DUAL INTEGRAL EQUATIONS

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1. Introduction. Erdélyi and Sneddon **(4)** have reduced the dual integral equations **(4**, (1.4)**)**

$$(1)\qquad \begin{cases} \displaystyle\int_0^\infty t^{-2\alpha}\{1 - E(t)\}\, J_\nu(\rho t)\, \Psi(t)\, dt = F(\rho) & \text{if } 0 < \rho < 1, \\ \displaystyle\int_0^\infty t^{-2\beta}\, J_\nu(\rho t)\, \Psi(t)\, dt = G(\rho) & \text{if } \rho > 1, \end{cases}$$

where Ψ is unknown, to a single Fredholm integral equation **(4**, (4.4)**)**, from the solution of which Ψ is explicitly obtainable. Their work extended and clarified an investigation by Cooke **(1)**, placing it in a context of standard integral transforms. Cooke's reduction was obtained after consideration of the Fredholm integral equation obtained by Love **(8)** in discussing Nicholson's problem of the electrostatic field of two equal circular coaxial conducting disks **(9)**.

I propose to show (§ 3) that Erdélyi and Sneddon's Fredholm equation is, after some recasting, a generalization of Love's; and (§§ 5–9) that it too possesses, under suitable conditions, a unique solution given by a Liouville–Neumann series. The work is restricted to cases in which E is a decaying exponential and G is zero, but the conditions on F, ν, α, and β are of a general character, more than wide enough to include applications that have been discussed in the literature.

Probably the most interesting and central part is § 6, where it is shown that the kernel, (25), of the recast Fredholm equation has a norm less than 1.

I am very grateful to Professor Erdélyi, who kindly showed me a copy of his paper **(4)** at the Summer Research Institute of the Australian Mathematical Society, and who later read my work and suggested a useful simplification in § 8.

2. Erdélyi and Sneddon's Fredholm equation **(4**, (4.4)**)** is

$$(2)\qquad h_1(u) - \int_0^1 K_0(u, v)\, h_1(v)\, dv = R_0(u) \qquad \text{if } 0 < u < 1,$$

where h_1 is the unknown function, K_0 and R_0 are known functions given by **(4**, (4.3)**)** and **(4**, (4.2)**)** respectively as follows:

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$$(3)\qquad K_0(u,v) = \left(\frac{v}{u}\right)^{\frac{1}{2}(\alpha+\beta)} \int_0^\infty J_{\nu+\beta-\alpha}(2\sqrt{uz})\, J_{\nu+\beta-\alpha}(2\sqrt{vz})\, e(z)\, dz$$

and

$$(4)\qquad R_0 = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha} f - S_{\frac{1}{2}\nu-\alpha,\beta+\alpha}\{eS_{\frac{1}{2}\nu+\beta,-\alpha-\beta}(K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, g)\},$$

where, according to **(4,** (1.6)**)** and **(4,** (1.2)**)**,

$$(5)\qquad \begin{cases} e(z) = E(2\sqrt{z}), & f(t) = 2^{2\alpha}t^{-\alpha}F(\sqrt{t}), \\ g(t) = 0 \text{ if } t < 1, & g(t) = 2^{2\beta}t^{-\beta}G(\sqrt{t}) \text{ if } t \geqslant 1; \end{cases}$$

while I K, and S are integral operators discussed in **(4,** (2.2)–(2.10)**)**. The original unknown function Ψ of the dual integral equations (1) is expressed in terms of h_1 by a complicated formula, of which we shall consider only the simplified version (18) appropriate to (6).

From here on we shall suppose that

$$(6)\qquad E(t) = \pm e^{-\kappa t} \qquad \text{and} \qquad G(t) = 0,$$

where κ is a positive constant.

