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MAPPINGS WHICH PRESERVE IDEMPOTENTS, LOCAL AUTOMORPHISMS, AND LOCAL DERIVATIONS

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ABSTRACT It is proved that linear mappings of matrix algebras which preserve idempotents are Jordan homomorphisms Applying this theorem we get some results concerning local derivations and local automorphisms As an another application, the complete description of all weakly continuous linear surjective mappings on standard operator algebras which preserve projections is obtained We also study local ring derivations on commutative semisimple Banach algebras

1. **Introduction.** Let $\mathcal{A}$ be an algebra and M an $\mathcal{A}$-bimodule. A linear mapping $\delta: \mathcal{A} \longrightarrow M$ is called a local derivation if for every $a \in \mathcal{A}$ there exists a derivation $\delta_a: \mathcal{A} \longrightarrow M$ (depending on a) such that $\delta(a) = \delta_a(a)$. There are three recent publications [1,8,9] where some conditions under which every local derivation is a derivation are given.

In [8], Kadison considered local derivations on von Neumann algebras and some polynomial algebras. He proved that each norm-continuous local derivation of a von Neumann algebra $\mathcal{A}$ into a dual $\mathcal{A}$-bimodule M is a derivation. This was generalized in [1] by showing that the same conclusion (moreover, M can be an arbitrary normed $\mathcal{A}$-bimodule) holds for a wider class of linear mappings; that is, for mappings δ of $\mathcal{A}$ into M which satisfy

$$\delta(p) = \delta(p)p + p\delta(p) \tag{1}$$

for every idempotent p in $\mathcal{A}$. Every local derivation satisfies (1) for any idempotent p (namely, we have $\delta(p) = \delta_p(p) = \delta_p(p^2) = \delta_p(p)p + p\delta_p(p) = \delta(p)p + p\delta(p)$). In this paper we will consider linear mappings satisfying (1) for each idempotent on some other algebras.

Larson and Sourour [9] proved that if $\mathcal{A} = \mathcal{B}(X)$, the algebra of all bounded linear operators on a complex Banach space X, then every local derivation of $\mathcal{A}$ into $\mathcal{A}$ is a derivation. They have also considered local automorphisms of $\mathcal{B}(X)$—a local automorphism of an algebra $\mathcal{A}$ is, of course, a linear mapping $\theta: \mathcal{A} \longrightarrow \mathcal{A}$ such that for every $a \in \mathcal{A}$, there is an automorphism θ_a of $\mathcal{A}$ such that $\theta(a) = \theta_a(a)$. They showed that in the case that X is infinite-dimensional, every surjective local automorphism of $\mathcal{B}(X)$ is an automorphism. In the finite-dimensional case, the situation is somewhat different. Namely, antiautomorphisms of $M_n(\mathbb{C})$, the algebra of all $n \times n$ complex matrices, are also local automorphisms (*cf.* [9, Theorem 2.2]). However, combining Theorems 2.1 and 2.2

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in [9] we see that in any case one can assert that a surjective local automorphism of $\mathcal{B}(X)$ onto $\mathcal{B}(X)$ is a Jordan homomorphism (recall that a linear mapping θ of an algebra $\mathcal{A}$ into an algebra $\mathcal{B}$ is called a Jordan homomorphism if $\theta(a^2) = \theta(a)^2$ for any $a \in \mathcal{A}$). Therefore, it seems natural to ask when a local Jordan homomorphism θ (the definition of this notion should be self-explanatory) of an algebra $\mathcal{A}$ into an algebra $\mathcal{B}$ is a Jordan homomorphism. In a similar fashion as we have showed that local derivations satisfy (1), one shows that a local Jordan homomorphism θ preserves idempotents, that is

$$\theta(p)^2 = \theta(p) \tag{2}$$

for any idempotent p in $\mathcal{A}$. In view of this observation, the question arises whether it is possible to describe linear mappings of $\mathcal{A}$ into $\mathcal{B}$ which preserve idempotents. In [10], Omladič determined a form of a linear bijective weakly continuous operator θ of $\mathcal{B}(X)$ onto $\mathcal{B}(X)$ which preserves projections of rank one in both directions, that is, θ and θ^{-1} both send projections of rank one into projections of rank one.

Let R be a commutative ring with an identity such that $1/2$ exists. We denote by $M_n(R)$ the algebra of all $n \times n$ matrices over R. In Section 2 we shall show that every linear mapping θ of $M_n(R)$ into an arbitrary R-algebra $\mathcal{B}$ which preserves idempotents is a Jordan homomorphism. As a consequence, an analogous result for mappings satisfying (1) for every idempotent is obtained.

In Section 3, we first show that the results from Section 2 can be easily extended to the case that $\mathcal{A} = \mathcal{F}(X)$, the algebra of all bounded finite rank operators on a real or complex Banach space X. These results have several applications. First of all, we obtain an analogue of Omladič's result by determining a form of a linear surjective weakly continuous operator $\theta\colon \mathcal{A} \longrightarrow \mathcal{B}$ which preserves projections. Here, $\mathcal{A}$ and $\mathcal{B}$ denote standard operator algebras on Banach spaces X and Y, respectively. Replacing the Omladič's condition that θ preserves projections of rank one by the condition that θ preserves projections, we do not need to assume that this property is satisfied in both directions. We have already mentioned the result of Larson and Sourour [9] which states that if X is a complex infinite-dimensional Banach space then every surjective local automorphism of $\mathcal{B}(X)$ is an automorphism. In the proof of this result they used a theorem concerning linear spectrum-preserving mappings on $\mathcal{B}(X)$ (see [5]). Consequently, their proof works only in the complex case. However, this result is also true in the real case as we shall see using our methods. We will also show that every linear weakly continuous mapping δ of a standard operator algebra $\mathcal{A}$ on X into $\mathcal{B}(X)$, which satisfies (1) for all linear bounded projections in $\mathcal{A}$, is of the form $\delta(T) = WT - TW$ for some bounded linear operator $W \in \mathcal{B}(X)$. At the end of Section 3 we shall obtain a new proof of Larson-Sourour's result which states that every local derivation of $\mathcal{B}(X)$ is a derivation.

Let $\mathcal{A}$ be a commutative semisimple Banach algebra. It is well-known that there are no nonzero derivations on $\mathcal{A}$ [6]; thus, the same is true for local derivations. But it can easily happen that there are nonzero ring derivations on $\mathcal{A}$. A ring derivation of an algebra $\mathcal{A}$ is an additive (not necessarily linear) mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that $\delta(ab) = \delta(a)b + a\delta(b)$ for all $a, b \in \mathcal{A}$. Relying on the fundamental theorem concerning ring derivations on semisimple Banach algebras due to Johnson and Sinclair [7], we give a description of all local ring derivations on $\mathcal{A}$ in Section 4.

