Proceedings of the Royal Society of Edinburgh, **151**, 863–884, *2021*
DOI:10.1017/prm.2020.42

Shadowing for infinite dimensional dynamics and exponential trichotomies

Lucas Backes[1] **and Davor Dragičević**[2]
[1]Departamento de Matemática, Universidade Federal do Rio Grande do Sul, Av. Bento Gonçalves 9500, CEP 91509-900 Porto Alegre, RS, Brazil (lucas.backes@ufrgs.br)
[2]Department of Mathematics, University of Rijeka, Croatia (ddragicevic@math.uniri.hr)

(MS received 21 June 2019; accepted 26 May 2020)

Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded linear maps acting on an arbitrary Banach space X and admitting an exponential trichotomy and let $f_m : X \to X$ be a Lispchitz map for every $m \in \mathbb{Z}$. We prove that whenever the Lipschitz constants of f_m, $m \in \mathbb{Z}$, are uniformly small, the nonautonomous dynamics given by $x_{m+1} = A_m x_m + f_m(x_m)$, $m \in \mathbb{Z}$, has various types of shadowing. Moreover, if X is finite dimensional and each A_m is invertible we prove that a converse result is also true. Furthermore, we get similar results for one-sided and continuous time dynamics. As applications of our results, we study the Hyers–Ulam stability for certain difference equations and we obtain a very general version of the Grobman–Hartman's theorem for nonautonomous dynamics.

Keywords: Shadowing; Nonautonomus systems; Exponential trichotomies; Nonlinear perturbations; Hyers–Ulam stability

2010 *Mathematics Subject Classification:* Primary: 37C50, 34D09
Secondary: 34D10

1. Introduction

The foundations of the theory of chaotic dynamical systems date back to the work of Poincaré [**26**] and is now a well-developed area of research. An important feature of chaotic dynamical systems, already observed by Poincaré, is the sensitivity to initial conditions: any small change to the initial condition may lead to a large discrepancy in the output. This fact makes somehow complicated or even impossible the task of predicting the real trajectory of the system based on approximations. On the other hand, many chaotic systems, like uniformly hyperbolic dynamical systems [**2**, **6**], exhibit an amazing property stating that, even though a small error in the initial condition may lead eventually to a large effect, there exists a true orbit with a slightly different initial condition that stays near the approximate trajectory. This property is known as the *shadowing property*.

The objective of this paper is to develop a shadowing theory for *nonautonomous* systems acting on an arbitrary Banach space X. More precisely, starting with a linear dynamics

$$x_{m+1} = A_m x_m \quad m \in \mathbb{Z}, \tag{1.1}$$

where the sequence $(A_m)_{m\in\mathbb{Z}}$ admits an *exponential trichotomy*, we prove that a small *nonlinear* perturbation of (1.1) has various types of the shadowing properties. Moreover, if X has finite dimension and the linear maps A_m are invertible, we prove that $(A_m)_{m\in\mathbb{Z}}$ admits an exponential trichotomy whenever (1.1) satisfies a certain type of shadowing. Furthermore, we partially extend these results to one-sided dynamics and to continuous time dynamics. As applications of our results, we provide a characterization of Hyers–Ulam stability for certain difference equations and also exhibit a very general version of the Grobman–Hartman's theorem for nonautonomous dynamics.

1.1. Relations with previous results

Our proof is inspired by the analytical proofs of the shadowing lemma by Palmer [**19**] and Mayer and Sell [**17**]. These proofs have also inspired versions of the shadowing lemma for maps acting on Banach spaces (see [**9**, **14**]). While all these previous results deal with *autonomous* dynamical systems, we on the other hand focus in the nonautonomous setting.

Our work was initiated in [**3**]. In that paper we have also dealt with the shadowing problem in the nonautonomous realm but in much less generality. In fact, our theorem 3 generalizes the main result of [**3**] in three directions:

- we allow the sequence $(A_m)_{m\in\mathbb{Z}}$ to admit an exponential trichotomy instead of more restrictive assumption made in [**3**] that $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy;
- the nonlinear perturbations of (1.1) allowed here are much more general (for instance, they do not need to be differentiable or bounded as in [**3**]);
- in the present paper, we do not assume that $\sup_{m\in\mathbb{Z}}\|A_m\| < \infty$.

Moreover, in the present paper, we treat the cases of one-sided dynamics and continuous time dynamics that were not considered in the previous work allowing us, for instance, to characterize Hyers–Ulam stability for certain difference equations.

In order to finish this introduction, we would also like to stress that there are other shadowing results for nonautonomous dynamics in Banach spaces in the literature (see for instance the nice monographs [**21**, **24**]) but in all those results there are some differentiability and boundedness and/or compactness assumptions that are not present in our results (see for instance condition (2.4) in § 1.3.4 of [**24**]). In particular, our work represents a nontrivial extension of these results. Moreover, our unified approach gives us several types of shadowing at once (see remark 7).

2. Preliminaries

2.1. Banach sequence spaces

In this subsection, we recall some basic definitions and properties from the theory of Banach sequence spaces. The material is taken from [**11**, **28**] where the reader can also find more details.

Let $\mathcal{S}(\mathbb{Z})$ be the set of all sequences $\mathbf{s} = (s_n)_{n\in\mathbb{Z}}$ of real numbers. We say that a linear subspace $B \subset \mathcal{S}(\mathbb{Z})$ is a *normed sequence space* (over $\mathbb{Z}$) if there exists a norm $\|\cdot\|_B : B \to \mathbb{R}_0^+$ such that if $\mathbf{s}' = (s'_n)_{n\in\mathbb{Z}} \in B$ and $|s_n| \leqslant |s'_n|$ for $n \in \mathbb{Z}$, then $\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in B$ and $\|\mathbf{s}\|_B \leqslant \|\mathbf{s}'\|_B$. If in addition $(B, \|\cdot\|_B)$ is complete, we say that B is a *Banach sequence space*.

Let B be a Banach sequence space over $\mathbb{Z}$. We say that B is *admissible* if:

1. $\chi_{\{n\}} \in B$ and $\|\chi_{\{n\}}\|_B > 0$ for $n \in \mathbb{Z}$, where χ_A denotes the characteristic function of the set $A \subset \mathbb{Z}$;
2. for each $\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in B$ and $m \in \mathbb{Z}$, the sequence $\mathbf{s}^m = (s_n^m)_{n\in\mathbb{Z}}$ defined by $s_n^m = s_{n+m}$ belongs to B and $\|\mathbf{s}^m\|_B = \|\mathbf{s}\|_B$.

Note that it follows from the definition that for each admissible Banach space B over $\mathbb{Z}$, we have that $\|\chi_{\{n\}}\|_B = \|\chi_{\{0\}}\|_B$ for each $n \in \mathbb{Z}$. Throughout this paper, we will assume for the sake of simplicity that $\|\chi_{\{0\}}\|_B = 1$.

We recall some explicit examples of admissible Banach sequence spaces over $\mathbb{Z}$ (see [**11**, **28**]).

Example 1. The set $l^\infty = \{\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in \mathcal{S}(\mathbb{Z}) : \sup_{n\in\mathbb{Z}}|s_n| < \infty\}$ is an admissible Banach sequence space when equipped with the norm $\|\mathbf{s}\| = \sup_{n\in\mathbb{Z}}|s_n|$.

Example 2. The set $c_0 = \{\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in \mathcal{S}(\mathbb{Z}) : \lim_{|n|\to\infty}|s_n| = 0\}$ is an admissible Banach sequence space when equipped with the norm $\|\cdot\|$ from the previous example.

Example 3. For each $p \in [1, \infty)$, the set

$$l^p = \left\{\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in \mathcal{S}(\mathbb{Z}) : \sum_{n\in\mathbb{Z}}|s_n|^p < \infty\right\}$$

is an admissible Banach sequence space when equipped with the norm

$$\|\mathbf{s}\| = \left(\sum_{n\in\mathbb{Z}}|s_n|^p\right)^{1/p}.$$

Example 4 Orlicz sequence spaces. Let $\varphi\colon (0, +\infty) \to (0, +\infty]$ be a nondecreasing nonconstant left-continuous function. We set $\psi(t) = \int_0^t \varphi(s)\, \mathrm{d}s$ for $t \geqslant 0$. Moreover,

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

for each $\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in \mathcal{S}(\mathbb{Z})$, let $M_\varphi(\mathbf{s}) = \sum_{n\in\mathbb{Z}} \psi(|s_n|)$. Then

$$B = \{\mathbf{s} \in \mathcal{S}(\mathbb{Z}) : M_\varphi(c\mathbf{s}) < +\infty \text{ for some } c > 0\}$$

is an admissible Banach sequence space when equipped with the norm

$$\|\mathbf{s}\| = \inf\{c > 0 : M_\varphi(\mathbf{s}/c) \leqslant 1\}.$$

We will also need the following auxiliary result (see [**28**, lemma 2.3.]).