Under these conditions we now simplify (3), first making the substitutions

$$(7)\qquad x = \sqrt{u}, \qquad y = \sqrt{v}, \qquad t = 2\sqrt{z}:$$

$$\begin{aligned}(8)\qquad \pm K_0(x^2, y^2) &= \frac{1}{2}\left(\frac{y}{x}\right)^{\alpha+\beta} \int_0^\infty J_{\nu+\beta-\alpha}(xt)\, J_{\nu+\beta-\alpha}(yt)\, e^{-\kappa t}\, t\, dt \\ &= -\frac{1}{2}\left(\frac{y}{x}\right)^{\alpha+\beta} \frac{\partial}{\partial\kappa} \int_0^\infty J_{\nu+\beta-\alpha}(xt)\, J_{\nu+\beta-\alpha}(yt)\, e^{-\kappa t}\, dt \\ &= -\frac{1}{2}\left(\frac{y}{x}\right)^{\alpha+\beta} \frac{\partial}{\partial\kappa} \frac{1}{\pi\sqrt{(xy)}}\, Q_{\nu+\beta-\alpha-\frac{1}{2}}\left(\frac{\kappa^2+x^2+y^2}{2xy}\right)\end{aligned}$$

provided $\nu+\beta-\alpha > -\frac{1}{2}$ for the validity of both steps taken **(10,** § 13.22**)**, Q being the Legendre function of the second kind. Thus

$$(9)\qquad \pm K_0(x^2, y^2) = -\frac{\kappa}{2\pi}\left(\frac{y}{x}\right)^{\alpha+\beta} \frac{1}{(xy)^{3/2}}\, Q'_{\nu+\beta-\alpha-\frac{1}{2}}\left(\frac{\kappa^2+x^2+y^2}{2xy}\right)$$

provided that

$$(10)\qquad \nu+\beta-\alpha+\tfrac{1}{2} > 0.$$

Instead of h_1 we take as unknown function

$$(11)\qquad \phi(x) = x^{\nu+2\beta+1}\, h_1(x^2);$$

the reason for choosing the first factor will appear at (18). Now (2) becomes

$$\phi(x) \mp \int_0^1 K(x, y)\, \phi(y)\, dy = R(x) \qquad \text{if } 0 < x < 1, \tag{12}$$

where

$$K(x, y) = \left(\frac{x}{y}\right)^{\nu+\beta-\alpha+\frac{1}{2}} \frac{-\kappa}{\pi x y} Q'_{\nu+\beta-\alpha-\frac{1}{2}}\left(\frac{\kappa^2 + x^2 + y^2}{2xy}\right)$$

and

$$R(x) = x^{\nu+2\beta+1} R_0(x^2).$$

Next we simplify (4) under conditions (6). Let n denote zero or a positive integer such that

$$\beta - \alpha + n \geqslant 0. \tag{13}$$

Omitting for the moment the case of equality, and using (**4**, (2.4) and (2.2)), we obtain

$$R_0(u) = I_{\frac{1}{2}\nu+\alpha,\beta-\alpha} f(u) = u^{-\frac{1}{2}\nu-\beta} \frac{d^n}{du^n} \int_0^u \frac{(u-t)^{\beta-\alpha+n-1}}{\Gamma(\beta-\alpha+n)} t^{\frac{1}{2}\nu+\alpha} f(t)\, dt. \tag{14}$$

It is convenient to define (compare (5)) the function

$$f_\nu(t) = (\sqrt{t})^\nu F(\sqrt{t}) = 2^{-2\alpha} t^{\frac{1}{2}\nu+\alpha} f(t); \tag{15}$$

then for the right side of (12) we have

$$R(u^{\frac{1}{2}}) = \begin{cases} 2^{2\alpha} u^{\frac{1}{2}} \dfrac{d^n}{du^n} \displaystyle\int_0^u \frac{(u-t)^{\beta-\alpha+n-1}}{\Gamma(\beta-\alpha+n)} f_\nu(t)\, dt & \text{if } \beta - \alpha + n > 0, \\ 2^{2\alpha} u^{\frac{1}{2}} f_\nu^{(n)}(u) & \text{if } \beta - \alpha + n = 0. \end{cases} \tag{16}$$

It is better not to replace u by x^2 in (16); the nth derivative expresses the $(\alpha - \beta)$th derivative of $f_\nu(u)$ with respect to u but not with respect to x.