2. **Linear mappings of matrix algebras which preserve idempotents.** We begin with

THEOREM 2.1. *Let R be a commutative ring with an identity such that $1/2$ exists (i.e., $1+1$ is invertible). Let $\mathcal{A} = M_n(R)$ and let $\mathcal{B}$ be an arbitrary algebra over R. If an R-linear mapping $\theta\colon \mathcal{A} \longrightarrow \mathcal{B}$ preserves idempotents (in particular, if θ is a local Jordan homomorphism), then θ is a Jordan homomorphism. Moreover, θ is the sum of a homomorphism and an antihomomorphism.*

PROOF. We denote by E_{ij} the matrix unit, that is, the matrix which has one in (i,j)-position and zeros elsewhere. Since each matrix in $\mathcal{A}$ can be represented as a linear combination of matrix units, the assertion that θ is a Jordan homomorphism will be established by showing that

$$\theta(E_{ij}E_{kl} + E_{kl}E_{ij}) = \theta(E_{ij})\theta(E_{kl}) + \theta(E_{kl})\theta(E_{ij}) \tag{3}$$

for all i,j,k,l. We have to consider several cases.

First, we consider the case that both matrix units in (3) are projections. Thus, we have $i = j$ and $k = l$. If $i = k$ there is nothing to prove, while in the case $i \neq k$ we have to show that

$$\theta(E_{ii})\theta(E_{kk}) + \theta(E_{kk})\theta(E_{ii}) = 0. \tag{4}$$

Let $P_1 = E_{ii} + E_{kk}$. Then P_1 is a projection, hence

$$\begin{aligned}\theta(E_{ii}) + \theta(E_{kk}) = \theta(P_1) = \theta(P_1)^2 &= \big(\theta(E_{ii}) + \theta(E_{kk})\big)^2 \\ &= \theta(E_{ii}) + \theta(E_{kk}) + \big(\theta(E_{ii})\theta(E_{kk}) + \theta(E_{kk})\theta(E_{ii})\big),\end{aligned}$$

which proves (4).

In the sequel we shall need the following relation

$$\theta(E_{ij})^2 = 0, \quad i \neq j. \tag{5}$$

In order to prove it we define projections $P_2 = E_{ii} + E_{ij}$ and $P_3 = E_{ii} - E_{ij}$. We have

$$\begin{aligned}2\theta(E_{ii}) = \theta(P_2) + \theta(P_3) &= \big(\theta(E_{ii}) + \theta(E_{ij})\big)^2 + \big(\theta(E_{ii}) - \theta(E_{ij})\big)^2 \\ &= 2\theta(E_{ii}) + 2\theta(E_{ij})^2,\end{aligned}$$

which implies (5).

Our next step will be to prove that (3) holds in the case that one matrix unit is a projection and the other one is a nilpotent, or in other words, $i = j$ and $k \neq l$. First we treat the case that $i = k$ is valid. We have to show

$$\theta(E_{il}) = \theta(E_{ii})\theta(E_{il}) + \theta(E_{il})\theta(E_{ii}). \tag{6}$$

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Since $P_4 = E_{ii} + E_{il}$ is a projection, we get

$$\begin{aligned}\theta(E_{ii}) + \theta(E_{il}) = \theta(P_4) &= \big(\theta(E_{ii}) + \theta(E_{il})\big)^2 \\ &= \theta(E_{ii}) + \theta(E_{il})^2 + \big(\theta(E_{ii})\theta(E_{il}) + \theta(E_{il})\theta(E_{ii})\big).\end{aligned}$$

Using (5) we get (6). A similar approach gives us

$$\theta(E_{ki}) = \theta(E_{ii})\theta(E_{ki}) + \theta(E_{ki})\theta(E_{ii}), \quad i \neq k. \tag{7}$$

In order to complete the proof of the relation (3) in the case that one of the matrix units is a projection and the other one is a nilpotent we have to consider the case that $k \neq i$ and $l \neq i$. As $E_{ii}E_{kl} + E_{kl}E_{ii} = 0$ we have to show that

$$\theta(E_{ii})\theta(E_{kl}) + \theta(E_{kl})\theta(E_{ii}) = 0. \tag{8}$$

Note that $P_5 = E_{kk} + E_{ii} + E_{kl}$ is an idempotent. Hence

$$\begin{aligned}\theta(E_{kk}) + \theta(E_{ii}) + \theta(E_{kl}) = \theta(P_5) &= \big(\theta(E_{kk}) + \theta(E_{ii}) + \theta(E_{kl})\big)^2 \\ &= \theta(E_{kk}) + \theta(E_{ii}) + \theta(E_{kl})^2 + \big(\theta(E_{kk})\theta(E_{ii}) + \theta(E_{ii})\theta(E_{kk})\big) \\ &\quad + \big(\theta(E_{kk})\theta(E_{kl}) + \theta(E_{kl})\theta(E_{kk})\big) \\ &\quad + \big(\theta(E_{ii})\theta(E_{kl}) + \theta(E_{kl})\theta(E_{ii})\big).\end{aligned}$$

Applying (4), (5), and (6) we obtain (8).

It remains to consider the case that in the relation (3) both matrix units are nilpotent. Then we have $i \neq j$ and $k \neq l$. Once again we have to distinguish several cases. We start with the assumption that $i = k$. If also $j = l$ then we get (3) using the relation (5). In the case $j \neq l$ we have to prove

$$\theta(E_{ij})\theta(E_{il}) + \theta(E_{il})\theta(E_{ij}) = 0 \tag{9}$$

Note that $P_6 = E_{ii} + E_{ij} + E_{il}$ is a projection. The relation (9) now follows easily from $\theta(P_6) = \theta(P_6)^2$, (5), and (6).

In order to complete the proof of the equation (3) we have to consider the case $i \neq j$, $k \neq l$, and $i \neq k$. If $j = l$ we can prove (3) in almost the same way as in the previous case. So, let us also assume that $j \neq l$. We have four possibilities:

(i) $i = l$ and $j = k$,
(ii) $i = l$ and $j \neq k$,
(iii) $i \neq l$ and $j = k$, and
(iv) $i \neq l$ and $j \neq k$.

