i$ is a (possibly empty) sum of roots in N . Therefore, $S \subset \mathbf{E}(U)$, and the inclusion is proper if $\mu \neq 0$.

Note that the following statements are equivalent: $\mu = 0$; ι is a bijection; $\dim V/U = \dim G/S$; \tilde{D}_{v_0} is an isomorphism of vector spaces.

Observe that the kernel of \tilde{D}_{v_0} is $D_{v_0}^{-1}(U)/\mathfrak{s}$ and we have a short exact sequence

$$0 \rightarrow \text{Ker } D_{v_0} / (\text{Ker } D_{v_0} \cap \mathfrak{s}) \rightarrow \text{Ker } \tilde{D}_{v_0} \rightarrow U/D_{v_0}(\mathfrak{s}) \rightarrow 0.$$

Thus, \tilde{D}_{v_0} is an isomorphism if and only if $\text{Ker } D_{v_0} \subset \mathfrak{s}$ and $D_{v_0}(\mathfrak{s}) = U$. Since $\text{Ker } D_{v_0} = \text{Lie } G_{v_0}$, the first condition means that $G_{v_0}^\circ \subset S$, while the second condition means that the S -orbit of v_0 is dense in U . This proves that conditions (33a)–(33d) are equivalent and that

the validity of (33d) does not depend on the choice of $v \in U \cap X$. Therefore, under these conditions, $U \cap X$ is a single S -orbit, since $U \cap X$ is irreducible. Since $U \cap X = \rho(G_{v_0,U})^{-1}v_0$, this also implies that $G_{v_0,U} = G_{v_0}S$, which is a finite union of left S -cosets since $G_{v_0}^\circ \subset S$.

Finally, if $G_{v_0,U} = \bigcup_{i=1}^n \eta_i S$ is the union of finitely many left S -cosets, then

$$U \cap X = \rho(G_{v_0,U})^{-1}v_0 = \bigcup_{i=1}^n \rho(S)\rho(\eta_i)^{-1}v_0,$$

is the union of finitely many S -orbits, at least one of which must be dense, while $G_{\rho(\eta_i)^{-1}v_0}^\circ \subset S$ for every $i = 1, \dots, n$, since $G_{\rho(\eta_i)^{-1}v_0} = \eta_i^{-1}G_{v_0}\eta_i \subset \eta_i^{-1}G_{v_0,U}$ is contained in a finite union of left S -cosets. Thus, condition (e) is also equivalent to the other ones. This concludes the proof of the proposition. \square

As we shall see, the spaces considered in the proposition above deserve special attention.

DEFINITION 6.5. We denote by $\Gamma_{\text{spcl}}(V)$ the set of spaces satisfying the conditions of Proposition 6.4 part 4. Thus, $\Gamma_{\text{spcl}}(V)$ consists of the subspaces $U \in \Gamma_{\text{reg}}(V)$ that are stabilized by P_0 and satisfy $\dim V/U = \dim G/\text{Stab}(U)$.

A subspace in $\Gamma_{\text{spcl}}(V)$ will be called *special*.

REMARK 6.6. Trivially, $V \in \Gamma_{\text{spcl}}(V)$. It follows from Proposition 6.4 that if V is basic (see Definition 2.8), then $\Gamma_{\text{spcl}}(V) = \{V\}$.

The converse is not true: $V = \text{Mat}_n$ is a PVS with respect to $G = \text{GL}_n$ acting by matrix multiplication. Here $H = 1$, but it is easy to see that $\Gamma_{\text{spcl}}(V) = \{V\}$.

In fact, the four examples in Table 2 of §3.3 provide counterexamples for irreducible PVS of DK-type. Indeed, by Proposition 6.4, if $U \in \Gamma_{\text{spcl}}(V)$, then it is stable under the maximal standard subgroup parabolic Q of G whose Levi part L satisfies $H^\circ = L^{\text{der}}$. In particular, U is stable under L . However, V decomposes into four irreducible subrepresentations of L , $V = U_1 \oplus U_2 \oplus U_3 \oplus U_4$ where U_1, U_2 are one-dimensional, Ψ_{U_1} consists of the highest weight of V , and $\dim U_3 = \dim U_4 = \dim G/Q$. Since every non-trivial P_0 -stable subspace of V contains U_1 , we cannot have $\dim V/U = \dim G/Q$. Hence $U = V$.

Consider the list of reduced, absolutely irreducible PVSs of Sato and Kimura in the appendix to [23]. Of the 29 regular cases, 23 are of DK-type.⁸ Among these, we have $\Gamma_{\text{spcl}}(V) \neq \{V\}$ only in the following eight cases: # 2 (Example 3.4 for $k = 1$ and $n_1 = 2$), # 4 (Example 3.7), # 8 (Example 3.8), # 9, # 11, # 12 (Example 3.9), # 15 (Example 3.5 for $k = 1$ and $n_1 = 2$ or $n_2 - 2$), # 28. (To verify the assertion $\Gamma_{\text{spcl}}(V) \neq \{V\}$ in the remaining cases # 9, # 11 and # 28, one may use the construction of §9.)

REMARK 6.7. In general, we may have $\mathbf{E}(U) \supsetneq \text{Stab}(U)$ even if $U \in \Gamma_{\text{spcl}}(V)$. See Examples 6.15 and 7.8 below.

In principle, using Proposition 6.4 one can enumerate $\Gamma_{\text{spcl}}(V)$, at least in small rank cases or when H is contained in only a few parabolic subgroups.

EXAMPLE 6.8. Consider Example 3.4 for $k = 1$, $n_1 = 2$ and Example 3.7. Write the simple roots of G' as $\{\alpha, \beta\}$ where α is the short root. Then $\Delta_0^G = \{\alpha\}$ and $\Psi_V = \{\beta, \beta + \alpha, \beta + 2\alpha, \beta + 3\alpha\}$ where root in square brackets occurs for $G' = G_2$. It is easy to see that $\Gamma_{\text{spcl}}(V) = \{V, U\}$ where $\Psi_U = \Psi_V \setminus \{\beta\}$. $\text{Stab}(U) = \mathbf{E}(U)$ is the Borel subgroup of G .

⁸ Case # 1 can be of DK-type or not, but it is always basic.

6.3

For $\alpha \in \Phi_G$ denote by $N_{(\alpha)}$ the unipotent subgroup of G with Lie algebra $\mathfrak{g}_\alpha + \mathfrak{g}_{2\alpha}$ if $2\alpha \in \Phi_G$ and \mathfrak{g}_α otherwise.

The following general fact about representations is probably well-known.

LEMMA 6.9. *Let $U \in \Gamma(V)$ and $\alpha \in \Phi_G$. Suppose that U is stable under $N_{(\alpha)}$. Then, for every $\beta \in \Psi_U$ such that $\alpha + \beta \in \Psi_V$ we have $\alpha + \beta \in \Psi_U$.*

This follows from the representation theory of SL_2 . To explain this, we need a standard consequence of the Jacobson–Morozov Theorem.

LEMMA 6.10. *Let $\alpha \in \Phi_G$ and let $0 \neq e \in \mathfrak{g}_\alpha$. Then, there exists a unique F -morphism $\varphi : SL_2 \rightarrow G$ such that $d\varphi\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e$, the image of the diagonal torus under φ is contained in T_0 and the co-character $t \mapsto \varphi\left(\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}\right)$ is α^\vee .*

Proof. This is well-known. For convenience we include the proof.

For the uniqueness of φ – see [8, Lemma 3.4.4].

By passing to the subgroup generated by $N_{(\alpha)}$ and $N_{(-\alpha)}$, we may assume that G is semisimple of F -rank one. We may also assume without loss of generality that $\alpha > 0$, so that $N_{(\alpha)} \subset N_0$.

By the Jacobson–Morozov Theorem, there exists an F -morphism $\varphi' : SL_2 \rightarrow G$ such that $d\varphi'\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = e$. Let a be the restriction of φ' to the diagonal torus, i.e., $a(s) = \varphi'\left(\begin{pmatrix} s & \\ & s^{-1} \end{pmatrix}\right)$, so that $\text{Ad}(a(s))e = s^2e$. Let T be the image of a , which is a maximal split torus in G . By the Bruhat decomposition, for any $g \notin P_0$ we have $gN_0g^{-1} \cap N_0 = 1$. Thus, $T \subset P_0$. It follows that $T = nT_0n^{-1}$ for some element n of $N_0(F)$.

We claim that n centralizes e . Indeed, for every $t \in T_0$, both $\text{Ad}(t)e$ and $\text{Ad}(ntn^{-1})e$ are scalar multiples of e . Fix $t \in T_0$ such that its centralizer in N_0 is trivial. Write $ntn^{-1} = n't$ where $n' = [n, t] \in N_0$. Then, $\text{Ad}(n')e$ is a scalar multiple of e , and hence $\text{Ad}(n')e = e$ since n' is unipotent. Note that the centralizer $N_{0,e}$ of e in N_0 is invariant under conjugation by T_0 (since $\text{Ad}(T_0)$ acts by scalars on e). By [2, Proposition III.9.3] the maps $N_0 \rightarrow N_0$ and $N_{0,e} \rightarrow N_{0,e}$ given by $n \mapsto n'$ are bijective. Since $n' \in N_{0,e}$, we infer that $n \in N_{0,e}$ as claimed.

We can now take $\varphi = \varphi'(n^{-1} \cdot n)$. □

Proof of Lemma 6.9. For any integer k let $V_k \in \Gamma(V)$ be such that

$$\Psi_{V_k} = \{\beta \in \Psi_V \mid \langle \beta, \alpha^\vee \rangle = k\}.$$

Fix $0 \neq e \in \mathfrak{g}_\alpha$. Using Lemma 6.10 and basic facts about representation theory of SL_2 , it follows that for any integer k , the map

$$d\rho(e) : V_{k-1} \rightarrow V_{k+1} \text{ is } \begin{cases} \text{surjective} & \text{if } k \geq 0, \\ \text{injective} & \text{if } k \leq 0, \end{cases}$$

Thus, for every $\beta \in \Psi_V$,

$$d\rho(e) : V_\beta \rightarrow V_{\beta+\alpha} \text{ is } \begin{cases} \text{surjective} & \text{if } \langle \beta, \alpha^\vee \rangle + 1 \geq 0, \\ \text{injective} & \text{if } \langle \beta, \alpha^\vee \rangle + 1 \leq 0. \end{cases}$$

In particular, $d\rho(e)(V_\beta) \neq 0$ if $V_{\beta+\alpha} \neq 0$. The lemma follows. □

LEMMA 6.11. *Suppose that $U_1, U_2 \in \Gamma_{\text{spl}}(V)$ and $U_1 \cap U_2 \in \Gamma_{\text{reg}}(V)$. Then, $U_1 \cap U_2 \in \Gamma_{\text{spl}}(V)$ and $\text{Stab}(U_1 \cap U_2) = \text{Stab}(U_1) \cap \text{Stab}(U_2)$.*

Proof. Let $U = U_1 \cap U_2$, $S = \text{Stab}(U)$, $N = N_S$, $S_k = \text{Stab}(U_k)$, $N_k = N_{S_k}$, $M_k = M_{S_k}$, $k = 1, 2$. Clearly, $\Psi_U = \Psi_{U_1} \cap \Psi_{U_2}$, P_0 stabilizes U and $S \supset S_1 \cap S_2$. In particular, $\Phi_N \subset \Phi_{N_1} \cup \Phi_{N_2}$.

By Proposition 6.4, there exists a one-to-one function

$$\iota : \mathcal{J}_{V/U} \rightarrow \mathcal{I}_N,$$

such that $\beta_j + \alpha_{\iota(j)} \in \Psi_U$ for all $j \in \mathcal{J}_{V/U}$.

We claim that if $\beta_j \in \Psi_V \setminus \Psi_{U_1}$, then $\alpha_{\iota(j)} \in \Phi_{N_1}$. Indeed, suppose otherwise. Then, $\alpha_{\iota(j)} \in \Phi_{M_1}$ (since $\alpha_{\iota(j)} > 0$ as $\alpha_{\iota(j)} \in \Phi_N$). Therefore, using Lemma 6.9, $\beta_j = (\beta_j + \alpha_{\iota(j)}) - \alpha_{\iota(j)} \in \Psi_{U_1}$ since U_1 is S_1 -stable, in contradiction to the assumption that $\beta_j \notin \Psi_{U_1}$.

Thus, the restriction of ι to \mathcal{J}_{V/U_1} is a bijection with \mathcal{I}_{N_1} (since $U_1 \in \Gamma_{\text{spcl}}(V)$). Similarly, the restriction of ι to \mathcal{J}_{V/U_2} is a bijection with \mathcal{I}_{N_2} .