Proposition 1. *Let B be an admissible Banach sequence space. For $\mathbf{s} = (s_n)_{n\in\mathbb{Z}} \in B$ and $\lambda > 0$, we define sequences $\mathbf{s}^i = (s^i_n)_{n\in\mathbb{Z}}$, $i = 1, 2$ by*

$$s^1_n := \sum_{m\geqslant 0} \mathrm{e}^{-\lambda m} s_{n-m} \quad \textit{and} \quad s^2_n := \sum_{m\geqslant 1} \mathrm{e}^{-\lambda m} s_{n+m},$$

for $n \in \mathbb{Z}$. Then, $\mathbf{s}^1, \mathbf{s}^2 \in B$ and in addition,

$$\|\mathbf{s}^1\|_B \leqslant \frac{1}{1-\mathrm{e}^{-\lambda}}\|\mathbf{s}\|_B \quad \textit{and} \quad \|\mathbf{s}^2\|_B \leqslant \frac{e^{-\lambda}}{1-e^{-\lambda}}\|\mathbf{s}\|_B.$$

2.2. Banach spaces associated to Banach sequence spaces

Let us now introduce sequence spaces that will play important role in our arguments. Let X be an arbitrary Banach space and B any Banach sequence space over $\mathbb{Z}$ with norm $\|\cdot\|_B$. Set

$$X_B := \left\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : (\|x_n\|)_{n\in\mathbb{Z}} \in B\right\}.$$

Finally, for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in X_B$ we define

$$\|\mathbf{x}\|_B := \|(\|x_n\|)_{n\in\mathbb{Z}}\|_B. \tag{2.1}$$

Remark 1. We emphasize that in (2.1) we slightly abuse the notation since norms on B and X_B are denoted in the same way. However, this will cause no confusion since in the rest of the paper we will deal with spaces X_B.

Example 5. Let $B = l^\infty$ (see example 1). Then,

$$X_B = \left\{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \subset X : \sup_{n\in\mathbb{Z}}\|x_n\| < \infty\right\}.$$

The proof of the following result is straightforward (see [**11**, **28**]).

Proposition 2. *$(X_B, \|\cdot\|_B)$ is a Banach space.*

2.3. Exponential dichotomy and trichotomy

In this subsection, we recall the crucial concepts of exponential dichotomy and trichotomy. Let $I \in \{\mathbb{Z}, \mathbb{Z}_0^+, \mathbb{Z}_0^-\}$ and take an arbitrary Banach space $X = (X, \|\cdot\|)$. Finally, let $(A_m)_{m\in I}$ be a sequence of bounded linear operators on X. For $m, n \in I$ such that $m \geqslant n$, set

$$\mathcal{A}(m,n) = \begin{cases} A_{m-1}\cdots A_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n. \end{cases}$$

We say that the sequence $(A_m)_{m\in I}$ admits an *exponential dichotomy* (on I) if:

1. there exists a sequence $(P_m)_{m\in I}$ of projections on X such that
$$P_{m+1}A_m = A_m P_m \tag{2.2}$$
for each $m \in I$ such that $m+1 \in I$;
2. $A_m|_{KerP_m} : KerP_m \to KerP_{m+1}$ is an invertible operator for each $m \in I$ such that $m+1 \in I$;
3. there exist $C, \lambda > 0$ such that for $m, n \in I$, we have
$$\|\mathcal{A}(m,n)P_n\| \leqslant C\,\mathrm{e}^{-\lambda(m-n)} \quad \text{if } m \geqslant n \tag{2.3}$$
and
$$\|\mathcal{A}(m,n)(\mathrm{Id}-P_n)\| \leqslant C\,\mathrm{e}^{-\lambda(n-m)} \quad \text{if} m \leqslant n, \tag{2.4}$$
where
$$\mathcal{A}(m,n) := \big(\mathcal{A}(n,m)|_{KerP_m}\big)^{-1} : KerP_n \to KerP_m,$$
for $m \leqslant n$.

We also introduce the notion of an exponential trichotomy. We say that a sequence $(A_m)_{m\in\mathbb{Z}}$ of bounded linear operators on X admits an *exponential trichotomy* (on $\mathbb{Z}$) if there exist $C, \lambda > 0$ and projections P_m^i, $m \in \mathbb{Z}$, $i \in \{1,2,3\}$ such that:

1. for $m \in \mathbb{Z}$, $P_m^1 + P_m^2 + P_m^3 = \mathrm{Id}$;
2. for $m \in \mathbb{Z}$ and $i, j \in \{1,2,3\}$, $i \neq j$ we have that $P_m^i P_m^j = 0$;
3. $$P_{m+1}^i A_m = A_m P_m^i \quad \text{for } m \in \mathbb{Z} \text{ and } i \in \{1,2,3\};$$
4. $A_m|_{KerP_m^1} : KerP_m^1 \to KerP_{m+1}^1$ is invertible for each $m \in \mathbb{Z}$;
5. for $m \geqslant n$,
$$\|\mathcal{A}(m,n)P_n^1\| \leqslant C\,\mathrm{e}^{-\lambda(m-n)}; \tag{2.5}$$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

6. for $m \leqslant n$,

$$\|\mathcal{A}(m,n)P_n^2\| \leqslant C\,\mathrm{e}^{-\lambda(n-m)}, \tag{2.6}$$

where

$$\mathcal{A}(m,n) := \big(\mathcal{A}(n,m)|_{KerP_m^1}\big)^{-1} \colon KerP_n^1 \to KerP_m^1;$$

7.

$$\|\mathcal{A}(m,n)P_n^3\| \leqslant C\,\mathrm{e}^{-\lambda(m-n)} \quad \text{for } m \geqslant n, \tag{2.7}$$

and

$$\|\mathcal{A}(m,n)P_n^3\| \leqslant C\,\mathrm{e}^{-\lambda(n-m)} \quad \text{for } m \leqslant n. \tag{2.8}$$

Obviously the notion of an exponential dichotomy on $\mathbb{Z}$ is a special case of the notion of an exponential trichotomy and corresponds to the case when $P_m^3 = 0$ for $m \in \mathbb{Z}$.

Remark 2. We stress that the notion of an exponential dichotomy was essentially introduced by Perron [**23**] and plays a central role in the qualitative theory of nonautonomous systems (see [**10**, **15**]). For the case of infinite-dimensional and noninvertible dynamics with discrete time, the notion of an exponential dichotomy was first studied by Henry [**15**].

Although extremely useful, the notion of an exponential dichotomy is sometimes restrictive and it is of interest to look for weaker concepts of asymptotic behaviour. The notion of an exponential trichotomy studied in this paper was introduced by Elaydi and Hajek [**12**] (with further contributions by Papaschinopoulos [**22**] and Alonso, Hong and Obaya [**1**]) and represents one of many possible meaningful and useful extensions of the notion of an exponential dichotomy. For the study of a similar but different concept of trichotomy, we refer to [**20**, **29**, **30**] and references therein.

The following result is a modification of [**1**, proposition 2.3] or [**22**, proposition 1]. More precisely, in contrast to [**1**, **22**] we do not restrict to the case when $B = l^\infty$.