In case (i) we have to show that

$$\theta(E_{ij})\theta(E_{ji}) + \theta(E_{ji})\theta(E_{ij}) = \theta(E_{ii}) + \theta(E_{jj}).$$

As $P_7 = (1/2)(E_{ii} + E_{jj} - E_{ij} - E_{ji})$ is a projection we have

$$\begin{aligned}
2\big(\theta(E_{ii}) + \theta(E_{jj}) - \theta(E_{ij}) - \theta(E_{ji})\big) \\
&= 4\theta(P_7) = 4\theta(P_7)^2 = \theta(E_{ii}) + \theta(E_{jj}) + \theta(E_{ij})^2 + \theta(E_{ji})^2 \\
&\quad + \big(\theta(E_{ii})\theta(E_{jj}) + \theta(E_{jj})\theta(E_{ii})\big) - \big(\theta(E_{ii})\theta(E_{ij}) + \theta(E_{ij})\theta(E_{ii})\big) \\
&\quad - \big(\theta(E_{ii})\theta(E_{ji}) + \theta(E_{ji})\theta(E_{ii})\big) - \big(\theta(E_{jj})\theta(E_{ij}) + \theta(E_{ij})\theta(E_{jj})\big) \\
&\quad - \big(\theta(E_{jj})\theta(E_{ji}) + \theta(E_{ji})\theta(E_{jj})\big) + \big(\theta(E_{ij})\theta(E_{ji}) + \theta(E_{ji})\theta(E_{ij})\big).
\end{aligned}$$

One can complete the proof of case (i) applying (4), (5), (6), and (7). In case (ii) we define a projection $P_8 = E_{ii} + E_{ij} + E_{ki} + E_{kj}$. Using $\theta(P_8) = \theta(P_8)^2$ and previous relations we get (3). Note that cases (ii) and (iii) coincide. Finally, in case (iv) we define a projection $P_9 = E_{ii} + E_{kk} + E_{ij} + E_{kl}$. We complete the proof of (3) in this last case in a similar way as in the previous ones.

It follows that θ is a Jordan homomorphism. According to [4, Theorem 7] θ is the sum of a homomorphism and an antihomomorphism. This completes the proof of the theorem.

REMARK 2.2. Let us present a brief proof of a special case of Theorem 2.1 where $R = \mathbb{C}$ is the field of complex numbers. Pick a Hermitian matrix $H \in M_n(\mathbb{C})$. Then $H = \sum_{i=1}^n t_i P_i$ where $t_i \in \mathbb{R}$ and P_i are projections such that $P_i P_j = P_j P_i = 0$ if $i \neq j$. Since $P_i + P_j$ is a projection if $i \neq j$, we have $\big(\theta(P_i) + \theta(P_j)\big)^2 = \theta(P_i) + \theta(P_j)$. This yields $\theta(P_i)\theta(P_j) + \theta(P_j)\theta(P_i) = 0$. Using this relation we see that $\theta(H^2) = \theta(H)^2$. Now, replacing H by $H+K$ where H and K are both Hermitian, we get $\theta(HK+KH) = \theta(H)\theta(K)+\theta(K)\theta(H)$. Since an arbitrary matrix $A \in M_n(\mathbb{C})$ can be written in the form $A = H + iK$ with H, K Hermitian, the last two relations imply that $\theta(A^2) = \theta(A)^2$.

An analogue of Theorem 2.1 for derivations is

THEOREM 2.3. *Let R be a commutative ring with an identity such that $1/2$ exists. Let $\mathcal{A} = M_n(R)$ and let M be an $\mathcal{A}$-bimodule. If a linear mapping $\delta\colon \mathcal{A} \longrightarrow M$ satisfies $\delta(P) = \delta(P)P + P\delta(P)$ for any idempotent P in $\mathcal{A}$ (in particular, if δ is a local derivation) then δ is a derivation.*

The proof of this theorem could be done by repeating and adapting the arguments given in the proof of Theorem 2.1. However, it turns out that there is a shorter way.

PROOF OF THEOREM 2.3. We define a multiplication in $\mathcal{B}' = \mathcal{A} \oplus \mathcal{A} \oplus M$ by

$$(a_1, b_1, m_1)(a_2, b_2, m_2) = (a_1a_2, b_1b_2, a_1m_2 + m_1b_2).$$

Note that $\mathcal{B}'$ then becomes an algebra over R (*cf.* [11, Example 1.1.9]). Let $\mathcal{B}$ be a subalgebra of $\mathcal{B}'$ generated by all elements of the form $\big(T, T, \delta(T)\big)$, $T \in \mathcal{A}$. Define a mapping $\theta\colon \mathcal{A} \longrightarrow \mathcal{B}$ by

$$\theta(T) = \big(T, T, \delta(T)\big).$$

Using the initial hypothesis on δ we see that θ maps idempotents in $\mathcal{A}$ into idempotents in $\mathcal{B}$. Thus θ is a Jordan homomorphism by Theorem 2.1.

Let us show that $\theta(I)$ is a unit element of $\mathcal{B}$. Replacing T by $T+I$ in the identity $\theta(T^2) = \theta(T)^2$ we arrive at $2\theta(T) = \theta(I)\theta(T) + \theta(T)\theta(I)$. Using the fact that $\theta(I)$ is an idempotent one can get using standard arguments that $\theta(T) = \theta(T)\theta(I) = \theta(I)\theta(T)$. Since the algebra $\mathcal{B}$ is generated by the image of θ, this proves our assertion.

Theorem 2.1 tells us that $\theta = \varphi + \psi$ where $\varphi: \mathcal{A} \longrightarrow \mathcal{B}$ is a homomorphism and $\psi: \mathcal{A} \longrightarrow \mathcal{B}$ is an antihomomorphism. We set $\pi = \varphi(I)$ and $\rho = \psi(I)$. Then π and ρ are idempotents and $\pi + \rho = \theta(I)$ is a unit element of $\mathcal{B}$. Consequently $\pi\rho = \rho\pi = 0$, which implies that

$$\psi(T) = \theta(T)\rho = \rho\theta(T) \text{ for all } T \in \mathcal{A}. \tag{10}$$

Since $\rho \in \mathcal{B}$, we have $\rho = (Q, Q, m)$ for some $Q \in \mathcal{A}$, $m \in M$. The relation $\rho^2 = \rho$ yields

$$Q^2 = Q, \quad Qm + mQ = m. \tag{11}$$

By (10), ρ commutes with $\theta(T)$ for any $T \in \mathcal{A}$. Hence Q commutes with all elements in $\mathcal{A}$. This and (11) tell us that $Q = aI$ for some idempotent $a \in R$. Since ψ is an antihomomorphism, (10) implies that $\theta(TS)\rho = \theta(S)\theta(T)\rho$ for any $S, T \in \mathcal{A}$. Hence we see that from $Q = aI$ it follows that $a(ST - TS) = 0$ for all $S, T \in \mathcal{A}$. Thus $a = 0$. Therefore $Q = 0$, and so (11) gives us that $m = 0$ too. This means that $\rho = 0$, and therefore, $\psi = 0$. Hence $\theta = \varphi$ is a homomorphism. From the definition of θ we see that this implies that δ is a derivation.