The original unknown function Ψ in equations (1) is expressible in terms of ϕ, the unknown function in (12), as follows. By (**4**, (3.3)),

$$\Psi(x) = \tfrac{1}{2} x\, \psi(\tfrac{1}{4}x^2) \qquad \text{and} \qquad \psi = S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}(h_1 + h_2) = S_{\frac{1}{2}\nu+\beta,-\alpha-\beta}\, h_1 \tag{17}$$

since $h_2 = K_{\frac{1}{2}\nu-\alpha,\alpha-\beta}\, g_2$ vanishes on $(1, \infty)$ because $g = 0$ by (6). Then by (**4**, (2.5))

$$\psi(u) = u^{\frac{1}{2}(\alpha+\beta)} \int_0^1 v^{\frac{1}{2}(\alpha+\beta)} J_{\nu+\beta-\alpha}(2\sqrt{uv})\, h_1(v)\, dv,$$

so

$$\Psi(x) = 2^{-\alpha-\beta} x^{\alpha+\beta+1} \int_0^1 y^{-\nu-\beta+\alpha} J_{\nu+\beta-\alpha}(xy)\, \phi(y)\, dy, \tag{18}$$

a Hankel transform. This formula has meaning, whether for all x or almost all, if and only if ϕ is integrable; for the other factor in the integrand is bounded, and its reciprocal is also bounded for any sufficiently small fixed x.

3. Special cases. Cooke **(1)** gave special attention to equations (1) in the cases

$$\begin{cases} \alpha = \tfrac{1}{2}, & \beta = 0, \qquad \nu = 0 \text{ and } 1, \\ E(t) = \pm e^{-\kappa t}, & F(t) = t^{\nu}, \qquad G(t) = 0. \end{cases} \tag{19}$$

For $\nu = 1$ this is the hydrodynamic problem of flow caused by two equal coaxial circular disks rotating in a viscous fluid, solved by Cooke; while for $\nu = 0$ it is the electrostatic problem of a circular disk condenser, considered by Nicholson **(9)** and Love **(8)**.

The reduction of (2) made in § 2 above is applicable to the hydrodynamic case $\nu = 1$ since the requirement (10) for the steps from (8) to (9) is fulfilled. But this requirement is just *not* fulfilled in the electrostatic case $\nu = 0$; we proceed to show that Erdélyi and Sneddon's equation then reduces to the Fredholm equation obtained by Love **(8**, (2)**)**. This equation is

$$\phi(x) \mp \frac{1}{\pi} \int_{-1}^{1} \frac{\kappa}{\kappa^2 + (x-y)^2}\, \phi(y)\, dy = 1 \qquad \text{if } -1 \leqslant x \leqslant 1. \tag{20}$$

To make the identification complete we must insert a constant multiplier $2/\sqrt{\pi}$ on the left side of (11).

Cooke makes a similar reduction of his Fredholm equation. We are largely repeating his work in the next dozen lines, but we do so in order to make an opportunity for some remarks.

We suppose that (19) holds with $\nu = 0$, then. The reductions of § 2 apply except for (8)–(12), which we replace. Proceeding from (8),

$$\begin{cases} \pm K_0(x^2, y^2) = \dfrac{1}{2}\sqrt{\dfrac{y}{x}} \displaystyle\int_0^{\infty} J_{-\frac{1}{2}}(xt)\, J_{-\frac{1}{2}}(yt)\, e^{-\kappa t}\, t dt \\ \qquad = \dfrac{1}{\pi x} \displaystyle\int_0^{\infty} \cos xt \cos yt\, e^{-\kappa t}\, dt \\ \qquad = \dfrac{1}{2\pi x} \left(\dfrac{\kappa}{\kappa^2 + (x+y)^2} + \dfrac{\kappa}{\kappa^2 + (x-y)^2} \right). \end{cases} \tag{21}$$

Also (13) is fulfilled by $n = 1$, (15) gives $f_\nu(t) = 1$, and so (16) becomes

$$R(u^{\frac{1}{2}}) = 2u^{\frac{1}{2}} \frac{d}{du} \left\{ \frac{1}{\Gamma(\frac{1}{2})} \int_0^u (u-t)^{-\frac{1}{2}} dt \right\} = \frac{2}{\sqrt{\pi}}.$$

Instead of h_1 we use a constant multiple of the unknown function defined in (11), extended to negative x so as to be even:

$$\phi(x) = \tfrac{1}{2}\sqrt{\pi}\, |x|\, h_1(x^2). \tag{22}$$

Now (2) becomes

$$\phi(x) \mp \int_0^1 \frac{1}{\pi} \left(\frac{\kappa}{\kappa^2 + (x+y)^2} + \frac{\kappa}{\kappa^2 + (x-y)^2} \right) \phi(y)\, dy = 1, \tag{23}$$

to hold for $0 < x < 1$ and consequently for $0 > x > -1$; and this becomes (20) when we separate the integral into a sum of two integrals and replace y by $-y$ in one of them.