It follows that ι is onto and $\Phi_N = \Phi_{N_1} \cup \Phi_{N_2}$. Thus, $U \in \Gamma_{\text{spcl}}(V)$ and $N = N_1 N_2$, which implies that $S = S_1 \cap S_2$. □

LEMMA 6.12. *Let $U \in \Gamma_{\text{reg}}(V)$ and $\tilde{U} = \mathbf{E}(U) \star U$. Then, $\tilde{U} \in \Gamma_{\text{spcl}}(V)$ and $\mathbf{E}(\tilde{U}) = \text{Stab}(\tilde{U}) = \mathbf{E}(U)$.*

Proof. Clearly $\tilde{U} \in \Gamma_{\text{reg}}(V)$, since $\tilde{U} \supset U$, and $\text{Stab}(\tilde{U}) \supset \mathbf{E} := \mathbf{E}(U)$. In view of Proposition 6.4, in order to prove the lemma we only need to show that $\tilde{\mathbf{E}} := \mathbf{E}(\tilde{U}) \subset \mathbf{E}$. We have $\Psi_{\tilde{U}} \supset \Psi_U$. Write

$$\lambda(\tilde{U}) = \sum_{\beta \in \Psi_{\tilde{U}}} c_\beta \beta + \sum_{\alpha \in \Delta_0^{\tilde{\mathbf{E}}}} d_\alpha \alpha,$$

with $c_\beta \geq 0$ for all $\beta \in \Psi_{\tilde{U}}$ and $d_\alpha > 0$ for all $\alpha \in \Delta_0^{\tilde{\mathbf{E}}}$. Then,

$$\lambda(U) = \sum_{\beta \in \Psi_U} c_\beta \beta + \sum_{\beta \in \Psi_{\tilde{U}} \setminus \Psi_U} (c_\beta + n_\beta) \beta + \sum_{\alpha \in \Delta_0^{\tilde{\mathbf{E}}}} d_\alpha \alpha.$$

Using Lemma 5.4 for each $\beta \in \Psi_{\tilde{U}} \setminus \Psi_U$, we can rewrite the above relation as

$$\lambda(U) = \sum_{\beta \in \Psi_U} c'_\beta \beta + \sum_{\alpha \in \Delta_0} d'_\alpha \alpha,$$

where $c'_\beta \geq 0$ for all $\beta \in \Psi_U$, $d'_\alpha \geq 0$ for $\alpha \in \Delta_0 \setminus \Delta_0^{\tilde{\mathbf{E}}}$ and $d'_\alpha \geq d_\alpha > 0$ for all $\alpha \in \Delta_0^{\tilde{\mathbf{E}}} \setminus \Delta_0^{\mathbf{E}}$. By Lemma 6.1 we deduce that $d'_\alpha = 0$ for all $\alpha \in \Delta_0 \setminus \Delta_0^{\tilde{\mathbf{E}}}$. This implies that $\tilde{\mathbf{E}} \subset \mathbf{E}$, as required. □

COROLLARY 6.13.

1. *If $U \in \Gamma_{\text{reg}}(V)$ is stable under $\mathbf{E}(U)$, then $U \in \Gamma_{\text{spcl}}(V)$.*
2. *If $U \in \Gamma_{\text{reg}}(V)$ and $\mathbf{E}(U) \subsetneq G$, then there exists $\tilde{U} \in \Gamma_{\text{spcl}}(V)$, $\tilde{U} \neq V$, containing U .*
3. *In particular, if $\Gamma_{\text{spcl}}(V) = \{V\}$, then $\mathbf{E}(U) = G$ for every $U \in \Gamma_{\text{reg}}(V)$.*

EXAMPLE 6.14. Consider Example 3.2 with $k = 3$ and $n_1 < n_2$. Then, it is easy to see that $\Gamma_{\text{spcl}}(V) = \{V, V_1, V_2\}$ where

$$\begin{aligned} V_1 &= \{(x_1, x_2) \mid \text{the first } n_2 - n_1 \text{ columns of } x_1 \text{ vanish}\}, \\ V_2 &= \{(x_1, x_2) \mid \text{the last } n_2 - n_3 \text{ rows of } x_2 \text{ vanish}\}. \end{aligned}$$

The general case of Example 3.2 is far more complicated and will not be considered here.

EXAMPLE 6.15. Consider Example 3.8 pertaining to F_4 . Write for simplicity $(i_1i_2i_3i_4)$ instead of $i_1\alpha_1 + i_2\alpha_2 + i_3\alpha_3 + i_4\alpha_4 \in X^*(T_0)$.

We explicate Ψ_V in a way that demonstrates its realization as pairs of 3×3 -symmetric matrices as follows:

$$\Psi_V = \left(\begin{matrix} (0100) & (0110) & (0111) \\ (0110) & (0120) & (0121) \\ (0111) & (0121) & (0122) \end{matrix} \right), \left(\begin{matrix} (1100) & (1110) & (1111) \\ (1110) & (1120) & (1121) \\ (1111) & (1121) & (1122) \end{matrix} \right).$$

Consider the subspaces U_1, U_2, U_3 in $\Gamma(V)$ given by

$$\begin{aligned} \Psi_{U_1} &= \Psi_V \setminus \{(0100), (1100)\}, \\ \Psi_{U_2} &= \Psi_V \setminus \{(0100), (0110), (0120)\}, \\ \Psi_{U_3} &= \Psi_V \setminus \{(0100), (0110), (0111)\}. \end{aligned}$$

In matrices,

$$U_1 = \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad U_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

Let $U_4 = U_1 \cap U_2, U_5 = U_2 \cap U_3$ so that

$$U_4 = \begin{pmatrix} 0 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}, \quad U_5 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & * \\ 0 & * & * \end{pmatrix}, \begin{pmatrix} * & * & * \\ * & * & * \\ * & * & * \end{pmatrix}.$$

Then, it is not difficult to check that

$$\Gamma_{\text{spl}}(V) = \{V, U_1, U_2, U_3, U_4, U_5\}.$$

Moreover,

$$\begin{aligned} \text{Stab}(U_1) = \mathbf{E}(U_1) &= (*00*), \quad \text{Stab}(U_2) = \mathbf{E}(U_2) = (00*0), \quad \text{Stab}(U_3) = \mathbf{E}(U_3) = (000*), \\ \text{Stab}(U_4) = \text{Stab}(U_5) = \mathbf{E}(U_5) &= (0000), \quad \mathbf{E}(U_4) = (00*0). \end{aligned}$$

§7. Proof of Theorem 4.3

In this section, we prove Theorem 4.3 by providing an estimate on Θ_U which is finer than (5.10) for spaces $U \in \Gamma_{\text{reg}}(V)$ containing non-isotropic elements (see Definition 2.7).

7.1

Let \mathcal{C} be a closed cone in a finite-dimensional real vector space W .

Let f be a continuous real-valued function on W and let $d > 0$. We say that f is positively- d -homogeneous if $f(rv) = r^d f(v)$ for all $r > 0, v \in W$. (In the cases at hand, $d = 1$.) In this case, f is positive on $\mathcal{C} \setminus \{0\}$ if (and of course, only if) it is positive on the intersection of \mathcal{C} with the unit sphere of W with respect to a prescribed norm.

The space of continuous positively- d -homogeneous functions on W is a Banach space with respect to the supremum norm on the unit ball and the positivity on $\mathcal{C} \setminus \{0\}$ is an open condition.

By integration using spherical coordinates, it is clear that if f is positively- d -homogeneous and positive on $\mathcal{C} \setminus \{0\}$, then the integral of e^{-f} over \mathcal{C} converges.

Suppose that f is positively-1-homogeneous and let $\lambda_j, j \in J$ be a finite set of vectors in W^* . Then, the positively-1-homogeneous function $f + N \sum_{j \in J} \langle \lambda_j, v \rangle_-$ is positive on $\mathcal{C} \setminus \{0\}$ for $N \gg 0$ if and only if f is positive on the subcone $\{v \in \mathcal{C} : \langle \lambda_j, v \rangle \geq 0 \forall j \in J\} \setminus \{0\}$.

These facts yield the following elementary convergence result.

LEMMA 7.1. *Let $\chi_i, i \in I$ and $\lambda_j, j \in J$ be finite sets of vectors in \mathfrak{a}_0^* . Let $W \subset \mathfrak{a}_0$ be a subspace and f a continuous positively-1-homogeneous function on W . Assume that f is positive on the cone*

$$\{x \in \mathfrak{a}_{0,+} \cap W \mid \langle \lambda_j, x \rangle \geq 0 \ \forall j \in J\} \setminus \{0\}.$$

Then, there exists $c > 0$ such that for all $N \gg 0$ the integral

$$\int_{\mathfrak{a}_{0,+} \cap W} e^{-f(H) - N \sum_{j \in J} \langle \lambda_j, H \rangle - \sum_{i \in I} \sigma_i \langle \chi_i, H \rangle} dH,$$

converges for $|\sigma_i| < c, i \in I$. Moreover, if $\chi_i \in \mathbb{R}_{\geq 0} \{\lambda_j\}_{j \in J}$ for all $i \in I$, then for all $\sigma_i > -c, i \in I$ the integral converges for sufficiently large N (depending on the maximum value of the σ_i 's).

7.2

We now turn to the support sets of non-isotropic elements.

DEFINITION 7.2. We denote by $\Gamma_{\text{niso}}(V) \subset \Gamma_{\text{reg}}(V)$ the set of subspaces in $\Gamma(V)$ that contain a non-isotropic element.

DEFINITION 7.3. Let $U \in \Gamma_{\text{spcl}}(V)$. We say that U is *exceptional* if there exists $U' \in \Gamma_{\text{reg}}(V), U' \subset U$ such that the linear span of $X^*(G) \cup \Delta_0^{\text{Stab}(U)} \cup \Psi_{U'}$ is a proper subspace of \mathfrak{a}_0^* . In this case, we will also say that (U, U') is an exceptional pair.

We denote by $\Gamma_{\text{excp}}(V)$ the set of exceptional subspaces of V .

PROPOSITION 7.4. *Let $U \in \Gamma_{\text{spcl}}(V) \cap \Gamma_{\text{niso}}(V)$. Then U is not exceptional.*

Conversely, suppose that V is regular. Then, for any isotropic $v \in X(F)$ there exists an exceptional pair (U, U') such that $v \in \rho(G(F))U'$.

Thus, if V is regular, then the set of isotropic elements in $X(F)$ is

$$\cup_{U \in \Gamma_{\text{excp}}(V)} \rho(G(F))U \cap X(F) = \cup_{(U, U')} \rho(G(F))U' \cap X(F),$$

where (U, U') range over the exceptional pairs.

REMARK 7.5. The first implication cannot be reversed. There is an example of a regular PVS V and a non-exceptional $U \in \Gamma_{\text{spcl}}(V)$ such that all elements of $U \cap X(F)$ are isotropic (see Example 7.8 below).

Proof. For the first part, let $U \in \Gamma_{\text{spcl}}(V)$ and $S = \text{Stab}(U)$. Assume that there exists $U' \in \Gamma_{\text{reg}}(V), U' \subset U$ such that the linear span of $X^*(G) \cup \Delta_0^S \cup \Psi_{U'}$ is a proper subspace of \mathfrak{a}_0^* . Equivalently, there exists a (split) torus $1 \neq T \subset T_0 \cap Z(M_S) \cap G^{\text{der}}$ fixing U' pointwise (namely, the identity component of $\cap_{\chi \in X^*(G) \cup \Delta_0^S \cup \Psi_{U'}} \text{Ker } \chi$). We will show that all elements of $U \cap X(F)$ are isotropic.

Fix a regular element u in U' . By Proposition 6.4, we have $G_u^\circ \subset S$. Since T lies in the center of $M_S, [T, G_u \cap S] \subset [Z(M_S), S] \subset N_S$. On the other hand, $T \subset G_u^\circ$ (since T stabilizes all elements of U'). Therefore, T centralizes $(G_u \cap S)/(G_u^\circ \cap N_S)$. In particular, the image of T in $G_u^\circ/(G_u^\circ \cap N_S)$ is central. Note that $G_u^\circ \cap N_S$ is contained in the unipotent radical of G_u° .

Now, let $v \in U \cap X(F)$. By Proposition 6.4, v is in the S -orbit (but not necessarily the $S(F)$ -orbit) of u . Let $x \in S(\bar{F})$ be such that $u = \rho(x)v$. Then, $x^{-1}Tx$ is a torus in $G^{\text{der}} \cap G_u^\circ$. Moreover, for any $\sigma \in \text{Gal}(\bar{F}/F)$ we have $\sigma(x)x^{-1} \in G_u \cap S$. Let T' be the image of $x^{-1}Tx$ in

$G_v^\circ/(G_v^\circ \cap N_S)$. Then, T' is central. Moreover, T' is a split torus. In fact, since T centralizes $\sigma(x)x^{-1}(G_v^\circ \cap N_S)$ for all $\sigma \in \text{Gal}(\bar{F}/F)$, the map $t \mapsto x^{-1}tx(G_v^\circ \cap N_S)$ is an F -isomorphism of T with T' . In particular, the image of T' in G_v^{red} is a non-trivial F -split central torus. Thus, v is isotropic by Lemma 2.6. This proves the first part.