3. **Mappings which preserve idempotents, local automorphisms, and local derivations of operator algebras.** Throughout this section, X and Y will be Banach spaces over $\mathbb{F}$ where $\mathbb{F}$ is either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. By $\mathcal{B}(X)$ we denote the algebra of all bounded linear operators on X, and $\mathcal{F}(X)$ denotes the algebra of all finite rank operators in $\mathcal{B}(X)$. Recall that a standard operator algebra is any subalgebra of $\mathcal{B}(X)$ which contains $\mathcal{F}(X)$. The dual of X will be denoted by X' and the adjoint of $A \in \mathcal{B}(X)$ by A'. For any $x \in X$ and $f \in X'$ we denote by $x \otimes f$ the bounded linear operator on X defined by $(x \otimes f)y = f(y)x$ for $y \in X$. Note that every operator of rank one can be written in this form. The operator $x \otimes f$ is a projection if and only if $f(x) = 1$. Note that $(x \otimes f)' = f \otimes (Kx)$, where K is the natural embedding of X into X''.

Let $\mathcal{A} \subset \mathcal{B}(X)$ and $\mathcal{B} \subset \mathcal{B}(Y)$ be standard operator algebras. In the sequel we shall need some facts about isomorphisms and antiisomorphisms on $\mathcal{A}$ onto $\mathcal{B}$. It is well-known that every isomorphism of $\mathcal{A}$ onto $\mathcal{B}$ is of the form $\theta(A) = TAT^{-1}$ for some bounded invertible linear operator $T: X \longrightarrow Y$ [2]. We also believe that the complete description of all antiisomorphisms of $\mathcal{A}$ onto $\mathcal{B}$ is known but to the best of our knowledge not yet published.

PROPOSITION 3.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on Banach spaces X and Y, respectively. Suppose that $\theta\colon \mathcal{A} \longrightarrow \mathcal{B}$ is an antiisomorphism. Then the spaces X and Y are reflexive and there exists a bounded invertible linear operator $T\colon X' \longrightarrow Y$ such that $\theta(A) = TA'T^{-1}$ for every $A \in \mathcal{A}$.*

PROOF. Let us fix a vector $z \in X$ and a functional $g \in X'$ satisfying $g(z) = 1$. The operator $z \otimes g$ is a projection of rank one. Consequently, $\theta(z \otimes g)$ is a projection. We claim that $\theta(z \otimes g)$ has rank one. Assume on the contrary that $\theta(z \otimes g)$ has rank greater than one. Then we can find nonzero projections Q_1 and Q_2 satisfying $\theta(z\otimes g) = Q_1 + Q_2$, Q_1 has rank one, and $Q_1Q_2 = Q_2Q_1 = 0$. From $Q_1 \in \mathcal{F}(Y) \subset \mathcal{B}$ it follows that $Q_2 = \theta(z \otimes g) - Q_1$ belongs to $\mathcal{B}$. Set $P_i = \theta^{-1}(Q_i)$, $i = 1, 2$. Obviously, we have $z \otimes g = P_1 + P_2$ and $P_1P_2 = P_2P_1 = 0$ which is a contradiction. Thus $\theta(z \otimes g) = u \otimes h$ for some $u \in Y$ and $h \in Y'$ satisfying $h(u) = 1$.

We define a linear mapping $T\colon X' \longrightarrow Y$ by

$$Tf = \theta(z \otimes f)u.$$

Assume that there exists nonzero $f \in X'$ such that $Tf = 0$. Let us choose $x \in X$ such that $f(x) = 1$. It follows that

$$0 = \theta(x \otimes g)\theta(z \otimes f)u = \theta\big((z \otimes f)(x \otimes g)\big)u = \theta(z \otimes g)u = (u \otimes h)u = u.$$

This contradiction yields that T is injective.

For an arbitrary $A \in \mathcal{A}$ we have

$$\begin{aligned}(TA')h = T(A'h) = \theta\big(z \otimes (A'h)\big)u &= \theta\big((z \otimes h)A\big)u \\ = \theta(A)\theta(z \otimes h)u &= \theta(A)Th, \quad h \in X'.\end{aligned}$$

One can easily prove that T is surjective using the above relation and the fact, that the set of all linear bounded rank one operators on Y is contained in $\operatorname{Im}\theta$.

Next, we shall prove that T is continuous. For every rank one operator $A \in \mathcal{F}(X)$ the operator TA' is continuous. The same must be true for the operator $\theta(A)T$. We have already showed that θ preserves projections of rank one. It can be easily seen that this implies that θ maps the set of all linear bounded rank one operators on X onto the set of all linear bounded rank one operators on Y. Thus, for every $v \in Y$ and $k \in Y'$ the operator $(v \otimes k)T$ is continuous. Let $(f_n) \subset X'$ be a sequence satisfying $f_n \longrightarrow f \in X'$ and $Tf_n \longrightarrow w \in Y$. It follows that

$$k(w)v = (v \otimes k)(w) = \lim_{n\to\infty}\big((v \otimes k)T\big)f_n = (v \otimes k)Tf = k(Tf)v,$$

and consequently, $Tf = w$. According to the closed graph theorem the operator T is continuous.

In order to complete the proof we must show that the spaces X and Y are reflexive. For every rank one operator $x \otimes f \in \mathcal{F}(X)$ we have

$$\theta(x \otimes f) = (Tf) \otimes (T^{-1})'Kx,$$

where K is the natural embedding of X into X''. As θ is an antiisomorphism, the operator $(T^{-1})'K\colon X \longrightarrow Y'$ is bijective. It follows that K is bijective, or equivalently, X is reflexive. The same must be true for Y which is isomorphic to X' via the isomorphism T.