4. Remarks on (12) **and** (20). We have just seen that (20) is a special case of (2). It is also the limiting case of (12) as $\nu + \beta - \alpha + \frac{1}{2}$ tends to zero. This is seen if we compare the right-hand sides of (12) and (23), interpreting $Q'_{-1}(\xi)$ by putting $\gamma = 0$ in the recurrence relation

$$Q'_{\gamma-1}(\xi) = \xi Q'_{\gamma}(\xi) - \gamma Q_{\gamma}(\xi).$$

Indeed, this interpretation makes (9) correct for $\nu + \beta - \alpha + \frac{1}{2} = 0$.

In **(8)** it is shown that (20) has just one solution, that this is continuous, and that it is the sum of a uniformly convergent Neumann series. The rest of the present paper is devoted to establishing like properties of (12). In fact we demonstrate the applicability of the following theorem associated with the Neumann series solution.

If $K(x, y)$ is continuous in $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$, and satisfies

$$\sup_{0 \leqslant x \leqslant 1} \int_0^1 |K(x, y)|\, dy < 1,$$

and $R(x)$ is integrable, then the Fredholm equation, for unknown ϕ,

$$\phi(x) - \int_0^1 K(x, y)\, \phi(y)\, dy = R(x) \qquad \textit{for} \quad 0 \leqslant x \leqslant 1,$$

has just one integrable solution. This solution shares with R any "properties of discontinuity" it may have (such as being in L^p, or behaving like $x^{-\lambda}$ as $x \to 0$) because it differs from R by a continuous function.

Using this theorem we conclude, in § 9, with conditions under which the dual integral equations (1), when specialized in accordance with (6), possess a unique solution, obtainable by solving (12).

We do not discuss, in this paper, any form of solution of (12) except the Neumann series. For purposes of *numerical* solution, however, a more convenient form of (12) may be the following integral equation, whose kernel is *symmetric*:

$$\chi(x) \mp \int_0^1 \frac{-\kappa}{\pi xy}\, Q'_{\nu+\beta-\alpha-\frac{1}{2}}\left(\frac{\kappa^2 + x^2 + y^2}{2xy}\right) \chi(y)\, dy = x^{\alpha+\beta+\frac{1}{2}}\, R_0(x^2), \tag{24}$$

to hold for $0 < x < 1$. This is obtained from (12) by using, instead of ϕ defined by (11), the unknown function

$$\chi(x) = x^{-\nu-\beta+\alpha-\frac{1}{2}}\, \phi(x) = x^{\alpha+\beta+\frac{1}{2}}\, h_1(x^2);$$

and its kernel is that of (12) without the factor $(x/y)^{\nu+\beta-\alpha+\frac{1}{2}}$. Like (12), (24) has (20) as limiting case as $\nu + \beta - \alpha + \frac{1}{2}$ tends to zero; so the numerical methods of Fox and Goodwin **(5)** and of Elliott **(2)** may be successful in solving (24) as they have been with cases of (20).