For the second part, suppose that V is regular and $v \in X(F)$ is isotropic. Let $T \neq 1$ be a split torus contained in $Z(G_v^\circ) \cap G^{\text{der}}$. The centralizer $L = C_G(T)$ contains G_v° . Upon replacing v by an element in its $G(F)$ -orbit, we can assume that T is contained in $T_0 \cap G^{\text{der}}$ and that L is a standard Levi subgroup of G . Thus, $X_*(T) \subset \mathfrak{a}_L$ and in the notation of [26, I.1.10] we have $X_*(T) \cap \mathfrak{a}_L^{\text{reg}} \neq \emptyset$. Fix an element $h \in X_*(T) \cap \mathfrak{a}_L^{\text{reg}} \subset \mathfrak{a}_0^G$. Upon conjugating by an element of the Weyl group of T_0 , we may assume that $\langle \alpha, h \rangle$ is non-negative precisely for the roots α of the (proper) standard parabolic subgroup Q of G with Levi subgroup L [26, Lemma I.1.10].

Consider the subspaces $U, U' \in \Gamma(V)$ given by

$$\Psi_U = \{\beta \in \Psi_V \mid \langle \beta, h \rangle \geq 0\}, \quad \Psi_{U'} = \{\beta \in \Psi_V \mid \langle \beta, h \rangle = 0\} \subset \Psi_U.$$

Clearly U is stable under Q and $U' \subset U$. Since T stabilizes v , $\text{supp } v \subset \Psi_{U'}$ so that $v \in U'$. In particular, $U' \in \Gamma_{\text{reg}}(V)$, and a fortiori $U \in \Gamma_{\text{reg}}(V)$. Recall the surjection (2.1). It satisfies

$$D_v(\mathfrak{g}_\alpha) \subset V_{\alpha + \Psi_{U'}}, \text{ for all } \alpha \in \Phi_G \cup \{0\},$$

by (5.3). In particular, $D_v(\mathfrak{q}) \subset U$ and $D_v(\bar{\mathfrak{n}}) \cap U = 0$ where $\mathfrak{q} = \text{Lie } Q$ and $\bar{\mathfrak{n}}$ is the Lie algebra of the unipotent radical of the parabolic subgroup of G opposite to Q . We claim that the preimage of U under D_v is \mathfrak{q} . It suffices to show that $D_v^{-1}(U) \cap \bar{\mathfrak{n}} = 0$. Suppose that $x \in \bar{\mathfrak{n}}$ and $D_v(x) \in U$. Then, by the above, $D_v(x) = 0$. Hence, x is in the Lie algebra of G_v , which is contained in \mathfrak{l} , and thus $x = 0$.

Hence, the induced map $\tilde{D}_v : \mathfrak{g}/\mathfrak{q} \rightarrow V/U$ is an isomorphism. It follows from Proposition 6.4 that $U \in \Gamma_{\text{spcl}}(V)$ and $\text{Stab}(U) = Q$.

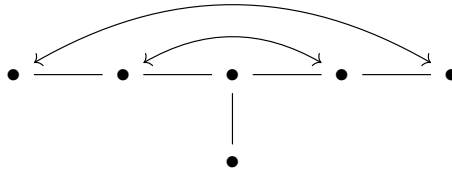
Since the nonzero vector $h \in \mathfrak{a}_0^G$ annihilates $\Psi_{U'} \cup \Delta_0^Q$, we obtain that (U, U') is exceptional.

Finally, the last part follows from the first two parts. □

REMARK 7.6. We do not know whether the second (and hence also the last) part of Proposition 7.4 hold without the regularity assumption on V .

EXAMPLE 7.7. Proposition 7.4 is best illustrated by the case where V is the PVS of binary quadratic forms with respect to the action of $G = \text{GL}_2$, i.e., the symmetric square representation. (Example 3.4 with $k = 1, n_1 = 2$.) Thus, X is the set of binary quadratic forms with nonzero discriminant. In this case, $\Psi_V = \{\beta, \beta + \alpha, \beta + 2\alpha\}$, where α is the simple root of G , and β the lowest weight of V . Note that $\beta + \alpha$ is the determinant character of G . Apart from V itself, there is a single space U in $\Gamma_{\text{spcl}}(V)$, namely $U = V_{\beta + \alpha} + V_{\beta + 2\alpha}$, and $\mathbf{E}(U) = \text{Stab}(U)$ is the standard Borel subgroup of G . The space U is exceptional. More precisely, (U, U') is an exceptional pair where $U' = V_{\beta + \alpha}$. Let $v \in X$. Then, G_v is an orthogonal group so that G_v° is a one-dimensional torus. Thus, Proposition 7.4 amounts to the fact that G_v° is split if and only if v can be represented as xy in suitable coordinates.

EXAMPLE 7.8. Let us revisit Example 3.9 pertaining to the simply connected group $G' = E_6$.



For brevity we will write $\binom{abcde}{f}$ for the linear combinations of simple roots with coefficients a, b, c, d, e, f .

We will also write standard parabolic subgroups as $\binom{*0**0}{*}$ where $*$ indicates that the simple root is in the Levi subgroup and 0 otherwise.

Let θ be the outer involution of G' acting on the simple roots as indicated.

Let P be the maximal standard parabolic subgroup $\binom{**0**}{*}$. The Levi subgroup G has derived group $SL_2 \times SL_3 \times SL_3$.

The unipotent radical N_P , which has dimension 29, has a three-step filtration, the first filtration step V (the abelianization) has dimension 18. We may write Ψ_V explicitly to reflect the realization of V as the space of pairs of 3×3 -matrices as follows.

$$\Psi_V = \left(\begin{pmatrix} 11100 \\ 0 \end{pmatrix}, \begin{pmatrix} 11110 \\ 0 \end{pmatrix}, \begin{pmatrix} 11111 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 11100 \\ 1 \end{pmatrix}, \begin{pmatrix} 11110 \\ 1 \end{pmatrix}, \begin{pmatrix} 11111 \\ 1 \end{pmatrix} \right), \left(\begin{pmatrix} 01100 \\ 0 \end{pmatrix}, \begin{pmatrix} 01110 \\ 0 \end{pmatrix}, \begin{pmatrix} 01111 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 01100 \\ 1 \end{pmatrix}, \begin{pmatrix} 01110 \\ 1 \end{pmatrix}, \begin{pmatrix} 01111 \\ 1 \end{pmatrix} \right), \left(\begin{pmatrix} 00100 \\ 0 \end{pmatrix}, \begin{pmatrix} 00110 \\ 0 \end{pmatrix}, \begin{pmatrix} 00111 \\ 0 \end{pmatrix} \right), \left(\begin{pmatrix} 00100 \\ 1 \end{pmatrix}, \begin{pmatrix} 00110 \\ 1 \end{pmatrix}, \begin{pmatrix} 00111 \\ 1 \end{pmatrix} \right).$$

We observe that for all proper subspaces $U \in \Gamma_{\text{spcl}}(V)$, $U \cap X(F)$ consists entirely of isotropic elements. Namely, if $U \in \Gamma_{\text{spcl}}(V)$ contains a non-isotropic element x , then G_x° has to be contained in $Q = \text{Stab}(U)$. But the projection of G_x° to any of the two SL_3 factors of G^{der} is an anisotropic maximal torus of this group, which implies that Q contains both SL_3 factors. Therefore, either $Q = G$ and $U = V$ or the intersection of Q with the SL_2 factor is the standard Borel subgroup. However, for this maximal parabolic subgroup Q of G the only non-trivial Q -invariant subspace of V does not intersect X .

Define $U_1, U_2, U_3 \in \Gamma(V)$ by

$$\begin{aligned} \Psi_{U_1} &= \Psi_V \setminus \left\{ \begin{pmatrix} 00100 \\ 0 \end{pmatrix}, \begin{pmatrix} 01100 \\ 0 \end{pmatrix}, \begin{pmatrix} 00100 \\ 1 \end{pmatrix}, \begin{pmatrix} 01100 \\ 1 \end{pmatrix} \right\}, \\ \Psi_{U_2} &= \Psi_V \setminus \left\{ \begin{pmatrix} 00100 \\ 0 \end{pmatrix}, \begin{pmatrix} 01100 \\ 0 \end{pmatrix}, \begin{pmatrix} 11100 \\ 0 \end{pmatrix} \right\}, \\ \Psi_{U_3} &= \Psi_V \setminus \left\{ \begin{pmatrix} 00100 \\ 0 \end{pmatrix}, \begin{pmatrix} 01100 \\ 0 \end{pmatrix}, \begin{pmatrix} 00110 \\ 0 \end{pmatrix}, \begin{pmatrix} 01110 \\ 0 \end{pmatrix}, \begin{pmatrix} 00100 \\ 1 \end{pmatrix}, \begin{pmatrix} 01100 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

In terms of matrices, these spaces are described as follows.

$$U_1 = \begin{pmatrix} *** \\ 0** \\ 0** \end{pmatrix}, \begin{pmatrix} *** \\ 0** \\ 0** \end{pmatrix}, \quad U_2 = \begin{pmatrix} 0** \\ 0** \\ 0** \end{pmatrix}, \begin{pmatrix} *** \\ *** \\ *** \end{pmatrix}, \quad U_3 = \begin{pmatrix} *** \\ 00* \\ 00* \end{pmatrix}, \begin{pmatrix} *** \\ 0** \\ 0** \end{pmatrix}.$$

Let $U'_i = \theta(U_i)$, $i = 1, 2, 3$ and $U''_i = U_i \cap U'_i$ so that

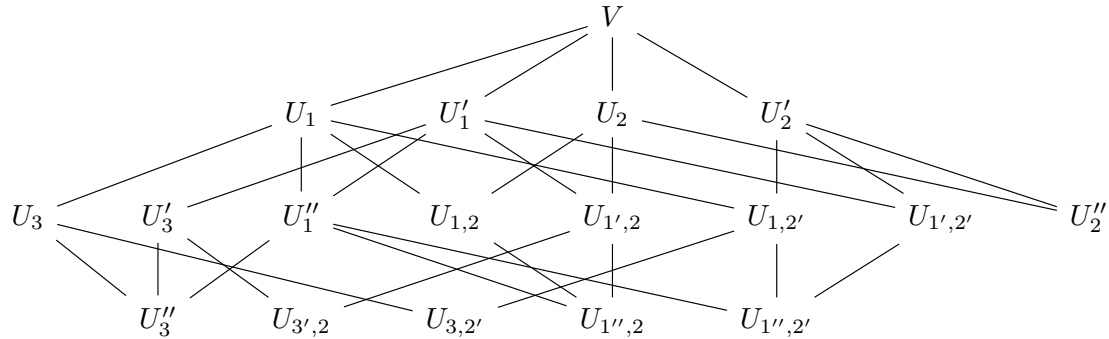
$$U'_1 = \begin{pmatrix} *** \\ *** \\ 00* \end{pmatrix}, \begin{pmatrix} *** \\ *** \\ 00* \end{pmatrix}, \quad U'_2 = \begin{pmatrix} *** \\ *** \\ 000 \end{pmatrix}, \begin{pmatrix} *** \\ *** \\ *** \end{pmatrix}, \quad U'_3 = \begin{pmatrix} *** \\ 00* \\ 00* \end{pmatrix}, \begin{pmatrix} *** \\ *** \\ 00* \end{pmatrix},$$

and

$$U''_1 = \begin{pmatrix} *** \\ 0** \\ 00* \end{pmatrix}, \begin{pmatrix} *** \\ 0** \\ 00* \end{pmatrix}, \quad U''_2 = \begin{pmatrix} 0** \\ 0** \\ 000 \end{pmatrix}, \begin{pmatrix} *** \\ *** \\ *** \end{pmatrix}, \quad U''_3 = \begin{pmatrix} *** \\ 00* \\ 00* \end{pmatrix}, \begin{pmatrix} *** \\ 0** \\ 00* \end{pmatrix}.$$

We write $U_{i,j} = U_i \cap U_j$, $U_{i',j} = U'_i \cap U_j$, $U_{i',j'} = U'_i \cap U'_{j'}$, $U_{i'',j} = U''_i \cap U_j$ and $U_{i'',j'} = U''_i \cap U'_{j'}$. Then, a straightforward but tedious computation gives the following Hasse diagram for

$\Gamma_{\text{spcl}}(V)$:



Moreover,

$$\begin{aligned} \text{Stab}(U_1) &= \begin{pmatrix} 0^* & 0^* \\ & * \end{pmatrix}, \text{Stab}(U_2) = \begin{pmatrix} ** & 0^* \\ & 0 \end{pmatrix}, \text{Stab}(U_3) = \begin{pmatrix} 0^* & 0^0 \\ & 0 \end{pmatrix} \\ \text{Stab}(U'_1) &= \begin{pmatrix} *0 & *0 \\ & ** \end{pmatrix}, \text{Stab}(U'_2) = \begin{pmatrix} *0 & ** \\ & 0^* \end{pmatrix}, \text{Stab}(U'_3) = \begin{pmatrix} 0^0 & *0 \\ & 0 \end{pmatrix} \\ \text{Stab}(U''_1) &= \begin{pmatrix} 0^0 & 0^0 \\ & * \end{pmatrix}, \text{Stab}(U''_2) = \begin{pmatrix} *0 & 0^* \\ & 0^* \end{pmatrix}, \text{Stab}(U''_3) = \begin{pmatrix} 0^0 & 0^0 \\ & 0 \end{pmatrix}. \end{aligned}$$

The stabilizers of the rest of the elements of $\Gamma_{\text{spcl}}(V)$ are given by Lemma 6.11. All spaces except U_2, U'_2 and V are exceptional.