THEOREM 3.2. *Let $\mathcal{B}$ be any algebra over $\mathbb{F}$. If a linear mapping $\theta\colon \mathcal{F}(X) \longrightarrow \mathcal{B}$ maps projections in $\mathcal{F}(X)$ into idempotents in $\mathcal{B}$ (in particular, if θ is a local Jordan homomorphism), then θ is a Jordan homomorphism.*

PROOF. Fix $A \in \mathcal{F}(X)$; we want to show that $\theta(A^2) = \theta(A)^2$. There is a projection P in $\mathcal{F}(X)$ such that $PAP = A$. Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of the range of P. Define linear functionals $f_1, f_2, \ldots, f_n$ on X by

$$f_i(x_j) = \delta_{ij},$$
$$f_i(z) = 0 \text{ for all } z \in \operatorname{Ker} P.$$

Let $\mathcal{C} \subset \mathcal{F}(X)$ be the algebra of all operators B of the form $B = \sum_{i,j=1}^{n} t_{ij} x_i \otimes f_j$, $t_{ij} \in \mathbb{F}$, and note that $\mathcal{C}$ is isomorphic to $M_n(\mathbb{F})$ via the isomorphism $B \longmapsto (t_{ij})$. Thus, for the restriction of θ to $\mathcal{C}$, Theorem 2.1 can be applied. Hence $\theta(T^2) = \theta(T)^2$ for every $T \in \mathcal{C}$. Since $A \in \mathcal{C}$, this proves that θ is a Jordan homomorphism.

THEOREM 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be standard operator algebras on real or complex Banach spaces X and Y, respectively. Suppose that $\theta\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a linear surjective mapping which is continuous in the weak operator topology. Assume further that θ preserves projections. Then either*

(a) there is a bounded bijective linear operator $T\colon X \longrightarrow Y$ such that $\theta(A) = TAT^{-1}$ for all $A \in \mathcal{A}$, or

(b) there is a bounded linear bijective operator $T\colon X' \longrightarrow Y$ such that $\theta(A) = TA'T^{-1}$ for all $A \in \mathcal{A}$. In this case X and Y must be reflexive.

PROOF OF THEOREM 3.3. According to Theorem 3.2 the restriction of θ to the subalgebra $\mathcal{F}(X)$ is a Jordan homomorphism. The subalgebra $\mathcal{F}(X)$ is dense in a weak operator topology in $\mathcal{A}$ and θ is continuous. Consequently, $\theta\colon \mathcal{A} \longrightarrow \mathcal{B}$ is a surjective Jordan homomorphism. Every standard algebra $\mathcal{A}$ is a prime ring, that is, if $A\mathcal{A}B = 0$ then $A = 0$ or $B = 0$. It follows that θ is a homomorphism or an antihomomorphism [3, Theorem 3.1]. We claim that θ is one-to-one. Suppose on the contrary that $\theta(A) = 0$ for some nonzero $A \in \mathcal{A}$. It follows that the restriction of θ to the two-sided ideal generated by A is zero. One can easily see that this ideal contains $\mathcal{F}(X)$. Since θ is continuous in the weak operator topology, this further implies that θ is zero on $\mathcal{A}$ which is a contradiction to the surjectivity of θ. This contradiction yields that θ is an isomorphism or an antiisomorphism. One can complete the proof using [2] and Proposition 3.1.

THEOREM 3.4. *Let X be an infinite-dimensional real or complex Banach space and let θ be a surjective local automorphism of $\mathcal{B}(X)$. Then θ is an automorphism.*

This result was proved by Larson and Sourour [9] for the special case that X is a complex Banach space. In the proof of this theorem we shall need the following lemma, which is an analogue of [5, Lemma 4].

LEMMA 3.5. *Let X be a real or complex Banach space. Assume that $A \in \mathcal{B}(X)$, $x \in X, f \in X'$, and $t \in \mathbb{R}$, $|t| > \|A\|$. Then t is an eigenvalue of $A + x \otimes f$ if and only if $f\big((t-A)^{-1}x\big) = 1$.*

PROOF. If $f\big((t-A)^{-1}x\big) = 1$, then

$$(A + x \otimes f)(t-A)^{-1}x = A(t-A)^{-1}x + x = t(t-A)^{-1}x,$$

and so t is an eigenvalue of $A+x\otimes f$. Conversely, if t is an eigenvalue of $A+x\otimes f$, then there exists a nonzero vector $y \in X$ such that $(A + x \otimes f)y = ty$. Therefore $y = f(y)(t-A)^{-1}x$. It follows from $y \neq 0$ that $f(y) \neq 0$ and consequently, $f\big((t-A)^{-1}x\big) = 1$.

PROOF OF THEOREM 3.4. Since θ is a local automorphism we can find for every $A \in \mathcal{B}(X)$ an invertible operator $T_A \in \mathcal{B}(X)$ such that $\theta(A) = T_A A T_A^{-1}$. It follows that θ maps every finite rank operator into a finite rank operator. θ is surjective, and so, for every finite rank operator B we can find $A \in \mathcal{B}(X)$ such that $\theta(A) = T_A A T_A^{-1} = B$. Obviously, A belongs to $\mathcal{F}(X)$ as well. Thus, the mapping $\theta_{|\mathcal{F}(X)}: \mathcal{F}(X) \longrightarrow \mathcal{F}(X)$ is a bijective linear mapping which preserves projections. As in the proof of Theorem 3.3 we see that either there exists a bounded bijective linear operator $T: X \longrightarrow X$ such that $\theta(A) = TAT^{-1}$ for every $A \in \mathcal{F}(X)$, or there exists a bounded bijective linear operator $T: X' \longrightarrow X$ such that $\theta(A) = TA'T^{-1}$ for every $A \in \mathcal{F}(X)$.

Our approach in the rest of the proof is similar to that of Jafarian and Sourour [5]. First we shall consider the case that the restriction of θ to $\mathcal{F}(X)$ is of the form $\theta(A) = TAT^{-1}$ for some invertible $T \in \mathcal{B}(X)$. It follows that $\theta(x \otimes f) = Tx \otimes (T^{-1})'f$ for every rank one operator $x \otimes f$. Let A be an arbitrary operator on X. Then

$$\theta(A + x \otimes f) = \theta(A) + Tx \otimes (T^{-1})'f.$$

Let t be a real number with $|t| > \max\{\|A\|, \|\theta(A)\|\}$. The mapping θ is a local automorphism, and so t is an eigenvalue of $B \in \mathcal{B}(X)$ if and only if t is an eigenvalue of $\theta(B)$. This yields together with Lemma 3.5 that $f\big((t-A)^{-1}x\big) = 1$ if and only if $\big((T^{-1})'f\big)\Big(\big(t-\theta(A)\big)^{-1}Tx\Big) = 1$, and so, by linearity, we have

$$f\big((t-A)^{-1}x\big) = f\Big(T^{-1}\big(t-\theta(A)\big)^{-1}Tx\Big)$$

for every $x \in X, f \in X'$, and real t, $|t| > \max\{\|A\|, \|\theta(A)\|\}$. Replacing t by $1/s$ we get

$$f\big((I-sA)^{-1}x\big) = f\Big(T^{-1}\big(I-s\theta(A)\big)^{-1}Tx\Big)$$

for every nonzero real number s in some neighbourhood $\{s : |s| < \varepsilon\}$. Thus,

$$\sum_{n=0}^{\infty} s^n f(A^n x) = \sum_{n=0}^{\infty} s^n f\big(T^{-1}\theta(A)^n Tx\big).$$

Comparing the terms at $n = 1$ we get $\theta(A) = TAT^{-1}$.