5. The kernel of (12). In applying the theorem quoted in § 4 we shall use the following lemma.

If $\kappa > 0$ and $\gamma = \nu + \beta - \alpha + \frac{1}{2} > 0$, *then the kernel of* (12), *namely*

$$K(x, y) = \left(\frac{x}{y}\right)^{\gamma} \frac{-\kappa}{\pi xy} Q'_{\gamma-1}\left(\frac{\kappa^2 + x^2 + y^2}{2xy}\right), \tag{25}$$

is positive in the quadrant $x > 0$, $y > 0$, *and continuous in the closed quadrant* $x \geqslant 0$, $y \geqslant 0$ *if suitably defined on the frontier.*

That K is positive in the open quadrant follows from the expansion

$$-Q'_{\gamma-1}(\xi) = \sum_{n=0}^{\infty} \frac{\Gamma(\gamma + 2n + 1)\Gamma(\frac{1}{2})}{\Gamma(\gamma + n + \frac{1}{2})\cdot n!} \cdot \frac{2}{(2\xi)^{\gamma+2n+1}}, \tag{26}$$

which is applicable with, and because,

$$\xi = \frac{\kappa^2 + x^2 + y^2}{2xy} \geqslant \frac{\kappa^2}{2xy} + 1 > 1. \tag{27}$$

That K is continuous in the open quadrant is apparent. For the frontier, as $x \to +0$ or $y \to +0$, $\xi \to +\infty$; so the first term of (26) gives the asymptotic behaviour of $-Q'_{\gamma-1}(\xi)$, and we find that

$$K(x, y) \sim \frac{2\kappa}{\pi} \cdot \frac{\Gamma(\gamma + 1)\Gamma(\frac{1}{2})}{\Gamma(\gamma + \frac{1}{2})} \cdot \frac{x^{2\gamma}}{(\kappa^2 + x^2 + y^2)^{\gamma+1}}. \tag{28}$$

This indicates the desired continuity on the frontier if we define

$$\begin{cases} K(0, y) = 0 & \text{for } y \geqslant 0, \\ K(x, 0) = \dfrac{2\kappa}{\pi} \cdot \dfrac{\Gamma(\gamma + 1)\Gamma(\frac{1}{2})}{\Gamma(\gamma + \frac{1}{2})} \cdot \dfrac{x^{2\gamma}}{(\kappa^2 + x^2)^{\gamma+1}} & \text{for } x > 0. \end{cases} \tag{29}$$

The above lemma is equally true of $(y/x)^{\gamma} K(x, y)$, *the kernel of* (24); the boundary values being 0 on both axes, instead of the values given by (29).

6. A norm of the kernel (25). In establishing the existence and uniqueness of the solutions of (12) we shall use the following:

If $\kappa > 0$ *and* $\gamma > 0$, *then* (25) *satisfies*

$$\sup_{0 \leqslant x \leqslant 1} \int_0^1 |K(x, y)| \, dy < 1. \tag{30}$$

On account of § 5 we may omit the modulus signs and the value $x = 0$. Substituting $x = \kappa s$ and $y = \kappa t$, s and t are positive and

$$\int_0^1 K(x, y) dy = \int_0^{1/\kappa} \left(\frac{s}{t}\right)^{\gamma} \frac{-1}{\pi st} Q'_{\gamma-1}\left(\frac{1 + s^2 + t^2}{2st}\right) dt$$

$$< \int_0^{\infty} \left(\frac{s}{t}\right)^{\gamma} \frac{-1}{\pi st} Q'_{\gamma-1}\left(\frac{1 + s^2 + t^2}{2st}\right) dt. \tag{31}$$

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The convergence of this integral will be assured by subsequent work, but we can also infer it directly from (27) and (28). For as $t \to \infty$, $y \to \infty$, and so $\xi \to \infty$ by (27); thus (28) applies with κ replaced by 1, giving the integrand of (31) as $O((1+t^2)^{-\gamma-1})$.

Substituting $t = (1+s^2)^{\frac{1}{2}} \tan \frac{1}{2}\theta$ and also $s = \cot \sigma$, both σ and $\frac{1}{2}\theta$ being in $(0, \frac{1}{2}\pi)$, and then using (26), (31) becomes

$$\int_0^\pi \left(\frac{\sqrt{(1+s^2)}}{s} \tan \tfrac{1}{2}\theta\right)^{-\gamma} \frac{-1}{\pi s} Q'_{\gamma-1}\left(\frac{\sqrt{(1+s^2)}}{s \sin \theta}\right) \frac{d\theta}{\sin \theta}$$

$$(32) \quad = \frac{\tan \sigma}{\pi} \int_0^\pi -Q'_{\gamma-1}\left(\frac{\sec \sigma}{\sin \theta}\right) (\cos \sigma \cot \tfrac{1}{2}\theta)^\gamma \frac{d\theta}{\sin \theta}$$