Theorem 1. *Assume that a sequence $(A_m)_{m\in\mathbb{Z}}$ admits an exponential trichotomy and let B be an arbitrary admissible Banach sequence space. Then, there exists a bounded operator $G \colon X_B \to X_B$ such that for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}, \mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in X_B$, the following assertions are equivalent*:

1. $G\mathbf{y} = \mathbf{x}$;
2. *for each $n \in \mathbb{Z}$,*

$$x_{n+1} - A_n x_n = y_{n+1}. \tag{2.9}$$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

Proof. For each $n, m \in \mathbb{Z}$, set

$$\mathcal{G}(n,m) := \begin{cases} \mathcal{A}(n,m)P_m^1 & \text{if } m \leqslant 0 \leqslant n \text{ or } m \leqslant n \leqslant 0; \\ -\mathcal{A}(n,m)(\mathrm{Id} - P_m^1) & \text{if } n < m \leqslant 0; \\ \mathcal{A}(n,m)(\mathrm{Id} - P_m^2) & \text{if } 0 < m \leqslant n; \\ -\mathcal{A}(n,m)P_m^2 & \text{if } 0 \leqslant n < m \text{ or } n \leqslant 0 < m. \end{cases}$$

Observe that it follows readily from (2.5)–(2.8) that

$$\|\mathcal{G}(n,m)\| \leqslant 2C\,\mathrm{e}^{-\lambda|m-n|} \quad \text{for } m,n \in \mathbb{Z}. \tag{2.10}$$

For $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ and $n \in \mathbb{Z}$, let

$$(G\mathbf{y})_n := \sum_{m\in\mathbb{Z}} \mathcal{G}(n,m+1)y_{m+1}.$$

Observe that (2.10) implies that

$$\begin{aligned} \|(G\mathbf{y})_n\| &\leqslant \sum_{m=-\infty}^{n-1} \|\mathcal{G}(n,m+1)y_{m+1}\| + \sum_{m=n}^{\infty} \|\mathcal{G}(n,m+1)y_{m+1}\| \\ &\leqslant 2C \sum_{m=-\infty}^{n-1} \mathrm{e}^{-\lambda(n-m-1)}\|y_{m+1}\| + 2C\sum_{m=n}^{\infty} \mathrm{e}^{-\lambda(m+1-n)}\|y_{m+1}\|, \end{aligned}$$

for $n \in \mathbb{Z}$. Hence, it follows from proposition 1 that $G\mathbf{y} \in Y_B$ and

$$\|G\mathbf{y}\|_B \leqslant 2C\frac{1+\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}\|\mathbf{y}\|_B. \tag{2.11}$$

Finally, in [**22**, proposition 1] it is proved that $\mathbf{x} = G\mathbf{y}$ satisfies (2.9). □

In the case of exponential dichotomy we can say more. More precisely, we have the following result established in [**28**, theorem 3.5].

THEOREM 2. *Assume that a sequence $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy and let B be an arbitrary admissible Banach sequence space. Then, for each $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \in X_B$ there exists a unique $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in X_B$ such that (2.9) holds. Furthermore, $\mathbf{x} = G\mathbf{y}$, where G is as in the statement of theorem 1.*

3. Main result

3.1. Setup

Let B be an admissible Banach sequence space, X a Banach space and $(A_m)_{m\in\mathbb{Z}}$ a sequence of bounded linear operators on X that admits an exponential trichotomy. Furthermore, let $f_n\colon X \to X$, $n \in \mathbb{Z}$ be a sequence of maps such that there exists $c > 0$ so that

$$\|f_n(x) - f_n(y)\| \leqslant c\|x-y\|, \tag{3.1}$$

for each $n \in \mathbb{Z}$ and $x, y \in X$.

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

We consider a nonautonomous and nonlinear dynamics defined by the equation

$$x_{n+1} = F_n(x_n), \quad n \in \mathbb{Z}, \tag{3.2}$$

where

$$F_n := A_n + f_n.$$

Let us now recall some notation introduced in [3]. Given $\delta > 0$, the sequence $(y_n)_{n\in\mathbb{Z}} \subset X$ is said to be an (δ, B)-*pseudotrajectory* for (3.2) if $(y_{n+1} - F_n(y_n))_{n\in\mathbb{Z}} \in X_B$ and

$$\|(y_{n+1} - F_n(y_n))_{n\in\mathbb{Z}}\|_B \leqslant \delta. \tag{3.3}$$

Remark 3. When $B = l^\infty$ (see example 1), condition (3.3) reduces to

$$\sup_{n\in\mathbb{Z}} \|y_{n+1} - F_n(y_n)\| \leqslant \delta.$$

The above requirement represents a usual definition of a pseudotrajectory in the context of smooth dynamics (see [21, 24]).

We say that (3.2) has a B-*shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for every (δ, B)-pseudotrajectory $(y_n)_{n\in\mathbb{Z}}$, there exists a sequence $(x_n)_{n\in\mathbb{Z}}$ satisfying (3.2) and such that $(x_n - y_n)_{n\in\mathbb{Z}} \in X_B$ together with

$$\|(x_n - y_n)_{n\in\mathbb{Z}}\|_B \leqslant \varepsilon. \tag{3.4}$$

Moreover, if there exists $L > 0$ such that δ can be chosen as $\delta = L\varepsilon$, we say that (3.2) has the B-*Lipschitz shadowing property*.

Let $G\colon X_B \to X_B$ be a linear operator given by theorem 1.

Theorem 3. *Assume that*

$$c\|G\| < 1. \tag{3.5}$$

Then, the system (3.2) *has the B-Lipschitz shadowing property.*

Proof. Take an arbitrary $\varepsilon > 0$ and let

$$K := \frac{\|G\|}{1 - c\|G\|}. \tag{3.6}$$

Finally, set $\delta := \varepsilon/K > 0$ and take an arbitrary (δ, B)-pseudotrajectory $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \subset X$ of (3.2). For $n \in \mathbb{Z}$, we define $g_n\colon X \to X$ by

$$g_n(v) = f_n(y_n + v) - f_n(y_n) + F_n(y_n) - y_{n+1}, \quad v \in X.$$

Furthermore, for $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in X_B$, let $S(\mathbf{x})$ be the sequence defined by

$$(S(\mathbf{x}))_n = g_{n-1}(x_{n-1}), \quad n \in \mathbb{Z}.$$

Observe that it follows from (3.1) and (3.3) that $S(\mathbf{x}) \in X_B$. Finally, set

$$T(\mathbf{x}) = GS(\mathbf{x}).$$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

We claim that T is a contraction on

$$D(\mathbf{0},\varepsilon) := \{\mathbf{x} \in X_B : \|\mathbf{x}\|_B \leqslant \varepsilon\}.$$

Indeed, let us choose $\mathbf{x}^1 = (x_n^1)_{n\in\mathbb{Z}}$ and $\mathbf{x}^2 = (x_n^2)_{n\in\mathbb{Z}}$ that belong to $D(\mathbf{0},\varepsilon)$. Observe that it follows from (3.1) that

$$\begin{aligned}\|g_n(x_n^1) - g_n(x_n^2)\| &= \|f_n(y_n + x_n^1) - f_n(y_n + x_n^2)\| \\ &\leqslant c\|x_n^1 - x_n^2\|,\end{aligned}$$

for $n \in \mathbb{Z}$. Hence,

$$\|S(\mathbf{x}^1) - S(\mathbf{x}^2)\|_B \leqslant c\|\mathbf{x}^1 - \mathbf{x}^2\|_B.$$

Consequently,

$$\|T(\mathbf{x}^1) - T(\mathbf{x}^2)\|_B \leqslant \|G\| \cdot \|S(\mathbf{x}^1) - S(\mathbf{x}^2)\|_B \leqslant c\|G\| \cdot \|\mathbf{x}^1 - \mathbf{x}^2\|_B.$$

Hence, (3.5) implies that T is a contraction on $D(\mathbf{0},\varepsilon)$.

We now show that T maps $D(\mathbf{0},\varepsilon)$ into itself. Take an arbitrary $\mathbf{x} \in D(\mathbf{0},\varepsilon)$. We have that

$$\begin{aligned}\|T(\mathbf{x})\|_B &\leqslant \|T(\mathbf{0})\|_B + \|T(\mathbf{x}) - T(\mathbf{0})\|_B \\ &\leqslant \|G\| \cdot \|S(\mathbf{0})\|_B + c\|G\| \cdot \|\mathbf{x}\|_B \\ &\leqslant \|G\| \cdot \|S(\mathbf{0})\|_B + \varepsilon c\|G\|.\end{aligned}$$

Since $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ is an (δ, B)-pseudotrajectory, we have that $\|S(\mathbf{0})\|_B \leqslant \delta = \varepsilon/K$ and consequently

$$\|T(\mathbf{x})\|_B \leqslant \varepsilon\left(\frac{\|G\|}{K} + c\|G\|\right) = \varepsilon,$$

where in the last equality we used (3.6).