It remains to consider the case that $\theta(x \otimes f) = Tf \otimes (T^{-1})'Kx$ for some invertible bounded linear operator $T: X' \longrightarrow X$. In a similar way as in the previous case we get that for every $A \in \mathcal{B}(X)$ we have $\theta(A) = TA'T^{-1}$. It was proved in [9] that in the case that X is infinite-dimensional, mappings of such form are not local automorphisms. This completes the proof.

THEOREM 3.6. *Let X be a real or complex Banach space and M an $\mathcal{F}(X)$-bimodule. If a linear mapping $\delta\colon \mathcal{F}(X) \longrightarrow M$ satisfies $\delta(P) = \delta(P)P + P\delta(P)$ for every projection P in $\mathcal{F}(X)$ (in particular, if δ is a local derivation), then δ is a derivation.*

PROOF. Pick $A, B \in \mathcal{F}(X)$, and let us show that $\delta(AB) = \delta(A)B + A\delta(B)$. There is a projection $Q \in \mathcal{F}(X)$ such that $QAQ = A$ and $QBQ = B$. Arguing as in the proof of Theorem 3.2 one shows that there is a subalgebra $\mathcal{D} \subset \mathcal{F}(X)$ isomorphic to $M_n(\mathbb{F})$ and containing A and B. By Theorem 2.3 it follows that the restriction of δ to $\mathcal{D}$ is a derivation. Thus, $\delta(AB) = \delta(A)B + A\delta(B)$.

COROLLARY 3.7. *Let $\mathcal{A}$ be a standard operator algebra on a real or complex Banach space X. Suppose that a linear mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{B}(X)$ satisfies $\delta(P) = \delta(P)P + P\delta(P)$ for every projection P in $\mathcal{A}$. If δ is continuous in a weak operator topology, then there exists $W \in \mathcal{B}(X)$ such that $\delta(T) = WT - TW$ for every $T \in \mathcal{A}$.*

PROOF. By Theorem 3.6 the restriction of δ to $\mathcal{F}(X)$ is a derivation. It is known that then there is an operator $W \in \mathcal{B}(X)$ such that $\delta(T) = WT - TW$ holds for every $T \in \mathcal{F}(X)$ [2]. However, since $\mathcal{F}(X)$ is dense in the weak operator topology in every standard operator algebra $\mathcal{A}$ and since δ is continuous, it follows that this relation holds for every $T \in \mathcal{A}$.

COROLLARY 3.8. (LARSON-SOUROUR) [9]. *Every local derivation of $\mathcal{B}(X)$ is a derivation.*

PROOF. As in the proof of Corollary 3.7 we see that there is an operator $W \in \mathcal{B}(X)$ such that $\delta(T) = WT - TW$ for every $T \in \mathcal{F}(X)$. We remark that, using quite different methods, Larson and Sourour also first proved this as an auxiliary result (see [9, Lemmas 1–5]).

In the rest of the proof we just repeat the arguments given in [9]. Define $\eta\colon \mathcal{B}(X) \longrightarrow \mathcal{B}(X)$ by $\eta(S) = WS - SW - \delta(S)$. Then η is a local derivation and $\eta(T) = 0$ for every finite rank operator T. The theorem will be proved by showing that $\eta = 0$. Take $T \in \mathcal{B}(X)$ and suppose that $S = \eta(T) \neq 0$. Pick $x \in X$ such that $y = Sx \neq 0$. Let P be a bounded projection from X onto $\operatorname{span}\{x, y\}$ along any closed complement of $\operatorname{span}\{x, y\}$ in X. Since η is a local derivation it follows at once that $P\eta\big((I-P)T(I-P)\big)P = 0$. The operator $T-(I-P)T(I-P)$ has finite rank, hence $\eta(T) = \eta\big((I-P)T(I-P)\big)$. This yields $PSP = 0$. But note that $PSPx = y \neq 0$. With this contradiction the theorem is proved.

4. **Local ring derivations on commutative semisimple Banach algebras.**

Let $\mathcal{A}$ be a complex algebra. A mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ is called a ring derivation if it is additive and satisfies $\delta(ab) = a\delta(b) + \delta(a)b$, $a, b \in \mathcal{A}$, so it is not assumed to be linear. The classical result concerning ring derivations due to Johnson and Sinclair [7] states that if $\mathcal{A}$ is a semi-simple Banach algebra and $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ is a ring derivation then $\mathcal{A}$ contains a central idempotent e such that $e\mathcal{A}$ and $(1-e)\mathcal{A}$ are invariant for δ, the restriction of δ to $(1-e)\mathcal{A}$ is continuous, and $e\mathcal{A}$ is finite dimensional.

A mapping $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ is called a local ring derivation if it is additive and if for every $x \in \mathcal{A}$ there exists a ring derivation $\delta_x\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that $\delta(x) = \delta_x(x)$. In the sequel we shall need some facts about local ring derivations on the field $\mathbb{C}$. We start with some known results concerning ring derivations on $\mathbb{C}$ [12]. Every such derivation vanishes at every algebraic number. On the other hand, if $t \in \mathbb{C}$ is transcendental and s an arbitrary complex number then there is a ring derivation $\delta\colon \mathbb{C} \longrightarrow \mathbb{C}$ with $\delta(t) = s$. Consider $\mathbb{C}$ as a vector space over the field of rational numbers $\mathbb{Q}$. It is easy to verify that a mapping $f\colon \mathbb{C} \longrightarrow \mathbb{C}$ is additive if and only if it is a $\mathbb{Q}$-linear mapping on $\mathbb{C}$. Let us denote by A the set of all algebraic numbers. Obviously, A is a $\mathbb{Q}$-linear subspace of $\mathbb{C}$. We choose a $\mathbb{Q}$-linear subspace $V \subset \mathbb{C}$ such that $\mathbb{C} = A \oplus V$. One can now easily verify that a mapping $\delta\colon \mathbb{C} \longrightarrow \mathbb{C}$ is a local ring derivation if and only if $\delta_{|A} \equiv 0$, $\delta_{|V}\colon V \longrightarrow \mathbb{C}$ is additive and $\delta(t+s) = \delta(t)$ for all pairs $t \in V$, $s \in A$.