$$= \tan \sigma \cos^\gamma \sigma \frac{\Gamma(\frac{1}{2})}{\pi} \int_0^\pi \sum_{n=0}^\infty \frac{\Gamma(\gamma+2n+1)}{\Gamma(\gamma+n+\frac{1}{2})\cdot n!} \left(\frac{\sin \theta}{2 \sec \sigma}\right)^{\gamma+2n+1} \frac{2 \cot^\gamma \frac{1}{2}\theta}{\sin \theta} d\theta$$

$$= \sin \sigma \cos^\gamma \sigma \frac{1}{\Gamma(\frac{1}{2})} \sum_{n=0}^\infty \frac{\Gamma(\gamma+2n+1)}{\Gamma(\gamma+n+\frac{1}{2})\cdot n!} \int_0^\pi \left(\frac{\sin \theta}{2 \sec \sigma}\right)^{\gamma+2n} \cot^\gamma \tfrac{1}{2}\theta \, d\theta$$

$$= \frac{\sin \sigma \cos^{2\gamma} \sigma}{\Gamma(\frac{1}{2})} \sum_{n=0}^\infty \frac{\Gamma(\gamma+2n+1)}{\Gamma(\gamma+n+\frac{1}{2})\cdot n!} \cos^{2n}\sigma \int_0^\pi (\tfrac{1}{2} \sin \theta)^{\gamma+2n} \cot^\gamma \tfrac{1}{2}\theta \, d\theta$$

$$= \frac{\sin \sigma \cos^{2\gamma} \sigma}{\Gamma(\frac{1}{2})} \sum_{n=0}^\infty \frac{\Gamma(\gamma+2n+1)}{\Gamma(\gamma+n+\frac{1}{2})\cdot n!} \cos^{2n}\sigma \int_0^\pi \sin^{2n} \tfrac{1}{2}\theta \cos^{2\gamma+2n} \tfrac{1}{2}\theta \, d\theta$$

$$= \frac{\sin \sigma \cos^{2\gamma} \sigma}{\Gamma(\frac{1}{2})} \sum_{n=0}^\infty \frac{\Gamma(\gamma+2n+1)}{\Gamma(\gamma+n+\frac{1}{2})\cdot n!} \cos^{2n}\sigma \frac{\Gamma(n+\frac{1}{2})\Gamma(\gamma+n+\frac{1}{2})}{\Gamma(\gamma+2n+1)}$$

$$= \sin \sigma \cos^{2\gamma} \sigma \sum_{n=0}^\infty \frac{1}{2}\cdot\frac{3}{4}\cdot \ldots \cdot\frac{2n-1}{2n} \cos^{2n}\sigma$$

$$= \sin \sigma \cos^{2\gamma}\sigma\, (1-\cos^2\sigma)^{-\frac{1}{2}}$$

$$(33) \quad = \cos^{2\gamma}\sigma.$$

The term-by-term integration is correct because all terms are positive, and the standard integral is quoted from (**3**, 1.5.1 (19)).

We now have (30) since, for $0 < x \leqslant 1$,

$$(34) \quad \int_0^1 K(x, y)\, dy < \cos^{2\gamma}\sigma = \left(\frac{s^2}{1+s^2}\right)^\gamma = \left(\frac{x^2}{\kappa^2+x^2}\right)^\gamma \leqslant \left(\frac{1}{\kappa^2+1}\right)^\gamma.$$

This proof actually holds at all stages except the last if $0 > \gamma > -\frac{1}{2}$.

7. Integrability of the right side of (12). The usefulness of (12) is limited by the necessity that the integral it involves should exist. Since the kernel $K(x, y)$ is continuous and positive for each $x > 0$, as shown in § 5, the integral exists if and only if ϕ is integrable. For this it is necessary and

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sufficient that $R(x)$ be integrable. We give sufficient conditions for this in terms of $f_\nu(t)$, the function defined by (15), and its derivatives.