The spaces $U \in \Gamma_{\text{spcl}}(V)$ with $\mathbf{E}(U) = \text{Stab}(U)$ are $V, U_1, U'_1, U_2, U'_2, U_{1,2}, U_{1',2}, U_{1,2'}, U_{1',2'}$. For $U = U_3, U_{3,2'}, U_{1,2'}$ we have $\mathbf{E}(U) \star U = U_1$. For $U = U'_3, U_{3',2}, U_{1',2}$ we have $\mathbf{E}(U) \star U = U'_1$. For $U = U''_3, U_{1'',2}, U_{1'',2'}$ we have $\mathbf{E}(U) \star U = U''_1$.

7.3

We can now formulate and prove the main geometric statement about $\lambda(U)$ where $U \in \Gamma_{\text{niso}}(V)$.

Consider the natural surjection

$$q : \mathfrak{a}_G = X_*(G^{\text{ab}}) \otimes \mathbb{R} \longrightarrow \bar{\mathfrak{a}}_G = X_*(G/G^*) \otimes \mathbb{R}.$$

The kernel of q is the image of $X_*(G^*/G^{\text{der}}) \otimes \mathbb{R}$ in \mathfrak{a}_G . The dual space $\bar{\mathfrak{a}}_G^* = X^*(G/G^*) \otimes \mathbb{R} \hookrightarrow \mathfrak{a}_G^*$ of $\bar{\mathfrak{a}}_G$ has basis \mathfrak{X} . Fix a section for q , i.e., a subspace \mathfrak{b} of \mathfrak{a}_G such that the restriction of q to \mathfrak{b} is an isomorphism. In other words, \mathfrak{b} is a complement of $\text{Ker } q$ in \mathfrak{a}_G . Let

$$W = (\mathfrak{a}_0^G) \oplus \mathfrak{b} \hookrightarrow (\mathfrak{a}_0^G) \oplus \mathfrak{a}_G = \mathfrak{a}_0.$$

Recall the cone \mathcal{C}_U defined in (6.1).

PROPOSITION 7.9. *Suppose that V is regular. Let $U \in \Gamma_{\text{niso}}(V)$. Then, for every $\mu \in \mathbb{R}_{>0}\mathfrak{X}$, the restriction of the linear form $\lambda(U) + \mu$ to W is positive on the cone $\mathcal{C}_U \cap W \setminus \{0\}$.*

Proof. Suppose that $x \in \mathcal{C}_U$ is such that $\langle \lambda(U) + \mu, x \rangle \leq 0$. By Corollary 5.3 part 1, we have $\langle \chi, x \rangle \geq 0$ for all $\chi \in \mathfrak{X}$. By (5.7), we obtain $\langle \lambda(U), x \rangle = 0$ and $\langle \chi, x \rangle = 0$ for all $\chi \in \mathfrak{X}$. Also, by the definition of $\mathbf{E} = \mathbf{E}(U)$, we have $\langle \alpha, x \rangle = 0$ for all $\alpha \in \Delta_0^{\mathbf{E}}$.

Let $U' \in \Gamma_{\text{reg}}(V)$ be as in Corollary 5.3 part 2. Then, $\langle \beta, x \rangle = 0$ for all $\beta \in \Psi_{U'}$.

Let $\tilde{U} = \mathbf{E}(U) \star U$. By Lemma 6.12, $\tilde{U} \in \Gamma_{\text{spcl}}(V)$ and $\mathbf{E}(\tilde{U}) = \text{Stab}(\tilde{U}) = \mathbf{E}$.

Since $\tilde{U} \supset U$, and $U \in \Gamma_{\text{nico}}(V)$ we may apply Proposition 7.4 to conclude that \tilde{U} is not exceptional. Thus, the set $\Psi_{U'} \cup \Delta_0^{\mathbb{E}} \cup \mathfrak{a}_G^*$ spans \mathfrak{a}_0^* . Let

$$p : \mathfrak{a}_0^* \rightarrow W^*,$$

be the restriction map, which is surjective. We have $\mathfrak{a}_G^* = \text{Ker } p \oplus \bar{\mathfrak{a}}_G^*$. Since \mathfrak{X} spans $\bar{\mathfrak{a}}_G^*$, we infer that $p(\Psi_{U'} \cup \Delta_0^{\mathbb{E}} \cup \mathfrak{X})$ spans W^* . Restricting to $x \in W$, we deduce $x = 0$. \square

7.4

We can now finish the proof of Theorem 4.3. As we have already noted in the discussion following (4.1), assuming convergence, the integral

$$a \mapsto \int_{G(F) \backslash G(\mathbb{A})^1} \theta^{\text{nico}}(ha) \, dh,$$

is $G^*(\mathbb{A})$ -invariant. It therefore remains to prove the convergence of

$$\int_{\mathfrak{b}} \int_{G(F) \backslash G(\mathbb{A})^1} \theta^{\text{nico}}(ha) \, dh \, e^{-\langle \delta_V + \mu, a \rangle} \, da,$$

where \mathfrak{b} is as in Proposition 7.9 and $\mu \in \mathbb{R}_{>0}\mathfrak{X}$.

Choose a Siegel set as in §5.1 such that $G(F)\mathfrak{S} = G(\mathbb{A})$. For any non-negative function f on $G(F) \backslash G(\mathbb{A})^1$ we have

$$\begin{aligned} \int_{G(F) \backslash G(\mathbb{A})^1} f(g) \, dg &\leq \int_K \int_{\mathfrak{a}_0^G} \int_{P_0(F) \backslash P_0(\mathbb{A})^1} f(pe^x k) \delta_{P_0}(e^x)^{-1} \tau_{c_0}(x) \, dp \, dx \, dk \\ &\leq \int_{\mathfrak{a}_0^G} \sup_{y \in \Omega} f(e^x y) \delta_{P_0}(e^x)^{-1} \tau_{c_0}(x) \, dx, \end{aligned} \tag{7.1}$$

where τ_{c_0} is the characteristic function of $\{x \in \mathfrak{a}_0^G \mid \langle \alpha, x \rangle > c_0 \, \forall \alpha \in \Delta_0\}$.

It is therefore enough to show the convergence of

$$\int_W \delta_{P_0}(e^x)^{-1} \sup_{y \in \Omega} |\theta_{\phi}^{\text{nico}}(e^x y)| \, e^{-\langle \delta_V + \mu, x \rangle} \tau_{c_0}(x) \, dx, \tag{7.2}$$

where $W = \mathfrak{a}_0^G \oplus \mathfrak{b} \hookrightarrow \mathfrak{a}_0$ and we extend τ_{c_0} trivially on \mathfrak{b} .

As in the proof of Proposition 5.1 we bound $\theta^{\text{nico}}(g)$ by splitting the sum according to support sets, i.e., we write

$$\theta_{\phi}^{\text{nico}}(g) = \sum_{U \in \Gamma(V)} \Theta_U^{\text{nico}}(g),$$

with

$$\Theta_U^{\text{nico}}(g) = \sum_{\xi \in X(F)^{\text{nico}} : \text{supp } \xi = \Psi_U} \phi(\rho(g)^{-1} \xi).$$

The integral (7.2) is then majorized by the sum over $U \in \Gamma(V)$ of

$$\int_W \delta_{P_0}(e^x)^{-1} \sup_{y \in \Omega} |\Theta_U^{\text{nico}}(e^x y)| \, e^{-\langle \delta_V + \mu, x \rangle} \tau_{c_0}(x) \, dx. \tag{7.3}$$

Moreover, by Lemma 5.8 for any $N \geq 0$ we have

$$\Theta_U^{\text{nico}}(g) \ll_{N, \phi}^{\text{rd}} \delta_{P_0}(m_0(g)) |\det \rho(g)| e^{-\langle \lambda(U), H_0(g) \rangle - N \sum_{\beta \in \Psi_U} \langle \beta, H_0(g) \rangle}, \quad g \in T_0(\mathbb{A})\Omega.$$

Therefore, (7.3) is majorized by

$$\nu_N(\phi) \int_W e^{-\langle \lambda(U) + \mu, x \rangle} e^{-N \sum_{\beta \in \Psi_U} \langle \beta, x \rangle} \tau_{c_0}(x) dx, \tag{7.4}$$

for suitable seminorms ν_N on $C_{\text{rd}}(V(\mathbb{A}))$. We may assume of course that $U \in \Gamma_{\text{nisso}}(V)$, for otherwise $\Theta_U^{\text{nisso}} \equiv 0$. By Proposition 7.9, the restriction of the linear form $\lambda(U) + \mu$ to W is then positive on the cone $\mathcal{C}_U \cap W \setminus \{0\}$. Therefore, the integral (7.4) converges by Lemma 7.1.

This finishes the proof of Theorem 4.3.

§8. PVSs without non-trivial special subspaces

In the section, we analyze the case where a regular PVS V has regular F -irreducible components and no non-trivial special subspaces. Under these conditions, we obtain a strengthening of our main result, namely meromorphic continuation of the zeta function to the left of the point $\lambda = 0$ with at most simple poles along the hyperplanes defined by the fundamental characters.

We continue to assume that G is reductive. In general, a subrepresentation of a regular PVS is not necessarily regular. We will say that a PVS V is *completely factorizable* (CF) if all its F -irreducible subrepresentations are regular. It is clear that every subrepresentation of a CF PVS is again CF.

For example, every basic PVS (see Definition 2.8) is CF, since any subrepresentation is basic, hence regular. Examples 3.2–3.6 are CF precisely when $n_1 = \dots = n_k$. (However, n_{k+1} may be bigger than n_k in cases 3.3 and 3.5.)

Note that in general, an F -irreducible representation π of G does not admit a central character. However, the restriction of π to the torus T_G acts by a character, which we denote by ω_π .

LEMMA 8.1 (cf. [35], Proposition 2.13). *Let (G, V, ρ) be a CF PVS and let $V = \bigoplus_{i \in I} (V_i, \rho_i)$ be a decomposition into irreducible subrepresentations over F . Let $\chi_i = \omega_{\rho_i}$, $i \in I$. Then,*

1. V is regular.
2. $X = \prod_{i \in I} X_i$, where X_i is the regular part of V_i .
3. The fundamental invariant polynomials of V are those of the V_i 's.
4. The restrictions of the fundamental characters of V to T_G are $n_i \chi_i$ for some positive integers n_i , $i \in I$.
5. The characters χ_i are linearly independent over \mathbb{Z} .
6. The union $\Psi_V = \cup \Psi_{V_i}$ is disjoint.
7. The irreducible components V_i are the isotypic components of V .
8. The set $\Gamma_{\text{reg}}(V)$ consists of the direct sums $U = \bigoplus_{i \in I} U_i$ with $U_i \in \Gamma_{\text{reg}}(V_i)$, $i \in I$.
9. The group $G^*/G^{\text{der}} \text{Ker } \rho$ is F -anisotropic.

Proof. First note that every F -irreducible regular PVS admits a unique fundamental invariant polynomial (over F).

Parts 1, 2, and 3 are contained in [35, Proposition 2.13]. (Note that part 3 over \bar{F} easily implies it over F .) Part 4 immediately follows. Part 5 follows since the fundamental characters of a PVS are always linearly independent. Parts 6, 7 and 8 are easy consequences of part 5.