Let $\mathcal{A}$ be a commutative algebra. Recall that idempotents $e,f \in \mathcal{A}$ are disjoint if $ef = 0$. The set of all idempotents is partially ordered by the relation: $e \leq f$ if and only if $ef = e$.

THEOREM 4.1. *Let $\mathcal{A}$ be a complex commutative semisimple Banach algebra. Suppose that $\delta\colon \mathcal{A} \longrightarrow \mathcal{A}$ is a local ring derivation. Then there exist nonzero disjoint minimal idempotents $e_1, e_2, \ldots, e_n \in \mathcal{A}$ such that*

$$\mathcal{A} = \operatorname{span}\{e_1, e_2, \ldots, e_n\} \oplus (1 - e_1 - e_2 - \cdots - e_n)\mathcal{A},$$
$$\delta_{|(1-e_1-e_2- \cdots -e_n)\mathcal{A}} \equiv 0, \text{ and}$$
$$\delta\Big(\sum_{i=1}^{n} t_i e_i\Big) = \sum_{i=1}^{n} \eta_i(t_i) e_i, \quad t_i \in \mathbb{C},$$

where $\eta_i\colon \mathbb{C} \longrightarrow \mathbb{C}$, $i = 1, 2, \ldots, n$, are local ring derivations.

PROOF. Pick an element $x \in \mathcal{A}$. According to our assumption there exists a ring derivation $\delta_x\colon \mathcal{A} \longrightarrow \mathcal{A}$ such that $\delta(x) = \delta_x(x)$. Using the result of Johnson and Sinclair [7] we can find an idempotent $e \in \mathcal{A}$ such that the restriction of δ_x to $(1-e)\mathcal{A}$ is continuous while $e\mathcal{A}$ is finite-dimensional. The mapping $\delta_{x|(1-e)\mathcal{A}} = \gamma$ is rational-linear continuous mapping, and therefore, it is real-linear. We shall prove that it is also complex-linear. For this purpose define a mapping $\alpha\colon (1-e)\mathcal{A} \longrightarrow (1-e)\mathcal{A}$ by $\alpha(x) = \gamma(ix) - i\gamma(x)$. A straightforward computation shows that $\alpha(ix) = -i\alpha(x)$, and consequently, $\alpha(tx) = \bar{t}\alpha(x)$ for all $t \in \mathbb{C}$ and all $x \in (1-e)\mathcal{A}$. For arbitrary $x, y \in (1-e)\mathcal{A}$ we have $\alpha(xy) = \gamma\big((ix)y\big) - i\gamma(xy) = ix\gamma(y) + y\gamma(ix) - i\big(x\gamma(y) + y\gamma(x)\big) = y\alpha(x)$. Using the fact that α is conjugate-linear we get $\alpha(t^2xy) = \overline{(t^2)}\alpha(xy)$ for arbitrary $t \in \mathbb{C}$, $x, y \in (1-e)\mathcal{A}$. On the other hand, we have $\alpha(t^2xy) = \alpha\big((tx)(ty)\big) = tx\alpha(ty) = |t|^2x\alpha(y)$. It follows that $x\alpha(y) = \alpha(xy) = 0$ for all $x, y \in (1-e)\mathcal{A}$. In particular, we get $\big(\alpha(x)\big)^2 = 0$ for every $x \in (1-e)\mathcal{A}$. As $(1-e)\mathcal{A}$ is semisimple the mapping α must be zero, or equivalently, the restriction of δ_x to $(1-e)\mathcal{A}$ is complex-linear. There are no nonzero complex-linear derivations on a commutative semisimple Banach algebra [6]. Thus,

$$\delta_{x|(1-e)\mathcal{A}} \equiv 0.$$

Let us denote by Δ the maximal ideal space of $\mathcal{A}$. The Gel'fand transformation will be denoted by Θ. For $z \in \mathcal{A}$ we shall denote by $\hat{z}$ the Gel'fand transform of z. Similarly, for any subset $\mathcal{C} \subset \mathcal{A}$ we use the notation $\hat{\mathcal{C}} = \Theta(\mathcal{C})$. The support of $\hat{z}$ is defined by $\operatorname{supp} \hat{z} = \overline{\{h \in \Delta : \hat{z}(h) \neq 0\}}$.

As e is an idempotent, the function $\hat{e}$ is a characteristic function of subset $K = \{h \in \Delta : \hat{e}(h) = 1\}$ of Δ. Obviously, we have $\Theta(e\mathcal{A}) = \hat{e}\hat{\mathcal{A}} = \{\hat{z} \in \hat{\mathcal{A}} : \operatorname{supp} \hat{z} \subset K\}$. It follows from $\dim \Theta(e\mathcal{A}) < \infty$ that for every $\hat{z} \in \Theta(e\mathcal{A})$ there exists a polynomial $p \in \mathbb{C}[X]$ such that $p(\hat{z}) = 0$. This further implies the existence of nonempty pairwise disjoint subsets $K_1, K_2, \ldots, K_m \subset K$ such that every $\hat{z} \in \Theta(e\mathcal{A})$ is of the form

$$\hat{z} = \sum_{i=1}^{m} t_i \chi_i, \quad t_i \in \mathbb{C},$$

where χ_i are characteristic functions of subsets K_i, $i = 1, 2, \ldots, m$. Let us recall that the algebra $\hat{\mathcal{A}}$ separates the points of Δ; that is, for every $h_1, h_2 \in \Delta$, $h_1 \neq h_2$, there exists $\hat{z} \in \hat{\mathcal{A}}$ satisfying $\hat{z}(h_1) \neq \hat{z}(h_2)$. Thus, $K_i = \{h_i\}$, $i = 1, 2, \ldots, m$, where $h_1, h_2, \ldots, h_m \in \Delta$ are isolated points. The algebra $\hat{e}\hat{\mathcal{A}}$ separates the points $\{h_1, h_2, \ldots, h_m\}$ and has an identity element $\hat{e}$. Consequently, there exist $\hat{e}_1, \hat{e}_2, \ldots, \hat{e}_m$ from $\hat{e}\hat{\mathcal{A}}$ with the property $\hat{e}_i(h_j) = \delta_{ij}$, which yields $\hat{e}\hat{\mathcal{A}} = \operatorname{span}\{\hat{e_1}, \hat{e_2}, \ldots, \hat{e_m}\} \approx \mathbb{C}^m$.