For $R(x)$, specified by (16), *to be integrable on* $0 \leqslant x \leqslant 1$, *either of the following conditions is sufficient:*

(i) *that $f_\nu(t)$ be integrable on $0 \leqslant t \leqslant 1$ and $\beta - \alpha \geqslant 0$;*

(ii) *that $f_\nu^{(n-1)}(t)$ be absolutely continuous and $\beta - \alpha \geqslant -n$, for some positive integer n; and, if $n > 1$ and $\beta - \alpha > -n$, also $f_\nu^{(r)}(0) = 0$ for $r = 0, 1, \ldots, n-2$.*

First suppose that conditions (i) hold. Then $n = 0$ fulfils (13), and (16) becomes

$$2^{-2\alpha}u^{-\frac{1}{2}}R(u^{\frac{1}{2}}) = \begin{cases} \displaystyle\int_0^u \frac{(u-t)^{\beta-\alpha-1}}{\Gamma(\beta-\alpha)} f_\nu(t)\,dt & \text{if } \beta - \alpha > 0, \\ f_\nu(u) & \text{if } \beta - \alpha = 0. \end{cases} \tag{35}$$

This shows that the left side is integrable on $0 \leqslant u \leqslant 1$, in the lower case by data, and in the upper case by (**6**, p. 10), since the right side is a convolution of integrable functions. So, putting $u = x^2$, $R(x)$ is integrable on $0 \leqslant x \leqslant 1$.

Next suppose that conditions (ii) hold. Then n is a positive integer satisfying (13). Supposing $\beta - \alpha > -n$, and writing $\delta = \beta - \alpha + n > 0$, the integral in (16) is, on integrating by parts n times,

$$\int_0^u \frac{(u-t)^{\delta-1}}{\Gamma(\delta)} f_\nu(t)\,dt = \sum_{r=0}^{n-1} \frac{u^{\delta+r}}{\Gamma(\delta+r+1)} f_\nu^{(r)}(0) + \int_0^u \frac{(u-t)^{\delta+n-1}}{\Gamma(\delta+n)} f_\nu^{(n)}(t)\,dt.$$

Differentiating this n times, (16) becomes

$$2^{-2\alpha}u^{-\frac{1}{2}}R(u^{\frac{1}{2}}) = \frac{u^{\delta-1}}{\Gamma(\delta)} f_\nu^{(n-1)}(0) + \int_0^u \frac{(u-t)^{\delta-1}}{\Gamma(\delta)} f_\nu^{(n)}(t)\,dt \tag{36}$$

almost everywhere, using a property of fractional integrals and also the data regarding $f_\nu^{(r)}(0)$. Both terms on the right of (36) are integrable, the convolution as before; so again $R(x)$ is integrable as required.

If $\beta - \alpha > -n$ and $n = 1$, (36) is obtained without need for any of the $f_\nu^{(r)}(0)$ to vanish, and the rest of the argument is unaffected.

If $\beta - \alpha = -n$, instead of (36) we have, directly from (16),

$$2^{-2\alpha}u^{-\frac{1}{2}}R(u^{\frac{1}{2}}) = f_\nu^{(n)}(u), \tag{37}$$

which is integrable by data. So again $R(x)$ is integrable, without need for any conditions on $f_\nu^{(r)}(0)$.

8. Continuity of the right side of (12). We supplement the conditions of § 7 by conditions sufficient for continuity of R, since this is necessary and sufficient for continuity of the solution ϕ. Such conditions are, as (16) shows, no more than conditions for $f_\nu(u)$ to have a continuous derivative of order $\alpha - \beta$, except possibly at $u = 0$. They might well be capable of refinement using the theorems of (**7**), as well as in other ways.