Finally, part 9 is equivalent to the statement that $T_G \cap \text{Ker } \rho$ is of finite index in $T_G \cap G^*$. Note that $T_G \cap \text{Ker } \rho$ is the intersection of $\text{Ker } \chi_i, i \in I$ and by part 4, this is of finite index in the common kernel of the restrictions of the characters in $X^*(G/G^*)$ to T_G . The lemma follows. \square

The following easy consequence about the special subspaces of V and its subrepresentations will be used repeatedly.

LEMMA 8.2. *Let V be a CF PVS and V' a subrepresentation of V (defined over F).*

1. *Let V'' be the (unique) complement to V' in V . Then the map $U \mapsto U \oplus V''$ defines an injection of $\Gamma_{\text{spcl}}(V')$ into $\Gamma_{\text{spcl}}(V)$.*
2. *If $\Gamma_{\text{spcl}}(V) = \{V\}$, then $\Gamma_{\text{spcl}}(V') = \{V'\}$.*

Proof. By Lemma 8.1 part 7, the map $U \mapsto U \oplus V''$ defines an injection of $\Gamma_{\text{reg}}(V')$ into $\Gamma_{\text{reg}}(V)$. The first part follows now from the definition of a special subspace. The second part is clear. \square

Let $\mathcal{S}(V(\mathbb{A}))$ be the space of Schwartz–Bruhat functions on $V(\mathbb{A})$. As a locally convex topological vector space, $\mathcal{S}(V(\mathbb{A}))$ can be identified with $\mathcal{S}(V(F_\infty)) \otimes \mathcal{S}(V(\mathbb{A}_{\text{fin}}))$ where $\mathcal{S}(V(F_\infty))$ is the usual Schwartz space and $\mathcal{S}(V(\mathbb{A}_{\text{fin}}))$ is the countable-dimensional space of locally constant compactly supported functions on $V(\mathbb{A}_{\text{fin}})$ with the finite topology. In other words, $\mathcal{S}(V(\mathbb{A}_{\text{fin}})) = \varinjlim U$ with the inductive limit topology, where U ranges over the finite-dimensional subspaces of $\mathcal{S}(V(\mathbb{A}_{\text{fin}}))$ with their canonical Hausdorff topology. Equivalently, $\mathcal{S}(V(\mathbb{A})) = \varinjlim \mathcal{S}(V(F_\infty)) \otimes U$ with the locally convex inductive limit topology where U is as before. The space $\mathcal{S}(V(\mathbb{A}))$ is Hausdorff, complete and nuclear. The embedding $\mathcal{S}(V(\mathbb{A})) \rightarrow C_{\text{rd}}(V(\mathbb{A}))$ is continuous.

Assume that (G, V, ρ) is CF with irreducible components $(\rho_i, V_i), i \in I$ over F and let $\chi_i = \omega_{\rho_i}, i \in I$. For any $J \subset I$ let $V^J = \bigoplus_{j \in J} V_j$ and $V_J = V/V^J = \bigoplus_{j \notin J} V_j$. Let ρ_J be the representation of G on V_J . These are again CF PVSs.

As usual, let $A_G = T_G(\mathbb{R})^\circ$ be the complement to $G(\mathbb{A})^1$ in $G(\mathbb{A})$ obtained by embedding \mathbb{R} into $F_\infty = \mathbb{R} \otimes F$ via $x \mapsto x \otimes 1$, and let $A_{\text{Ker } \rho} = A_G \cap \text{Ker } \rho$. Since $G^*/G^{\text{der}} \text{Ker } \rho$ is F -anisotropic, we have $G(\mathbb{A})^1 G^*(\mathbb{A}) \backslash G(\mathbb{A}) \simeq A_{\text{Ker } \rho} \backslash A_G$ and we can write the zeta function of $\phi \in \mathcal{S}(V(\mathbb{A}))$ as

$$Z(\phi, \lambda) = \int_{G(F)A_{\text{Ker } \rho} \backslash G(\mathbb{A})} \theta_\phi^{\text{niso}}(g) g^{-\lambda} |\det \rho(g)|^{-1} dg,$$

for $\lambda \in (\mathfrak{a}_{G, \mathbb{C}}/\mathfrak{a}_{\text{Ker } \rho, \mathbb{C}})^* = \mathbb{C}\mathfrak{X}$. We know from Theorem 4.3 that the integral converges absolutely for $\Re \lambda \in \mathbb{R}_{>0}\mathfrak{X}$. In the following, we will use the isomorphism $A_{\text{Ker } \rho} \backslash A_G \simeq \mathbb{R}^I, a \mapsto (\log |\chi_i(a)|)_{i \in I}$ and the Lebesgue measure on the vector space \mathbb{R}^I to normalize the measure on the quotient $G(F)A_{\text{Ker } \rho} \backslash G(\mathbb{A})$.

For any $\phi \in \mathcal{S}(V(\mathbb{A}))$, let $\phi_J \in \mathcal{S}(V_J(\mathbb{A}))$ be given by

$$\phi_J(v) = \int_{V^J(\mathbb{A})} \phi(u+v) du.$$

For $\lambda = \sum_{i \in I} \lambda_i \chi_i$ we write $\lambda_J = \sum_{j \notin J} \lambda_j \chi_j$. We will consider the zeta functions $Z^{V_J}(\phi_J, \lambda_J)$ simultaneously. In the limiting case $J = I$, where $V_I = 0$, the term $Z^{V_I}(\phi_I, \lambda_I)$ is of course interpreted as the constant $\text{vol}(G(F) \backslash G(\mathbb{A})^1) \hat{\phi}(0)$.

The main result of this section is the following.

THEOREM 8.3. *Let (G, V, ρ) be a CF PVS such that $\Gamma_{\text{spcl}}(V) = \{V\}$. Then, there exists $c > 0$ such that for any $\phi \in \mathcal{S}(V(\mathbb{A}))$ the zeta function $Z(\phi, \lambda)$ admits a meromorphic continuation to the region*

$$\mathcal{D} = \left\{ \lambda = \sum_{i \in I} \lambda_i \chi_i \mid \Re \lambda_i > -c \right\},$$

and $Z(\phi, \lambda) \prod_{i \in I} \lambda_i$ is holomorphic in \mathcal{D} . Moreover, with the notation above,

$$\sum_{J \subset I} (-1)^{\#J} \frac{Z^{V_J}(\phi_J, \lambda_J)}{\prod_{j \in J} \lambda_j}, \tag{8.1}$$

is holomorphic in \mathcal{D} .

Theorem 8.3 will be proved in the remainder of this section. Applying it to V_J for every $J \subset I$ and using inclusion–exclusion we obtain

COROLLARY 8.4. *Under the above conditions, for every subset $J \subset I$ the function*

$$f_J(\lambda_J) = \sum_{K \subset I \setminus J} (-1)^{\#K} \frac{Z^{V_{J \cup K}}(\phi_{J \cup K}, \lambda_{J \cup K})}{\prod_{j \in K} \lambda_j},$$

is holomorphic in $\{\lambda_J = \sum_{j \notin J} \lambda_j \chi_j \mid \Re \lambda_j > -c\}$ and we have

$$Z(\phi, \lambda) = \sum_{J \subset I} \frac{f_J(\lambda_J)}{\prod_{j \in J} \lambda_j},$$

in \mathcal{D} .

Theorem 8.3 and its corollary apply in particular to any basic PVS, as well as to the “magic square” examples of §3.3 – see Remark 6.6. Recall that the basic PVSs obtained from nilpotent orbits for the classical groups are the following:

1. GL_n^k acting on Mat_n^{k-1} (Example 3.2).
2. $\text{GL}_n^k \times \text{Sp}_{n+m}$ acting on $\text{Mat}_n^{k-1} \oplus \text{Mat}_{n, n+m}$ for n, m even (Example 3.3).
3. GL_n^k acting on $\text{Mat}_n^{k-1} \oplus \text{Sym}_n$ for $n \neq 2$ (Example 3.4).
4. $\text{GL}_n^k \times \text{SO}_{n+m}$ acting on $\text{Mat}_n^{k-1} \oplus \text{Mat}_{n, n+m}$ for $n, m \neq 2$ (Example 3.5).
5. GL_n^k acting on $\text{Mat}_n^{k-1} \oplus \text{Skew}_n$ (Example 3.6).

For F -irreducible V such that $\Gamma_{\text{spcl}}(V) = \{V\}$, we obtain that the zeta function has a simple pole at $\lambda = 0$ with residue $\text{vol}(G(F) \backslash G(\mathbb{A})^1) \hat{\phi}(0)$, as in Tate’s thesis (cf. Proposition 8.5 below). Example 3.7 (treated in [50], building on [44]) and Example 3.8 (treated in [53]) show that without the assumption $\Gamma_{\text{spcl}}(V) = \{V\}$, the structure of the pole of the zeta function at $\lambda = 0$ may be more complicated.

Note that the assumption $\Gamma_{\text{spcl}}(V) = \{V\}$ implies (by Lemma 8.2 and Proposition 7.4) that all V_J are non-isotropic, i.e. that $\theta_{\phi_J}^{\text{niiso}}(g) = \theta_{\phi_J}(g)$. (In our proof below we will simply work with $\theta_{\phi_J}(g)$ throughout.)

We first consider the case where ρ is irreducible over F . For simplicity we write $x_{<1} = \min(x, 1)$ and $x_{\geq 1} = \max(1, x)$ for any $x \in \mathbb{R}_{>0}$. Denote by $\mathbf{1}_A$ the characteristic function of a set A .

For brevity, we will use the notation $A(\cdot, \phi) \ll_{N, \phi} B_N(\cdot)$ to mean that for every $N \geq 0$ there exists a continuous seminorm ν_N on $\mathcal{S}(V(\mathbb{A}))$ such that $|A(\cdot, \phi)| \leq B_N(\cdot) \nu_N(\phi)$ for

every $\phi \in \mathcal{S}(V(\mathbb{A}))$. Recall that we have already used the notation $\ll_{N,\phi}^{\text{rd}}$ in §5.5 with respect to the space $C_{\text{rd}}(V(\mathbb{A}))$. Clearly $\ll_{N,\phi}^{\text{rd}}$ implies $\ll_{N,\phi}$.

PROPOSITION 8.5. *Suppose that (G, V, ρ) is an F -irreducible PVS such that $\Gamma_{\text{spcl}}(V) = \{V\}$ and let $\chi = \omega_\rho$. For any $\phi \in \mathcal{S}(V(\mathbb{A}))$ let*

$$\mathfrak{I}_\phi(g) = |\det \rho(g)|^{-1} \theta_\phi(g) - \mathbf{1}_{\mathbb{R}_{>1}}(|\chi(g)|) \hat{\phi}(0), \quad g \in G(F) \backslash G(\mathbb{A}).$$

Let β_0 be the lowest weight of V . Then,

$$\mathfrak{I}_\phi(g) \ll_{N,\phi} \delta_{P_0}(m_0(g)) e^{-\langle \mu, H_0(g) \rangle - \langle \beta_0, H_0(g) \rangle_+ - N \langle \chi, H_0(g) \rangle_-}, \quad g \in \mathfrak{S}, \tag{8.2}$$

for a suitable $\mu \in \mathbb{R}_{>0}(\Delta_0)$. Therefore, there exists $c > 0$ such that the integral

$$\int_{G(F) \backslash A_{\text{Ker } \rho} \backslash G(\mathbb{A})} \mathfrak{I}_\phi(g) |\chi(g)|^{-s} dg = Z(\phi, s\chi) - \text{vol}(G(F) \backslash G(\mathbb{A})^1) \frac{\hat{\phi}(0)}{s}, \tag{8.3}$$

converges absolutely for $\Re s > -c$.

Proof. We will estimate \mathfrak{I}_ϕ using the decomposition (5.8) of $\theta_\phi(g)$. We write

$$\mathfrak{I}_\phi(g) = |\det \rho(g)|^{-1} \sum_{U \in \Gamma_{\text{reg}}(V), U \neq V} \Theta_U(g) + |\det \rho(g)|^{-1} \Theta_V(g) - \mathbf{1}_{\mathbb{R}_{>1}}(|\chi(g)|) \hat{\phi}(0),$$

and separately bound the summands Θ_U , $U \in \Gamma_{\text{reg}}(V)$, $U \neq V$ and $|\det \rho|^{-1} \Theta_V - \mathbf{1}_{\mathbb{R}_{>1}}(|\chi|) \hat{\phi}(0)$ on \mathfrak{S} .