We conclude that T has a fixed point $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in D(\mathbf{0},\varepsilon)$. Hence, $\mathbf{x} = GS(\mathbf{x})$. In a view of theorem 1, we deduce that

$$x_n = A_{n-1}x_{n-1} + g_{n-1}(x_{n-1}) \quad \text{for } n \in \mathbb{Z}.$$

Therefore, $\mathbf{x} + \mathbf{y} = (x_n + y_n)_{n\in\mathbb{Z}}$ is a solution of (3.2) and

$$\|\mathbf{x} + \mathbf{y} - \mathbf{y}\|_B = \|\mathbf{x}\|_B \leqslant \varepsilon.$$

This completes the proof of the theorem. □

REMARK 4. Observe that it follows from (2.11) that (3.5) holds for any c such that

$$0 < c < \frac{1 - \mathrm{e}^{-\lambda}}{2C(1 + \mathrm{e}^{-\lambda})},$$

where $C, \lambda > 0$ are the constants associated with the trichotomy of $(A_m)_{m\in\mathbb{Z}}$.

Remark 5. Let us now briefly describe the relationship between theorem 3 and the results dealing with the shadowing of structurally stable diffeomorphisms discussed in [**24**, chapter 2]. Let M be a compact Riemannian manifold and let $f\colon M \to M$ be a C^1-structurally stable diffeomorphism. We recall that this means that there exists $\delta > 0$ with the property that for each C^1-diffeomorphism g on M such that $d_{C^1}(f, g) < \delta$, there exists a homeomorphism $h\colon M \to M$ satisfying $h \circ f = g \circ h$. Here, $d_{C^1}(f, g)$ is given by

$$d_{C^1}(f,g) := \sup_{p\in M} d(f(p), g(p)) + \sup_{p\in M} \|Df(p) - Dg(p)\|,$$

where d is a distance on M induced by the Riemannian metric. Furthermore, we recall (see [**24**, theorem 2.2.4]) that a C^1-diffeomorphism on M is structurally stable if and only if the following two conditions hold:

- f is uniformly hyperbolic on the set $\Omega(f)$ of its nonwandering points and the set of periodic points of f is dense in $\Omega(f)$;
- for every $p, q \in \Omega(f)$, the stable manifold $W^s(p)$ and the unstable manifold $W^u(q)$ are transverse.

Take now a structurally stable C^1-diffeomorphism f on M. It is known (see [**24**, lemma 2.2.16]) that for each $p \in M$ there are subspaces $S(p)$ and $U(p)$ of the tangent space T_pM with the following properties:

- $T_pM = S(p) \oplus U(p)$;
- $Df(p)S(p) \subset S(f(p))$ and $Df^{-1}(p)U(p) \subset U(f^{-1}(p))$;
- there exist $C, \lambda > 0$ with the property that

$$\|Df^k(p)\Pi^s(p)v\| \leqslant C\,\mathrm{e}^{-\lambda k}\|v\| \quad \text{for } v \in T_pM \text{ and } k \geqslant 0 \tag{3.7}$$

  and

$$\|Df^{-k}(p)\Pi^u(p)v\| \leqslant C\,\mathrm{e}^{-\lambda k}\|v\| \quad \text{for } v \in T_pM \text{ and } k \geqslant 0, \tag{3.8}$$

  where $\Pi^s(p)$ is a projection onto $S(p)$ and $\Pi^u(p)$ is a projection onto $U(p)$.

In other words, f possesses a structure similar to that of an Anosov diffeomorphism with the only difference being that $S(p)$ and $U(p)$ are not invariant under the action of Df. Using these and some additional properties, one can show (see [**24**, theorem 2.2.7]) that f has the Lipschitz shadowing property. In addition, Pilyugin and Tikhomirov [**25**] proved that the converse statement also holds. More precisely, if f is a C^1-diffeomorphism with the Lipschitz shadowing property, then f is structurally stable.

Let us now comment on how these results relate to ours. We continue to consider a structurally stable C^1-diffeomorphism f on M. In addition, assume that the tangent bundle TM is isomorphic to $\mathbb{R}^k$, where $k = \dim M$. Take $p \in M$ and let

$$A_n = Df(f^n(p)), \quad n \in \mathbb{Z}.$$

We claim that the sequence $(A_n)_{n\in\mathbb{Z}}$ admits an exponential trichotomy. Indeed, take an arbitrary $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \subset \mathbb{R}^k$ such that $\|\mathbf{y}\|_\infty := \sup_{n\in\mathbb{Z}}\|y_n\| < \infty$. For $n \in \mathbb{Z}$, set

$$x_n = \sum_{k=0}^{\infty} Df^k(f^{n-k}(p))\Pi^s(f^{n-k}(p))y_{n-k} - \sum_{k=1}^{\infty} Df^{-k}(f^{n+k}(p))\Pi^u(f^{n+k}(p))y_{n+k}.$$

By (3.7) and (3.8), we have that

$$\|\mathbf{x}\|_\infty = \sup_{n\in\mathbb{Z}}\|x_n\| \leqslant C\frac{1+\mathrm{e}^{-\lambda}}{1-\mathrm{e}^{-\lambda}}\|\mathbf{y}\|_\infty,$$

where $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$. In addition, it is easy to verify that

$$x_{n+1} - A_n x_n = y_{n+1}, \quad \text{for } n \in \mathbb{Z}.$$

Hence, it follows from [**22**, proposition 1] that $(A_n)_{n\in\mathbb{Z}}$ admits an exponential trichotomy. By taking into account that we can regard f as the perturbation of Df, we observe that this setting is similar to the one studied in the present paper. However, there are important differences. Namely, in order to study the shadowing property for a dynamics acting on a noncompact phase space, we first start with a linear dynamics that admits an exponential trichotomy and then introduce a suitable class of nonlinear perturbations which exhibit the shadowing property. This class is completely determined by the constants in the notion of exponential trichotomy (associated to the linear part). Besides this important difference, in contrast to the above mentioned results from [**24**], in the present paper we consider the general case of nonautonomous and noninvertible dynamics that acts on an arbitrary Banach space.

Our results in particular apply to the case of linear dynamics

$$x_{n+1} = A_n x_n, \quad n \in \mathbb{Z}. \tag{3.9}$$

Corollary 1. *System (3.9) has the B-Lipschitz shadowing property, for any admissible Banach sequence space B.*

Proof. The desired conclusion follows by applying theorem 3 in the particular case when $f_n = 0$, $n \in \mathbb{Z}$. □

Now we obtain a partial converse to the previous corollary.

Proposition 3. *Assume that X is finite-dimensional and that $(A_m)_{m\in\mathbb{Z}}$ is a sequence of linear operators on X such that (3.9) has the l^∞-shadowing.*

Furthermore, suppose that A_m is invertible for each $m \in \mathbb{Z}$. Then, $(A_m)_{n\in\mathbb{Z}}$ admits an exponential trichotomy.

Proof. Choose $\delta > 0$ that corresponds to $\varepsilon = 1$ in the notion of l^∞-shadowing. We will prove that for every $\mathbf{z} = (z_n)_{n\in\mathbb{Z}} \in X_{l^\infty}$ there exists $\mathbf{w} = (w_n)_{n\in\mathbb{Z}} \in X_{l^\infty}$ such that

$$w_{n+1} - A_n w_n = z_{n+1}, \quad n \in \mathbb{Z}. \tag{3.10}$$

Choose a sequence $\mathbf{y} = (y_n)_{n\in\mathbb{Z}} \subset X$ (which is completely determined with y_0) such that

$$y_{n+1} = A_n y_n + \frac{\delta}{\|\mathbf{z}\|_{l^\infty}} z_{n+1}, \quad n \in \mathbb{Z}.$$

Then, $\mathbf{y}$ is an (δ, l^∞)-pseudotrajectory. Hence, there exists a solution $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ of (3.9) such that $\sup_{n\in\mathbb{Z}}\|x_n - y_n\| \leqslant 1$. Set

$$w_n = \frac{\|\mathbf{z}\|_{l^\infty}}{\delta}(y_n - x_n) \quad \text{for } n \in \mathbb{Z}.$$

Obviously, $\mathbf{w} = (w_n)_{n\in\mathbb{Z}} \in X_{l^\infty}$ and it is easy to verify that (3.10) holds. The conclusion of the proposition now follows directly from [**22**, proposition 1.]. □

Remark 6. We observe that proposition 3 is false (in general) for infinite-dimensional dynamics even in the autonomous case when all A_m coincide (see for example [**5**, remark 9(a)]).