For $R(x)$, specified by (16), to be continuous in $0 \leqslant x \leqslant 1$, any one of the following three sets of conditions is sufficient:

(i) *f_ν is integrable on $(0, 1)$,*
$f_\nu(t) = O(t^{\eta-1})$ as $t \to +0$, for some $\eta > 0$,
f_ν is in $L^p(\epsilon, 1)$ for each small positive ϵ, and some $p > 1$,
$\beta - \alpha > \max(\frac{1}{2} - \eta, 1/p)$;

(ii) *$f_\nu^{(n-1)}$ is absolutely continuous on $(0, 1)$, for some positive integer n,*
$f_\nu^{(r)}(0) = 0$ for $r = 0, 1, \ldots, n-2, n-1$,
$f_\nu^{(n)}(t) = O(t^{\eta-1})$ as $t \to +0$, for some $\eta > 0$,
$f_\nu^{(n)}$ is in $L^p(\epsilon, 1)$ for each small positive ϵ, and some $p > 1$,
$\beta - \alpha > -n + \max(\frac{1}{2} - \eta, 1/p)$;

(iii) *$t^{\frac{1}{2}} f_\nu^{(n)}(t)$ is continuous in $0 \leqslant t \leqslant 1$ for some non-negative integer n, and $\beta - \alpha = -n$.*

Under conditions (iii) the result is immediate; for, by (16) or (37), $R(u^{\frac{1}{2}})$ is a continuous function of u, and $u = x^2$ is a continuous function of x.

Suppose conditions (i) hold. Then $\beta - \alpha > 1/p > 0$, so § 7(i) applies; writing $\delta = \beta - \alpha > 0$, (35) gives

$$2^{-2\alpha}\Gamma(\delta)R(u^{\frac{1}{2}}) = u^{\frac{1}{2}} \int_0^u (u-t)^{\delta-1} f_\nu(t)\, dt. \tag{38}$$

This is continuous at $u = 0$; in fact it tends to zero as $u \to +0$, because for small u its modulus is majorized by

$$u^{\frac{1}{2}} \int_0^u (u-t)^{\delta-1} t^{\eta-1}\, dt = u^{\delta-\frac{1}{2}+\eta} \int_0^1 (1-s)^{\delta-1} s^{\eta-1}\, ds \qquad (t = us).$$

In proving continuity of $R(u^{\frac{1}{2}})$ at $u = U$, where $0 < U \leqslant 1$, we may omit the factor $u^{\frac{1}{2}}$ in (38). Our aim will be achieved by showing that

$$\int_{U/4m}^u (u-t)^{\delta-1} f_\nu(t)\, dt \qquad (m = 1, 2, 3, \ldots)$$

is a uniformly convergent sequence of continuous functions on $\frac{1}{2}U \leqslant u \leqslant 1$.

For the continuity, f_ν is in $L^p(U/4m, 1)$ and $t^{\delta-1}$ is in the conjugate Lebesgue class since $(\delta - 1)/(1 - 1/p) > -1$ by the datum regarding $\beta - \alpha$. So we have a convolution which, by (**6**, p. 11), is continuous on $(U/4m, 1)$ and hence on $(\frac{1}{2}U, 1)$.

For the uniform convergence on $(\frac{1}{2}U, 1)$, we have for sufficiently large m:

$$\left| \int_0^{U/4m} (u-t)^{\delta-1} f_\nu(t)\, dt \right| \leqslant \int_0^{U/4m} (u-t)^{\delta-1} M t^{\eta-1}\, dt$$

$$\leqslant \max\left(1, (\tfrac{1}{4}U)^{\delta-1}\right) \cdot \frac{M}{\eta} \left(\frac{U}{4m}\right)^{\eta},$$

whether $\delta - 1$ is positive or negative.

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Finally, suppose conditions (ii) hold. Then $\beta - \alpha + n > 1/p > 0$, so § 7(ii) applies; writing $\delta = \beta - \alpha + n > 0$, (36) gives

$$2^{-2\alpha}\Gamma(\delta)R(u^{\frac{1}{2}}) = u^{\frac{1}{2}} \int_0^u (u-t)^{\delta-1} f_\nu^{(n)}(t)\, dt,$$

and this is continuous in $0 \leqslant u \leqslant 1$ by the arguments used above.

9. Conclusion. The theorems of §§ 4–8 establish the conclusions:

For the integral equation (12) *to have just one solution it is sufficient that* $\kappa > 0$, $\beta - \alpha > -\nu - \frac{1}{2}$, *and that R be integrable on* (0, 1). *Conditions sufficient for the last requirement are given in* § 7.

For the solution to be continuous it is sufficient that R also be continuous on [0, 1]. *Conditions sufficient for this are given in* § 8.

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