Consider the contribution from $U \in \Gamma_{\text{reg}}(V)$, $U \neq V$. By Lemma 5.8 we have

$$|\det \rho(g)|^{-1} \Theta_U(g) \ll_{N,\phi}^{\text{rd}} \delta_{P_0}(m_0(g)) e^{-\langle \lambda(U), H_0(g) \rangle - N \sum_{\beta \in \Psi_U} \langle \beta, H_0(g) \rangle_-}, \quad g \in \mathfrak{S}.$$

Recall that $\lambda(U) = \delta_0 + \sum_{\beta \in \Psi_V \setminus \Psi_U} n_\beta \beta \in \mathbb{R}_{\geq 0}(\Psi_U \cup \Delta_0)$ by (5.6) and (5.7). By Corollary 6.13 part 3 and our assumption on V we have $\mathbf{E}(U) = G$. Thus, the positive envelope of $\lambda(U)$ with respect to $\Delta_0 \cup \Psi_U$ (see §6.1) contains Δ_0 and we have

$$\lambda(U) = \sum_{\beta \in \Psi_U} c_\beta \beta + \sum_{\alpha \in \Delta_0} d_\alpha \alpha,$$

with suitable coefficients $c_\beta \geq 0$ and $d_\alpha > 0$. Moreover, by considering $\langle \lambda(U), H \rangle$ for $H \in \mathfrak{a}_G$ we see that $\sum_{\beta \in \Psi_U} c_\beta = \sum_{\beta \in \Psi_V \setminus \Psi_U} n_\beta \geq 1$ since $U \neq V$. Using that on the Siegel domain $\langle \beta_0, H_0(g) \rangle - \langle \beta, H_0(g) \rangle$ is bounded from above for all $\beta \in \Phi_V$, we obtain

$$|\det \rho(g)|^{-1} \Theta_U(g) \ll_{N,\phi}^{\text{rd}} \delta_{P_0}(m_0(g)) e^{-\langle \mu, H_0(g) \rangle - \langle \beta_0, H_0(g) \rangle_+ - N \sum_{\beta \in \Psi_U} \langle \beta, H_0(g) \rangle_-}, \quad g \in \mathfrak{S}.$$

By Corollary 5.3 part 1, $\chi \neq 0$ is a non-negative linear combination of Ψ_U . Therefore, we can replace here $\sum_{\beta \in \Psi_U} \langle \beta, H_0(g) \rangle_-$ by a suitable positive multiple of $\langle \chi, H_0(g) \rangle_-$, proving the required bound for the contribution of any $U \neq V$.

As for $U = V$, we claim that

$$|\det \rho(g)|^{-1} \Theta_V(g) - \mathbf{1}_{\mathbb{R}_{>1}}(|\chi(g)|) \hat{\phi}(0) \ll_{N,\phi} e^{-\langle \beta_0, H_0(g) \rangle_+ - N \langle \chi, H_0(g) \rangle_-}, \quad g \in \mathfrak{S}. \tag{8.4}$$

We may assume that $g \in T_0(\mathbb{A}) \cap \mathfrak{S}$, or even that $g \in T_0(\mathbb{R}) \cap \mathfrak{S}$. We split into cases according to the size of $\langle \beta_0, H_0(g) \rangle$. Fix a constant c_1 .

If $\langle \beta_0, H_0(g) \rangle \leq c_1$, then (8.4) holds because $|\det \rho(g)|^{-1} \Theta_V(g) \ll_{N,\phi}^{\text{rd}} e^{-N \langle \chi, H_0(g) \rangle_-}$ by Lemma 5.8.

Assume that $\langle \beta_0, H_0(g) \rangle > c_1$. Then $\langle \beta, H_0(g) \rangle$ is bounded away from 0 for every $\beta \in \Psi_V$, since $g \in \mathfrak{S}$. By a standard argument, e.g., using the Poisson summation formula, we have

$$|\det \rho(g)|^{-1} \sum_{v \in V(F)} \phi(\rho(g)^{-1}v) - \hat{\phi}(0) \ll_{\phi} e^{-\langle \beta_0, H_0(g) \rangle}. \tag{8.5}$$

(Here we cannot replace \ll_{ϕ} by \ll_{ϕ}^{rd} .) On the other hand,

$$\left| \Theta_V(g) - \sum_{v \in V(F)} \phi(\rho(g)^{-1}v) \right| \leq \sum_{v \in V(F) | P(v)=0} |\phi(\rho(g)^{-1}v)|,$$

for some non-zero polynomial P on V . (For instance, we can take the relative invariant times the product over $\beta \in \Psi_V$ of arbitrary nonzero linear forms on V_{β} .) It follows from [11, Lemma 6.4] that

$$|\det \rho(g)|^{-1} (\Theta_V(g) - \sum_{v \in V(F)} \phi(\rho(g)^{-1}v)) \ll_{\phi} e^{-\langle \beta_0, H_0(g) \rangle}. \tag{8.6}$$

(In fact, the argument gives the stronger assertion with \ll_{ϕ}^{rd} .)

Since $\langle \chi, H_0(g) \rangle - \langle \beta_0, H_0(g) \rangle$ is bounded from below for $g \in \mathfrak{S}$, we may choose c_1 such that $|\chi(g)| > 1$ whenever $\langle \beta_0, H_0(g) \rangle > c_1$. Thus, (8.4) for $\langle \beta_0, H_0(g) \rangle > c_1$ follows from (8.5) and (8.6). This finishes the proof of (8.2).

The convergence of

$$\int_{P_0(F)A_{\text{Ker } \rho} \backslash \mathfrak{S}} |\mathfrak{I}_{\phi}(g)| |\chi(g)|^{-\sigma} dg,$$

for $\sigma > -c$ for a suitable $c > 0$ follows from (8.2) and Lemma 7.1, since for any complement W of $\mathfrak{a}_{\text{Ker } \rho}$ in \mathfrak{a}_0 the positively-1-homogeneous function $\langle \mu, x \rangle + \langle \beta_0, x \rangle_+$ on W is positive on the cone $\{x \in \mathfrak{a}_{0,+} \cap W \mid \langle \chi, x \rangle \geq 0\} \setminus \{0\}$.

This implies the absolute convergence of (8.3) for $\Re s > -c$. □

Proof of Theorem 8.3. By induction on the cardinality of $I \setminus J$, starting with the trivial case $J = I$ (and, if we like, Proposition 8.5 for $\#(I \setminus J) = 1$), we may assume holomorphic continuation of the functions $Z^{V_J}(\phi_J, \lambda_J) \prod_{j \notin J} \lambda_j$ to $\{\lambda_J = \sum_{j \notin J} \lambda_j \chi_j \mid \Re \lambda_j > -c\}$ for all $J \neq \emptyset$. To prove the theorem, we need to show that (8.1) is holomorphic on \mathcal{D} .

For every $i \in I$ let β_i be the lowest weight of V_i . For any $J \subset I$ let $\chi_J = \prod_{j \notin J} \chi_j$ and let G^J be the subset of $G(\mathbb{A})$ defined by the conditions $|\chi_j(g)| > 1$ for all $j \in J$.

Consider

$$\mathfrak{T}_{\phi}(g) = \sum_{J \subset I} (-1)^{\#J} |\det \rho_J(g)|^{-1} \theta_{\phi_J}^{V_J}(g) \mathbf{1}_{G^J}(g).$$

We claim that there exists $\mu \in \mathbb{R}_{>0} \Delta_0$ such that

$$\mathfrak{T}_{\phi}(g) \ll_{N, \phi} \delta_{P_0}(m_0(g)) e^{-\langle \mu, H_0(g) \rangle - \sum_{i \in I} \langle \beta_i, H_0(g) \rangle_+ - N \sum_{i \in I} \langle \chi_i, H_0(g) \rangle_-}, \quad g \in \mathfrak{S}. \tag{8.7}$$

We may identify $\mathcal{S}(V(\mathbb{A}))$ with the completed tensor product $\hat{\otimes}_{i \in I} \mathcal{S}(V_i(\mathbb{A}))$ with respect to, say, the projective tensor product topology (the spaces $\mathcal{S}(V_i(\mathbb{A}))$ are nuclear). Therefore, it is enough to prove (8.7) for ϕ of the form $\phi = \otimes \psi_i$ where $\psi_i \in \mathcal{S}(V_i(\mathbb{A}))$. We then have

$$\mathfrak{T}_{\phi}(g) = \prod_{i \in I} (|\det \rho_i(g)|^{-1} \theta_{\psi_i}^{V_i}(g) - \mathbf{1}_{\mathbb{R}_{>1}}(|\chi_i(g)|) \hat{\psi}_i(0)).$$

If we simply apply (8.2) to each factor, the bound that we get will not be good enough. This is not surprising since we have not yet used that V itself is a PVS, only the individual factors. Instead, we write for each $i \in I$

$$\theta_{\psi_i}^{V_i} = \Theta_{\psi_i}^{V_i, \text{fs}} + \Theta_{\psi_i}^{V_i, \text{nfs}},$$

where

$$\Theta_{\psi_i}^{V_i, \text{fs}} = \sum_{\xi \in X_i(F) : \text{supp } \xi = \Psi_{V_i}} \phi(\rho_i(g)^{-1}\xi),$$

is the summand corresponding to $U = V_i$ (full support) in the decomposition (5.8) for $\theta_{\psi_i}^{V_i}$. Then, we can decompose

$$\Upsilon_\phi = \sum_{J \subset I} \prod_{j \in J} |\det \rho_j|^{-1} \Theta_{\psi_j}^{V_j, \text{nfs}} \prod_{j \notin J} (|\det \rho_j|^{-1} \Theta_{\psi_j}^{V_j, \text{fs}} - \mathbf{1}_{\mathbb{R}_{>1}}(|\chi_j|) \hat{\psi}_j(0)). \tag{8.8}$$

For any subset $J \subset I$ let $\phi^J = \otimes_{j \in J} \psi_j$. Then,

$$\prod_{j \in J} \Theta_{\psi_j}^{V_j, \text{nfs}} = \sum_{U \in \Gamma_{\text{reg}}(V^J), U \not\supset V_j \forall j \in J} \Theta_{\phi^J; U}^{V^J}, \tag{8.9}$$

is the contribution to $\Theta_{\phi^J}^{V^J}$ with respect to the decomposition (5.8) of the subspaces U not containing any $V_j, j \in J$. As in the proof of Proposition 8.5, there exists $\mu \in \mathbb{R}_{>0} \Delta_0$ (which we may take to be independent of J) such that

$$\prod_{j \in J} |\det \rho_j(g)|^{-1} \Theta_{\psi_j}^{V_j, \text{nfs}}(g) \ll_{N, \phi^J} \delta_{P_0}(m_0(g)) e^{-\langle \mu, H_0(g) \rangle - \sum_{j \in J} \langle \beta_j, H_0(g) \rangle_+ - N \sum_{j \in J} \langle \chi_j, H_0(g) \rangle_-}, \tag{8.10}$$

on \mathfrak{S} . Indeed, this estimate holds for each individual contribution $\Theta_{\phi^J; U}^{V^J}$ to (8.9). Since $\mathbf{E}(U) = G$ by Lemma 8.2 and Corollary 6.13 part 3, the positive envelope of $\lambda(U)$ with respect to $\Psi_U \cup \Delta_0$ contains Δ_0 . Writing

$$\lambda(U) = \sum_{\beta \in \Psi_U} c_\beta \beta + \sum_{\alpha \in \Delta_0} d_\alpha \alpha,$$

with $c_\beta \geq 0$ and $d_\alpha > 0$, and considering $\langle \lambda(U), H \rangle$ for $H \in \mathfrak{a}_G$, we obtain

$$\sum_{j \in J} \sum_{\beta \in \Psi_U \cap \Psi_{V_j}} c_\beta \chi_j = \sum_{j \in J} \sum_{\beta \in \Psi_{V_j} \setminus \Psi_U} n_\beta \chi_j,$$

and therefore $\sum_{\beta \in \Psi_U \cap \Psi_{V_j}} c_\beta = \sum_{\beta \in \Psi_{V_j} \setminus \Psi_U} n_\beta \geq 1$ for all $j \in J$, since the χ_j are linearly independent over \mathbb{Z} and $U \not\supset V_j$ for all $j \in J$.

Using Lemma 5.8 together with Corollary 5.3 part 1 and the fact that $\langle \beta_j, H_0(g) \rangle - \langle \beta, H_0(g) \rangle$ is bounded from above for any $\beta \in \Phi_{V_j}$ and $g \in \mathfrak{S}$, we conclude (8.10) as in the irreducible case.

Combining (8.10) with the estimate (8.4) for every $j \notin J$ and using the decomposition (8.8) we obtain the required estimate (8.7).