It turns out that in the case when $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy, we can say more. We first recall (see theorem 2) that in that case, for each $\mathbf{y} \in X_B$, $\mathbf{x} := G\mathbf{y}$ is the unique sequence in X_B such that (2.9) holds.

Theorem 4. *Assume that the sequence $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy and that (3.5) holds. Then, (3.2) has an B-Lipschitz shadowing property. Furthermore, trajectory that shadows each pseudotrajectory is unique.*

Proof. We will use the same notation as in the proof of theorem 3. In a view of theorem 3, we only need to establish the uniqueness part. Let $\mathbf{y}$ be an (δ, B)-pseudotrajectory for (3.2) and assume that $\mathbf{z}^1 = (z_n^1)_{n\in\mathbb{Z}}, \mathbf{z}^2 = (z_n^2)_{n\in\mathbb{Z}}$ are trajectories of (3.2) such that

$$\|\mathbf{z}^i - \mathbf{y}\|_B \leqslant \varepsilon \quad \text{for} i = 1, 2.$$

Then,

$$z_n^i - y_n = A_{n-1}(z_{n-1}^i - y_{n-1}) + g_{n-1}(z_{n-1}^i - y_{n-1}),$$

for $n \in \mathbb{Z}$ and $i \in \{1, 2\}$. Consequently,

$$z_n^1 - z_n^2 = A_{n-1}(z_{n-1}^1 - z_{n-1}^2) + w_n, \tag{3.11}$$

where

$$w_n := g_{n-1}(z_{n-1}^1 - y_{n-1}) - g_{n-1}(z_{n-1}^2 - y_{n-1}), \quad n \in \mathbb{Z}.$$

Let $\mathbf{w} = (w_n)_{n\in\mathbb{Z}}$. It follows from (3.1) that

$$\|\mathbf{w}\|_B \leqslant c\|\mathbf{z}^1 - \mathbf{z}^2\|_B.$$

On the other hand, (3.11) implies that

$$\|\mathbf{z}^1 - \mathbf{z}^2\|_B \leqslant \|G\| \cdot \|\mathbf{w}\|_B.$$

By combining the last two inequalities, we conclude that

$$\|\mathbf{z}^1 - \mathbf{z}^2\|_B \leqslant c\|G\| \cdot \|\mathbf{z}^1 - \mathbf{z}^2\|_B.$$

By (3.5), we conclude that $\|\mathbf{z}^1 - \mathbf{z}^2\|_B = 0$ and thus $\mathbf{z}^1 = \mathbf{z}^2$. The proof of the theorem is completed. □

REMARK 7. Observe that our unified approach gives us all the usual types of shadowing simply by considering different types of admissible Banach sequence spaces B. For instance, for $B = l^\infty$ we get the usual notion of Lipschitz shadowing. For $B = l^p$ as in example 3 we get the notion of l^p-shadowing and so on.

4. One-sided dynamics

Let us now consider the case of one-sided dynamics on $\mathbb{Z}_0^+$. We stress that the dynamics on $\mathbb{Z}_0^-$ can be treated analogously.

For $\mathbf{x} = (x_n)_{n\geqslant 0} \subset X$, we define $\bar{\mathbf{x}} = (\bar{x}_n)_{n\in\mathbb{Z}} \subset X$ by

$$\bar{x}_n := \begin{cases} x_n & \text{if } n \geqslant 0; \\ 0 & \text{if } n < 0. \end{cases}$$

For an admissible Banach sequence space B, let

$$X_B^+ := \left\{ \mathbf{x} = (x_n)_{n\geqslant 0} \subset X : \bar{\mathbf{x}} \in X_B \right\}.$$

Then, X_B^+ is the Banach space with respect to the norm $\|\mathbf{x}\|_B^+ := \|\bar{\mathbf{x}}\|_B$.

Assume that $(A_m)_{m\geqslant 0}$ is a sequence of bounded linear operators on X and let $f_n \colon X \to X$, $n \geqslant 0$ be the sequence of maps such that (3.1) holds for $n \geqslant 0$ (and with some $c > 0$). We consider the associated nonlinear dynamics

$$x_{n+1} = F_n(x_n) \quad n \geqslant 0, \tag{4.1}$$

where $F_n := A_n + f_n$. Given $\delta > 0$, the sequence $(y_n)_{n\geqslant 0} \subset X$ is said to be an (δ, B)-*pseudotrajectory* for (4.1) if $(y_{n+1} - F_n(y_n))_{n\geqslant 0} \in X_B^+$ and

$$\|(y_{n+1} - F_n(y_n))_{n\geqslant 0}\|_B^+ \leqslant \delta. \tag{4.2}$$

We say that (4.1) has an B-*shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ so that for every (δ, B)-pseudotrajectory $(y_n)_{n\geqslant 0}$, there exists a sequence $(x_n)_{n\geqslant 0}$ satisfying (4.1) and such that $(x_n - y_n)_{n\geqslant 0} \in X_B^+$ together with

$$\|(x_n - y_n)_{n\geqslant 0}\|_B^+ \leqslant \varepsilon. \tag{4.3}$$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

THEOREM 5. *Assume that $(A_m)_{m\geqslant 0}$ admits an exponential dichotomy and let B be an admissible Banach sequence space. Then, if $c>0$ is sufficiently small (4.1) has a B-shadowing property.*

Proof. We extend the sequence $(A_m)_{m\geqslant 0}$ to a sequence over $\mathbb{Z}$ in the following manner: choose an invertible, hyperbolic linear operator A on X such that $KerP_0$ coincides with the unstable subspace of A and let $A_m := A$ for $m<0$. Then (see [**27**, lemma 1.]), $(A_m)_{m\in\mathbb{Z}}$ admits an exponential dichotomy. Consider G as in theorems 1 and 2 and let $c>0$ be such that $c\|G\|<1$. Finally, set $f_n=0$ for $n<0$ and consider the nonlinear system

$$x_{n+1}=F_n(x_n) \quad n\in\mathbb{Z}, \tag{4.4}$$

where $F_n=A_n+f_n=A$ for $n<0$. Take an arbitrary $\varepsilon>0$, define K as in (3.6) and let $\delta:=\varepsilon/K>0$. Furthermore, choose an (δ,B)-*pseudotrajectory* $\mathbf{y}=(y_n)_{n\geqslant 0}$ for (4.1). We consider $\hat{\mathbf{y}}=(\hat{y}_n)_{n\in\mathbb{Z}}\subset X$ defined by

$$\hat{y}_n := \begin{cases} y_n & \text{if} n\geqslant 0;\\ A^n y_0 & \text{if} n<0.\end{cases}$$

Clearly, $\hat{\mathbf{y}}$ is an (δ,B)-*pseudotrajectory* for (4.4). Hence, it follows from the proof of theorem 3 that there exists a sequence $\mathbf{x}=(x_n)_{n\in\mathbb{Z}}\subset X$ that solves (4.4) and such that $\|\mathbf{x}-\hat{\mathbf{y}}\|_B\leqslant\varepsilon$. Then, $\mathbf{z}=(x_n)_{n\geqslant 0}$ is a solution of (4.1) such that $\|\mathbf{z}-\mathbf{y}\|_B^+\leqslant\varepsilon$ and the proof is complete. □

As in the case of two-sided dynamics, our results in particular apply to the case of linear dynamics

$$x_{n+1}=A_n x_n, \quad n\geqslant 0. \tag{4.5}$$

COROLLARY 2. *Assume that $(A_m)_{m\geqslant 0}$ admits an exponential dichotomy and let B be an admissible Banach sequence space. Then, (4.5) has an B-Lipschitz shadowing property.*

Proof. The desired conclusion follows directly from theorem 5 applied to the case when $f_n=0$ for $n\geqslant 0$. □

We also have the following partial converse to corollary 2.