The algebra $\hat{e}\hat{\mathcal{A}}$ is invariant under $\widehat{\delta_x}$ where $\widehat{\delta_x}\colon \hat{\mathcal{A}} \longrightarrow \hat{\mathcal{A}}$ is defined by $\widehat{\delta_x}(\hat{z}) = \Theta\big(\delta_x(z)\big)$ for all $z \in \mathcal{A}$. We define mappings $\zeta_{ij}\colon \mathbb{C} \longrightarrow \mathbb{C}$, $i, j = 1, 2, \ldots, m$, by

$$\widehat{\delta_x}(t\hat{e_i}) = \sum_{j=1}^{m} \zeta_{ij}(t)\hat{e_j}, \quad t \in \mathbb{C}.$$

For $i \neq j$ we have

$$0 = \widehat{\delta_x}(\hat{0}) = \widehat{\delta_x}(t\hat{e}_i\hat{e}_j) = t\hat{e}_i\widehat{\delta_x}(\hat{e}_j) + \hat{e}_j\widehat{\delta_x}(t\hat{e}_i).$$

It follows from $\hat{e}_i\widehat{\delta_x}(\hat{e}_j) = \hat{e}_i\widehat{\delta_x}(\hat{e}_j{}^2) = 2\hat{e}_i\hat{e}_j\delta_x(\hat{e}_j) = 0$ that

$$0 = \sum_{k=1}^{m} \big(\zeta_{ik}(t)\hat{e_k}\big)\hat{e}_j = \zeta_{ij}(t)\hat{e}_j,$$

which implies $\zeta_{ij} = 0$ for $i \neq j$. Set $\zeta_i = \zeta_{ii}$. It is easy to verify that ζ_i, $i = 1, 2, \ldots, m$, are ring derivations.

Thus, the element $x \in \mathcal{A}$ can be written as $x = ex + (1-e)x = \sum_{j=1}^{m} s_j e_j + (1-e)x$, $s_j \in \mathbb{C}$, and the following relation holds

$$\delta(x) = \delta_x(x) = \delta_x\Big(\sum_{j=1}^{m} s_j e_j + (1-e)x\Big) = \sum_{j=1}^{m} \zeta_j(s_j)e_j. \tag{12}$$

At the beginning of our proof the element $x \in \mathcal{A}$ was arbitrarily chosen. So, we have proved that for an arbitrary element $z \in \mathcal{A}$ we have

$$\operatorname{supp} \hat{\delta}(\hat{z}) \subset \operatorname{supp} \hat{z}, \tag{13}$$

and $\operatorname{supp}\hat{\delta}(\hat{z})$ is a finite set. From $\operatorname{supp}\hat{e_j} = \{h_j\}$ and (13) it follows that $\delta(te_j) = \eta_j(t)e_j$, $t \in \mathbb{C}$. Here, the mapping $\eta_j \colon \mathbb{C} \longrightarrow \mathbb{C}$ is a local ring derivation. The relations (12) and (13) also imply that $\delta\big((1-e)x\big) = 0$.

Hence, for every $x \in \mathcal{A}$ we can find a finite collection of minimal idempotents $e_1, e_2, \ldots, e_m \in \mathcal{A}$ such that

(i) $\hat{e}_i$ is a characteristic function of $\{h_i\}$ for some isolated point $h_i \in \Delta$,

(ii) $\delta(te_i) = \eta_i(t)e_i$, $i = 1, 2, \ldots, m$, where η_i is a nonzero local ring derivation of $\mathbb{C}$,

(iii) $\delta\big(x - (e_1 + e_2 + \cdots + e_m)x\big) = 0$.

Next, we shall show that the set of minimal idempotents in $\mathcal{A}$ satisfying (i) and (ii) is finite. Assume on the contrary, that there are infinitely many idempotents e_i, $i \in \mathbb{N}$, satisfying (i) and (ii). As η_i, $i \in \mathbb{N}$, are nonzero additive mappings, they must be nonzero on an arbitrary neighborhood of zero. Thus, we can find a sequence $(t_n) \subset \mathbb{C}$ such that $|t_i| < 2^{-i}\|e_i\|^{-1}$ and $\eta_i(t_i) \neq 0$ for all $i \in \mathbb{N}$. Set $y = \sum_{i=1}^{\infty} t_i e_i \in \mathcal{A}$. Then we have

$$\hat{\delta}(\hat{y}) = \hat{\delta}(t_n\hat{e_n}) + \hat{\delta}\Big(\sum_{i\neq n} t_i\hat{e}_i\Big).$$

Using (13) we get $\operatorname{supp}\hat{\delta}(\sum_{i\neq n} t_i\hat{e}_i) \subset \overline{\{h_i : i \neq n\}}$. Obviously, $h_n \notin \overline{\{h_i : i \neq n\}}$, and consequently, by (13),

$$\hat{\delta}\Big(\sum_{i\neq n} t_i\hat{e}_i\Big)(h_n) = 0.$$

This yields

$$\hat{\delta}(\hat{y})(h_n) = \hat{\delta}(t_n\hat{e_n})(h_n) = \eta_n(t_n) \neq 0.$$

As a consequence we have $h_n \in \operatorname{supp}\hat{\delta}(\hat{y})$ for all positive integers n. This is a contradiction to the fact that $\operatorname{supp}\hat{\delta}(\hat{x})$ is a finite set for all $x \in \mathcal{A}$. It follows that there are only finitely many idempotents $\{e_1, e_2, \ldots, e_n\} \subset \mathcal{A}$ satisfying (i) and (ii).

In order to complete the proof we have to show that $\delta(x - \sum_{i=1}^{n} e_i x) = 0$ for every $x \in \mathcal{A}$. After an appropriate renumeration (iii) implies that

$$\delta(x) = \delta\Big(\sum_{i=1}^{m} e_i x\Big)$$

for some positive integer $m \leq n$. Thus, by (13),

$$\delta\Big(x - \sum_{i=1}^{n} e_i x\Big) = \delta\Big(-\sum_{i=m+1}^{n} e_i x\Big) = 0,$$

which completes the proof.

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