Note that the expression (8.1) is equal to the integral

$$\int_{A_{\text{Ker } \rho} G(F) \backslash G(\mathbb{A})} \Upsilon_\phi(g) e^{-\langle \lambda, H_0(g) \rangle} dg.$$

Therefore, Theorem 8.3 follows from (8.7) and Lemma 7.1 by noting that for any complement W of $\mathfrak{a}_{\text{Ker } \rho}$ in \mathfrak{a}_0 , the positively-1-homogeneous function $\langle \mu, x \rangle + \sum_{i \in I} \langle \beta_i, x \rangle_+$ on W is positive on the cone $\{x \in \mathfrak{a}_{0,+} \cap W \mid \langle \chi_i, x \rangle \geq 0 \ \forall i \in I\} \setminus \{0\}$. \square

REMARK 8.6. Let V be a regular PVS with $\Gamma_{\text{spcl}}(V) = \{V\}$. The example of $V = \text{Mat}_n$ with $G = \text{GL}_n$ acting by matrix multiplication shows that V is not necessarily CF. One may ask whether disjointness of the sets of weights Ψ_{V_i} of the irreducible components V_i of V is sufficient to imply that V is CF. This would follow if one could show that an F -irreducible PVS V such that $\Gamma_{\text{spcl}}(V) = \{V\}$ is necessarily regular.⁹

In the case of PVSs V of DK-type, the construction of §9 below shows that the condition $\Gamma_{\text{spcl}}(V) = \{V\}$ implies that V is CF.

§9. A construction of special subspaces for PVSs of DK-type

We finish the paper by providing a general construction of special subspaces for regular PVSs coming from nilpotent orbits (see §3).

Let G' be a reductive group over F . Recall that any nilpotent $e \in \mathfrak{g}'$ gives rise to a filtration $\mathcal{F}(e) = (\mathcal{F}_i(e))_{i \in \mathbb{Z}}$ of \mathfrak{g}' whose stabilizer $\text{Stab}(\mathcal{F}(e))$ in G' is a parabolic subgroup of G' with Lie algebra $\mathcal{F}_0(e)$.

Clearly, $\mathcal{F}(\text{Ad}(g)e) = \text{Ad}(g)\mathcal{F}(e)$ for all $g \in G'$. In particular, e and $\text{Ad}(g)e$ define the same filtration if and only if $g \in \text{Stab}(\mathcal{F}(e))$.

If \mathfrak{o} is the nilpotent orbit of e , then we say that $\mathcal{F}(e)$ is an \mathfrak{o} -filtration. Thus, the variety of \mathfrak{o} -filtrations is a G' -orbit whose stabilizer is the canonical parabolic subgroup pertaining to \mathfrak{o} .

9.1

We recall the notion of induction of nilpotent orbits [25].

Let Q be a parabolic F -subgroup of G' with Lie algebra \mathfrak{q} . Let L be the Levi quotient of Q , with Lie algebra \mathfrak{l} . Let $\text{pr}_Q : Q \rightarrow L$ (resp., $\text{pr}_\mathfrak{q} : \mathfrak{q} \rightarrow \mathfrak{l}$) be the canonical projections. Let \mathfrak{o}^L be a nilpotent orbit in \mathfrak{l} and let $\tilde{\mathfrak{o}}^L$ be its inverse image under $\text{pr}_\mathfrak{q}$. It is an irreducible subvariety of $\mathfrak{q} \subset \mathfrak{g}'$ consisting of nilpotent elements. By definition, the induced orbit $\text{Ind}_Q \mathfrak{o}^L$ is the nilpotent orbit in \mathfrak{g}' intersecting $\tilde{\mathfrak{o}}^L$ in an open subset. In fact, it depends only on L (and \mathfrak{o}^L), and not on Q , but it will be important for us to keep track of Q .

DEFINITION 9.1. Suppose that $\mathfrak{o} = \text{Ind}_Q \mathfrak{o}^L$ where Q is a parabolic F -subgroup of G' with Levi quotient L and \mathfrak{o}^L is a nilpotent orbit of \mathfrak{l} . Let \mathcal{F}^L be an \mathfrak{o}^L -filtration of \mathfrak{l} . We then say that the triple $\mathfrak{d} = (Q, \mathfrak{o}^L, \mathcal{F}^L)$ is an *induced filtration data* (IFD) for \mathfrak{o} .

For instance, $(G', \mathfrak{o}, \mathcal{F})$ is a (trivial) IFD for any \mathfrak{o} -filtration \mathcal{F} of \mathfrak{g}' .

The variety $\mathfrak{D} = \mathfrak{D}_\mathfrak{o}$ of IFDs for \mathfrak{o} comprises finitely many G' -orbits. (If $\mathfrak{D}_\mathfrak{o}$ is a single orbit, i.e., if \mathfrak{o} is not induced non-trivially, then \mathfrak{o} is called *rigid*.)

LEMMA 9.2. Let $\mathfrak{d} = (Q, \mathfrak{o}^L, \mathcal{F}^L) \in \mathfrak{D}$. Define inductively

$$\tilde{\mathcal{F}}_i = \begin{cases} \text{pr}_\mathfrak{q}^{-1}(\mathcal{F}_i^L), & i = 0, 1, 2, \\ [\tilde{\mathcal{F}}_1, \tilde{\mathcal{F}}_{i-1}] + [\tilde{\mathcal{F}}_2, \tilde{\mathcal{F}}_{i-2}], & i > 2, \end{cases}$$

⁹ This can be easily checked when F is algebraically closed by classification.

and

$$\tilde{\mathcal{F}}_i = \{x \in \mathfrak{g}' \mid [x, \tilde{\mathcal{F}}_{-i}] \subset \tilde{\mathcal{F}}_0\}, \quad i < 0.$$

Then $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_i)_i$ is a filtration of \mathfrak{g}' of V -type.

Note that $\text{pr}_{\mathfrak{q}}(\tilde{\mathcal{F}}_i) = \mathcal{F}_i^L$ for all i but that the preimage $\text{pr}_{\mathfrak{q}}^{-1}(\mathcal{F}_i^L)$ strictly contains $\tilde{\mathcal{F}}_i$ for $i > 2$.

Proof. Without loss of generality, we may assume that Q is a standard parabolic subgroup of G' and \mathcal{F}^L the standard \mathfrak{o}^L -filtration defined by a dominant element $h^L \in \mathfrak{a}_0^L$, i.e. $\mathcal{F}_i^L = \bigoplus_{\alpha \in \Phi_L \cup \{0\}: \langle \alpha, h^L \rangle \geq i} \mathfrak{g}'_{\alpha}$.

Let $h = h^L + h_L \in \mathfrak{a}_0^{G'}$ be the unique vector with $h_L \in \mathfrak{a}_L^{G'}$ and $\langle \alpha, h \rangle = 2$ for all $\alpha \in \Delta_0^{G'} \setminus \Delta_0^Q$. We claim that the inductively defined spaces $\tilde{\mathcal{F}}_i$ coincide with the filtration obtained from the gradation of \mathfrak{g}' defined by h : $\tilde{\mathcal{F}}_i = \bigoplus_{\alpha \in \Phi_{G'} \cup \{0\}: \langle \alpha, h \rangle \geq i} \mathfrak{g}'_{\alpha}$.

To see this, note that we have $\langle \alpha, h \rangle = \langle \alpha, h^L \rangle \in \{0, 1, 2\}$ for $\alpha \in \Delta_0^Q$ (and therefore $\langle \alpha, h \rangle \in \{0, 1, 2\}$ for all $\alpha \in \Delta_0^{G'}$). This implies the claim for $i = 0, 1, 2$. The claim for $i \geq 3$ follows, since any positive root $\alpha \in \Phi_{G'}$ with $\langle \alpha, h \rangle \geq 3$ can be written as $\alpha = \beta + \gamma$ with positive roots β and γ and $\langle \beta, h \rangle \in \{1, 2\}$.

The relation between $\tilde{\mathcal{F}}_i$ and $\tilde{\mathcal{F}}_{-i}$ holds for any filtration obtained from a gradation. \square

DEFINITION 9.3. We call the filtration $(\tilde{\mathcal{F}}_i)_i$ of Lemma 9.2 the I -filtration associated to \mathfrak{d} .

We say that a filtration $\tilde{\mathcal{F}}$ of \mathfrak{g}' is an I -filtration compatible with \mathfrak{o} if it arises from some $\mathfrak{d} \in \mathfrak{D}$ by this construction.

The stabilizer $\text{Stab}(\mathfrak{d})$ of $\mathfrak{d} = (Q, \mathfrak{o}^L, \mathcal{F}^L) \in \mathfrak{D}$ is the parabolic subgroup of Q whose image in L is $\text{Stab}_L(\mathcal{F}^L)$. It is also the stabilizer of the associated I -filtration $\tilde{\mathcal{F}}$. Its Lie algebra is $\tilde{\mathcal{F}}_0$.

\mathfrak{o} -filtrations are I -filtrations compatible with \mathfrak{o} , but in general there will be others. For example, if $\mathfrak{o} = \text{Rich}(Q)$, there is an I -filtration compatible with \mathfrak{o} with $\tilde{\mathcal{F}}_2 = \mathfrak{n}_Q$, which is an \mathfrak{o} -filtration only if \mathfrak{o} is even and Q is conjugate to the canonical parabolic subgroup of \mathfrak{o} . On the other hand, different IFDs may give rise to the same I -filtration.

LEMMA 9.4. Let $\mathfrak{d} = (Q, \mathfrak{o}^L, \mathcal{F}^L)$ be an IFD for \mathfrak{o} . Let $P' = \text{Stab}(\mathfrak{d})$ and $(\tilde{\mathcal{F}}_i)_i$ be the associated I -filtration. Then,

1. $\mathfrak{o}_{\tilde{\mathcal{F}}} = \mathfrak{o} \cap \tilde{\mathcal{F}}_2$ is open in $\tilde{\mathcal{F}}_2$.
2. $\text{pr}_{\mathfrak{q}}(\mathfrak{o}_{\tilde{\mathcal{F}}}) = \mathfrak{o}^L \cap \mathcal{F}_2^L$.
3. $\mathfrak{o}_{\tilde{\mathcal{F}}}$ is a P' -orbit.
4. Let $v \in \mathfrak{o}_{\tilde{\mathcal{F}}}$. Then,

$$\{g \in G' \mid \text{Ad}(g)^{-1}v \in \mathfrak{o}_{\tilde{\mathcal{F}}}\} = G'_v P',$$

and $(G'_v)^{\circ} \subset P'$.

Proof. For simplicity, we sometimes write $\bar{\cdot}$ for the image under pr_Q or $\text{pr}_{\mathfrak{q}}$.

The first part follows since $\text{pr}_{\mathfrak{q}}^{-1}(u) \cap \mathfrak{o}$ is open in $\text{pr}_{\mathfrak{q}}^{-1}(u)$ for every $u \in \mathfrak{o}^L$ and $\mathfrak{o}^L \cap \mathcal{F}_2^L$ is open in \mathcal{F}_2^L .

For the second part, suppose that $u \in \mathfrak{q}$ with $\bar{u} \in \mathcal{F}_2^L \setminus \mathfrak{o}^L$. Then, \bar{u} lies in a nilpotent orbit $\mathfrak{o}' \neq \mathfrak{o}^L$ of \mathfrak{l} in the closure of \mathfrak{o}^L and u lies in the closure of $\text{Ind}_Q \mathfrak{o}'$. The latter has smaller dimension than \mathfrak{o} . Thus, $u \notin \mathfrak{o}$.

For the last part, suppose that $u = \text{Ad}(g)^{-1}v \in \mathfrak{o}_{\tilde{\mathcal{F}}}$. Then, by the second part $\bar{u} \in \mathfrak{o}^L$. By [25, Theorem 1.3(d)], $g \in G'_v Q$. Write $g = hq$ where $h \in G'_v, q \in Q$. Then, $\bar{u} = \text{Ad}(\bar{q})^{-1}\bar{v}$. Hence, $\text{Ad}(\bar{q})^{-1}\bar{v} \in \mathfrak{o}^L \cap \mathcal{F}_2^L$. By Theorem 3.1 part 4 (applied to \mathfrak{o}^L), we infer that $\bar{q} \in \text{pr}_Q(P')$, so that $q \in P'$. Hence $g \in G'_v P'$.

Also, $(G'_v)^\circ \subset Q$ again by [25, Theorem 1.3(d)], since $\bar{v} \in \mathfrak{o}^L$ by the second part. Thus, $(G'_v)^\circ = Q_v^\circ$. Since $\text{pr}_Q(Q_v) \subset L_{\bar{v}} \subset \text{pr}_Q(P')$, by Theorem 3.1 we deduce that $(G'_v)^\circ \subset P'$.

Finally, the third part follows from the last part. □

9.2

We fix P'_0, T'_0 as in §3. The following standard fact is an easy consequence of the Bruhat decomposition.

REMARK 9.5. Let P be a standard parabolic F -subgroup of G' with standard Levi subgroup M . Then, for any parabolic F -subgroup Q of G' there exists a unique parabolic subgroup Q' of the form $Q' = Q^p, p \in P(F)$ such that $Q' \cap M$ is a standard parabolic subgroup of M .