PROPOSITION 4. *Assume that X is finite-dimensional and that (4.5) has an l^∞-Lipschitz shadowing property. Then, $(A_m)_{m\geqslant 0}$ admits an exponential dichotomy.*

Proof. By proceeding as in the proof of proposition 3, one can show that for every $\mathbf{z}=(z_n)_{n\geqslant 0}\in X_B^+$ such that $z_0=0$, there exists $\mathbf{w}=(w_n)_{n\geqslant 0}\in X_B^+$ satisfying

$$w_{n+1}-A_n w_n=z_{n+1} \quad \text{for } n\geqslant 0.$$

Hence, [**16**, theorem 3.2] implies that $(A_m)_{m\geqslant 0}$ admits an exponential dichotomy. □

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

5. A case of continuous time

In this section, we will apply our previous results in order to develop shadowing theory for continuous time dynamics.

For the sake of simplicity, in this section, we will study only the classical l^∞-shadowing. We consider a nonlinear differential equation

$$x' = A(t)x + f(t, x), \tag{5.1}$$

where A is a continuous map from $\mathbb{R}$ to the space of all bounded linear operators on X satisfying

$$N := \sup_{t\in\mathbb{R}} \|A(t)\| < \infty,$$

and $f\colon \mathbb{R} \times X \to X$ is a continuous map. We assume that $f(\cdot, 0) = 0$ and that there exists $c > 0$ such that

$$\|f(t, x) - f(t, y)\| \leqslant c\|x - y\| \quad \text{for } t \in \mathbb{R} \textit{ and } x, y \in X. \tag{5.2}$$

We consider the associated linear equation

$$x' = A(t)x. \tag{5.3}$$

Let $T(t, s)$ be the (linear) evolution family associated to (5.3). We will suppose that it admits an exponential trichotomy, i.e. that there exists a family of projections $P^i(s)$, $s \in \mathbb{R}$, $i \in \{1, 2, 3\}$ on X and $C, \lambda > 0$ such that:

1. for $s \in \mathbb{R}$, $P^1(s) + P^2(s) + P^3(s) = \mathrm{Id}$;
2. for $s \in \mathbb{R}$, $i, j \in \{1, 2, 3\}$, $i \neq j$ we have that $P^i(s)P^j(s) = 0$;
3. for $t, s \in \mathbb{R}$ and $i \in \{1, 2, 3\}$,
$$T(t, s)P^i(s) = P^i(t)T(t, s); \tag{5.4}$$
4. for $t \geqslant s$,
$$\|T(t, s)P^1(s)\| \leqslant C\,\mathrm{e}^{-\lambda(t-s)}; \tag{5.5}$$
5. for $t \leqslant s$,
$$\|T(t, s)P^2(s)\| \leqslant C\,\mathrm{e}^{-\lambda(s-t)}; \tag{5.6}$$
6.
$$\|T(t, s)P^3(s)\| \leqslant C\,\mathrm{e}^{-\lambda(t-s)} \quad \text{for } t \geqslant s, \tag{5.7}$$
and
$$\|T(t, s)P^3(s)\| \leqslant C\,\mathrm{e}^{-\lambda(s-t)} \quad \text{for } t \leqslant s. \tag{5.8}$$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

Since we assumed that $N < \infty$, it is easy to show using Gronwall's lemma that there exist $D, b > 0$ such that

$$\|T(t,s)\| \leqslant D\,\mathrm{e}^{b(t-s)} \quad t \geqslant s. \tag{5.9}$$

Recall that the nonlinear evolution family associated with (5.1) is given by

$$U(t,s)x = T(t,s)x + \int_s^t T(t,\tau)f(\tau, U(\tau,s)x)\,\mathrm{d}\tau, \tag{5.10}$$

for $x \in X$ and $t, s \in \mathbb{R}$. By applying Gronwall's lemma, it is easy to prove that (5.2) and (5.9) imply that there exist $K, a > 0$ such that

$$\|U(t,s)\| \leqslant K\,\mathrm{e}^{a(t-s)} \quad \text{for } t \geqslant s. \tag{5.11}$$

We now introduce the concept of shadowing in this setting. Let $\delta > 0$. A differentiable function $y \colon \mathbb{R} \to X$ is said to be a *δ-pseudotrajectory* for (5.1) if

$$\sup_{t\in\mathbb{R}} \|y'(t) - A(t)y(t) - f(t, y(t))\| \leqslant \delta.$$

We say that (5.1) has the *shadowing property* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudotrajectory $y \colon \mathbb{R} \to X$, there exists a solution $x \colon \mathbb{R} \to X$ of (5.1) satisfying

$$\sup_{t\in\mathbb{R}} \|x(t) - y(t)\| \leqslant \varepsilon.$$

Moreover, if there exists $L > 0$ such that δ can be chosen as $\delta = L\varepsilon$, we say that (5.1) has the *Lipschitz shadowing property*. We now formulate and prove the main result of this section.

THEOREM 6. *If $c > 0$ is sufficiently small, then* (5.1) *has the Lipschitz shadowing property.*

Proof. Set

$$A_n := T(n+1, n), \quad \text{for } n \in \mathbb{Z}.$$

It follows readily from (5.4)– (5.8) that $(A_m)_{m\in\mathbb{Z}}$ admits an exponential trichotomy with projections $P_n^i = P^i(n)$, $n \in \mathbb{Z}$, $i \in \{1,2,3\}$. Furthermore, set

$$f_n(x) = \int_n^{n+1} T(n+1,\tau) f(\tau, U(\tau,n)x)\,\mathrm{d}\tau, \quad \text{for } x \in X \text{ and } n \in \mathbb{Z}.$$

It follows from (5.2), (5.9) and (5.11) that

$$\begin{aligned}\|f_n(x) - f_n(y)\| &\leqslant \int_n^{n+1} \|T(n+1,\tau)\| \cdot \|f(\tau, U(\tau,n)x) - f(\tau, U(\tau,n)y)\|\,\mathrm{d}\tau \\ &\leqslant c\int_n^{n+1} \|T(n+1,\tau)\| \cdot \|U(\tau,n)(x-y)\|\,\mathrm{d}\tau \\ &\leqslant cDK\,\mathrm{e}^{a+b}\|x-y\|,\end{aligned}$$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

and thus there exists $c' > 0$ such that

$$\|f_n(x) - f_n(y)\| \leqslant cc'\|x - y\| \quad \text{for } n \in \mathbb{Z} \text{ and } x, y \in X. \tag{5.12}$$

Set $F_n := A_n + f_n$ and consider the system

$$x_{n+1} = F_n(x_n), \quad n \in \mathbb{Z}. \tag{5.13}$$

Observe that it follows from (5.10) that $F_n = U(n+1, n)$ for each $n \in \mathbb{Z}$.

Since $(A_m)_{m\in\mathbb{Z}}$ admits an exponential trichotomy, it follows from theorem 3 and (5.12) that for sufficiently small c, (5.13) has the l^∞-Lipschitz shadowing. Let $L > 0$ be the constant as in the definition of Lipschitz shadowing related to (5.13). Take $\varepsilon > 0$ and let $\delta := L'\varepsilon$, where

$$L' := \frac{1}{\left(1 + \frac{\mathrm{e}^{N+c}}{L}\right)\mathrm{e}^{N+c}}.$$

Furthermore, let y be the δ-pseudotrajectory for (5.1). Then,

$$y' = A(t)y + f(t, y) + h,$$

for some $h\colon \mathbb{R} \to X$ such that $\|h(t)\| \leqslant \delta$ for $t \in \mathbb{R}$. Take $n \in \mathbb{Z}$ and let z be the solution of (5.1) such that $z(n) = y(n)$. Then, for all $t \in [n, n+1]$ we have [see (5.2)] that

$$\begin{aligned}\|y(t) - z(t)\| &\leqslant \left\| \int_n^t (A(s)(y(s) - z(s)) + f(s, y(s)) - f(s, z(s)) + h(s))\,\mathrm{d}s \right\| \\ &\leqslant \delta + (N + c)\int_n^t \|y(s) - z(s)\|\,\mathrm{d}s.\end{aligned}$$

Hence, it follows from Gronwall's lemma that

$$\|y(t) - z(t)\| \leqslant \delta\,\mathrm{e}^{N+c}, \quad \text{for } t \in [n, n+1].$$

In particular,

$$\|y(n+1) - F_n(y(n))\| = \|y(n+1) - z(n+1)\| \leqslant \delta\,\mathrm{e}^{N+c},$$

for every $n \in \mathbb{Z}$. Hence, the sequence $(y_n)_{n\in\mathbb{Z}} \subset X$ defined by $y_n := y(n)$ is an $(\delta\,\mathrm{e}^{N+c},\ l^\infty)$-pseudotrajectory for (5.13). Hence, there exists a solution $(x_n)_{n\in\mathbb{Z}}$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

of (5.13) such that $\sup_{n\in\mathbb{Z}}\|x_n - y_n\| \leqslant (\delta\, \mathrm{e}^{N+c})/L$. We define $x\colon \mathbb{R} \to X$ by

$$x(t) = U(t,n)x_n \quad \mathrm{n} \in \mathbb{Z},\ t \in [n, n+1).$$

Then, x is a solution of (5.1). Finally, observe that for $n \in \mathbb{Z}$ and $t \in [n, n+1)$ we have that

$$\begin{aligned}\|x(t) - y(t)\| &\leqslant \|x_n - y_n\| \\ &\quad + \left\| \int_n^t (A(s)(x(s) - y(s)) + f(s, x(s)) - f(s, y(s)) - h(s))\,\mathrm{d}s \right\| \\ &\leqslant \delta\left(1 + \frac{\mathrm{e}^{N+c}}{L}\right) + (N + c)\int_t^n \|x(s) - y(s)\|\,\mathrm{d}s.\end{aligned}$$

Hence, Gronwall's lemma implies that

$$\sup_{t\in\mathbb{R}}\|x(t) - y(t)\| \leqslant \delta\left(1 + \frac{\mathrm{e}^{N+c}}{L}\right)\mathrm{e}^{N+c} = \varepsilon. \qquad \square$$

One can now easily formulate and prove continuous time versions of all other results we established in § 3. We refrain from doing this since it represents a very simple exercise and requires only simple modification of the arguments we developed.

6. Applications

6.1. Hyers–Ulam stability

It turns out that our results are closely related to the so-called Hyers–Ulam stability and in fact, can be used to obtain new results related to this concept. We will not attempt to survey various results in the literature regarding the Hyers–Ulam stability but will rather focus on the recent papers [**4**, **7**, **8**] and the results obtained there.

It seems that there are various flavours of the Hyers–Ulam stability studied in the literature. However, the concept studied in [**4**, **7**, **8**] precisely corresponds to our notion of shadowing. In a series of remarks, we will now show how our results extend and unify those established in the papers we mentioned.

Remark 8. In [**4**], the authors prove that if $X = \mathbb{C}^m$ and if A is an hyperbolic operator on X (i.e its spectrum does not intersect the unit circle), then (4.5) with $A_n = A$, $n \geqslant 0$ has the l^∞-Lipschitz shadowing property. This result is a particular case of our corollary 2 since the constant sequence $(A_n)_{n\geqslant 0}$ admits an exponential dichotomy.

Remark 9. In [**8**], the authors study the system (4.5) when $(A_n)_{n\geqslant 0}$ is a q-periodic sequence of linear operators on $X = \mathbb{C}^m$. They prove that (4.5) has the l^∞-shadowing property if $\mathcal{A}(q, 0) = A_{q-1}\cdots A_0$ is hyperbolic. Since the hyperbolicity of $\mathcal{A}(q, 0)$ implies that $(A_n)_{n\geqslant 0}$ admits an exponential dichotomy, this result is also a particular case of our corollary 2.

Remark 10. Consider two sequences $(a_n)_{n\geqslant 0}$ and $(b_n)_{n\geqslant 0}$ in $\mathbb{C}$ and the associated linear recurrence

$$x_{n+2} = a_n x_{n+1} + b_n x_n, \quad n \geqslant 0. \tag{6.1}$$

Set

$$A_n = \begin{pmatrix} 0 & 1 \\ b_n & a_n \end{pmatrix} \quad \text{for } n \geqslant 0,$$

and consider the associated linear system in $\mathbb{C}^2$ given by

$$y_{n+1} = A_n y_n, \quad n \geqslant 0. \tag{6.2}$$

Observe that if $(x_n)_{n\geqslant 0} \subset \mathbb{C}$ is a solution of (6.1) then $(y_n)_{n\geqslant 0}$ given by $y_n = \begin{pmatrix} x_n \\ x_{n+1} \end{pmatrix}$ is a solution of (6.2). Conversely, if $(y_n)_{n\geqslant 0}$, $y_n = \begin{pmatrix} y_n^1 \\ y_n^2 \end{pmatrix}$ is a solution of (6.2), then $(x_n)_{n\geqslant 0}$ given by $x_n = y_n^1$ is a solution of (6.1) and $y_n^2 = y_{n+1}^1$ for each $n \geqslant 0$.

Assume that the sequence $(A_m)_{m\geqslant 0}$ admits an exponential dichotomy and let us consider the norm $\|\cdot\|$ on $\mathbb{C}^2$ given by

$$\|(z_1, z_2)\| := \max\{|z_1|, |z_2|\}.$$

Take $\varepsilon > 0$ and let us consider $\delta > 0$ that corresponds to l^∞-Lipschitz shadowing of (6.2). We now take a sequence $(w_n)_{n\geqslant 0} \subset \mathbb{C}$ such that

$$\sup_{n\geqslant 0} |w_{n+2} - a_n w_{n+1} - b_n w_n| \leqslant \delta.$$

Set $z_n = \begin{pmatrix} w_n \\ w_{n+1} \end{pmatrix}$, $n \geqslant 0$. Then, $(z_n)_{n\geqslant 0}$ is an (δ, l^∞)-pseudotrajectory. Hence, corollary 2 implies that there exists $(y_n)_{n\geqslant 0}$ solution of (6.2) such that $\sup_{n\geqslant 0}\|y_n - z_n\| \leqslant \varepsilon$. Hence, $(x_n)_{n\geqslant 0}$ given by $x_n = y_n^1$ is a solution of (6.1) and $\sup_{n\geqslant 0}|x_n - w_n| \leqslant \sup_{n\geqslant 0}\|x_n - z_n\| \leqslant \varepsilon$. We conclude that (6.1) also has an l^∞-Lipschitz shadowing property. Consequently, since we have not assumed that the sequences $(a_n)_{n\geqslant 0}, (b_n)_{n\geqslant 0}$ are periodic, this gives a partial generalization of [**7**, theorem 2.3].

We hope that the results and the ideas developed in the present paper could be of use to establish additional results related to Hyers–Ulam stability.

6.2. Grobman–Hartman's theorem

As an other application of our results, we obtain a new proof of the nonautonomous version of the classical Grobman–Hartman theorem [**13**]. More precisely, we revisit [**3**, § 4.1] to apply our new results in order to show that our ideas can be used to obtain a less restrictive version of [**3**, theorem 4.1].

Let $(A_m)_{m\in\mathbb{Z}}$ be a sequence of bounded linear operators on X as in § 3.1. Furthermore, suppose that each A_m is invertible and that $\sup_{m\in\mathbb{Z}}\|A_m^{-1}\| < \infty$. Associated to these parameters by theorem 4 (applied to $B = l^\infty$ and $f_n \equiv 0$), consider $\varepsilon > 0$

https://doi.org/10.1017/prm.2020.42 Published online by Cambridge University Press

sufficiently small and $\delta = L\varepsilon > 0$. Let $(g_n)_{n\in\mathbb{Z}}$ be a sequence of maps $g_n\colon X \to X$ satisfying (3.1) with c sufficiently small and such that

$$\|g_n\|_{\sup} \leqslant \delta \quad \text{for each } n \in \mathbb{Z}.$$

We consider a difference equation

$$y_{n+1} = G_n(y_n) \quad n \in \mathbb{Z}, \tag{6.3}$$

where $G_n := A_n + g_n$. By decreasing c (if necessary), we have that G_n is a homeomorphism for each $n \in \mathbb{Z}$ (see [**3**]). Then, we have the following result.