Namely, if Q is conjugate to the standard parabolic F -subgroup Q'' of G , then we can write $Q = (Q'')^{np}$ with $n \in N_{G'(F)}(T'_0)$ and $p \in P(F)$ by the Bruhat decomposition, and moreover the Weyl group element $\bar{n} \in W = N_{G'(F)}(T'_0)/T'_0(F)$ defined by n can be taken of minimal length in its $(W_{Q''}, W_P)$ -double coset. Then $Q' = (Q'')^n$ works by [6, Proposition 2.8.9], and uniqueness follows from [6, Proposition 2.3.3, 2.8.1].

Now let \mathfrak{o} be a nilpotent orbit in \mathfrak{g}' and let (G, V, ρ) be the regular PVS of DK-type attached to \mathfrak{o} (cf. §3). Thus, G is the standard Levi subgroup of the (standard) canonical parabolic subgroup P of \mathfrak{o} . We write $\mathfrak{o}_{\mathcal{F}} = \mathfrak{o} \cap \mathcal{F}_2$, where \mathcal{F} is the standard \mathfrak{o} -filtration (i.e., with $\text{Stab}(\mathcal{F}) = P$).

Denote by $[\mathfrak{F}]$ the (finite) set of G' -orbits of I-filtrations compatible with \mathfrak{o} . The elements of $[\mathfrak{F}]$ give rise to special subspaces of V . More precisely,

PROPOSITION 9.6. *For any G' -orbit $\mathfrak{C} \in [\mathfrak{F}]$ there exists a unique I-filtration $\tilde{\mathcal{F}} \in \mathfrak{C}$ such that $\text{Stab}(\tilde{\mathcal{F}}) \cap G$ is a standard parabolic subgroup of G and $\mathfrak{o} \cap \mathcal{F}_2 \cap \tilde{\mathcal{F}}_2 \neq \emptyset$. Moreover, $U := \tilde{\mathcal{F}}_2 \cap V \in \Gamma_{\text{spcl}}(V)$ and $\text{Stab}(U) = \text{Stab}(\tilde{\mathcal{F}}) \cap G$.*

Proof. We first show that the set

$$\mathfrak{C}' = \{\tilde{\mathcal{F}} \in \mathfrak{C} \mid \mathfrak{o} \cap \mathcal{F}_2 \cap \tilde{\mathcal{F}}_2 \neq \emptyset\},$$

is a P -orbit. Clearly, \mathfrak{C}' is stable under P , and it is non-empty since $\mathfrak{o} \cap \tilde{\mathcal{F}}_2 \neq \emptyset$ for all $\tilde{\mathcal{F}} \in \mathfrak{C}$. Assume that $\tilde{\mathcal{F}} \in \mathfrak{C}'$ and $\text{Ad}(g)\tilde{\mathcal{F}} \in \mathfrak{C}'$ for some $g \in G'$. We need to show that $g \in PP'$ for $P' = \text{Stab}(\tilde{\mathcal{F}})$. By assumption, $\mathfrak{o}_{\tilde{\mathcal{F}}}$ intersects both $\mathfrak{o}_{\mathcal{F}}$ and $\text{Ad}(g)^{-1}\mathfrak{o}_{\mathcal{F}}$. Since $\mathfrak{o}_{\tilde{\mathcal{F}}}$ is a P' -orbit by Lemma 9.4 part 3, there exists $p' \in P'$ such that

$$\text{Ad}(p')(\mathfrak{o}_{\mathcal{F}}) \cap \text{Ad}(g)^{-1}(\mathfrak{o}_{\mathcal{F}}) \neq \emptyset.$$

By Theorem 3.1 we conclude that $gp' \in P$, as required.

The first assertion follows now from Remark 9.5. Let $\tilde{\mathcal{F}} = (\tilde{\mathcal{F}}_i)_i \in \mathfrak{C}'$ be the unique I-filtration provided by this construction. As before, let $P' = \text{Stab}(\tilde{\mathcal{F}})$. Since P' contains T'_0 , we have

$$\tilde{\mathcal{F}}_2 \cap \mathcal{F}_2 = (\tilde{\mathcal{F}}_2 \cap V) + (\tilde{\mathcal{F}}_2 \cap \mathcal{F}_3).$$

Since $\mathfrak{o}_{\mathcal{F}} = X + \mathcal{F}_3$ by Theorem 3.1, we conclude that

$$\mathfrak{o}_{\tilde{\mathcal{F}}} \cap \mathfrak{o}_{\mathcal{F}} = \mathfrak{o} \cap \tilde{\mathcal{F}}_2 \cap \mathcal{F}_2 = (\mathfrak{o}_{\tilde{\mathcal{F}}} \cap X) + (\tilde{\mathcal{F}}_2 \cap \mathcal{F}_3).$$

In particular, $\mathfrak{o}_{\tilde{\mathcal{F}}} \cap X \neq \emptyset$, so that $U \in \Gamma_{\text{reg}}(V)$.

Clearly, $\text{Stab}(U)$ contains the standard parabolic subgroup $P' \cap G$ of G . Let $v_0 \in \mathfrak{o} \cap U$. It follows from Lemma 9.4 part 4 that there exist $\eta_1, \dots, \eta_k \in G'$ such that

$$\{g \in G \mid \text{Ad}(g)^{-1}v_0 \in U\} = \{g \in G \mid \text{Ad}(g)^{-1}v_0 \in \mathfrak{o}_{\tilde{\mathcal{F}}}\} = \bigcup_{1 \leq i \leq k} \eta_i P' \cap G.$$

Each non-empty intersection $\eta_i P' \cap G$ is a left coset of $P' \cap G$ in G . The second assertion of the proposition now follows from Proposition 6.4 part 4. \square

For instance, if $\mathfrak{o} = \text{Rich}(Q)$, where Q is a parabolic F -subgroup of G' , then Proposition 9.6 says that there is a unique $G(F)$ -conjugate Q' of Q such that $\mathfrak{n}_{Q'} \cap \mathfrak{o}_{\mathcal{F}} \neq \emptyset$ and $Q' \cap G$ is a standard parabolic subgroup of G . In general, $Q' \cap P$ is not a parabolic subgroup of G' . In other words, if Q is standard, it is not necessary that $Q' = Q$.

Consider the following examples.

- Rank two cases (see Example 6.8).

We may write the subregular orbit \mathfrak{o} as $\text{Rich}(Q)$ where $\Delta_0^Q = \{\beta\}$. This IFD gives rise to $U = V \cap \mathfrak{n}_Q$, the unique non-trivial special subspace.

- Example 6.14 (= Example 3.2 with $k = 3$ and $n_1 < n_2$).

We have $\mathfrak{o} = \text{Rich}(Q_1) = \text{Rich}(Q_2)$ where Q_1 and Q_2 are the standard parabolic subgroups of type (n_2, n_1, n_1) and (n_1, n_1, n_2) . These IFDs give rise to the spaces $V_i = V \cap \mathfrak{n}_{Q_i}$ in $\Gamma_{\text{spcl}}(V)$.

On the other hand, if $k = 5$ and $n_1 < n_2 < n_3$ and Q is the standard parabolic subgroup of type $(n_3, n_2, n_2, n_1, n_1)$, then $\mathfrak{o} = \text{Rich}(Q)$ but $V \cap \mathfrak{n}_Q \notin \Gamma_{\text{reg}}(V)$.

- Example 6.15 (= Example 3.8). In the following two examples, for standard parabolic subgroups $Q_1 \supset Q_2$ we will use the notation $\text{Rich}^{Q_1}(Q_2)$ for the Richardson orbit of the standard parabolic subgroup $L_1 \cap Q_2$ inside the standard Levi subgroup L_1 of Q_1 .

We have $\mathfrak{o} = \text{Rich}((**0*)) = \text{Rich}((0**0))$. This will give rise to the subspaces U_1 and U_2 , respectively. Using induction by stages, we may also write $\mathfrak{o} = \text{Ind}_Q \mathfrak{o}'$ where $Q = (0***)$ and $\mathfrak{o}' = \text{Rich}^Q((0**0))$.

The weighted Dynkin diagram of \mathfrak{o}' is $\overset{0}{\bullet} \Rightarrow \overset{1}{\bullet} - \overset{0}{\bullet}$. The standard \mathfrak{o}' -filtration satisfies $\Phi_{\mathcal{F}'_2} = \{(0120), (0121), (0122)\}$, and the associated I-filtration $\Phi_{\tilde{\mathcal{F}}_2} = \Phi_{G'}^+ \setminus \{(0100), (0010), (0001), (0110), (0011), (0111)\}$. Therefore, this IFD gives rise to the space U_3 .

Note that writing $\mathfrak{o} = \text{Ind}_{(***0)} \mathfrak{o}''$ where $\mathfrak{o}'' = \text{Rich}^{(***0)}((0**0))$ will give rise to U_2 again, since the weighted Dynkin diagram of \mathfrak{o}'' is $\overset{2}{\bullet} - \overset{0}{\bullet} \Rightarrow \overset{0}{\bullet}$ and therefore $\tilde{\mathcal{F}}_2 = \mathfrak{n}_{(0**0)}$ again.

4. Example 7.8 (= Example 3.9)

We have $\mathfrak{o} = \text{Rich}(Q'_1) = \text{Rich}(Q'_2)$, where $Q'_1 = \begin{pmatrix} 0 & * & * & 0 & * \\ * & & & & \end{pmatrix}$, $Q'_2 = \begin{pmatrix} * & * & * & 0 & * \\ & & & & 0 \end{pmatrix}$ (which are associate). These give rise to U_1 and U_2 , respectively.

They also give three presentations $\mathfrak{o} = \text{Ind}_{Q_i} \mathfrak{o}'_i$, $i = 1, 2, 3$ where

$$\begin{aligned} Q_1 &= \begin{pmatrix} * & * & * & 0 & * \\ & & & & * \end{pmatrix}, & \mathfrak{o}'_1 &= \text{Rich}^{Q_1}(Q'_1) = \text{Rich}^{Q_1}(Q'_2), \\ Q_2 &= \begin{pmatrix} * & * & * & * & * \\ & & & & 0 \end{pmatrix}, & \mathfrak{o}'_2 &= \text{Rich}^{Q_2}(Q'_2), \\ Q_3 &= \begin{pmatrix} 0 & * & * & * & * \\ & & & & * \end{pmatrix}, & \mathfrak{o}'_3 &= \text{Rich}^{Q_3}(Q'_1). \end{aligned}$$

Note that $L_1^{\text{der}} = \text{SL}_5 \times \text{SL}_2$ and \mathfrak{o}'_1 is the minimal orbit of the SL_5 -factor. The weighted Dynkin diagram is $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. We have $\Phi_{\mathcal{F}'_2} = \{ \begin{pmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & & & & \end{pmatrix} \}$. We get the space $U_1 \cap U_2$.

The weighted Dynkin diagram of \mathfrak{o}'_2 is $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \end{pmatrix}$ and

$$\Phi_{\mathcal{F}'_2} = \{ \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & & & & \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & & & & \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & & & & \end{pmatrix} \}.$$

We get the space U''_2 .

The weighted Dynkin diagram of \mathfrak{o}'_3 is $\begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & & & \end{pmatrix}$, and we get U_1 again.

Of course, we can apply the involution θ to all of the above, obtaining in addition U'_1 , U'_2 and $U'_1 \cap U'_2$.

Finally, we also have $\mathfrak{o} = \text{Ind}_Q \mathfrak{o}'$ where $Q = \begin{pmatrix} 0 & * & * & * & 0 \\ * & & & & \end{pmatrix}$ and \mathfrak{o}' is the minimal orbit of the Levi subgroup of Q (whose derived group is Spin_8). The weighted Dynkin diagram of \mathfrak{o}' is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$ and $\Phi_{\mathcal{F}'_2} = \{ \begin{pmatrix} 0 & 1 & 2 & 1 & 0 \\ 1 & & & & \end{pmatrix} \}$. Here, $\tilde{\mathcal{F}}_2 \cap \mathfrak{o}_{\mathcal{F}} = \emptyset$ for the standard I-filtration $\tilde{\mathcal{F}}$. However, it is easy to see that if w is the simple reflection with respect to $\begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & & & & \end{pmatrix}$ (the simple root defining G), then $\text{Stab}(w\tilde{\mathcal{F}}) \cap G$ is the standard Borel subgroup of G and $w\tilde{\mathcal{F}}_2 \cap \mathfrak{o}_{\mathcal{F}} \neq \emptyset$. The intersection $w\tilde{\mathcal{F}}_2 \cap V$ is U''_3 .

Note that U_3 is not obtained from this construction (either directly or by intersection).

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