Theorem 7. *There exists a unique sequence $h_m\colon X \to X$, $m \in \mathbb{Z}$, of homeomorphisms such that for each $m \in \mathbb{Z}$,*

$$h_{m+1} \circ G_m = A_m \circ h_m \tag{6.4}$$

and

$$\|h_m - \mathrm{Id}\|_{\sup} = \sup_{x\in X}\|h_m(x) - x\| \leqslant \varepsilon. \tag{6.5}$$

The family of homeomorphism $h_m\colon X \to X, m \in \mathbb{Z}$, satisfying (6.4) and (6.5) is constructed 'explicitly' using the l^∞-Lipschitz shadowing property. In fact, fix $m \in \mathbb{Z}$. Given $y \in X$, let us consider the sequence $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ given by $y_n = \mathcal{G}(n, m)y$ for $n \in \mathbb{Z}$ where

$$\mathcal{G}(m,n) = \begin{cases} G_{m-1} \circ \ldots \circ G_n & \text{if } m > n, \\ \mathrm{Id} & \text{if } m = n, \\ G_m^{-1} \circ \ldots \circ G_{n-1}^{-1} & \text{if } m < n. \end{cases}$$

Then, $\mathbf{y}$ is a solution of (6.3). Moreover,

$$\sup_{n\in\mathbb{Z}}\|y_{n+1} - A_n y_n\| = \sup_{n\in\mathbb{Z}}\|g_n(y_n)\| \leqslant \delta.$$

In particular, $\mathbf{y} = (y_n)_{n\in\mathbb{Z}}$ is a (δ, l^∞)-pseudotrajectory for (3.9). Hence, it follows from theorem 4 applied to the case when $B = l^\infty$ and $f_n \equiv 0$ that there exists a unique sequence $\mathbf{x} = (x_n)_{n\in\mathbb{Z}}$ such that $x_{n+1} = A_n x_n$ for $n \in \mathbb{Z}$ and $\sup_{n\in\mathbb{Z}}\|x_n - y_n\| \leqslant \varepsilon$. Set

$$h_m(y) = h_m(y_m) := x_m.$$

It is easy to verify that (6.4) holds. Moreover,

$$\|h_m(y) - y\| = \|x_m - y_m\| \leqslant \varepsilon,$$

proving (6.5). It remains to show that each h_m is a homeomorphism. The proof of this fact is similar to the proof of theorem 4.1 of [**3**] and thus is left as an exercise. We also refer to remark 4.3 of [**3**] for references to related results. The difference from the aforementioned result and our theorem 7 is that this last result works under less restrictive assumptions since the nonlinear perturbations allowed in theorem 4 are much more general than the ones in [**3**]. Indeed, the assumptions in theorem 7 coincide with those in [18] where the first nonautonomous version of the Grobman–Hartman theorem was obtained (although Palmer studied dynamics with continuous time).

Acknowledgements

We would like to thank the anonymous referee for his/hers constructive comments that helped us to improve the quality of the presentation. L.B. was partially supported by a CNPq-Brazil PQ fellowship under Grant No. 306484/2018-8. D.D. was supported in part by Croatian Science Foundation under the project IP-2019-04-1239 and by the University of Rijeka under the projects uniri-prirod-18-9 and uniri-prprirod-19-16.

References

1 A. I. Alonso, J. Hong and R. Obaya. Exponential dichotomy and trichotomy for difference equations. *Comput. Math. Appl.* **38** (1999), 41–49.

2 D. Anosov. On a class of invariant sets of smooth dynamical systems (in Russian). *Proc. 5th Int. Conf. Nonlinear Oscill.* **2. Kiev** (1970), 39–45.

3 L. Backes and D. Dragičević. Shadowing for nonautonomous dynamics. *Adv. Nonlinear Stud.* **19** (2019), 425–436.

4 D. Barbu, C. Buse and A. Tabassum. Hyers–Ulam stability and discrete dichotomy. *J. Math. Anal. Appl.* **423** (2015), 1738–1752.

5 N. Bernardes Jr., P. R. Cirilo, U. B. Darji, A. Messaoudi and E. R. Pujals. Expansivity and shadowing in linear dynamics. *J. Math. Anal. Appl.* **461** (2018), 796–816.

6 R. Bowen. *Equilibrium states and the ergodic theory of anosov diffeomorphisms*. Lecture Notes in Mathematics, vol. 470 (Berlin, Heidelberg: Springer-Verlag, 1975).

7 C. Buse, V. Lupulescu and D. O'Regan. Hyers–Ulam stability for equations with differences and differential equations with time-dependent and periodic coefficients. Proc. R. Soc. Edinburgh Sect. A (to appear). https://doi.org/10.1017/prm.2019.12

8 C. Buse, D. O'Regan, O. Saierli and A. Tabassum. Hyers–Ulam stability and discrete dichotomy for difference periodic systems. *Bull. Sci. Math.* **140** (2016), 908–934.

9 S. N. Chow, X. B. Lin and K. J. Palmer. A shadowing lemma with applications to semilinear parabolic equations. *SIAM J. Math. Anal.* **20** (1989), 547–557.

10 W. A. Coppel. *Dichotomies in stability theory* (Berlin, Heidelberg, New-York: Springer Verlag, 1978).

11 D. Dragičević. Admissibility, a general type of Lipschitz shadowing and structural stability. *Commun. Pure Appl. Anal.* **14** (2015), 861–880.

12 S. Elaydi and O. Hajek. Exponential trichotomy of differential systems. *J. Math. Anal. Appl.* **129** (1988), 362–374.

13 P. Hartman. On local homeomorphisms of Euclidean spaces. *Bol. Soc. Mat. Mex.* **5** (1960), 220–241.

14 D. B. Henry. Exponential dichotomies, the shadowing lemma and homoclinic orbits in Banach spaces. *Resenhas* **1** (1994), 381–401.

15 D. Henry. *Geometric theory of semilinear parabolic equations*. Lecture Notes in Mathematics, vol. 840 (Berlin-New York: Springer-Verlag, 1981).

16 N. T. Huy and N. V. Minh. Exponential dichotomy of difference equations and applications to evolution equations on the half-line. *Comput. Math. Appl.* **42** (2001), 301–311.

17 K. R. Meyer and G. R. Sell. An analytic proof of the shadowing lemma. *Funkc. Ekvac.* **30** (1987), 127–133.

18 K. Palmer. A generalization of Hartman's linearization theorem. *J. Math. Anal. Appl.* **41** (1973), 753–758.

19 K. J. Palmer. Exponential dichotomies, the shadowing lemma, and transversal homoclinic points. *Dyn. Rep.* **1** (1988), 266–305.

20 K. J. Palmer. Shadowing and silnikov chaos. *Nonlinear Anal.* **27** (1996), 1075–1093.

21 K. Palmer. *Shadowing in dynamical systems. Theory and applications* (Dordrecht: Kluwer, 2000).

22 G. Papaschinopoulos. On exponential trichotomy of linear difference equations. *Appl. Anal.* **40** (1991), 89–109.

23 O. Perron. Die stabilitätsfrage bei differentialgleichungen. *Math. Z.* **32** (1930), 703–728.

24 S. Yu. Pilyugin. *Shadowing in dynamical systems.* Lecture Notes in Mathematics, vol. 1706 (Berlin: Springer-Verlag, 1999).

25 S. Yu. Pilyugin and S. Tikhomirov. On lipschitz shadowing and structural stability. *Nonlinearity* **23** (2010), 2509–2515.

26 J. H. Poincaré. Sur le problème des trois corps et les équations de la dynamique. Divergence des séries de M. Lindstedt. *Acta Math.* **13** (1890), 1–270.

27 C. Potzsche. Corrigendum on: a note on the dichotomy spectrum. *J. Differ. Equation Appl.* **18** (2012), 1257–1261.

28 A. L. Sasu. Exponential dichotomy and dichotomy radius for difference equations. *J. Math. Anal. Appl.* **344** (2008), 906–920.

29 A. L. Sasu and B. Sasu. Input–output admissibility and exponential trichotomy of difference equations. *J. Math. Anal. Appl.* **380** (2011), 17–32.

30 A. L. Sasu and B. Sasu. Discrete admissibility and exponential trichotomy of dynamical systems. *Discrete Cont. Dyn. Syst.* **34** (2014), 2929–2962.